GERRYMANDERING IMPOSSIBILITY

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1. Introduction

The practice of gerrymandering, or the manipulation of electoral district boundaries for political gain, has long been a contentious issue in the realm of democratic governance. It raises significant concerns about the fairness and integrity of electoral processes. The term "gerrymandering" itself dates back to 1812, named after Governor Elbridge Gerry's salamander-shaped electoral district in Massachusetts, highlighting the historical roots of this practice. Despite its longstanding presence in political strategy, only in recent years has there been a concerted effort to analyze and quantify gerrymandering using advanced computational methods and mathematical models *add ref.

The core challenge in combatting gerrymandering lies in the development of objective, quantifiable measures to evaluate the fairness of district boundaries. While the visual oddity of district shapes often triggers suspicions of gerrymandering, the underlying issue is more complex, involving the strategic distribution of voters to dilute or concentrate political power. This manuscript builds upon the foundational concepts laid out in "An Impossibility Theorem for Gerrymandering" by Alexeev and Mixon, refining the established formula to enhance the analysis of electoral districting. Rather than introducing a new method, we focus on the computational enhancement of existing models to simulate and scrutinize the effects of varied district configurations on election results.

Our research is driven by two primary objectives: firstly, to develop a sophisticated simulation model that can generate a range of districting scenarios based on specific parameters and constraints; and secondly, to apply this model in evaluating the implications of various districting strategies on the efficiency and fairness of the electoral process. This study is particularly timely and relevant, considering the increasing scrutiny of electoral fairness in democracies worldwide and the advancement of computational tools that can offer more transparent and objective analyses.

In this paper, we build upon the existing body of research in electoral studies and computational modeling, aiming to contribute a nuanced understanding of how district boundaries can influence electoral outcomes. Our approach combines mathematical rigor with practical applicability, offering insights that are valuable for policymakers, political analysts, and citizens alike.

Say something about redistricting in general terms; about the notion of gerrymandering and the ways it may be identified (geometry, efficiency gap).

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Date: December 2023.

Define "districting system" here in a way that matches Alexeev and Mixon's definition.

Setting aside all technical considerations for a moment, one might imagine that a "fair" districting system would yield approximately the same proportion of districts won by the minority party as the overall statewide proportion of minority votes to total votes. Accordingly, any districting system for which this is not the case might, on the surface, seem to have been subject to meddling, and thus be seen as an example of gerrymandering. Naively, this could prove a simple test for detecting gerrymandering.

However, in [Alexeev and Mixon], Alexeev and Mixon prove that with a sufficiently homogeneous distribution of voters split between two political parties, it is impossible for the minority party to win any districts provided the two most fundamental features of districts are preserved; namely, balanced populations and reasonable geometric compactness. The present paper refines the result of [Alexeev and Mixon], with a particular focus placed on the precise interplay between the level of homogeneity, the degree of population imbalance, and the allowable amount of non-compactness leading to impossibility.

2. Related Work

The study of gerrymandering and its quantification has evolved significantly, with various methodologies proposed and debated within the academic community. The literature on this subject is extensive, reflecting a range of approaches from mathematical models to computational simulations.

One foundational approach in the analysis of gerrymandering is the use of geometric and compactness measures. Polsby and Popper [1] proposed a compactness test, where districts are evaluated based on how closely they resemble a circle. This method, while intuitive, has its limitations, as it does not account for natural geographic features or community boundaries.

Moving beyond geometric analysis, the concept of the "efficiency gap" was introduced by Stephanopoulos and McGhee [2] as a means to quantify wasted votes and measure partisan advantage. This measure calculates the difference in the parties' respective wasted votes, divided by the total number of votes, providing a clear and straightforward metric. However, Bernstein and Duchin [3] noted that the efficiency gap could be misleading in certain contexts, particularly in non-competitive states or districts.

The application of computational methods in redistricting has opened new avenues for analysis. Cho and Liu [4] utilized computational algorithms to generate a multitude of possible district maps, assessing the extent of partisan bias in each. This method allows for a more exhaustive exploration of redistricting possibilities, although it requires significant computational resources.

Recent advancements have also been made in the use of machine learning and data analytics in electoral studies. For instance, Fifield et al. [5] developed an ensemble of districting plans to evaluate the representativeness of current legislative maps, employing machine learning algorithms to assess thousands of potential redistricting configurations.

In our study, we build on these existing methodologies, integrating geometric, statistical, and computational approaches. Our model aims to provide a more holistic and nuanced analysis of districting strategies, evaluating not only their geometric properties but also their impact on electoral fairness and representativeness.

3. Methodology

This study employs a multifaceted computational approach to simulate and analyze the effects of gerrymandering on electoral outcomes. The methodology is divided into several key components, each contributing to the comprehensive analysis of districting systems.

3.1. **Simulation Model Development.** The core of our analysis is a Python-based simulation model designed to generate electoral districts within a predefined area, akin to a state or municipality. The model operates on a grid system, where each grid point represents a voting unit, such as a neighborhood or a small town. Voter distribution across this grid is based on real demographic data, allowing for the simulation of various voting patterns.

3.2. **Parameter Definition.** Key parameters influencing district shapes and sizes include:

- **Delta** (δ) and **Gamma** (γ): These parameters define the constraints for district compactness and voter equality. Delta represents the allowable deviation in population size among districts, while Gamma sets the standard for geographic compactness.
- **Proportion of Voters** (p_0), **Grid Size** (n), and **District Count** (k): p_0 determines the ratio of voters from two major parties across the grid. The grid size (n) and the number of districts (k) are defined to simulate various districting scenarios.

3.3. **District Generation Algorithm.** The districting system is formulated as a function that partitions the grid into k districts based on the defined parameters. This function takes into account voter locations and aims to create districts that comply with the one-person-one-vote principle and Polsby-Popper compactness measure while maintaining a bounded efficiency gap.

3.4. **Efficiency Gap Calculation.** To assess the potential for partisan gerrymandering, we compute the efficiency gap for each districting plan. This involves determining the proportion of wasted votes for both parties across all districts. A small efficiency gap indicates a more equitable distribution of votes.

3.5. **Iterative Analysis and Optimization.** The model iteratively generates numerous districting plans, adjusting the centers of districts based on voter distributions and optimizing according to the efficiency gap and compactness measures. This iterative process allows us to explore a wide range of districting possibilities and their implications.

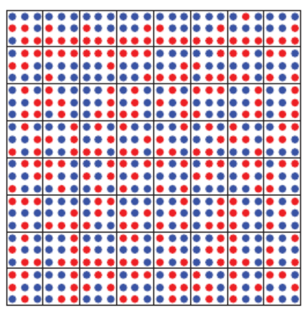
3.6. **Data Analysis and Visualization.** The output from the simulations is analyzed to identify patterns and anomalies in districting plans. Key metrics such as the number of districts won by each party, the efficiency gap, and compactness scores are visualized to provide an intuitive understanding of the results.

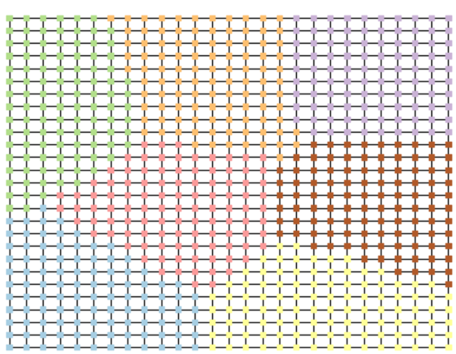
3.7. **Sensitivity Analysis.** We conduct sensitivity analyses to understand how variations in key parameters impact the districting outcomes. This analysis helps in identifying the robustness of our findings across different scenarios.

The methodology outlined here combines computational rigor with practical insights, making it a valuable tool for policymakers and researchers in understanding and addressing the challenges posed by gerrymandering.

4. Illustrative Examples

Following [Alexeev and Mixon], our geographic unit (think a state) will be a square grid with a distribution of voters at lattice points, such as in the following figure.





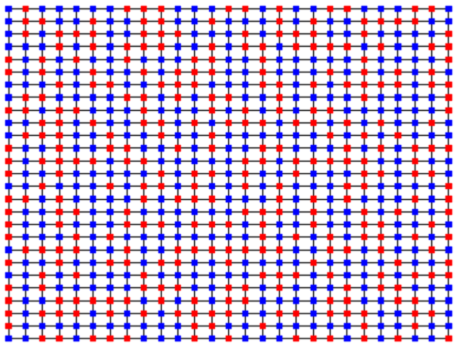


Table 1. Part 1 of Data from Excel File

votesA	votesB	area	perimeter	pp_compactness	eg	win_count_a	win_count_b	total_votes	sta
46	59	26.25	37	0.238539	0.248285	1	6	729	

Table 2. Part 2 of Data from Excel File

P_a	P_b	p_a	p_b	pOverP_a	pOverP_b	proportion_a	left_side	right_side	refine
0.444444	0.555556	0.142857	0.857143	3.111111	0.648148	0.555	0.8	-0.981923	

We define n to be the number of subsquares along each side of the region, and l the number of lattice points along each side of these subsquares. Accordingly, the region is divided into n^2 subregions, each of which contains l^2 voters, for a total of $(nl)^2$ voters in the state.

We further assume that each of the n^2 subsquares contains the same number of red and the same number of blue points (i.e., voters of each political party). This condition establishes the level of voter homogeneity within the state; small values of n permit more heterogeneous "clumping" of same-party voters; large values of n the opposite. Compare the two figures below – each contains the same number of voters, but with $n = 2$ in the former case, and $n = 8$ in the latter, there is a marked difference in the permissible levels of voter homogeneity in these examples.

NOTE: Include, say, 4 examples – each with 24^2 total voters (consistent with diagram above), but with $n = 2$, $l = 12$ for two of them (with one having very clumped coloration in each of the 4 subsquares) and the other $n = 8$, $l = 3$ (with maximum clumpiness in one).

In [Alexeev and Mixon], the level of homogeneity for which impossibility is guaranteed arises as a function of δ, γ, k, and b/a. In fact, they show that whenever

$$\frac{b}{a} \leq \frac{F^2 - 4\pi\gamma^{-1}F\sqrt{2}\epsilon - 8\pi^2\gamma^{-1}\epsilon^2}{F^2 + 4\pi\gamma^{-1}F\sqrt{2}\epsilon + 8\pi^2\gamma^{-1}\epsilon^2},$$

the majority party will win all k districts. This may be rearranged to obtain a quadratic expression in the variable ϵ $(= 1/n)$. There is one positive and one negative root of this expression, from which one can readily deduce that all positive values of ϵ less than or equal to the positive root will result in a situation in which the majority party wins all districts. Of course, this can then be translated into

δ	γ	k	b	a	n
0.05	0.2	2	4	5	1651
0.05	0.2	2	24	25	8944
0.05	0.2	5	4	5	2609
0.05	0.2	5	24	25	14141
0.05	0.2	10	4	5	3690
0.05	0.2	10	24	25	19998

TABLE 3. Caption

a conclusion about those values of n for which the same conclusion holds. Since n establishes the degree of heterogeneity that is possible in a given situation, this result of Alexeev and Mixon establishes a threshold level of homogeneity at which it becomes impossible for the minority party to win a single district given constraints we might choose to place on δ and γ, and the statewide ratio of voters for parties A and B. The table below provides these threshold values for certain combinations of δ, γ, a, and b.

These values for n are quite large, and signify very high levels of homogeneity necessary for impossibility. In fact, even the most modest of these would mean a state divided into 1651^2 subunits all of which must have the same ratio of voters for each party. [Check to determine whether this value exceeds the number of census blocks in any state in the US.]

In the present paper we prove a much more refined impossibility theorem, one which illustrates the level of homogeneity necessary for impossibility is dramatically lower than in Alexeev and Mixon. We then demonstrate, through computational results, that our thresholds are close to (perhaps even actually) best possible. We end with some practical implications for the matter of drawing actual district boundaries.

NOTE: Somewhere here or above, perhaps show some examples of districts that have different values for γ, perhaps ranging from 0.05 up through 0.8 or so.

5. Refined Impossibility Theorem

NOTE: Run computational tests on much lower levels of homogeneity than those from Alexeev and Mixon. Note the fraction that allow at least one district win for the minority party.

Theorem 1. *Suppose A and B represent the number of statewide voters for two political parties with $A > B$. Let n be such that*

$$\frac{B}{A} \leq 1 - \frac{\pi\sqrt{k+1}}{2\gamma n^2\sqrt{1-\delta}}.$$

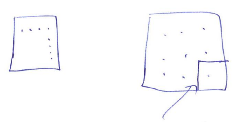
If the region of the state can be divided into an $n \times n$ grid for which each subregion contains A/n^2 and B/n^2 voters, then it is impossible for B to win any districts.

NOTE: Technically $(A+B)/n^2$ needs to be a perfect square for the proof details to work. See if there's a way to avoid this assumption.

Proof. NOTE: The initial part is verbatim from Alexeev and Mixon – I'll rewrite later.

Fix a positive integer k. For a large positive integer n, partition $[0,1]^2$ into squares of edge length $\epsilon = 1/n$. Consider any positive integers a, b, and l such that $a > b$ and $a + b = l^2$, and let L denote the lattice $(\frac{1}{nl}(\mathbb{Z} + \frac{1}{2}))^2$. Define A to have a voters in each ϵ-square intersect L, and define B to have b voters in each ϵ-square intersect L.

Now take a partition into districts $D_1 \sqcup \cdots \sqcup D_k = [0,1]^2$ that satisfies one-person/one-vote and Polsby-Popper. Pick $i \in [k]$. Consider an ϵ-square that contains a portion of the boundary ∂D_i of D_i.



If part of ∂D_i, then the amount of ∂D_i contained in a given ε-square is at least $\frac{\varepsilon}{l}$.

If a portion of ∂D_i lies within an ϵ-square, then we can, without a loss of generality, assume it lies along a grid as illustrated in the above. Given that, the portion of ∂D_i that lies in any ϵ-square is at least of length $2\epsilon/l$ since each ϵ-square has side length ϵ and the number of subsegments is equal to l. This means ∂D_i is contained in at most

$$E = \frac{|\partial D_i|}{2\epsilon/l} = \frac{|\partial D_i| l}{2\epsilon}$$

ϵ-squares.

Meanwhile, D_i contains at least

$$\frac{|D_i|}{\epsilon^2} - E$$

and at most $|D_i|/\epsilon^2$ ϵ-squares. Since $|A \cap D_i|$ and $|B \cap D_i|$ represent the number of votes for A and B, respectively, in D_i, it follows that

$$|A \cap D_i| \geq a \left(\frac{|D_i|}{\epsilon^2} - E \right),$$

and

$$b \left(\frac{|D_i|}{\epsilon^2} \right) \geq |B \cap D_i|.$$

If we can show that

$$a\left(\frac{|D_i|}{\epsilon^2} - E\right) \geq b\left(\frac{|D_i|}{\epsilon^2}\right),$$

then we will be able to conclude that A wins every district over B.

To investigate the circumstances that would allow us to reach this conclusion, we rearrange the above to obtain $a(|D_i| - \epsilon^2 E) \geq b|D_i|$, which in turn can be written as

$$\frac{|D_i| - \epsilon^2 E}{|D_i|} \geq \frac{b}{a},$$

or

$$\frac{b}{a} \leq 1 - \frac{\epsilon^2 E}{|D_i}.$$

Using E as defined above,

$$\frac{b}{a} \leq 1 - \frac{\epsilon^2 |\partial D_i| l}{2\epsilon |D_i|}.$$

The Polsby-Popper condition is that

$$\gamma \leq \frac{4\pi |D_i|}{|\partial D_i|^2},$$

and therefore

$$\frac{|\partial D_i|}{|D_i|} \leq \frac{4\pi}{\gamma |\partial D_i|},$$

which leads to

$$1 - \frac{\epsilon^2 |\partial D_i| l}{2\epsilon |D_i|} = 1 - \frac{\epsilon l}{2} \frac{|\partial D_i|}{|D_i|} \geq 1 - \frac{\epsilon l}{2} \frac{4\pi}{\gamma |\partial D_i|} = 1 - \frac{2\epsilon l \pi}{\gamma |\partial D_i|}.$$

Thus, the "A wins every district" conclusion is derivable from the inequality

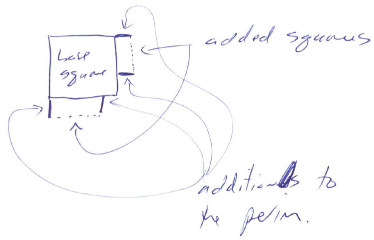
$$\frac{b}{a} \leq 1 - \frac{2\epsilon l \pi}{\gamma |\partial D_i|}.$$

But, as it stands, this inequality still depends on i. To remove all i dependence (and thus reach a conclusion which applies to all districts), we seek some expression X such that

$$X \leq 1 - \frac{2\epsilon l \pi}{\gamma |\partial D_i|}.$$

This can be rearranged to lead to an expression of the form $Y \leq |\partial D_i|$. In other words, if we can find an i-independent lower bound on the length of the perimeter of the districts, then we will have a suitable point from which to proceed.

Since the boundaries of D_i can be taken to lie in a grid, a lower bound for $|\partial D_i|$ comes by assuming D_i is a square.



Note that however many sub-squares are left over from the square of side length $\lfloor\sqrt{|D_i|}\rfloor$, these can be added to the edges of the "base" square (as in the figure above). The total extra contribution to the perimeter is no more than 4. Thus, we can conclude

$$|\partial D_i| \geq 4\lfloor\sqrt{|D_i|}\rfloor + 4 = 4\left(\lfloor\sqrt{|D_i|}\rfloor + 1\right) \geq 4\sqrt{|D_i|}.$$

If we now use the one-person/one-vote condition (with a tolerance of δ), we conclude that

$$|(A\cup B)\cap D_i| \geq (1-\delta)\left\lfloor\frac{|A\cup B|}{k}\right\rfloor \geq (1-\delta)\frac{|A\cup B|}{k+1},$$

(check the last of these inequalities...if nothing else $k+1$ replaced by $2k$ will certainly work). Because $A\cup B$ contains n^2l^2 voters, we have

$$|\partial D_i| \geq 4\sqrt{\frac{1-\delta}{k+1}}nl.$$

From this inequality we get

$$1 - \frac{2\epsilon l\pi}{\gamma|\partial D_i|} \geq 1 - \frac{2\epsilon l\pi}{4\gamma\sqrt{\frac{1-\delta}{k+1}}nl}.$$

Thus if we can show

$$\frac{b}{a} \leq 1 - \frac{2\epsilon l\pi}{4\gamma\sqrt{\frac{1-\delta}{k+1}}nl},$$

we will have concluded that A wins all districts.

$\square$

δ	γ	k	b	a	n
0.05	0.2	2	4	5	9
0.05	0.2	2	24	25	19
0.05	0.2	5	4	5	10
0.05	0.2	5	24	25	23
0.05	0.2	10	4	5	12
0.05	0.2	10	24	25	26

Table 4. Caption

6. Computational Results

Using the inequality above to obtain an expression for threshold values of n at which impossibility occurs leads to the table below – which provides a notable contrast to the values in the table presented earlier.

These results make clear that the level of homogeneity necessary for impossibility is substantially less than implied by the result of Alexeev and Mixon. We now show that the bound obtained in the theorem above is close to optimal (or perhaps we can demonstrate it is actually optimal?) through examples generated computationally.

NOTE: Using the first row above to illustrate how to approach the computational step, run simulations with $k = 2$, $n = 9$, $b = 4$, and $a = 5$. For each, keep only those with $delta \leq 0.05$ and $\gamma \geq 0.2$. Record all such instances – both in terms of whether minority party wins a district and, if so, the resulting map. Can any be found?

7. Practical Implications

Say something about possible situations in which this may be implementable.

8. Future Work

Among other things, perhaps questions related to what happens if a different measure of compactness is used, such as Reock, etc.

References

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