The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

An Impossibility Theorem for Gerrymandering

Boris Alexeev & Dustin G. Mixon

To cite this article: Boris Alexeev & Dustin G. Mixon (2018) An Impossibility Theorem for Gerrymandering, The American Mathematical Monthly, 125:10, 878-884, DOI: 10.1080/00029890.2018.1517571

To link to this article: https://doi.org/10.1080/00029890.2018.1517571

Published online: 01 Dec 2018.

An Impossibility Theorem for Gerrymandering

Boris Alexeev and Dustin G. Mixon

Abstract. In 2018, the U.S. Supreme Court considered a proposed mathematical formula to help detect unconstitutional partisan gerrymandering. We show that in some cases, this formula only flags bizarrely-shaped districts as potentially constitutional.

In 1812, the *Boston Gazette* published a political cartoon that likened the contorted shape of a Massachusetts state senate election district to the profile of a salamander [6]. The cartoon insinuated that Governor Elbridge Gerry approved this district's shape for his party's benefit, thereby coining the portmanteau "gerrymander." Ever since, it has been common practice to use geometry as a signal for gerrymandering, with the most egregious districts exhibiting bizarre-shapes; see [7, 12] for example. To help bring this geometric signal to gerrymandering court cases, a team of Boston-based mathematicians known as the Metric Geometry and Gerrymandering Group has been offering expert witness training in a sequence of Geometry of Redistricting workshops across the country [9].

Partisan gerrymandering was recently the subject of a U.S. Supreme Court case [5]. This case considered a completely different quantitative approach to detect gerrymandering. Instead of flagging districts with irregular shapes, the proposed method attempted to detect the intended consequence of partisan gerrymandering: one party wasting substantially more votes than the other party. This method summarizes the disproportion of wasted votes in a tidy statistic known as the efficiency gap.

Recently, Bernstein and Duchin [2] provided a helpful discussion of the efficiency gap, in which they mention that it sometimes incentivizes bizarrely-shaped districts. This is perhaps counterintuitive considering the geometry-infused history of gerrymandering. In this note, we demonstrate an extreme version of this observation:

Sometimes, a small efficiency gap is only possible with bizarrely-shaped districts.

Specifically, we show that every districting system must violate one of three well-established desiderata that we make explicit later: one person, one vote; Polsby–Popper compactness; and bounded efficiency gap. As such, our result is reminiscent of Arrow's impossibility theorem concerning ranked voting electoral systems [1]; while Arrow's theorem requires an analysis of discrete functions, ours involves planar geometry.

Definition 1. A **districting system** is a function that receives disjoint finite sets $A, B \subseteq [0, 1]^2$ and a positive integer k, and then outputs a partition $D_1 \sqcup \cdots \sqcup D_k = [0, 1]^2$.

Here, A and B correspond to voter locations from two major parties, respectively; we do not consider third-party voters. In practice, districts are drawn given the locations of the entire population from census data, but A and B can be estimated

doi.org/10.1080/00029890.2018.1517571
MSC: Primary 91D99
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uamm.

geo
Pmlus

using past election data; in particular, these estimates enable partisan gerrymandering. We focus on districting systems for the unit square largely for convenience, and without loss of generality. Indeed, one may partition any state with such a districting system by first inscribing the state in a square, and conversely, a districting system on any state determines a system for the square by inscribing a square in that state.

When evaluating a given districting system $f\colon (A, B, k) \mapsto \{D_i\}_{i=1}^k$, one may test for any number of desirable characteristics. What follows is a list of such characteristics. Note that while each of these has the form "there exist constant(s) such that the districts always satisfy some condition," the court may assign values for the constants (namely, δ, γ, α, and β), explicitly requiring the districts to satisfy (1)–(3) below with these parameters. (Here and throughout, we denote by $[k]$ the set $\{1, \dots, k\}$.)

(i) **One person, one vote.** There exists $\delta \in [0, 1)$ such that the districts always satisfy

δ always less than 1.

odd to tolerance?

$$(1-\delta)\left\lfloor \frac{|A \cup B|}{k} \right\rfloor \leq \left|(A \cup B) \cap D_i\right| \leq (1+\delta)\left\lceil \frac{|A \cup B|}{k} \right\rceil \qquad \forall i \in [k]. \qquad (1)$$

Higher δ : higher voter disproportion.

In words, the districts are drawn to contain roughly equal numbers of voters, with $\delta = 0$ forcing near-exact equality. Assuming equal voter turnout, this is equivalent to the districts containing roughly equally represented populations. The latter has been a guiding principle for all levels of redistricting in the United States following a series of U.S. Supreme Court decisions in the 1960s, namely *Gray v. Sanders*, *Reynolds v. Sims*, *Wesberry v. Sanders*, and *Avery v. Midland County* [15].

(ii) **Polsby–Popper compactness.** There exists $\gamma > 0$ such that the districts always satisfy

IF $\gamma \leq 0.02$ gerrymanderer!

$$\mathrm{PP}(D_i) := \frac{4\pi |D_i|}{|\partial D_i|^2} \geq \gamma \qquad \forall i \in [k]. \qquad (2)$$

Here, $|\partial D_i|$ denotes the perimeter of D_i, and $|D_i|$ denotes its area. The isoperimetric inequality [11] gives that $|\partial D_i|^2 \geq 4\pi |D_i|$, with equality precisely when D_i is a disk, and so we necessarily have $\gamma \leq 1$. In 1991, Polsby and Popper [13] introduced their score, which we denote by $\mathrm{PP}(\cdot)$ above, as a measure of geographic compactness. Their intent was to allow for an enforceable standard (e.g., no district shall score below 0.2 without additional scrutiny) that would "make the gerrymanderer's life a living hell." In this spirit, Arizona's redistricting commission in 2000 used the Polsby–Popper score to ensure geographic compactness among their voting districts [10]. The exceedingly long perimeters of the 1st and 12th congressional districts of North Carolina were cited in the recent U.S. Supreme Court case *Cooper v. Harris*, in which the Court ruled that both districts were the result of unconstitutional racial gerrymandering [3]. At the time, these were two of the three congressional districts across the country with the smallest Polsby–Popper scores [12].

(iii) **Bounded efficiency gap.** There exist $\alpha, \beta > 0$ such that the districts always satisfy

$$\left|\mathrm{EG}(D_1, \dots, D_k; A, B)\right| < \frac{1}{2} - \alpha \quad \text{when} \quad \big||A| - |B|\big| < \beta|A \cup B|. \qquad (3)$$

Here, the so-called **efficiency gap** $\mathrm{EG}(\cdot; A, B)$ quantifies the extent to which votes are disproportionately "wasted" by the districting. Suppose $|A \cap D_i| > |B \cap D_i|$. Then the number of wasted votes in $A \cap D_i$ is the excess

$$|A \cap D_i| - \left\lceil \frac{1}{2}\big|(A \cup B) \cap D_i\big| \right\rceil,$$

Why $\times \frac{1}{2}$

considering A did not need these votes to carry the district D_i. Meanwhile, all of the votes in $B \cap D_i$ were wasted, since they did not contribute to winning the district D_i. Letting $w_{A,i}$ and $w_{B,i}$ denote the wasted votes in $A \cap D_i$ and $B \cap D_i$, respectively, then the efficiency gap is defined by

$$\mathrm{EG}(D_1, \ldots, D_k; A, B) := \frac{1}{|A \cup B|} \sum_{i=1}^{k} \Big(w_{A,i} - w_{B,i}\Big).$$

Stephanopoulos and McGhee introduced the efficiency gap in [17], and it played a key role in the U.S. Supreme Court case *Gill v. Whitford* [5]. We note that the efficiency gap can range anywhere from 0 to 1/2, and Stephanopoulos and McGhee suggest that a gap of 8% or more (i.e., $|\,\mathrm{EG}\,| \geq 0.08$) is sufficient to flag potential partisan gerrymandering. Bernstein and Duchin [2] observe that such a bounded efficiency gap is only possible when neither A nor B make up more than 79% of the vote. In particular, there already exist A and B for which no choice of districts satisfies (3) with $\alpha = 0.50 - 0.08 = 0.42$ and $\beta > 0.79 - (1 - 0.79) = 0.58$. By comparison, our notion of "bounded efficiency gap" is particularly weak since we allow α and β to be arbitrarily small. In particular, we only ask that the efficiency gap be bounded away from 1/2 (by α) whenever A and B are sufficiently similar in size (controlled by β).

Observe that (i) and (ii) are agnostic to the voting preferences of A and B; in particular, (i) only depends on $A \cup B$, whereas (ii) only sees the resulting districts. Meanwhile, (iii) explicitly distinguishes between A and B. In practice, (iii) would be evaluated after election day, though it could be predicted with the help of past election data. We are now ready to state the main result:

Theorem 2. *There is no districting system that simultaneously satisfies all three desiderata above. In particular, for every $\underbrace{\delta, \gamma, \alpha, \beta, \text{and } k}$, there exist A and B such that every choice of districts $\{D_i\}_{i=1}^k$ violates one of (1), (2), and (3).*

Notably, our result does not require the districts to be contiguous. The idea behind the proof is straightforward: We consider homogeneous mixtures of voters in the unit square, where just over half of the voters belong to A, and just under half belong to B. In this extreme case, one would need to surgically design a district in order for B to be the majority while simultaneously being large enough to satisfy the left-hand inequality in (i). This surgery would in turn force the district to exhibit a bizarre shape, i.e., its Polsby–Popper score would be quite small (see Figure 1). As such, districts satisfying both (i) and (ii) are necessarily majority-A. All told, A wastes a tiny portion of its votes, whereas B wastes all of its votes, thereby violating (iii).

Example 3. The left panel of Figure 1 illustrates voter locations in $[0, 1]^2$. The blue and red dots correspond to A and B, respectively. This example was generated using a method outlined later in the proof of Theorem 2. Here, the unit square is partitioned strictly used to measure how homogeneous the population is.

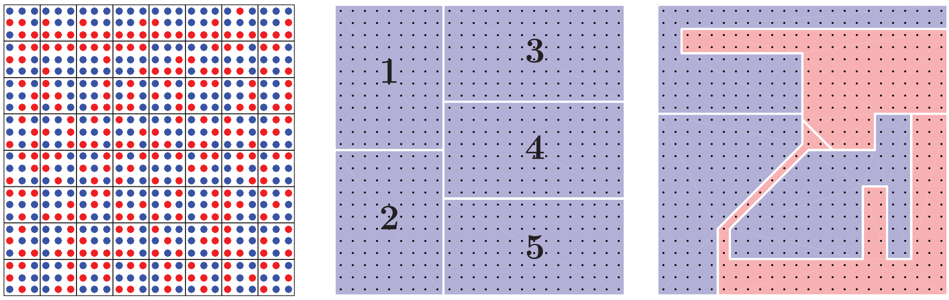

Figure 1. Instance of the impossibility formulated in Theorem 2; see Example 3 for details.

into smaller squares of side length $\epsilon = 1/8$. Each ϵ-square contains 9 voters in an $\ell \times \ell$ subset of a lattice with $\ell = 3$. Of these voters, $a = 5$ belong to A and $b = 4$ belong to B.

Next, the middle panel of Figure 1 draws five districts according to the shortest splitline algorithm proposed by the Center for Range Voting in [16]. This algorithm iteratively draws the shortest possible line segments that split a region into subregions while respecting the proportions of the population. The shortest splitline algorithm was specifically designed to produce districts that satisfy desiderata (i) and (ii). In particular, it ignores voter preference. In this case, the number of voters in each district is within $\delta = 0.07$ of the average and the smallest Polsby–Popper score $4\pi|D_i|/|\partial D_i|^2$ is over 0.70. (Recall that 1 is the largest score possible by the isoperimetric inequality [11].) However, despite B making up 44% of the vote, A won every district (illustrated by all districts being colored in blue). This is reflected in the efficiency gap being over 38% in favor of A. Indeed, one may verify the following figures:

i	$\lvert A \cap D_i \rvert$	$\lvert B \cap D_i \rvert$	$w_{A,i}$	$w_{B,i}$	$w_{A,i} - w_{B,i}$
1	60	48	6	48	−42
2	60	48	6	48	−42
3	68	52	8	52	−44
4	67	53	7	53	−46
5	65	55	5	55	−50

Then combining with $|A \cup B| = 576$ gives

$$\mathrm{EG}(D_1, D_2, D_3, D_4, D_5; A, B) = \frac{1}{|A \cup B|} \sum_{i=1}^{5} \left(w_{A,i} - w_{B,i} \right) = -\frac{224}{576}.$$

One may attempt to decrease the magnitude of the efficiency gap by exploiting clusters in B. In this spirit, we hand drew districts of similar size in the right panel of Figure 1. The result is five districts within $\delta = 0.04$ of the average and an efficiency gap of about 2% in favor of A. In exchange for this bounded efficiency gap, the smallest Polsby–Popper score is now about 0.12. For reference, Case [4] suggests flagging Polsby–Popper scores below 0.20 as instances of possible gerrymandering. Our main result (Theorem 2) establishes that this tradeoff is unavoidable with any districting system.

Before presenting the formal proof of Theorem 2, we take a moment to discuss the result. First, one could argue that our result does not necessarily preclude the real-world utility of (i), (ii), and (iii), as the A and B we construct are perhaps not likely to arise in practice. Indeed, our result does not suggest that impossibility arises from every possible distribution of voters. For example, if the western half of the state were all blue and the eastern half were all red, then it would be straightforward to draw nice-looking districts with a bounded efficiency gap. So now we have a spectrum of distribution possibilities (from purely homogeneous to completely separated), and the real-world partisan distributions reside somewhere along this spectrum. How does impossibility depend on this spectrum, and where does reality reside? These are both interesting questions that warrant further investigation.

Along these lines, we point out that—perhaps surprisingly—the issue is not necessarily resolved by the presence of partisan clusters. For instance, you can modify the example in Figure 1 by selecting a contiguous portion of the map and changing all the red votes in that region to blue. While this would result in a blue partisan cluster, the modification would not change the fact that all districts in the middle panel are majority blue, and so the efficiency gap, though smaller, would still be quite large. One could argue that the congressional districts in Massachusetts present a real-world example of this phenomenon, as all of the seats in this state are occupied by Democrats even though the districts do not exhibit bizarre shape.

While our impossibility result identifies a tension between Polsby–Popper compactness and bounded efficiency gap, we suspect there is a more general meta-theorem dictating a fundamental trade-off between geographic compactness and simple quantifications of partisan gerrymandering. For example, the A and B we construct demonstrate impossibility with any alternative to the efficiency gap that would disallow a slight majority winning every district. One such alternative is proportionality, which requires the number of seats won across the state by a given party to be roughly proportional to the number of votes cast for that party; impossibility here is perhaps a moot point since the U.S. Supreme Court already established in *Davis v. Bandemer* that proportionality is not a valid constitutional standard.

Many states have laws requiring voting districts to exhibit geographic compactness [**8**], but there is no standard approach to measure compactness. For example, as an alternative to Polsby–Popper, one could ask that a district's area be sufficiently large compared to either the smallest circle containing the district or the district's convex hull [**14**]. It appears that the techniques in our proof do not easily transfer to these alternatives. In particular, forcing the districts to be convex already presents what appears to be an interesting problem in additive combinatorics, which we leave for future work.

Proof of Theorem 2. Fix a positive integer k. For a large positive integer n, partition $[0, 1]^2$ into squares of edge length $\epsilon = 1/n$. Consider any positive integers a, b, and ℓ such that $a > b$ and $a + b = \ell^2$, and let L denote the lattice $(\frac{1}{n\ell}(\mathbb{Z} + \frac{1}{2}))^2$. Define A to have a voters in each ϵ-square intersect L, and define B to have b voters in each ϵ-square intersect L. See Figure 1 (left) for an illustration.

Now take a partition into districts $D_1 \sqcup \cdots \sqcup D_k = [0, 1]^2$ that satisfies (i) and (ii). Pick $i \in [k]$. The ϵ-squares that contain ∂D_i are in turn contained in an $\epsilon\sqrt{2}$-thickened version of ∂D_i, which has area at most $\sqrt{2}|\partial D_i|\epsilon + 2\pi\epsilon^2$. As such, ∂D_i is contained in at most $E := \sqrt{2}|\partial D_i|/\epsilon + 2\pi$ different ϵ-squares. Meanwhile, D_i contains at least $|D_i|/\epsilon^2 - E$ and at most $|D_i|/\epsilon^2$ different ϵ-squares. Overall, we may conclude that A wins the district D_i if the second inequality below holds:

$$|A \cap D_i| \geq a\left(\frac{|D_i|}{\epsilon^2} - E\right) \geq b\left(\frac{|D_i|}{\epsilon^2} + E\right) \geq |B \cap D_i|.$$

If n big enough
ε small
so we know

$\frac{b}{a}$ 1

want RHS to be here

we know A wins

Specifically, it suffices to have

$$\frac{b}{a} \leq \frac{|D_i| - \epsilon^2 E}{|D_i| + \epsilon^2 E},$$

which by (ii) is implied by

$$\frac{b}{a} \leq \frac{|\partial D_i|^2 - 4\pi\gamma^{-1}\epsilon^2 E}{|\partial D_i|^2 + 4\pi\gamma^{-1}\epsilon^2 E}. \tag{4}$$

small ε
we'll need population
A wins every dist

lim ≤ 0
ε→0

Next, (i) gives that

$$\big|(A \cup B) \cap D_i\big| \geq (1-\delta)\left\lfloor \frac{|A \cup B|}{k} \right\rfloor \geq (1-\delta)\frac{|A \cup B|}{2k} = (1-\delta)\frac{n^2\ell^2}{2k},$$

and since the points in $(A \cup B) \cap D_i$ lie in L, two of them must be of distance at least $\sqrt{(1-\delta)/(2k)}$ from each other. As such, $|\partial D_i| \geq \sqrt{(1-\delta)/(2k)} =: F$, and so (4) is implied by

effecty n.

How big has n to be to break impossibility

$\frac{b}{a}$ 1 sufficiently close to 1 or > $\frac{b}{a}$

$$\frac{b}{a} \leq \frac{F^2 - 4\pi\gamma^{-1}F\sqrt{2}\epsilon - 8\pi^2\gamma^{-1}\epsilon^2}{F^2 + 4\pi\gamma^{-1}F\sqrt{2}\epsilon + 8\pi^2\gamma^{-1}\epsilon^2}. \tag{5}$$

How many green squares we need to break impossibility

Means of estimating ε

establish fix values for blue/red

we want it to be > $\frac{b}{a}$ so A wins every district

we know less than 1

Function of $\epsilon = 1/n$

Observe that (5) is independent of our choice of i, and so (5) implies that A wins every district D_i.

we then know EG can't be satisfied

At this point, since n was chosen to be arbitrarily large, ϵ is arbitrarily small, and so we may pick a and b so that $c = 1 - b/a > 0$ is arbitrarily small while still satisfying (5). As such, B loses every district, thereby wasting all $bn^2 = (1-c)an^2$ of its votes, whereas A narrowly wins every district, thereby wasting at most $an^2 - bn^2 = can^2$ of its votes. Overall, the efficiency gap is

Conditions might exist to violate one of the three rules:
fix
k:
δ:
γ:
$\frac{b}{a}$:

$$\mathrm{EG}(D_1, \ldots, D_k; A, B) \leq \frac{can^2 - bn^2}{an^2 + bn^2} = \frac{2c-1}{2-c},$$

which is arbitrarily close to $-1/2$ despite

Q: How homogenous does the population distribution have to be so to guarantee A wins every district.

$$\frac{\big||A| - |B|\big|}{|A \cup B|} = \frac{an^2 - bn^2}{an^2 + bn^2} = \frac{c}{2-c}$$

being arbitrarily small. Therefore, every districting system that satisfies both (i) and (ii) necessarily violates (iii). ∎

sufficiently fn.

largest n for 5 to hold

How to present homogeneity?

ACKNOWLEDGMENTS. This work was conceived from an episode of the podcast More Perfect while DGM was visiting New York University to speak in the Math and Data Seminar. The authors thank Joey Iverson and John Jasper for enlightening conversations, as well as Mira Bernstein, Ben Blum-Smith, Moon Duchin, Vladimir Kogan, and the anonymous reviewers for constructive feedback that substantially improved the discussion of our results. DGM was partially supported by AFOSR F4FGA06060J007 and AFOSR Young Investigator Research Program award F4FGA06088J001. The views expressed in this article are those of the authors and do not reflect the official policy or position of the authors' employers, the United States Air Force, Department of Defense, or the U.S. Government.

Simulation parameters.
k = 6
δ = 0.05
γ = 0.2
$\frac{b}{a}$ = 0.8
set

When does (5) tells find ε s.t. A wins every district.
How big n has to be

n can be smaller
so voters don't have to be as mixed to suggest impossibility?

biggest n where minority can't win

[1] Arrow, K. J. (1950). A difficulty in the concept of social welfare. *J. Political Econ.* 58(4): 328–346. jstor.org/stable/1828886

[2] Bernstein, M., Duchin, M. (2017). A formula goes to court: Partisan gerrymandering and the efficiency gap. *Notices Amer. Math. Soc.* 64(9): 1020–1024.

[3] Blythe, A. (2017). U.S. Supreme Court agrees NC lawmakers created illegal congressional district maps in 2011. *Charlotte Observer*, May 22. charlotteobserver.com/article151912142.html

[4] Case, J. (2007). Flagrant gerrymandering: Help from the isoperimetric theorem? *SIAM News.* 40(9). archive.siam.org/pdf/news/1237.pdf

[5] Cohn, N., Bui, Q. (2017). How the new math of gerrymandering works. *New York Times*, October 3. nytimes.com/interactive/2017/10/03/upshot/how-the-new-math-of-gerrymandering-works-supreme-court.html

[6] Griffith, E. C. (1907). *The Rise and Development of the Gerrymander*. Chicago, IL: Scott, Foresman and Company.

[7] Ingraham, C. (2014). America's most gerrymandered congressional districts. *Washington Post*, May 15. washingtonpost.com/news/wonk/wp/2014/05/15/americas-most-gerrymandered-congressional-districts

[8] Levitt, J. (2018). Where are the lines drawn? redistricting.lls.edu/where-state.php

[9] Tufts University (2018). Metric Geometry and Gerrymandering Group. sites.tufts.edu/gerrymandr/project/

[10] Moncrief, G. F. (2011). *Reapportionment and Redistricting in the West*. Plymouth, UK: Lexington Books.

[11] Osserman, R. (1978). The isoperimetric inequality. *Bull. Amer. Math. Soc.* 84(6): 1182–1238. projecteuclid.org/euclid.bams/1183541466

[12] Pisanty, E. (2016). How can I use Mathematica to sort the US Congressional Districts by perimeter÷area. Mathematica Stack Exchange. mathematica.stackexchange.com/questions/119944

[13] Polsby, D. D., Popper, R. D. (1991). The third criterion: Compactness as a procedural safeguard against partisan gerrymandering. *Yale Law Policy Rev.* 9(2): 301–353.

[14] Redistricting the Nation (2009). Redrawing the map on redistricting 2010: A national study. Azavea. redistrictingthenation.com/whitepaper.aspx

[15] Smith, J. D. (2014). *On Democracy's Doorstep: The Inside Story of how the Supreme Court Brought "One Person, One Vote" to the United States*. New York, NY: Hill and Wang.

[16] Smith, W. D. (2018). Gerrymandering and a cure—shortest splitline algorithm. rangevoting.org/GerryExamples.html

[17] Stephanopoulos, N. O., McGhee, E. M. (2015). Partisan gerrymandering and the efficiency gap. *Univ. Chic. Law Rev.* 82(2): 831–900.

BORIS ALEXEEV received his Ph.D. in mathematics from Princeton University. He is interested in some mix of algebra, combinatorics, and complexity.
New York, NY 10011
boris.alexeev@gmail.com

DUSTIN G. MIXON received his Ph.D. in applied and computational mathematics from Princeton University. He served as an officer in the U.S. Air Force before joining the faculty at Ohio State in 2017.
Department of Mathematics, The Ohio State University, Columbus, OH 43210
mixon.23@osu.edu

[1] Arrow, K. J. (1950). A difficulty in the concept of social welfare. *J. Political Econ.* 58(4): 328–346. jstor.org/stable/1828886

[2] Bernstein, M., Duchin, M. (2017). A formula goes to court: Partisan gerrymandering and the efficiency gap. *Notices Amer. Math. Soc.* 64(9): 1020–1024.

[3] Blythe, A. (2017). U.S. Supreme Court agrees NC lawmakers created illegal congressional district maps in 2011. *Charlotte Observer*, May 22. charlotteobserver.com/article151912142.html

[4] Case, J. (2007). Flagrant gerrymandering: Help from the isoperimetric theorem? *SIAM News.* 40(9). archive.siam.org/pdf/news/1237.pdf

[5] Cohn, N., Bui, Q. (2017). How the new math of gerrymandering works. *New York Times*, October 3. nytimes.com/interactive/2017/10/03/upshot/how-the-new-math-of-gerrymandering-works-supreme-court.html

[6] Griffith, E. C. (1907). *The Rise and Development of the Gerrymander*. Chicago, IL: Scott, Foresman and Company.

[7] Ingraham, C. (2014). America's most gerrymandered congressional districts. *Washington Post*, May 15. washingtonpost.com/news/wonk/wp/2014/05/15/americas-most-gerrymandered-congressional-districts

[8] Levitt, J. (2018). Where are the lines drawn? redistricting.lls.edu/where-state.php

[9] Tufts University (2018). Metric Geometry and Gerrymandering Group. sites.tufts.edu/gerrymandr/project/

[10] Moncrief, G. F. (2011). *Reapportionment and Redistricting in the West*. Plymouth, UK: Lexington Books.

[11] Osserman, R. (1978). The isoperimetric inequality. *Bull. Amer. Math. Soc.* 84(6): 1182–1238. projecteuclid.org/euclid.bams/1183541466

[12] Pisanty, E. (2016). How can I use Mathematica to sort the US Congressional Districts by perimeter÷area. Mathematica Stack Exchange. mathematica.stackexchange.com/questions/119944

[13] Polsby, D. D., Popper, R. D. (1991). The third criterion: Compactness as a procedural safeguard against partisan gerrymandering. *Yale Law Policy Rev.* 9(2): 301–353.

[14] Redistricting the Nation (2009). Redrawing the map on redistricting 2010: A national study. Azavea. redistrictingthenation.com/whitepaper.aspx

[15] Smith, J. D. (2014). *On Democracy's Doorstep: The Inside Story of how the Supreme Court Brought "One Person, One Vote" to the United States*. New York, NY: Hill and Wang.

[16] Smith, W. D. (2018). Gerrymandering and a cure—shortest splitline algorithm. rangevoting.org/GerryExamples.html

[17] Stephanopoulos, N. O., McGhee, E. M. (2015). Partisan gerrymandering and the efficiency gap. *Univ. Chic. Law Rev.* 82(2): 831–900.

BORIS ALEXEEV received his Ph.D. in mathematics from Princeton University. He is interested in some mix of algebra, combinatorics, and complexity.
New York, NY 10011
boris.alexeev@gmail.com

DUSTIN G. MIXON received his Ph.D. in applied and computational mathematics from Princeton University. He served as an officer in the U.S. Air Force before joining the faculty at Ohio State in 2017.
Department of Mathematics, The Ohio State University, Columbus, OH 43210
mixon.23@osu.edu