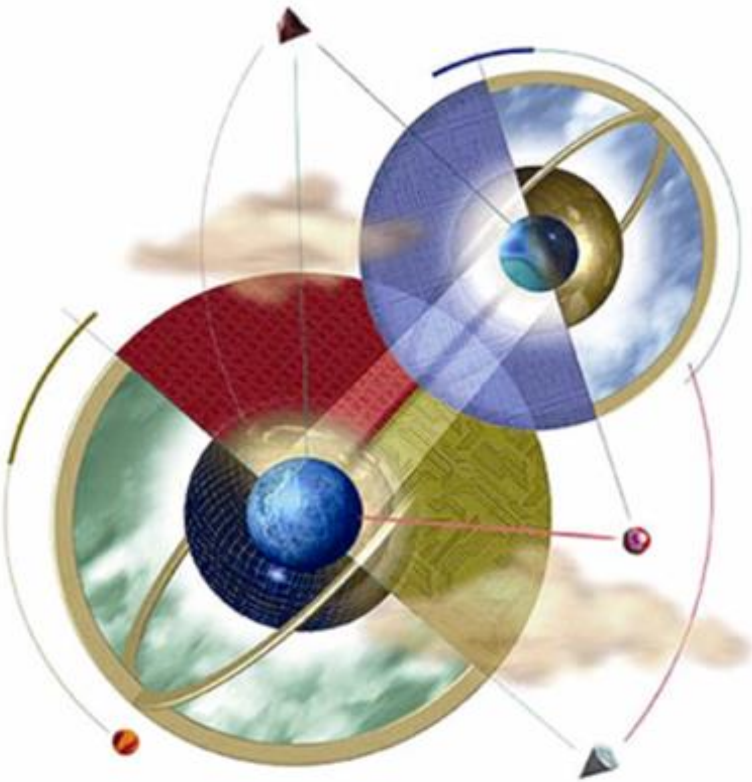
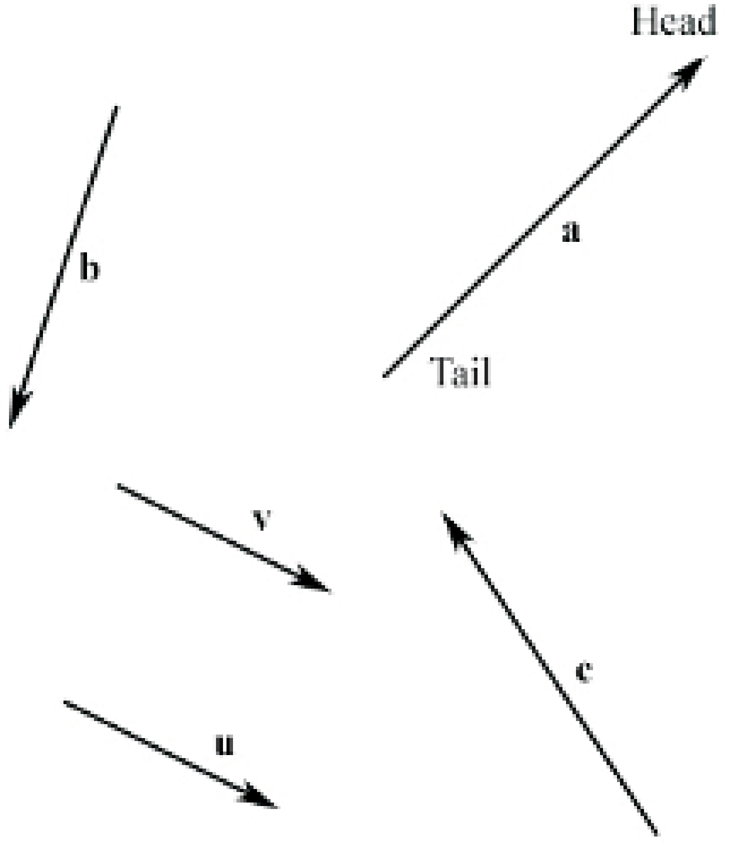


Mathematical Prerequisites

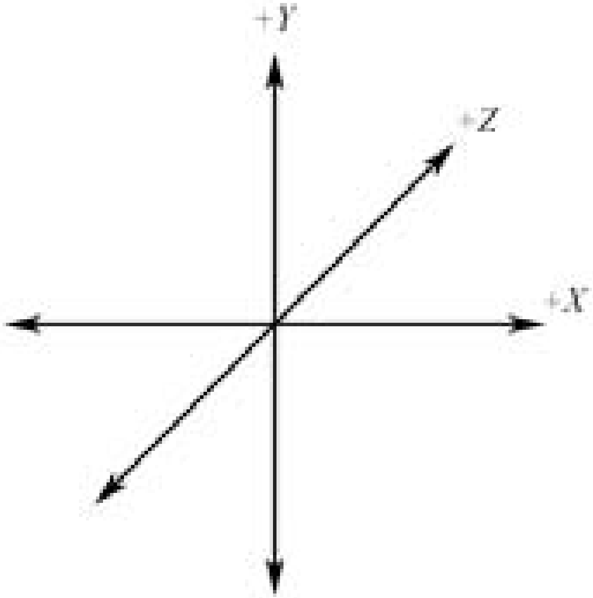
Objectives

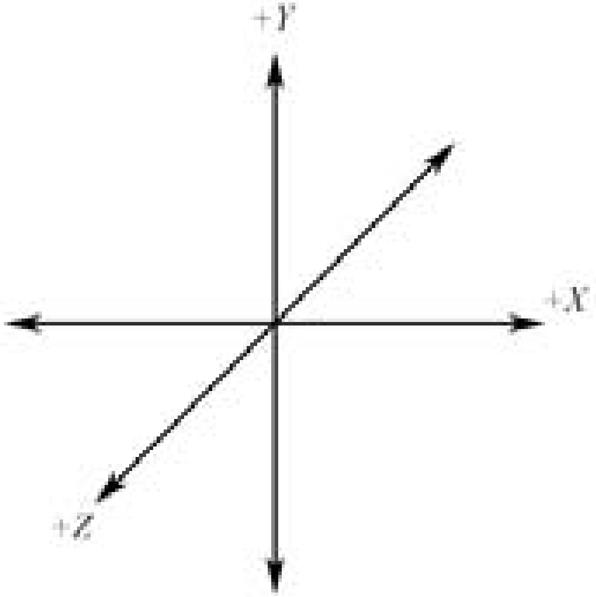
- To learn the geometry and algebra of vectors and their applications to 3D computer graphics
- To learn about matrices, their algebra, and how we use them to transform 3D geometry
- To learn how to model planes and rays algebraically and their applications to 3D graphics
- To become familiar with a subset of the classes and functions provided by the D3DX library that are used for 3D math operations

Vectors in 3-Space

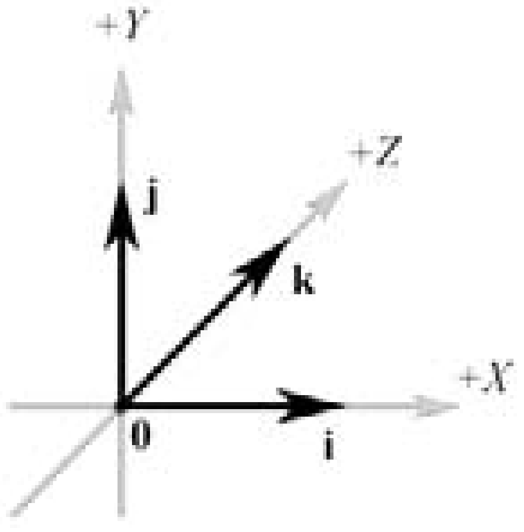


Coordinate system





Special 3D vectors



Implementation Class

- D3DXVECTOR3
- This class inherits its component data from D3DVECTOR, which is defined as:

```
typedef struct  D3DVECTOR {
    float x;
    float y;
    float z;
} D3DVECTOR;
```

Vector Equality

$$(u_x, u_y, u_z) = (v_x, v_y, v_z)$$

- If

$$u_x = v_x, u_y = v_y, \text{ and } u_z = v_z$$

Magnitude of a Vector

$$\|\mathbf{u}\| = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2}$$

- Example: Find the magnitude of the vectors **u = (1, 2, 3) and v = (1, 1).**
- Solution:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Normalizing a Vector

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \left(\frac{u_x}{\|\mathbf{u}\|}, \quad \frac{u_y}{\|\mathbf{u}\|}, \quad \frac{u_z}{\|\mathbf{u}\|} \right)$$

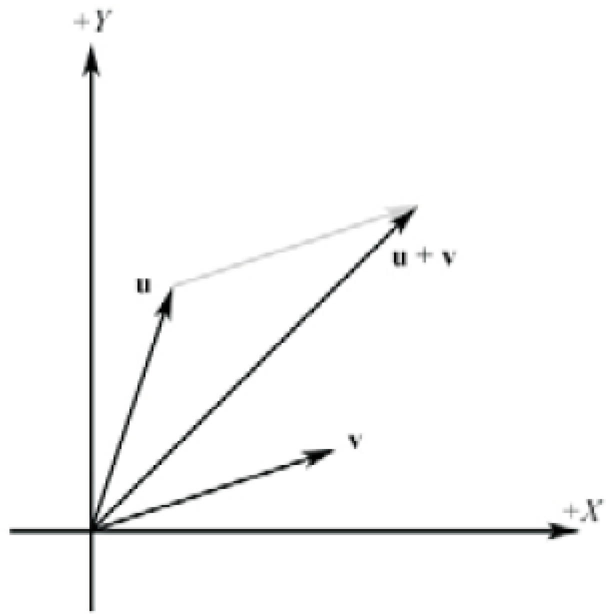
- Example: Normalize the vectors **u = (1, 2, 3) and v = (1, 1).**
- Solution:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\sqrt{14}} = \left(\frac{1}{\sqrt{14}}, \quad \frac{2}{\sqrt{14}}, \quad \frac{3}{\sqrt{14}} \right)$$

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}} \right)$$

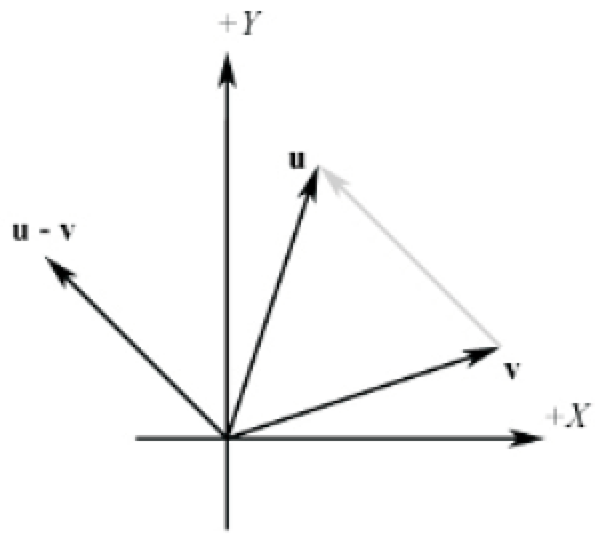
Vector Addition

$$\mathbf{u} + \mathbf{v} = (u_x + v_x, \quad u_y + v_y, \quad u_z + v_z)$$



Vector Subtraction

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_x - v_x, \quad u_y - v_y, \quad u_z - v_z)$$



Scalar Multiplication

$$k\mathbf{u} = \left(ku_x, \quad ku_y, \quad ku_z \right)$$

```
D3DXVECTOR3 u(1.0f, 1.0f, -1.0f);
D3DXVECTOR3 scaledVec = u * 10.0f; // = (10.0f, 10.0f, -10.0f)
```

Dot Products

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z = s$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \vartheta$$

Some useful properties of the dot product:

- If u · v = 0, then u v
- If u · v > 0, then the angle, , between the two vectors is less than 90 degrees.
- If u · v < 0, then the angle, , between the two vectors is greater than 90 degrees.

Cross Products

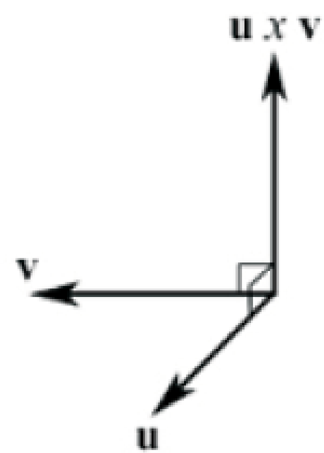
$$\mathbf{p} = \mathbf{u} \times \mathbf{v} = [(u_y v_z - u_z v_y), \quad (u_z v_x - u_x v_z), \quad (u_x v_y - u_y v_x)]$$

- Example: Find j = k i = (0, 0, 1) (1, 0, 0) and verify that j is orthogonal to both k and i.
- Solution:

$$j_x = (0(0) - 1(0)) = 0$$

$$j_y = (1(1) - 0(0)) = 1$$

$$j_z = (0(0) - 0(1)) = 0$$



Basic Transformations

Homogeneous space

Point

$\mathbf{p} = (p_1, p_2, p_3)$

$[p_1, p_2, p_3, 1]$

Vector

$\mathbf{v} = (v_1, v_2, v_3)$

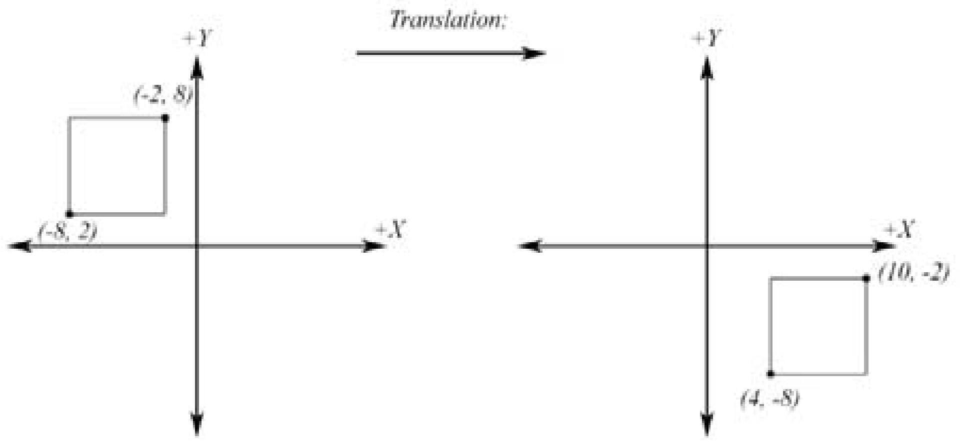
$[v_1, v_2, v_3, 0]$

perspective projections

$$\mathbf{p} = \begin{bmatrix} p_1, & p_2, & p_3, & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} p_1, & p_2, & p_3, & p_3 \end{bmatrix} = \mathbf{p}', \text{ for } p_3 \neq 0$$

$$\left(\frac{x}{w}, \quad \frac{y}{w}, \quad \frac{z}{w}, \quad \frac{w}{w} \right) = \left(\frac{x}{w}, \quad \frac{y}{w}, \quad \frac{z}{w}, \quad 1 \right) = \left(\frac{x}{w}, \quad \frac{y}{w}, \quad \frac{z}{w} \right) = x$$

Translation Matrix

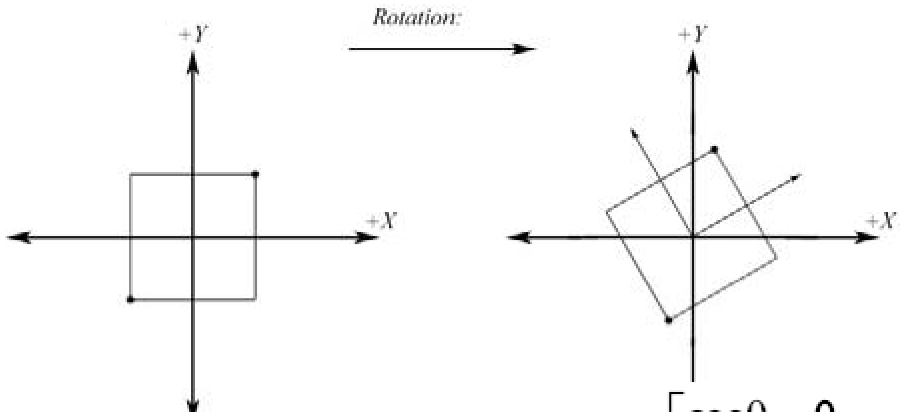


$$\mathbf{T}(\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_x & p_y & p_z & 1 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \mathbf{T}(-\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p_x & -p_y & -p_z & 1 \end{bmatrix}$$

```
D3DXMATRIX *D3DXMatrixTranslation(
     D3DXMATRIX* pOut,     // Result.
     FLOAT x,              // Number of units to translate on x-axis.
     FLOAT y,              // Number of units to translate on y-axis.
     FLOAT z               // Number of units to translate on z-axis.
);
```

Rotation Matrices

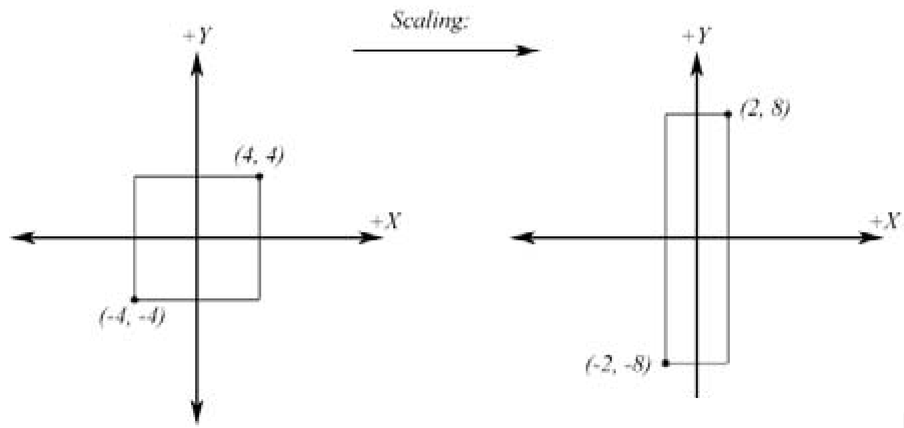


$$\mathbf{X}(\vartheta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Y}(\vartheta)=\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Z}(\vartheta)=\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling Matrix



$$\mathbf{S}(\mathbf{q}) = \begin{bmatrix} q_x & 0 & 0 & 0 \\ 0 & q_y & 0 & 0 \\ 0 & 0 & q_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S}^{-1} = \mathbf{S}\left(\frac{1}{q_x}, \quad \frac{1}{q_y}, \quad \frac{1}{q_z}\right) = \begin{bmatrix} \frac{1}{q_x} & 0 & 0 & 0 \\ 0 & \frac{1}{q_y} & 0 & 0 \\ 0 & 0 & \frac{1}{q_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining Transformations

$$\mathbf{S}\left(\frac{1}{5}, \quad \frac{1}{5}, \quad \frac{1}{5}\right) = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_y\left(\frac{\pi}{4}\right) = \begin{bmatrix} .707 & 0 & -.707 & 0 \\ 0 & 1 & 0 & 0 \\ .707 & 0 & .707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{p} = [5, 0, 0, 1]$$

$$\mathbf{T}(1, \quad 2, \quad -3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & -3 & 1 \end{bmatrix}$$

$$\mathbf{pS} = [1, \quad 0, \quad 0, \quad 1] = \mathbf{p}'$$

$$\mathbf{p}'\mathbf{R}_y = [.707, \quad 0, \quad -.707, \quad 1] = \mathbf{p}''$$

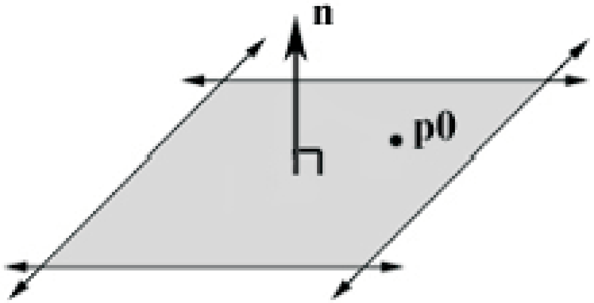
$$\mathbf{p}''\mathbf{T} = [1.707, \quad 2, \quad -3.707, \quad 1]$$

Combining Transformations

$$\mathbf{SR}_y\mathbf{T} = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .707 & 0 & -.707 & 0 \\ 0 & 1 & 0 & 0 \\ .707 & 0 & .707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .1414 & 0 & -.1414 & 0 \\ 0 & 1 & 0 & 0 \\ .1414 & 0 & .1414 & 0 \\ 1 & 2 & -3 & 1 \end{bmatrix} = \mathbf{Q}$$

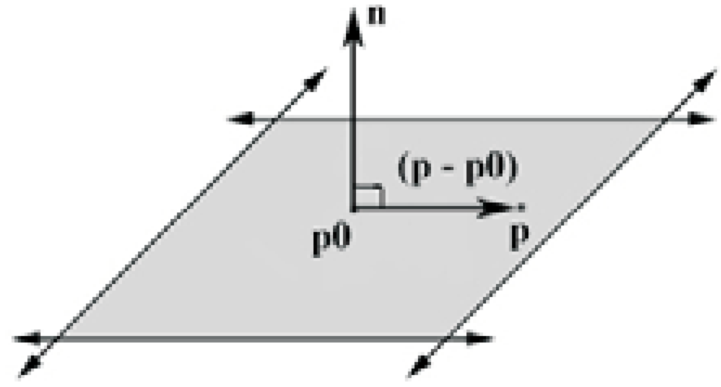
Plane



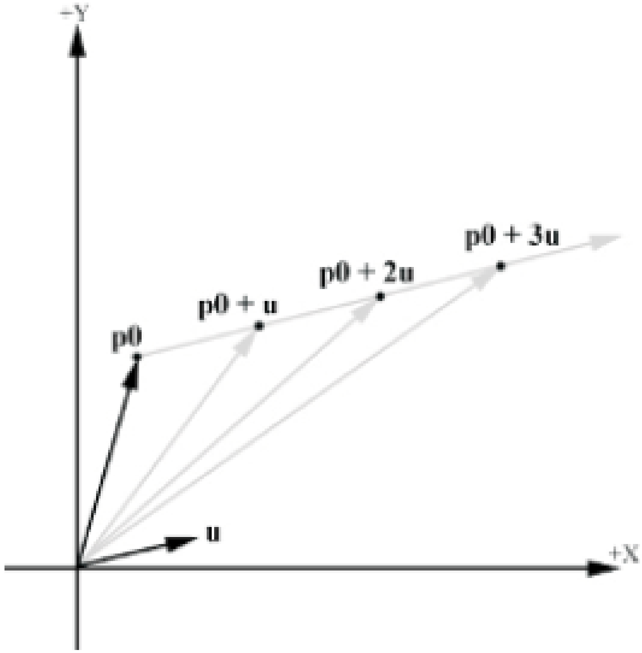
$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$$

$$\mathbf{n} \cdot \mathbf{p} + d = 0$$

$$d = -\mathbf{n} \cdot \mathbf{p}_0$$



Ray



$$\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{u}$$

Ray/Plane Intersection

$$\mathbf{n}\cdot\mathbf{p}(t)+d=0$$

$$\mathbf{n}\cdot(\mathbf{p}_0+t\mathbf{u})+d=0$$

$$\mathbf{n}\cdot\mathbf{p}_0+\mathbf{n}\cdot t\mathbf{u}+d=0$$

$$\mathbf{n}\cdot t\mathbf{u}=-d-(\mathbf{n}\cdot\mathbf{p}_0)$$

$$t(\mathbf{n}\cdot\mathbf{u})=-d-(\mathbf{n}\cdot\mathbf{p}_0)$$

$$t=\frac{-d-(\mathbf{n}\cdot\mathbf{p}_0)}{(\mathbf{n}\cdot\mathbf{u})}$$

$$\mathbf{p}\left(\frac{-d-(\mathbf{n}\cdot\mathbf{p}_0)}{(\mathbf{n}\cdot\mathbf{u})}\right)=\mathbf{p}_0+\frac{-d-(\mathbf{n}\cdot\mathbf{p}_0)}{(\mathbf{n}\cdot\mathbf{u})}\mathbf{u}$$