AnalyticalProblems

July 3, 2024

0.1 Question 1

Part I: Analytical problems

1.

In a two-class one-dimensional problem, the pdfs are the Gaussians $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(1, \sigma^2)$ for the two classes, respectively. Show that the threshold x_0 minimizing the average risk is equal to

$$x_0 = 1/2 - \sigma^2 \ln \frac{\lambda_{21} P(\omega_2)}{\lambda_{12} P(\omega_1)}$$

where $\lambda_{11} = \lambda_{22} = 0$ has been assumed.

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$$1.\ P(x|w_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$P(x|w_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-1)^2}{2\sigma^2}\right) \qquad \lambda = \begin{pmatrix} 0 & \lambda_{12} \\ \lambda_{21} & 0 \end{pmatrix}$$

$$\lambda_{21} P(x|w_2) P(w_2) = \lambda_{12} P(x|w_1) P(w_1) \rightarrow$$

$$\ln \lambda_{21} P(w_2) - \frac{1}{2} \ln 2\pi\sigma^2 - \frac{(x-1)^2}{2\sigma^2} =$$

$$\ln \lambda_{12} P(w_1) - \frac{1}{2} \ln 2\pi\sigma^2 - \frac{x^2}{2\sigma^2} \rightarrow$$

$$\frac{-x^2+2x-1}{2\sigma^2} + \frac{x^2}{2\sigma^2} = \ln \frac{\lambda_{12} P(w_1)}{\lambda_{21} P(w_2)} \rightarrow$$

$$\frac{2x-1}{2} = -\sigma^2 \ln \frac{\lambda_{21} P(w_2)}{\lambda_{12} P(w_1)} \rightarrow x = \frac{1}{2} - \sigma \ln \frac{\lambda_{21} P(w_2)}{\lambda_{12} P(w_1)}$$

0.2 Question 2

2.

Prove that the covariance estimate

$$\hat{\Sigma} = \frac{1}{N-1}\sum_{k=1}^{N}(\boldsymbol{x}_k - \hat{\boldsymbol{\mu}})(\boldsymbol{x}_k - \hat{\boldsymbol{\mu}})^T$$

is an unbiased one, where

$$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{k=1}^{N}\boldsymbol{x}_k$$

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2. $\hat{\Sigma} = \frac{1}{N-1}\sum_{K=1}^{N}(x_K-\hat{\mu})(x_K-\hat{\mu})^T$ $\quad \hat{\mu} = \frac{1}{N}\sum_{K=1}^{N} x_K$

$\forall_{1,j\le l}$ $\hat{\sigma}_{ij} = \frac{1}{N-1}\sum_{K=1}^{N}(x_K-\hat{\mu}_i)(x_K-\hat{\mu}_j) =$

$\frac{1}{N-1}\sum_{K=1}^{N} x_K^2 - x_K\hat{\mu}_i - x_K\hat{\mu}_j + \hat{\mu}_i\hat{\mu}_j =$

$\frac{1}{N-1}\left(\sum_{K=1}^{N}(x_K)^2 - N\hat{\mu}_i\hat{\mu}_j - N\hat{\mu}_i\hat{\mu}_j + N\hat{\mu}_i\hat{\mu}_j\right) \to$

$E(\sigma_{ij}) = \frac{1}{N-1}\left(\sum_{K=1}^{N} E(x_K^2) - N.E(\hat{\mu}_i\hat{\mu}_j)\right) =$

$\frac{1}{N-1} = N(\sigma_{ij}^2+\mu^2) - N\left(\frac{\sigma_{ij}^2}{N}+\mu^2\right) = \frac{N-1}{N-1}\sigma_{ij}^2 = \sigma_{ij}^2$

note that $E(X^2) = Var(X) + E(X)^2$

0.3 Question 3

3.

Show that for the lognormal distribution

$$p(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \theta)^2}{2\sigma^2}\right), \quad x > 0$$

the ML estimate is given by

$$\hat{\theta}_{ML} = \frac{1}{N}\sum_{k=1}^{N} \ln x_k$$

3- $P(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \theta)^2}{2\sigma^2}\right) \quad x > 0$

$P(x;\theta) = \prod^{N} P(x) \to L(x,\theta) = \sum \ln P(x) =$

$\sum_{K=1}^{N} -\ln\sigma - \ln x - \frac{1}{2}\ln 2\pi - \frac{(\ln x_K - \theta)^2}{2\sigma^2} \to$

$\frac{\partial L}{\partial\theta} = \sum_{K=1}^{N} \frac{2(\ln x_K - \theta)}{2\sigma^2} = 0 \to \frac{1}{\sigma^2}\sum_{K=1}^{N} \ln x_K - \theta = 0 \to$

$\sum_{K=1}^{N} \ln x_K - N\theta = 0 \to \hat{\theta} = \frac{1}{N}\sum_{K=1}^{N} \ln x_K$

0.4 Question 4

4.

Consider Cauchy distributions in a two-class one-dimensional classification problem

$$f(x\,|\,\omega_i) = \frac{1}{\pi b}\cdot\frac{1}{1+(\frac{x-a_i}{b})^2} \qquad i = 1,2 \qquad a_2 > a_1$$

(a) By explicit integration, check that the distributions are indeed normalized.

(b) Assuming $P(\omega_1) = P(\omega_2)$, show that $P(\omega_1\,|\,x) = P(\omega_2\,|\,x)$ if $x = (a_1+a_2)/2$. Plot $P(\omega_1\,|\,x)$ for the case $a_1 = 3$, $a_2 = 5$ and $b = 1$. How do $P(\omega_1\,|\,x)$ and $P(\omega_2\,|\,x)$ behave as $x \to -\infty$? $x \to +\infty$? Explain.

(c) Show that the minimum probability of error is given by

$$P(\text{error}) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left|\frac{a_2 - a_1}{2b}\right|$$

Plot this as a function of $|a_2 - a_1|/b$.

(d) What is the maximum value of P(error) and under which conditions can this occur? Explain.

(e) Design the Bayes minimum error classifier in terms of a_i and b if $P(\omega_1) = P(\omega_2)$. Show the decision boundaries in this case. What is the probability of error?

(f) Design the Bayes minimum risk classifier with the following error weights

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Show the decision boundaries in this case. What is the probability of error? Compare the results in (e) and (f).

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4- $f(x|w_i) = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{x-a_i}{b}\right)^2}$ $\quad a_2 > a_1$

a) $\int_{-\infty}^{+\infty} f(x|w_i)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{x-a_i}{b}\right)^2} dx$

$\left\{ \frac{x-a_i}{b} = x' \rightarrow dx = b\,dx' \right\}$

$= \int_{-\infty}^{+\infty} \frac{1}{\pi b} \times \frac{1}{1+x'^2} b\,dx' = \frac{1}{\pi} \int_{-\infty} \frac{1}{1+x'^2} dx' = \frac{1}{\pi} (\arctan(x)]^{+\infty} -$

$\arctan(x)]^{-\infty}) = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$

b) $x = \frac{a_1+a_2}{2} \rightarrow P(x|w_1) = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{\frac{a_1+a_2}{2} - a_1}{b}\right)^2} =$

$\frac{1}{\pi b} \times \frac{1}{1+\left(\frac{a_2-a_1}{b}\right)^2}$

$P(x|w_2) = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{\frac{a_1+a_2}{2} - a_2}{b}\right)^2} = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{a_1-a_2}{b}\right)^2}$

$\rightarrow P(x|w_1) = P(x|w_2)$, $P(w_1) = P(w_2) \rightarrow P(w_2|x) = P(w_1|x)$

$P(w_1|x) = \frac{P(x|w_1)P(w_1)}{P(x)} = \frac{\frac{1}{2}P(x|w_1)}{\frac{1}{2}P(x|w_1) + \frac{1}{2}P(x|w_2)} =$

$\frac{\frac{1}{\pi} \cdot \frac{1}{1+(x-3)^2}}{\frac{1}{\pi} \cdot \frac{1}{1+(x-3)^2} + \frac{1}{\pi} \cdot \frac{1}{1+(x-5)^2}} = \frac{x^2-10x+26}{2x^2-16x+36} \rightarrow P(w_2|x) = \frac{x^2-6x+10}{2x^2-16x+36}$

$\lim_{x \to +\infty} P(w_1|x) = \frac{1}{2}$ $\qquad \lim_{x \to -\infty} P(w_1|x) = \frac{1}{2}$

$\lim_{x \to +\infty} P(w_2|x) = \frac{1}{2}$ $\qquad \lim_{x \to -\infty} P(w_2|x) = \frac{1}{2}$

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c) $x_0 \to P(x_0|w_1)=P(x_0|w_2) \to \frac{1}{\pi b}\cdot\frac{1}{1+(\frac{x_0-a_1}{b})^2} = \frac{1}{\pi b}\cdot\frac{1}{1+(\frac{x_0-a_2}{b})^2}$

$\to (\frac{x_0-a_1}{b})^2 = (\frac{x_0-a_2}{b})^2 \to x_0-a_1=\pm(x_0-a_2) \to x_0=\frac{a_1+a_2}{2}$

$P_e=\int_{-\infty}^{x_0} P(x|w_1)P(w_1)\,dx+\int_{x_0}^{+\infty}P(x|w_2)P(w_2)=$

$P(w_1)\int_{-\infty}^{\frac{a_1+a_2}{2}} \frac{1}{\pi b}\cdot\frac{1}{1+(\frac{x-a_1}{b})^2}\,dx+\int_{\frac{a_1+a_2}{2}}^{+\infty}\frac{1}{\pi b}\cdot\frac{1}{1+(\frac{x-a_2}{b})^2}\,dx\,P(w_2)=$

$P(w_1)\left[\frac{1}{\pi}\arctan(\frac{a_2-a_1}{2b})-\frac{1}{\pi}\times(\frac{-\pi}{2})\right]+P(w_2)\left[\frac{1}{\pi}\times\frac{\pi}{2}-\right.$

$\left.\frac{1}{\pi}\arctan(\frac{a_1-a_2}{2b})\right] = P(w_1)\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(\frac{a_2-a_1}{2b})\right]+$

$(1-P(w_1))\left[\frac{1}{2}-\frac{1}{\pi}\tan^{-1}(\frac{a_1-a_2}{2b})\right]=\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$

d) $P(error)=\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$

$a_2=a_1 \to \min(P(error))=\frac{1}{2}-0=\frac{1}{2}$

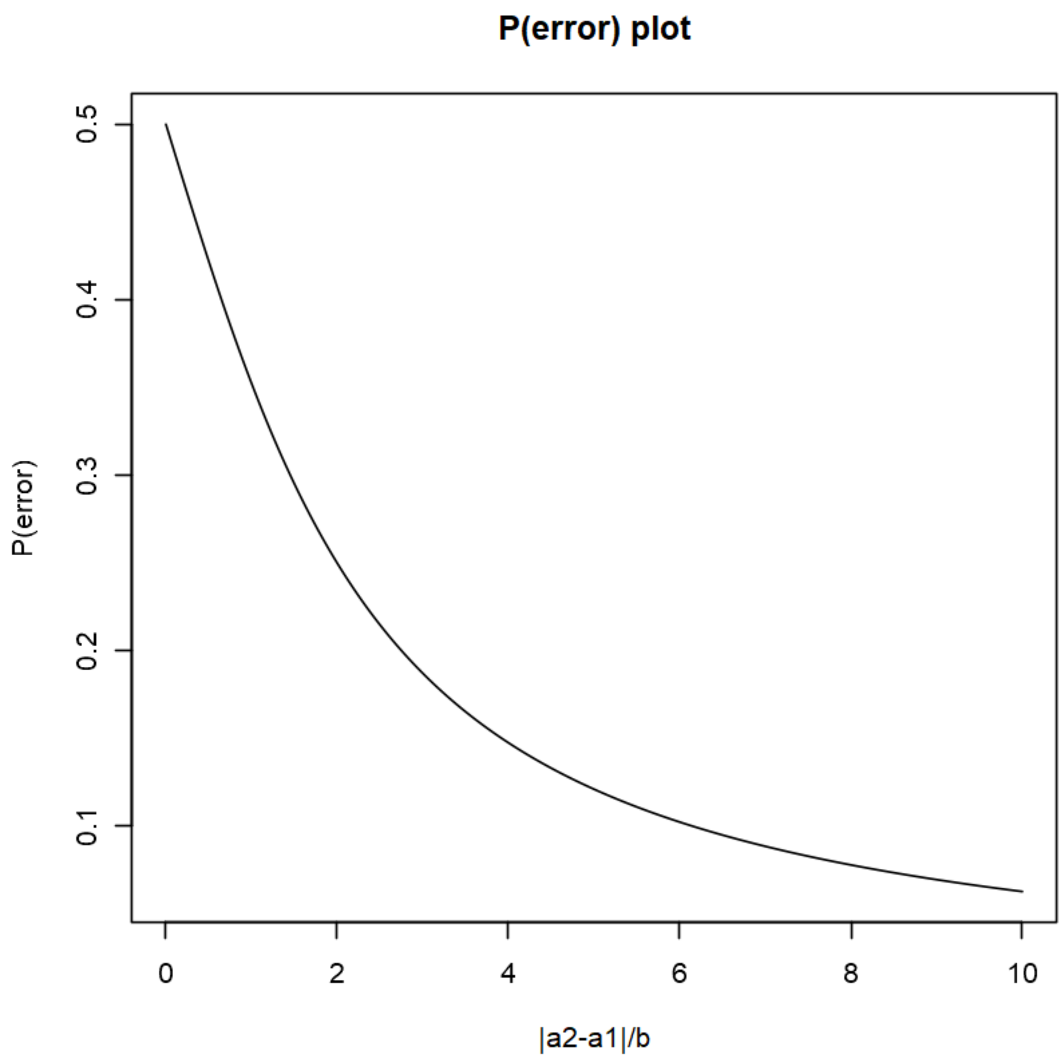
e) $P_e=P(w_2)\int_{-\infty}^{x_0}P(x|w_2)\,dx+P(w_1)\int_{x_0}^{+\infty}P(x|w_1)\,dx$

F) $P_e=\lambda_{21}P(w_2)\int_{-\infty}^{x_0}P(x|w_2)\,dx+\lambda_{12}P(w_1)\int_{x_0}^{+\infty}P(x|w_1)\,dx=$

$2P(w_2)\int_{-\infty}^{x_0}P(x|w_2)\,dx+P(w_1)\int_{x_0}^{+\infty}P(x|w_1)\,dx$

[8]:
```
x <- seq(0, 10, length.out = 1000)
y <- 0.5 - (1/pi) * atan(x/2)

plot(x, y, type = "l", main = "P(error) plot",
     xlab = "|a2-a1|/b", ylab = "P(error)")
```


P(error) plot
P(error)
0.5
0.4
0.3
0.2
0.1
0
2
4
6
8
10
|a2-a1|/b

0.5 Question 5

5. We have a SVM to be trained on a set of inputs X_n, where $n = 1, \ldots, N$, together with a corresponding set of target values ($t_n = +1$ or -1). The goal is to maximize the margin and at the same time allows some small misclassifications. An example of classify as follows:

$$y(\mathbf{x}_n) = \mathbf{w}^T \phi(\mathbf{x}_n) + b; \qquad \hat{t}_n = \begin{cases} +1, & \text{if } y(\mathbf{x}_n) \geq 0 \\ -1, & \text{otherwise} \end{cases}$$

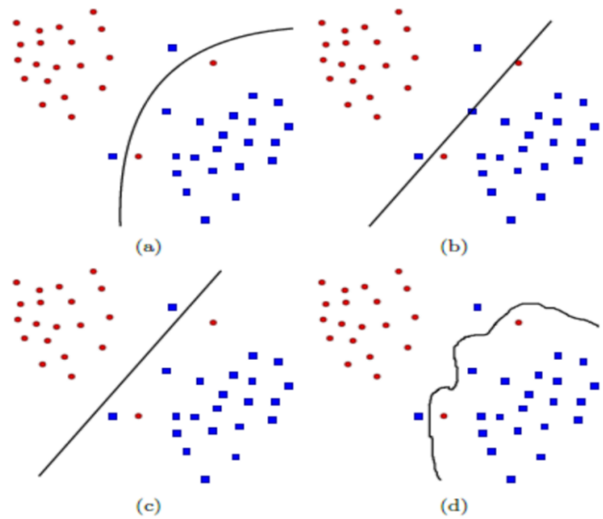
We therefore minimize:

$$\begin{aligned} & C \sum_{n=1}^{N} \xi_n + \frac{1}{2}\|\mathbf{w}\|^2 & \\ \text{s.t.} \quad & t_n y(\mathbf{x}_n) \geq 1 - \xi_n, & n = 1, \ldots, N \\ & \xi_n \geq 0, & n = 1, \ldots, N \end{aligned}$$

Where $C > 0$ is a parameter that controls the trade-off between the stack variable penalty and the margin. Figure below illustrates the decision boundaries for four different SVMs

using different values of C and different kernels. For each, specify which setting was used to get the result.

(Note: You should explain why you have chosen each specific choice for the corresponding figures)



Decision boundaries corresponding to the four different SVMs.

1. $C = 1$ and no kernel is used,
2. $C = 0.1$ and no kernel is used,
3. $C = 0.1$ and the kernel is $\mathbf{K}(\mathbf{x}_i, \mathbf{x}_j) = \exp -\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{2}$.
4. $C = 0.1$ and the kernel is $\mathbf{K}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^T \mathbf{x}_j + (\mathbf{x}_i^T \mathbf{x}_j)^2$

1.c The decision boundary is a straight line with minimal margin, focusing on classifying all training points correctly

2.b The decision boundary is still a straight line but allows some misclassifications to achieve a larger margin

3.a The decision boundary is non-linear and flexible, accommodating the data points more smoothly.

The margin is wider, allowing some misclassifications

4.d The decision boundary is more complex, capturing quadratic relationships between the features. It also allows some misclassifications for a larger margin