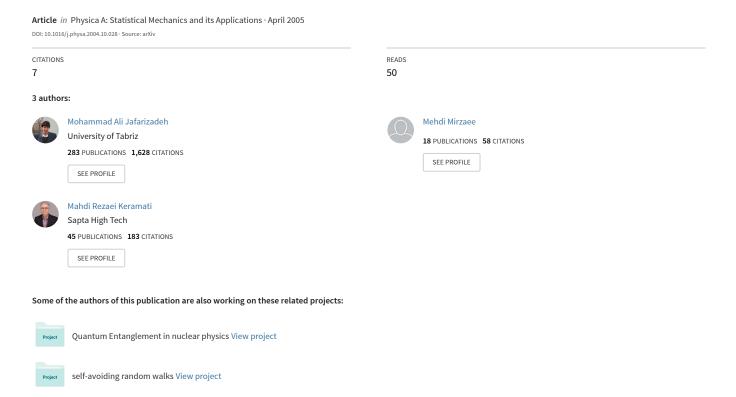# Evaluation of relative Entropy of Entanglement and derivation of Optimal lewenstein- Sanpera Decomposition of Bell decomposable states via convex optimization

3 authors:

Mohammad Ali Jafarizadeh
University of Tabriz
283 PUBLICATIONS   1,628 CITATIONS

SEE PROFILE

Mehdi Mirzaee
18 PUBLICATIONS   58 CITATIONS

SEE PROFILE

Mahdi Rezaei Keramati
Sapta High Tech
45 PUBLICATIONS   183 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Quantum Entanglement in nuclear physics View project

self-avoiding random walks View project

Evaluation of relative entropy of entanglement and derivation of optimal Lewenstein-Sanpera decomposition of Bell decomposable states via convex optimization

M. A. Jafarizadeh[a,b,c] *, M.Mirzaee[a,b] †, M.Rezaee[a,b] ‡

[a]Department of Theoretical Physics and Astrophysics, Tabriz University, Tabriz 51664, Iran.

[b]Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran.

[c]Research Institute for Fundamental Sciences, Tabriz 51664, Iran.

February 1, 2008

*E-mail:jafarizadeh@tabrizu.ac.ir

†E-mail:mirzaee@tabrizu.ac.ir

‡E-mail:karamaty@tabrizu.ac.ir

Abstract

We provide an analytical expression for optimal Lewenstein-Sanpera decomposition of Bell decomposable states by using semi-definite programming. Also using the Karush-Kuhn-Tucker optimization method, the minimum relative entropy of entanglement of Bell decomposable states has been evaluated and it is shown that the same separable Bell decomposable state lying at the boundary of convex set of separable Bell decomposable states, optimizes both Lewenstein-Sanpera decomposition and relative entropy of entanglement. **Keywords: Minimum relative entropy of entanglement, Semi-definite programming, Convex optimization, Lewenstein-Sanpera decomposition, Bell decomposable states.**

1 INTRODUCTION

Quantum entanglement is a powerful property that has attracted much attention, since it provides new means of communication, such as quantum cryptography[1], quantum teleportation[2] superdense coding, and quantum computation. Therefore, the measure of entanglement of composite systems becomes crucial [3, 4, 5, 6, 7]. Even though, for pure bipartite states the Von-Neumann entropy is a good measure and it is easily calculated [3]; however, to quantify the entanglement of mixed states is a harder task. Some measures have been proposed, among them one based on relative entropy[8]. This measure needs a minimization procedure[6, 8, 9, 10].

In order to obtain the relative entropy of entanglement of a density matrix , the crucial point is to find a separable state ρ_s such that minimizes the relative entropy of entanglement. In general case, to find ρ_s is a laborious work.

Additionally, there is another useful tool to study entanglement, i.e., Lewenstein-Sanpera decomposition (LSD) [11]. Among them is the unique optimal decomposition, the one with the largest λ in LSD.

From the above considerations, we see that there are some specific separable states playing an important role in determining the entanglement measure. Then it is meaningful to determine the properties of these separable states.

On the other hand, over the past years, semidefinite programming (SDP) has been recognized as valuable numerical tools for control system analysis and design. In (SDP) one minimizes a linear function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. SDP, has been studied (under various names) as far back as the 1940s. Subsequent research in SDP during the 1990s was driven by applications in combinatorial optimization[12], communications and signal processing [13, 14, 15], and other areas of engineering[16]. Although SDP is designed to be applied in numerical methods it

can be used for analytic computations, too. Some authors try to use the SDP to construct an explicit entanglement witness [17, 18, 19]. Kitaev used SDP duality to prove the impossibility of quantum coin flipping [20], and Rains gave bounds on distillable entanglement using SDP [21]. In the context of quantum computation, Barnum, Saks and Szegedy reformulated quantum query complexity in terms of SDP[22]. The problem of finding the optimal measurement to distinguish between a set of quantum states was first formulated as a SDP in 1972 by Holevo, who gave optimality conditions equivalent to the complementary slackness conditions [23]. Recently, Eldar, Megretski and Verghese showed that the optimal measurements can be found efficiently by solving the dual followed by the use of linear programming [24]. Audenaert at al. [25, 26] applied convex analysis method and Lagrange duality to calculate the relative entropy of entanglement for Werner states. Also Lawrence in [27] used SDP to show that the standard algorithm implements the optimal set of measurements. All of the above mentioned applications indicate that the method of SDP is very useful.

In this paper, by using the convex optimization method, we find optimal LSD and minimum relative entropy of entanglement for BD states. Finally, we show that the separable states that minimize relative entropy of entanglement are the same as states which optimize the LSD which lie at the boundary of convex and compact set of separable states [28].

The paper is organized as follows:
In section-2 we give a brief review of convex optimization. Section -3 is about the Optimal LSD and also by using SDP we find optimal LSD for Bell decomposable (BD) states. In section-4 we explain how one can find the minimum relative entropy of entanglement by using the Karush-Kuhn-Tucker(KKT) optimization method, we find the minimum relative entropy of entanglement of BD states. Finally it is shown that the same separable BD state lying at the boundary of convex set of separable BD states, optimizes both LSD and relative entropy of entanglement. The paper is ended with a brief conclusion.

2 Convex Optimization

2.1 Semi-definite programming

A SDP is a particular type of convex optimization problem [29]. A SDP problem requires minimizing a linear function subject to a linear matrix inequality (LMI) constraint [30]:

$$
\begin{aligned}
\text{minimize} \quad & \mathcal{P} = c^T x \\
\text{subject to} \quad & F(x) \geq 0,
\end{aligned}
\tag{2-1}
$$

where c^T is a given vector, $x = (x_1, ..., x_n)$, and $F(x) = F_0 + \sum_i x_i F_i$, for some fixed hermitian matrices F_i. The inequality sign in $F(x) \geq 0$ means that $F(x)$ is positive semidefinite.

This problem is called the primal problem. Vectors x whose components are the variables of the problem and satisfy the constraint $F(x) \geq 0$ are called primal feasible points, and if they satisfy $F(x) > 0$ they are called strictly feasible points. The minimal objective value $c^T x$ is by convention denoted by $\mathcal{P}^*$ and is called the primal optimal value.

Due to the convexity of set of feasible points, SDP has a nice duality structure, with, the associated dual program being:

$$
\begin{aligned}
\text{maximize} \quad & -Tr[F_0 Z] \\
& Z \geq 0 \\
& Tr[F_i Z] = c_i.
\end{aligned}
\tag{2-2}
$$

Here the variable is the real symmetric (or Hermitean) matrix Z, and the data c, F_i are the same as in the primal problem. Correspondingly, matrices Z satisfying the constraints are called dual feasible (or strictly dual feasible if $Z > 0$). The maximal objective value of $-Tr F_0 Z$, i.e., the dual optimal value, is denoted by d^*.

The objective value of a primal(dual) feasible point is an upper (lower) bound on $\mathcal{P}^*(d^*$. The main reason why one is interested in the dual problem is that one can prove that $d^* \leq \mathcal{P}^*$, and under relatively mild assumptions, we can have $\mathcal{P}^* = d^*$. if the equality holds, one can prove the following optimality condition on x:

A primal feasible x and a dual feasible Z are optimal which is denoted by $\hat{x}$ and $\hat{Z}$ if and only if

$$F(\hat{x})\hat{Z} = \hat{Z}F(\hat{x}) = 0. \tag{2-3}$$

This latter condition is called the complementary slackness condition.

In one way or another, numerical methods for solving SDP problems always exploit the inequality $d \leq d^* \leq \mathcal{P}^* \leq \mathcal{P}$, where d and $\mathcal{P}$ are the objective values for any dual feasible point and primal feasible point, respectively. The difference

$$\mathcal{P}^* - d^* = c^T x + Tr[F_0 Z] = Tr[F(x)Z] \geq 0 \tag{2-4}$$

is called the duality gap. If the equality holds $d^* = \mathcal{P}^*$, i.e., the optimal duality gap is zero, then we say that strong duality holds.

2.2 Karush-Kuhn-Tucker Theorem:

All problems treated in this paper are convex optimization problems: minimizing a convex objective function subject to upper bound inequality constraints on other convex functions. Lagrange duality theory is also well developed for convex optimization. For example, the duality gap is zero under mild technical conditions such as Slater 's condition that requires the existence of a strictly feasible point.

We seek first order conditions for local optimality for the following formulation of nonlinear programming:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h(x) = 0 \\ & g(x) \leq 0, \end{aligned} \tag{2-5}$$

where x is a vector in R^n, h maps into R^k, and g maps into R^m.

Karush-Kuhn-Tucker (KKT) Theorem: Assuming that functions g_i, h_i are differentiable and that strong duality holds, there exists vectors $\zeta \in R^k$, and $y \in R^m$, such that the

gradient of dual Lagrangian $L(x^*, \zeta^*, y^*) = f(x^*) + \sum_i \zeta_i^* h_i(x^*) + \sum_i y_i^* g_i(x^*)$ over x vanishes at x^* :

$$h_i(x^*) = 0 \text{(primal feasible)}$$

$$g_i(x^*) \leq 0 \text{(primal feasible)}$$

$$y_i^* \geq 0 \text{(dual feasible)} \tag{2-6}$$

$$y_i^* g_i(x^*) = 0$$

$$\nabla f(x^*) + \sum_i \zeta_i^* \nabla h_i(x^*) + \sum_i y_i^* \nabla g_i(x^*) = 0$$

Then x^* and (ζ^*, y^*) are primal and dual optimal, with zero duality gap. In summary, for any convex optimization problem with differentiable objective and constraint functions, any points that satisfy the KKT conditions are primal and dual optimal, and have zero duality gap. Necessary KKT conditions satisfied by any primal and dual optimal pair and for convex problems, KKT conditions are also sufficient. If a convex optimization problem with differentiable objective and constraint functions satisfies Slater's condition, then the KKT conditions provide necessary and sufficient conditions for optimality: Slater's condition implies that the optimal duality gap is zero and the dual optimum is attained, so x is optimal if and only if there are (ζ^*, y^*) that, together with x, satisfy the KKT conditions.

Slater's condition: Suppose x^* solves

$$\text{minimize} \quad f(x) \tag{2-7}$$

$$g_i(x) \geq b_i \ , \quad i = 1, ..., m$$

and the feasible set is non empty. Then there is a non-negative vector ζ such that for all x

$$L(x, \zeta) = f(x) + \zeta^T (b - g(x)) \leq f(x^*) = L(x^*, \zeta). \tag{2-8}$$

If in addition $f(.), g_i(.) \ , \quad i = 1, ..., m$ are continuously differentiable

$$\frac{\partial f}{\partial x_j}(x^*) - \zeta \frac{\partial g}{\partial x}(x^*) = 0. \tag{2-9}$$

In the spatial case the vector x^* is a solution of the linear program

$$\text{minimize} \quad c^T x$$

$$\text{subject to} \quad Ax = b \tag{2-10}$$

$$x \geq 0,$$

if and only if there exist vectors $\zeta \in R^k$, and $y \in R^m$ for which the following conditions hold for $(x, \zeta, y) = (x^*, \zeta^*, y^*)$

$$A^T \zeta + y = c$$

$$Ax = b \tag{2-11}$$

$$x_i \geq 0; \quad y_i \geq 0; \quad x_i y_i = 0 \ , \quad i = 1, ..., m.$$

A solution (x^*, ζ^*, y^*) of (2-11) is called strictly complementary, if $x^* + y^* > 0$, i. e., if there exists no index $i \in \{1, ..., m\}$ such that $x_i^* = y_i^* = 0$.

3   Lewenstein-Sanpera decomposition

According to LSD [11], any bipartite density matrix ρ can be written as

$$\rho = \lambda \rho_s + (1 - \lambda)\rho_e, \qquad \lambda \in [0, 1], \tag{3-1}$$

where ρ_s is a separable density matrix and ρ_e is an entangled state. The LSD of a given density matrix ρ is not unique and, in general, there is a continuum set of LSD to choose from. The decomposition with the largest weight λ of the separable part is the optimal LSD [36, 37], which can be determined uniquely.

Furthermore, in the case of a two-qubit state it can be shown [11] that the entangled part ρ_e of the optimal LSD is always a pure state i.e,

$$\rho = (1 - \lambda)|\psi\rangle\langle\psi| + \lambda \rho_s' \tag{3-2}$$

3.1 Optimal Lewenstein-Sanpera decomposition for Bell-decomposable states via convex optimization

A BD state is defined by

$$\rho = \sum_{i=1}^{4} p_i \left| \psi_i \right\rangle \left\langle \psi_i \right|, \qquad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{4} p_i = 1, \tag{3-3}$$

where $\left| \psi_i \right\rangle$ are Bell states given by

$$\left| \psi_1 \right\rangle = \left| \phi^+ \right\rangle = \frac{1}{\sqrt{2}}(\left| 00 \right\rangle + \left| 11 \right\rangle), \qquad \left| \psi_2 \right\rangle = \left| \phi^- \right\rangle = \frac{1}{\sqrt{2}}(\left| 00 \right\rangle - \left| 11 \right\rangle),$$

$$\left| \psi_3 \right\rangle = \left| \psi^+ \right\rangle = \frac{1}{\sqrt{2}}(\left| 01 \right\rangle + \left| 10 \right\rangle), \qquad \left| \psi_4 \right\rangle = \left| \psi^- \right\rangle = \frac{1}{\sqrt{2}}(\left| 01 \right\rangle - \left| 10 \right\rangle). \tag{3-4}$$

In terms of Pauli's matrices, ρ can be written as,

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i), \tag{3-5}$$

where

$$t_1 = p_1 - p_2 + p_3 - p_4,$$
$$t_2 = -p_1 + p_2 + p_3 - p_4, \tag{3-6}$$
$$t_3 = p_1 + p_2 - p_3 - p_4.$$

From positivity of ρ we get

$$1 + t_1 - t_2 + t_3 \geq 0, \qquad 1 - t_1 + t_2 + t_3 \geq 0,$$
$$1 + t_1 + t_2 - t_3 \geq 0, \qquad 1 - t_1 - t_2 - t_3 \geq 0. \tag{3-7}$$

These equations form a tetrahedral with its vertices located at $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, -1, -1)$ [38]. In fact these vertices denote the Bell states given in (3-4).

On the other hand ρ given in (3-5) is separable if and only if t_i satisfy Eq. (3-7) and,

$$1 + t_1 + t_2 + t_3 \geq 0, \qquad 1 - t_1 - t_2 + t_3 \geq 0,$$
$$1 + t_1 - t_2 - t_3 \geq 0, \qquad 1 - t_1 + t_2 - t_3 \geq 0. \tag{3-8}$$

inequalities (3-7) and (3-8) form an octahedral with its vertices located at $O_1^{\pm} = (\pm 1, 0, 0)$, $O_2^{\pm} = (0, \pm 1, 0)$ and $O_3^{\pm} = (0, 0, \pm 1)$. So, tetrahedral is divided into five regions. Central regions, defined by octahedral, are separable states $(p_k \leq \frac{1}{2})$. There are also four smaller equivalent tetrahedral corresponding to entangled states$(p_k > \frac{1}{2}$ for only one of $k = 1, ..., 4)$, where $p_k = \frac{1}{2}$ denote to boundary between separable and entangled region. Each tetrahedral takes one Bell state as one of its vertices (see Fig-1).

Now in order to obtain optimal LSD via convex optimization of entangled BD state given in (3-3), with $p_1 > \frac{1}{2}$, we first choose an arbitrary separable state as

$$\rho_s' = \sum_{i=1}^{4} p_i' |\phi_i\rangle \langle \phi_i|, \qquad \sum_{i=1}^{4} p_i' = 1, \quad p_1' < \frac{1}{2} \qquad (3\text{-}9)$$

in the separable region. Then according to strict SDP optimization prescription of section (2), we need to optimize $Tr(\Lambda \rho_s')$ with respect to $\rho - \Lambda \rho_s' > 0$, where the feasible solution corresponds to

$$\Lambda_{max} = min\{\frac{p_1}{p_1'}, \frac{p_2}{p_2'}, \frac{p_3}{p_3'}, \frac{p_4}{p_4'}\}. \qquad (3\text{-}10)$$

One can show that the only possible choice of Λ_{max} consistent with positivity of $\rho - \Lambda_{max}\rho_s'$ is

$$\Lambda_{max} = \frac{(1 - p_1)}{(1 - p_1')} \quad \text{and} \quad p_i' = \frac{p_i}{\Lambda_{max}}, i = 2, 3, 4. \qquad (3\text{-}11)$$

The equation (3-11) indicates that Λ_{max} is a monotonic increasing function of p_1' and its maximum value corresponds to $p_1' = \frac{1}{2}$, with

$$\Lambda_{max} = 2(1 - p_1) \quad \text{and} \quad p_i' = \frac{p_i}{2(1 - p_1)}, i = 2, 3, 4, \qquad (3\text{-}12)$$

where this separable state lies at the boundary of separable region, see Fig-1. Substituting the results that obtained in $\rho - \Lambda_{max}\rho_s'$ we get

$$\rho - \Lambda_{max}\rho_s' = (2p_1 - 1)|\phi_1\rangle \langle \phi_1|, \qquad (3\text{-}13)$$

which is a pure states in agreement with theorem (2) of Ref.[11]. Equation (3-13) with Λ_{max} given in (3-12) is nothing but the optimal LSD of BD states.

4 Evaluation of relative entropy of entanglement

4.1 Minimizing relative entropy via KKT method

Fundamental to our understanding of distinguishability is the measure of uncertainty in a given probability distribution. The uncertainty in a collection of a probability mass function (pmf) $\omega \in \mathcal{R}^k$ is given by its entropy,

$$S(p) = -\sum_i \omega_i \log(\omega_i) \tag{4-14}$$

called the Shannon entropy. This measure is suitable for the states of systems described by the laws of classical physics, but it will have to be changed, along with other classical measures, when we present the quantum information theory. We ultimately wish to be able to talk about storing and processing information. For this we require a means of comparing two different probability distributions, which is why we introduce the notion of relative entropy (first introduced by Kullback and Leibler, 1951 [32]).

Definition of the relative entropy:

Suppose that we consider the problem of estimating a probability mass function (pmf) $\omega \in \mathcal{R}^k$ given a strictly positive prior $q \in \mathcal{R}^k$. The relative entropy between these two distributions is defined as [8]

$$\mathcal{I}(\omega; q) = \sum_{i=1}^{k} \omega_i \log \frac{\omega_i}{q_i} \tag{4-15}$$

This function is a measure of the distance between ω_i and q_i, even though, strictly speaking, it is not a mathematical metric since it fails to be symmetric $\mathcal{I}(\omega_i \parallel q_j) \neq \mathcal{I}(q_j \parallel \omega_i)$.

A common restriction on ω is a mean constraint, so that if there are k observations $A = [a_1, ..., a_k] \in \mathcal{R}^{dk}$ and a mean $b \in \mathcal{R}^d$, then ω must satisfy $A\omega = b$. A standard approach [31, 33, 34] is to minimize the relative entropy function (4-15) over the constrained probability simplex

$$1^T \omega = 1, \quad \omega \geq 0$$
$$A\omega = b, \tag{4-16}$$

where the first two constraints as due to the fact that ω represents a distribution and the symbol 1 denotes a vector of ones. Consider the relative entropy optimization problem

$$\text{minimize} \quad \mathcal{I}(\omega; q) \tag{4-17}$$

subject to constraint (4-16). The optimization problem (4-17) is convex; in other words its objective function is convex, and the equality constraints are linear. Under a suitable constraint qualification, such as Slaters condition, the first-order optimality conditions of (4-17) are in fact both necessary and sufficient (see, for example, [30, 35]).

One simple constraint qualification is Slater's condition, i.e., there exists an ω in the relative interior of the feasible set, such that

$$\omega > 0, \quad 1^T \omega = 1, \quad A\omega = b. \tag{4-18}$$

Let ζ and y be the Lagrange multipliers associated with the first and second constraints of (4-17), respectively.

Definition: (KKT optimality conditions): A triple (ω^*, ζ^*, y^*) is a first-order KKT point of (4-17) if it satisfies the following conditions:

$$1^T \omega = 1$$
$$A\omega = b \tag{4-19}$$
$$\nabla_\omega \mathcal{I}(\omega, q) + \zeta 1 + A^T y = 0.$$

Theorem: Suppose that Slaters condition holds. Then the vector with components

$$\omega_j^* = \frac{u_j}{\sum_{j=1}^{k} u_j}, \tag{4-20}$$

for $j = 1, ..., k$, solves (4-17), where

$$u_j = q_j \exp(-a_j^T y^*), \tag{4-21}$$

and y^* is the Lagrange multiplier corresponding to the constraint $A\omega = b$ [10].

4.2 Evaluation of relative entropy of entanglement of Bell decomposable states via KKT method

In order to obtain minimal relative entropy of entanglement we choose entangled BD state (given in (3-3)), with $p_1 > \frac{1}{2}$ and then we choose an arbitrary separable states

$$\rho'_s = \sum_{i=1}^{4} p'_i |\phi_i\rangle \langle \phi_i|, \qquad 0 \le p'_i \le 1, \ \sum_{i=1}^{4} p'_i = 1, \ \ p'_1 = \frac{1}{2}, \tag{4-22}$$

at the boundary of separable region.

Then according to the prescription of the minimization of the relative entropy given in section (2), all we need is to minimize $\mathcal{I}(\omega; q)$ with $\omega^T = (p'_1, p'_2, p'_3, p'_4)$, $q^T = (p_1, p_2, p_3, p_4)$, $p'_1 \le \frac{1}{2}$ and $p_1 > \frac{1}{2}$ subject to the constraints (4-16)with matrix A and vector b of the following form:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b^T = \begin{pmatrix} b_1 & 0 & 0 & 0 \end{pmatrix}, \tag{4-23}$$

where $b_1 < \frac{1}{2}$. Now, using the solutions (4-20) and (4-21) of KKT (optimality conditions) or relative entropy optimization given in (4-19) we obtain

$$u_1 = p_1 \exp(-y_1^*) \ \ , \ \ u_2 = p_2 \ \ , \ \ u_3 = p_3 \ \ , \ \ u_4 = p_4 \ , \tag{4-24}$$

now, using the slater's constraint we obtain

$$\omega_1^* = \frac{p_1 \exp(-y_1^*)}{p_1 \exp(-y_1^*) + p_2 + p_3 + p_4} = b_1. \tag{4-25}$$

After some calculation one can find the Lagrange multiplier y_1^* and the other components of optimal ω^*, where we quote only the optimal values ω_j^* , $j = 2, 3, 4$ in the following:

$$\omega_1^* = b_1 \ \ , \ \ \omega_2^* = (\tfrac{1-b_1}{1-p_1})p_2 \ \ , \ \ \omega_3^* = (\tfrac{1-b_1}{1-p_1})p_3 \ \ , \ \ \omega_4^* = (\tfrac{1-b_1}{1-p_1})p_4 \ . \tag{4-26}$$

Finally inserting to optimal ω^* in (4-26), we obtain the following expression for the relative entropy:

$$\mathcal{I}(\omega^*; q) = \omega_1^* \log \frac{\omega_1^*}{p_1} + \omega_2^* \log \frac{\omega_2^*}{p_2} + \omega_3^* \log \frac{\omega_3^*}{p_3} + \omega_4^* \log \frac{\omega_4^*}{p_4} = b_1 \log(\frac{b_1}{p_1}) + (1 - b_1) \log(\frac{1 - b_1}{1 - p_1}).$$

$$(4\text{-}27)$$

The equation (4-27) indicates that $\mathcal{I}(\omega; q)$ is a monotonic increasing function of b_1 and its maximum value corresponds to $p_1' = b_1 = \frac{1}{2}$, with

$$\mathcal{I}(\omega; q) = \frac{1}{2} \log(\frac{1}{2p_1}) + \frac{1}{2} \log(\frac{1}{2(1 - p_1)}). \tag{4-28}$$

and

$$p_i' = \omega_i^* = \frac{p_i}{2(1 - p_1)} \, , i = 2, 3, 4. \tag{4-29}$$

The BD separable state with $p_i' = \omega_i^* = \frac{p_i}{2(1-p_1)}$ which minimizes the relative entropy of entanglement of entangled BD state $\rho = \sum_{i=1}^{4} p_i < \psi \mid \psi >$ with $p_1 > \frac{1}{2}$ is the same as the one that has been found by Vedral and Plenio in Ref. [5]. Also one should notice that $c = 2p_1 - 1$, where c is concurrence of BD states and we have

$$\mathcal{I}(\omega; q) = -\frac{1}{2} \log (1 - c^2), \tag{4-30}$$

which is in agreement with Ref.[6].

Finally comparing the results of sections 3 and 4 we see that the same separable BD state lying at the boundary of convex set of separable BD states, optimizes both LSD and relative entropy of entanglement.

5   conclusion

Here in this work, using the convex optimization method, we have been able to obtain optimal LSD of BD states (by using the semidefinite programming method) and we have also evaluated the relative entropy of entanglement of BD states with respect to convex set of BD separable states (by using KKT method).

At the end it is shown that the same separable BD state lying at the boundary of convex set of separable BD states, optimizes both LSD and relative entropy of entanglement.

References

[1] A. K. Ekert, Phys. Rev. Lett. 67, 661, (1991).

[2] M. B. Plenio and V.Vedral, Contemp. Phys. 39, 431, (1998).

[3] S. Popescu and D. Rohrlich, Phys. Rev. A 56, R3319, (1997).

[4] C. H. Bennet, Phys. Scrip. T76, 210, (1998).

[5] V. Vedral, M. B. Plenio, M. A. Ripping, and P. L. Knight, Phys. Rev. Lett. 78, 2275, (1997).

[6] V. Vedral and M. B. Plenio, Phys. Rev. A.57, 1619, (1998).

[7] C. H. Bennet, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824, (1996).

[8] V. Vedral, Rev. Mod. Phys, 74, 197-234, (2002)

[9] R. V. Ramos, and R. F. Souzay, Journal of Computational Physics 175, 576583, (2002).

[10] M. P. Friedlander and M. R. Gupta, On minimizing distortion and relative entropy, Preprint ANL/MCS-P1110-1203; Mathematics and Computer Science Division; Argonne National Laboratory(Submitted to IEEE Transactions on Information Theory.)(2003).

[11] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. 80, 2261-2264, (1998).

[12] M. X. Goemans and D. P. Williamson. Journal of the Association for Computing Machinery, 42(6): 1115-1145, (1995).

[13] Z. Q. Luo. Mathematical Programming Series B, 97, 177-207, (2003).

[14] T. N. Davidson, Z.-Q. Luo, and K. M. Wong. IEEE Transactions on Signal Processing, 48(5): 1433-1445, (2000).

[15] W. K. Ma, T. N. Davidson, K. M. Wong, Z.-Q. Luo, and P. C. Ching. IEEE Transactions on Signal Processing, 50: 912-922, (2002).

[16] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory, vol. 15 of Studies in Applied Mathematics, SIAM, Philadelphia, PA, (1994).

[17] A. C. Doherty, P.A. Parrilo, F. M. Spedalieri, Phys. Rev. A, 69, 022308, (2004).

[18] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri. Phys. Rev. Lett. 88, 187904, (2002).

[19] P. A. Parrilo, A. C. Doherty, and F. M. Spedalieri. Entanglement witnesses and semidefinite programming, Proceedings of the 41st IEEE Conference of Decision and Control, (2002).

[20] Alexei Kitaev. Quantum coin flipping. Talk at Quantum Information Processing 2003, slides and video at http://www.msri.org, December (2002).

[21] E. M. Rains. IEEE Transactions on Information Theory, 47(7): 29212933, (2001).

[22] H. Barnum, M. Saks, and M. Szegedy. In Proceedings of the 18th IEEE Annual Conference on Computational Complexity, pages 179193, (2003).

[23] C. W. Helstrom. Quantum Detection and Estimation Theory. Academic Press, (1976).

[24] Y. C. Eldar, A. Megretski, and G. C. Verghese. IEEE Transactions on Information Theory, 49(4): 10071012, (2003).

[25] K. Audenaert, J. Eisert, E. Jane, M.B. Plenio, S. Virmani and B. De Moor, Phys. Rev. Lett. 87, 217902, (2001).

[26] K. Audenaert, B. De Moor, K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A. 66, 032310, (2002).

[27] Lawrence. Ip, Shor's Algorithm is Optimal, University of California, Berkeley, USA MU Seminar, http://www.qcaustralia.org, (2003).

[28] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A223, 1,(1996).

[29] L. Vandenberghe, S. Boyd SIAM Review 38, 49 - 95,(1996).

[30] S. Boyd and L. Vandenberghe, Convex Optimization. New York: Cambridge University Press, (2004).

[31] J. Shore and R. Johnson, Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy, IEEE Trans. Inform. Theory, vol. 33, no. 1, 2637, (1980).

[32] S. Kullback, and R. A. Leibler, Ann. Math. Stat. 22, 79-86, (1951).

[33] S. Kullback, Information Theory and Statistics, Dover Publications, NY, (1967).

[34] I. Csiszar, Why least squares and maximum entropy? An axiomatic approach to inference for linear inverse problems, The Annals of Statistics, vol. 19, no. 4, pp. 20322066, (1991).

[35] O. L. Mangasarian, Nonlinear Programming, ser. Classics in Applied Mathematics, G. Golub, Ed. Philadelphia: SIAM, (1994), vol. 10, originally published: New York, McGraw-Hill, (1969).

[36] S. J. Akhtarshenas, and M. A. Jafarizadeh, Phys. A: Math. Gen. 37, 2965-2982, (2004).

[37] B.-G Englert and N. Metwally, J. Mod. Opt. 47, 2221, (2000).

[38] P. Horodecki, M. Horodecki and R. Horodecki, Phys. Rev. Lett, 82, 1056-1059, (1999).

Figure Captions

Figure-1: All BD states are defined as points interior to tetrahedral. Vertices P_1, P_2, P_3 and P_4 denote projectors corresponding to Bell states given in Eqs. (3-4) . Interior points of Octahedral correspond to separable states.