Exercise session on order of convergence and Richardson extrapolation

September 27 2021

Exercise 1

Consider the function $f(x) = \exp(-x^2)$ on the interval $[-3, 3]$.

(a) Using the centered finite difference approximation $\delta^0 f_i$, compute an approximation of $f'(x) = -2x\exp(-x^2)$ at the interior points $x_1, \ldots, x_{n-1}$ of a uniform mesh $x_0, = -3 \ldots, x_n = 3$ with $n = 10, 20, 40, 80, 160, 320, 640$. In each case, compute the l_2 and l_∞ relative error with respect to the exact solution. For each approximation pair $n = 10, 20, n = 20, 40, \ldots, n = 320, 640$ compute the empirical convergence order of the centered finite difference approximation. Explain if the results are coherent with the theory and why.

(b) Repeat the computation for the function $f(x) = \exp(-100x^2)$, whose first derivative is $f'(x) = -200x\exp(-100x^2)$. Explain the differences between the results in this case and those in the previous case on the basis of the theory of finite difference approximations.

Exercise 2

Consider the function $f(x) = x^3 - 2x^2 + x - 1$ on the interval $[-1, 1]$.

(a) Using the fourth order finite difference approximation $\delta^{(4)} f_i$ compute an approximation of $f'(x) = 3x^2 - 4x + 1$ at the interior points $x_2, \ldots, x_{n-2}$ of a uniform mesh $x_0, = -1 \ldots, x_n = 1$ with $n = 15, 30, 60, 120$. In each case, compute the l_2 and l_∞ relative error with respect to the exact solution. For each approximation pair compute the empirical convergence order. Explain if the results are coherent with the theory and why.

(b) Repeat the computation for the function $f(x) = x^6 + x^4 - x + 1$ whose first derivative is $f'(x) = 6x^5 + 4x^3 - 1$. Explain the differences between the results in this case and those in the previous case on the basis of the theory of finite difference approximations.

Exercise 3

Repeat the previous two exercises for the forward and backward finite difference approximations.

Exercise 4

Consider on the interval $[0, \frac{\pi}{2}]$ the function $f(x) = x \cos x$, whose second derivative is given by $f^{(2)}(x) = -x \cos x - 2 \sin x$.

(a) Compute a centered finite difference approximation of the exact second derivative at the internal points at the interior points $x_1, \ldots, x_{n-1}$ of a uniform mesh $x_0, = 0 \ldots, x_n = \frac{\pi}{2}$ with $n = 50, 100, 200, 400, 800$. For each approximation pair compute the empirical convergence order. Explain if the results are coherent with the theory and why.

(b) Repeat the previous computation in the case $f(x) = x^{\frac{5}{2}}$, $f''(x) = \frac{15}{4} x^{\frac{1}{2}}$. Explain the difference between the results in the two cases on the basis of the theory of finite difference approximation.

Exercise 5

Consider function $f(x) = \arctan(2x)$, whose first derivative is given by $f'(x) = 2/(4x^2 + 1)$.

(a) Compute a forward finite difference approximation of the derivative for $x = 1$ with steps $h = 0.1$ and $h = 0.05$. Build the Richardson extrapolation from the previously computed values. For all three approximations, compute the absolute error with respect to the true derivative values.

(b) Repeat the computation for $h = 10^{-10}$ and $h = 0.5 \times 10^{-10}$. Discuss the differences in the results with respect to the previous case.

(c) Repeat point a) for the function $f(x) = \arctan(200x)$, whose first derivative is given by $f'(x) = 200/(40000x^2 + 1)$. Explain the differences in the results on the basis of the theory of finite difference approximation.

Exercise 6

Consider the function $f(x) = x^{\frac{15}{2}}$, whose second derivative is given by $f''(x) = \frac{195}{4} x^{\frac{11}{2}}$.

(a) Compute a centered finite difference approximation of the exact second derivative at $x = 1$ with steps $h = 10^{-2}$ and $h = 0.5 \times 10^{-2}$. Build the Richardson extrapolation from the previously computed values. For all three approximations, compute the absolute error with respect to the true derivative values.

(b) Repeat point a) for $x = 10$. Explain the differences in the results on the basis of the theory of finite difference approximation.

Exercise 7

Consider the function $f(x) = \exp(-x)$, whose first derivate is given by compute an approximation of $f'(x) = -\exp(-x)$.

(a) Compute a centered finite difference approximation of the exact first derivative at $x = 3$ with steps $h = 10^{-3}$ and $h = 0.5 \times 10^{-3}$. Build the Richardson extrapolation from the previously computed values. For all three approximations, compute the absolute error with respect to the true derivative values.

(b) Compute a fourth order finite difference approximation of the first derivative at $x = 3$ with step $h = 0.5 \times 10^{-3}$. Compare the accuracy with that of previously computed approximations and say which method is the most accurate.

(c) Repeat the previous points computing all the approximations at $x = -30$. Explain the differences in the results on the basis of the theory of finite difference approximation.