Optimal Unambiguous Discrimination of Quantum States

Article *in* Physical review A, Atomic, molecular, and optical physics · April 2008

DOI: 10.1103/PhysRevA.77.042314 · Source: arXiv

4 authors, including:

Mohammad Ali Jafarizadeh
University of Tabriz

Mahdi Rezaei Keramati
Sapta High Tech

Optimal Unambiguous Discrimination of Quantum States

M. A. Jafarizadeh[a,b,c] *, M. Rezaei[b] †, N. Karimi[a] ‡, A. R. Amiri[a] §

[a]The Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz, 51664, Iran.

[b]The Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran.

[c]The Research Institute for Fundamental Sciences, Tabriz 51664, Iran.

February 5, 2008

*E-mail:Jafarizadeh@tabrizu.ac.ir

†E-mail:Karamaty@tabrizu.ac.ir

‡E-mail:karimi@tabrizu.ac.ir

§E-mail:amiri@tabrizu.ac.ir

Abstract

Unambiguously distinguishing between nonorthogonal but linearly independent quantum states is a challenging problem in quantum information processing. In this work, an exact analytic solution to an optimum measurement problem involving an arbitrary number of pure linearly independent quantum states is presented. To this end, the relevant semi-definite programming task is reduced to a linear programming one with a feasible region of polygon type which can be solved via simplex method. The strength of the method is illustrated through some explicit examples. Also using the close connection between the Lewenstein-Sanpera decomposition(LSD) and semi-definite programming approach, the optimal positive operator valued measure for some of the well-known examples is obtain via Lewenstein-Sanpera decomposition method.

Keywords: Optimal Unambiguous State Discrimination, Linear Programming, Lewenstein-Sanpera decomposition.

PACs Index: 03.67.Hk, 03.65.Ta, 42.50.-p

1 Introduction

In quantum information and quantum computing, the carrier of information is some quantum system and information is encoded in its state. A quantum state describes what we know about a quantum system. Given a single copy of a quantum system which can be prepared in several known quantum states, our aim is to determine in which state the system is. This can be well-understood in a communication context where only a single copy of the system is given and only a single shot-measurement is performed. This is in contrast with usual experiments in physics where many copies of a system are measured to get the probability distribution of the system. In quantum state discrimination, no statistics is built since only a single-shot measurement is performed on a single copy of the system. Actually there are fundamental limitations to the precision with which the state of the system can be determined with a single measurement. A fundamental aspect of quantum information theory is that non-orthogonal quantum states cannot be perfectly distinguished. Therefore, a central problem in quantum mechanics is to design measurements optimized to distinguish between a collection of non-orthogonal quantum states. The topic of quantum state discrimination was firmly established in 1970s by pioneering work of Helstrom [1], who considered a minimum error discrimination of two known quantum states. In this case, the state identification is probabilistic. Another possible discrimination strategy is the so-called unambiguous state discrimination (USD) where the states are successfully identified with non-unit probability, but without error. USD was originally formulated and analyzed by Ivanovic, Dieks and Peres [2, 3, 4] in 1987. The solution for unambiguous discrimination of two known pure states appearing with arbitrary prior probabilities was obtained by Jaeger and Shimony[5]. Although the two-state problem is well developed, the problem of unambiguous discrimination between multiple quantum states has received considerably less attention. The problem of discrimination among three nonorthogonal states was first considered by Peres and Terno [4]. They developed a geometric approach and applied it numerically on several examples. A different method was considered by Duan and Guo [6] and Y. Sun and et al [7]. Chefles [8] showed that a necessary and sufficient condition for the existence of unambiguous measurements for distinguishing between N quantum states is that the states are linearly independent. He also proposed a simple suboptimal measurement for unambiguous discrimination for which the probability of an inconclusive result is the same regardless of the state of the system. Equivalently, the measurement yields an equal probability of correct detection of each one of the ensemble states.

Over the past years, semidefinite programming (SDP) has been recognized as a valuable numerical tool for control system analysis and design. In SDP, one minimizes a linear function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. SDP has been studied (under various names) as far back as the 1940s. Subsequent research in semidefinite programming during the 1990s was driven by applications in combinatorial optimization[9], communications and signal processing [10, 11, 12], and other areas of engineering[13]. Although semidefinite programming is designed to be applied in numerical methods, it can be used for analytic computations, too. In the context of quantum computation and quantum information, Barnum, Saks and Szegedy have reformulated quantum query complexity and in terms of a semidefinite program [14], while M. A. Jafarizadeh, M. Mirzaee and M. Rezaee have revealed the close connection between Lewenstein-Sanpera decomposition, robustness of entanglement, finite quantum tomography and semi-definite programming algorithm. [15, 16, 17, 18].

The problem of finding the optimal measurement to distinguish between a set of quantum states was first formulated as a semidefinite program in 1972 by Holevo, who gave optimality conditions equivalent to the complementary slackness conditions [1]. Recently, Eldar, Megretski and Verghese showed that the optimal measurements can be found efficiently by solving the dual followed by the use of linear programming [19]. Also in [20], SDP has been used to show that the standard algorithm implements the optimal set of measurements. All of the above mentioned applications indicate that the method of SDP is very useful. The reason why the area has shown relatively slow progress until recently within the rapidly evolving field of quantum information is that it poses quite formidable mathematical challenges. Except for a handful of very special cases, no general exact solution has been available involving more than two arbitrary states and mostly numerical algorithms are proposed for finding optimal measurements for quantum-state discrimination, where the theory of the semi-definite programming provides a simple check of the optimality of the numerically obtained results.

In this study, we obtain the feasible region in terms of the inner product of the states which enables us to solve the problem analytically without using dual states. Exact analytical solution for optimal unambiguous discrimination of N linearly independent pure states is calculated and a geometrical interpretation for minimum inconclusive result for unambiguous discrimination of two pure states is presented. For more than three states, the analytical calculations is very complicated to write down and therefore we will consider the spacial cases such as geometrical uniform states and Welch bound equality (WBE) sequences. To solve the problem in general form, following prescription of Refs [21, 22, 23, 24, 25], we have reduced it to LP one, where the computation can be done at a very fast pace and with high precision. Moreover, we obtain the feasible region in terms of the inner product of the dual states and show that LSD is equivalent to optimal unambiguous state discrimination, and thus one can use LSD to solve the problem of optimal unambiguous state discrimination. This method is illustrated for two and three linearly independent states explicitly.

The organization of the paper is as follows: First the definition of the unambiguous quantum state discrimination are given. After that in section 3 unambiguous discrimination of quantum states by introducing feasible region and using linear programming are discussed. Then Lewnstein-sanpera decomposition is studied as an optimal unambiguous discrimination of quantum states. Finally, discrimination of non-orthogonal quantum states using approximated linear programming are discussed. The paper is ended with a brief conclusion and an appendix.

2 Unambiguous quantum state discrimination

In quantum theory, measurements are represented by positive operator valued measures (POVMs). A measurement is described by a collection $\{M_k\}$ of measurement operators. These operators are acting on the state space of the system being measured. The index k refers to the measurement outcomes that may occur in the experiment. In quantum information theory the measurement operators $\{M_k\}$ are often called Kraus operators [26]. If we define the operator

$$\Pi_k = M_k^\dagger M_k, \tag{2-1}$$

the probability of obtaining the outcome k for a given state ρ_i is given by $p(k|i) = Tr(\Pi_k \rho_i)$. Thus, the set of operators Π_k is sufficient to determine the measurement statistics.

Definition (POVM). A set of operators $\{\Pi_k\}$ is named a positive operator valued measure (POVM) if and only if the following two conditions are met:

1. Each operator Π_k is positive $\quad \Leftrightarrow \quad \langle\psi \mid \Pi_k \mid \psi\rangle \geq 0, \quad \forall \quad \mid \psi\rangle$.

2. The completeness relation is satisfied, i.e.,

$$\sum_k \Pi_k = 1. \tag{2-2}$$

The elements of $\{\Pi_k\}$ are called effects or POVM elements. On its own, a given POVM $\{\Pi_k\}$ is enough to give complete knowledge about the probabilities of all possible outcomes; measurement statistics is the only item of interest. Consider a set of known states $\rho_i, i = 1, ..., N$, with their prior probabilities η_i. We are looking for a measurement that either identifies a state uniquely (conclusive result) or fails to identify it (inconclusive result). The goal is to minimize the probability of inconclusive result. The measurements involved are typically generalized measurements. A measurement described by a POVM $\{\Pi_k\}_{k=1}^N$ is called unambiguous state discrimination measurement(USDM) on the set of states $\{\rho_i\}_{i=1}^N$ if and only if the following conditions are satisfied:

1. The POVM contains the elements $\{\Pi_k\}_{k=0}^N$ where N is the number of different signals in the set of states .The element Π_0 is related to an inconclusive result, while the other elements correspond to an identification of one of the states ρ_i, $i = 1, ..., N$.

2. No states are wrongly identified, that is, $Tr(\rho_i \Pi_k) = 0 \qquad \forall i \neq k \quad i, k = 1, ..., N$.

Each USD measurement gives rise to a failure probability, that is, the rate of inconclusive result. This can be calculated as

$$Q = \sum_i \eta_i Tr(Tr(\rho_i \Pi_0)). \tag{2-3}$$

And the success probability can be calculated as

$$P = 1 - Q = \sum_i \eta_i Tr(Tr(\rho_i \Pi_i)). \tag{2-4}$$

A measurement described by a POVM $\{\Pi_k^{opt}\}$ is called an optimal unambiguous state discrimination measurement (OptUSDM) on a set of states $\{\rho_i\}$ with the corresponding prior probabilities $\{\eta_i\}$ if and only if the following conditions are satisfied:

1. The POVM $\{\Pi_k^{opt}\}$ is a USD measurement on $\{\rho_i\}$

2. The probability of inconclusive result is minimal, that is, $Q(\{\Pi_k^{\mathrm{opt}}\}) = \min Q(\{\Pi_k\})$ where the minimum is taken over all USDM.

Unambiguous state discrimination is an error-free discrimination. This implies a strong constraint on the measurement. Suppose that a quantum system is prepared in a pure quantum state drawn from a collection of given states $\{|\psi_i\rangle, 1 \leq i \leq N\}$ in d-dimensional complex Hilbert space $\mathcal{H}$ with $d \geq N$. These states span a subspace $\mathcal{U}$ of $\mathcal{H}$. In order to detect the state of the system, a measurement is constructed comprising $N+1$ measurement operators $\{\Pi_i, 0 \leq i \leq N\}$. Given that the state of the system is $|\psi_i\rangle$, the probability of obtaining outcome k is $\langle\psi_i|\Pi_k|\psi_i\rangle$. Therefore, in order to ensure that each state is either correctly detected or an inconclusive result is obtained, we must have

$$\langle\psi_i|\Pi_k|\psi_i\rangle = p_i\delta_{ik}, \quad 1 \leq i, k \leq N, \tag{2-5}$$

for some $0 \leq p_i \leq 1$. Since $\Pi_0 = I_d - \sum_{i=1}^{N} \Pi_i$, we have $\langle\psi_i|\Pi_0|\psi_i\rangle = 1 - p_i$. So a system with given state $|\psi_i\rangle$, the state of the system is correctly detected with probability p_i and an inconclusive result is obtained with probability $1-p_i$. It was shown in [8] that (2-5) is satisfied if and only if the vectors $|\psi_i\rangle$ are linearly independent, or equivalently, $\dim\mathcal{U} = \dim\mathcal{H}$. Therefore, we will take this assumption throughout the paper. In this case, we may choose [27]

$$\Pi_i = p_i|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|, \quad 1 \leq i \leq N, \tag{2-6}$$

where, the vectors $|\tilde{\psi}_i\rangle \in \mathcal{U}$ are the reciprocal states associated with the states $|\psi_i\rangle$, i.e., there are unique vectors in $\mathcal{U}$ such that

$$\langle\tilde{\psi}_i|\psi_k\rangle = \delta_{ik}, \quad 1 \leq i, k \leq N. \tag{2-7}$$

With Φ and $\tilde{\Phi}$ we denote the matrices such that their columns are $|\psi_i\rangle$ and $|\tilde{\psi}_i\rangle$, respectively. Then, one can show that $\tilde{\Phi}$ is

$$\tilde{\Phi} = \Phi(\Phi\Phi^*)^{-1}. \tag{2-8}$$

Since the vectors $|\psi_i\rangle$, $i = 1, ..., N$ are linearly independent, $\Phi\Phi^*$ is always invertible. Alternatively,

$$\tilde{\Phi} = (\Phi\Phi^*)^{\ddagger}\Phi, \tag{2-9}$$

so that

$$|\tilde{\psi}_i\rangle = (\Phi\Phi^*)^{\ddagger}|\psi_i\rangle \tag{2-10}$$

where, $(.)^{\ddagger}$ denotes the Moore-Penrose pseudo-inverse [28]. The inverse is taken on the subspace spanned by the columns of the matrix. If the state $|\psi_i\rangle$ is prepared with prior probability η_i, then the total probability of correctly detecting the state is

$$P = \sum_{i=1}^{N} \eta_i \langle\psi_i|\Pi_i|\psi_i\rangle = \sum_{i=1}^{N} \eta_i p_i \tag{2-11}$$

and the probability of the inconclusive result is given by

$$Q = 1 - P = \sum_{i=1}^{N} \eta_i \langle\psi_i|\Pi_0|\psi_i\rangle = 1 - \sum_{i=1}^{N} \eta_i p_i. \tag{2-12}$$

In general, an optimal measurement for a given strategy depends on the quantum states and the prior probabilities of their appearance. In the unambiguous discrimination for a given strategy and a given ensemble of states, the goal is to find a measurement which minimizes the inconclusive result. In fact, it is known that USD (of both pure and mixed states) is a convex optimization problem . Mathematically, this means that the quantity which is to be optimized as well as the constraints on the unknowns, are convex functions. Practically, this implies that the optimal solution can be computed in an extremely efficient way. This is therefore a very useful tool. Nevertheless our aim is to understand the structure of USD in order to relate it with neat and relevant quantities and to find feasible region for numerical and analytical solutions. So, by using SDP we determine feasible region via reciprocal states and reduce the theory of the SDP to a linear programming one with a feasible region of polygon type which can be solved via simplex method exactly or approximately.

3 Unambiguous discrimination of quantum states using linear programming

The method presented in this section, seems to be a powerful method which enables us to analytically discriminate N linearly independent pure quantum states. We naturally want to the probabilities p_i be as large as possible in order to increase the detection probabilities, but their values are bounded by the demand of positivity of Π_0. Let

$$|\psi\rangle := \sum_{i=1}^{N} a_i|\psi_i\rangle + |\psi^{\perp}\rangle$$

with $\langle\psi_i \mid \psi^{\perp}\rangle = 0$, and normalization condition is defined by

$$\langle\psi \mid \psi\rangle = \sum_{i,j=1}^{N} a_i^* a_j \langle\psi_i \mid \psi_j\rangle + \langle\psi^{\perp} \mid \psi^{\perp}\rangle = 1. \tag{3-13}$$

The relation (3-13) can be written as

$$\sum_{i,j=1}^{N} a_i^* a_j G_{ij} \leq 1, \tag{3-14}$$

where, $G_{ij} = \langle\psi_i \mid \psi_j\rangle$ are matrix elements of the Gram matrix. Now we define the following vector representation

$$X := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \tag{3-15}$$

then, the Eq. (3-14) can be rewritten as the following constraint

$$X^{\dagger} G X \leq 1. \tag{3-16}$$

Positivity of $Tr(\Pi_0)$ gives

$$\sum_i a_i^2 p_i \leq 1.$$

This last condition is a decisive one that actually determines the domain of acceptable values of p_i. This result leads us to the optimization problem defined as

$$\begin{array}{ll} & \text{maximize} \quad \sum_i \eta_i p_i \leq 1 \\ \text{s.t} & \begin{cases} \sum_i a_i^2 p_i \leq 1 \\ X^{\dagger} G X \leq 1. \end{cases} \end{array} \tag{3-17}$$

If we write $p_i = \lambda^{'} \xi_i$ with $0 \leq \xi_i \leq 1$, then we will have

$$\sum_i a_i^2 p_i = \lambda^{'} \sum_i a_i^2 \xi_i \leq 1 \Rightarrow \lambda^{'} \leq \frac{1}{\sum_i a_i^2 \xi_i}. \tag{3-18}$$

Then, we must compute the maximum value of $\sum_i a_i^2\xi_i$ such that λ' possesses its lowest possible value. That is,

$$\begin{aligned} &\text{maximize} \;\; \textstyle\sum_i a_i^2\xi_i \\ &\text{s.t} \quad X^\dagger G X \leq 1. \end{aligned} \tag{3-19}$$

By defining a diagonal matrix as follows

$$D := \begin{pmatrix} \xi_1 & 0 & \dots & 0 \\ 0 & \xi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_d \end{pmatrix}, \tag{3-20}$$

we can write $\sum_i a_i^2\xi_i = X^\dagger D X$. This leads us to the following optimization problem

$$\begin{aligned} &\text{maximize} \;\; X^\dagger D X \\ &\text{s.t} \;\; X^\dagger G X \leq 1. \end{aligned} \tag{3-21}$$

Now, let $Y = \sqrt{D}X$. Then, (3-21) can be rewritten as

$$\begin{aligned} &\text{maximize} \;\; Y^\dagger Y \\ &\text{s.t} \;\; Y^\dagger D^{-\frac{1}{2}} G D^{-\frac{1}{2}} Y \leq 1. \end{aligned} \tag{3-22}$$

Suppose $D^{-\frac{1}{2}} G D^{-\frac{1}{2}} = \widehat{G}$. Then, we have

$$\begin{aligned} &\text{maximize} \;\; Y^\dagger Y \\ &\text{s.t} \;\; Y^\dagger \widehat{G} Y \leq 1. \end{aligned} \tag{3-23}$$

The determinant of $\tilde{G} - \lambda I = 0$ determines the feasible region provided that, λ coincides with λ'. Now, in order to show the ability of our method, we calculate the optimal failure probabilities corresponding to unambiguous discrimination of two and three linearly independent states with arbitrary prior probabilities. In the simple case of two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with arbitrary prior probabilities η_1 and η_2, $\widehat{G}$ is given by

$$\widehat{G} = \begin{pmatrix} \frac{1}{\sqrt{\xi_1}} & 0 \\ 0 & \frac{1}{\sqrt{\xi_2}} \end{pmatrix} \begin{pmatrix} 1 & a_{12} \\ a_{12}^* & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\xi_1}} & 0 \\ 0 & \frac{1}{\sqrt{\xi_2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\xi_1} & \sqrt{\frac{1}{\xi_1(\xi_2)}}a_{12} \\ \sqrt{\frac{1}{\xi_1(\xi_2)}}a_{12}^* & \frac{1}{\xi_2} \end{pmatrix}. \tag{3-24}$$

The characteristic equation is given by

$$\lambda^2 - (\frac{1}{\xi_1} + \frac{1}{\xi_2})\lambda + \frac{1}{\xi_1\xi_2}(1 - |a_{12}^2|) = 0. \tag{3-25}$$

Thus,

$$\xi_1\xi_2\lambda^2 - \lambda(\xi_1 + \xi_2) + (1 - |a_{12}^2|) = 0. \tag{3-26}$$

In the feasible region we have $p_1 = \lambda\xi_1$ and $p_2 = \lambda\xi_2$, then the Eq. (3-26) is equivalent to

$$p_1p_2 - (p_1 + p_2) + (1 - |a_{12}^2|) = 0. \tag{3-27}$$

This equation determines the feasible region (see Figure. 1). To calculate the minimum probability of inconclusive result we put equal the gradient of line $\eta_1p_1 + \eta_2p_2$ to the gradient of equation (3-27), then we will have

$$p_2 - 1 = \Lambda\eta_1 \quad and \quad p_1 - 1 = \Lambda\eta_2 \tag{3-28}$$

By substituting the equation (3-28) into (3-27), we obtain

$$(1 + \Lambda\eta_1)(1 + \Lambda\eta_2) - (2 + \Lambda) + (1 - a_{12}^2) = 0, \tag{3-29}$$

which implies that $\Lambda = \pm\frac{|a_{12}|}{\sqrt{\eta_1\eta_2}}$. Substituting Λ into equation (3-28), gives the following solutions:

$$p_1 = 1 \pm \sqrt{\frac{\eta_2}{\eta_1}}|a_{12}| \quad and \quad p_2 = 1 \pm \sqrt{\frac{\eta_1}{\eta_2}}|a_{12}|. \tag{3-30}$$

Since $p_i \leq 1$, thus we conclude that

$$p_1 = 1 - \sqrt{\frac{\eta_2}{\eta_1}}|a_{12}| \quad and \quad p_2 = 1 - \sqrt{\frac{\eta_1}{\eta_2}}|a_{12}|.$$

From the positivity of p_1 and p_2, we have:

$$|a_{12}| \leq \sqrt{\frac{\eta_2}{\eta_1}} \leq \frac{1}{|a_{12}|}.$$

In this case, the minimum probability of inconclusive result is:

$$Q = 1 - (\eta_1p_1 + \eta_2p_2) = 2\sqrt{\eta_1\eta_2}|a_{12}|. \tag{3-31}$$

If $\sqrt{\frac{\eta_2}{\eta_1}} < |a_{12}|$, then $p_2 = 0$ and if $\frac{1}{|a_{12}|} < \sqrt{\frac{\eta_2}{\eta_1}}$, then $p_1 = 0$. Then, by using (3-27), we obtain

$$\begin{cases} Q = \eta_2 + \eta_1 |a_{12}|^2 \quad if \quad \sqrt{\frac{\eta_2}{\eta_1}} < |a_{12}| \\ Q = 2\sqrt{\eta_1\eta_2}|a_{12}| \quad if \quad |a_{12}| \leq \sqrt{\frac{\eta_2}{\eta_1}} \leq \frac{1}{|a_{12}|} \\ Q = \eta_1 + \eta_2 |a_{12}|^2 \quad if \quad \frac{1}{|a_{12}|} < \sqrt{\frac{\eta_2}{\eta_1}} \end{cases} \tag{3-32}$$

Here, we discuss geometrical interpretation of optimal unambiguous discrimination of two pure states on Bloch sphere. One can show that the minimum inconclusive result for unambiguous discrimination of two pure states is equivalent to distance between sphere center and the line connecting ρ_1 to ρ_2 (see Figure 2). Density matrix for a pure qubit state is defined in the Bloch form as follows

$$\begin{aligned} \rho_1 &= \tfrac{1}{2}(I_2 + n_1.\sigma) \\ \rho_2 &= \tfrac{1}{2}(I_2 + n_2.\sigma). \end{aligned} \tag{3-33}$$

For unambiguous discrimination we will have

$$\begin{aligned} Tr(\Pi_1\rho_2) \to 0, \quad Tr(\Pi_2\rho_1) \to 0 \\ Tr(\Pi_1\rho_1) \to 1, \quad Tr(\Pi_2\rho_2) \to 1, \end{aligned} \tag{3-34}$$

where Π_1 and Π_2 are the POVM elements in the pure Bloch form

$$\begin{aligned} \Pi_1 &= \tfrac{1}{2}(I_2 + n'_1.\sigma) \\ \Pi_2 &= \tfrac{1}{2}(I_2 + n'_2.\sigma). \end{aligned} \tag{3-35}$$

This is clear from (3-34) that $Tr(\Pi_1\rho_2)$ is minimal if $n'_1.n_2 < 0$. Then the optimal case is attained for $n'_1 = -n_2$ and $Tr(\Pi_2\rho_1)$ is minimal if $n'_2 = -n_1$ then the POVM elements are given by

$$\begin{aligned} \Pi_1 &= \tfrac{1}{2}(I_2 - n_2.\sigma) = | \tilde{\psi}_1\rangle\langle\tilde{\psi}_1 |, \\ \Pi_2 &= \tfrac{1}{2}(I_2 - n_1.\sigma) = | \tilde{\psi}_2\rangle\langle\tilde{\psi}_2 | . \end{aligned} \tag{3-36}$$

For $\eta_1 = \eta_2 = 1/2$, using (3-32) optimum p's coefficients corresponds to $\phi_1 = \phi_2$ and p's are given by

$$p_1 = p_2 = 1 - |a_{12}| = 1 - \cos\theta, \tag{3-37}$$

where $2\theta = n_1.n_2$ and then minimum inconclusive result is equal $Q = \cos\theta$.

Regarding to figure 2 we see that $R = \cos\theta = Q$. That is the minimum inconclusive result for unambiguous discrimination of two pure states is equivalent to distance between sphere center and the line which connecting ρ_1 to ρ_2.

Now, we give analytical solution for three linearly independent normalized state vectors $|\psi_1\rangle$, $|\psi_2\rangle$ and ψ_3 in the three-dimensional complex vector space with arbitrary prior probabilities η_1, η_2 and η_3. If we consider $a_{ij} = \langle\psi_i|\psi_j\rangle$, then $\widehat{G}$ is given by

$$\widehat{G} = \begin{pmatrix} \frac{1}{\sqrt{\xi_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\xi_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\xi_3}} \end{pmatrix} \begin{pmatrix} 1 & a_{12} & a_{13} \\ a_{12}^* & 1 & a_{23} \\ a_{13}^* & a_{23}^* & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\xi_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\xi_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\xi_3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\xi_1} & \frac{a_{12}}{\sqrt{\xi_1\xi_2}} & \frac{a_{13}}{\sqrt{\xi_1\xi_3}} \\ \frac{a_{12}^*}{\sqrt{\xi_1\xi_2}} & \frac{1}{\xi_2} & \frac{a_{23}}{\sqrt{\xi_2\xi_3}} \\ \frac{a_{13}^*}{\sqrt{\xi_1\xi_3}} & \frac{a_{23}^*}{\sqrt{\xi_2\xi_3}} & \frac{1}{\xi_3} \end{pmatrix} \tag{3-38}$$

The characteristic equation is given by

$$\lambda^3 - (\frac{1}{\xi_1} + \frac{1}{\xi_2} + \frac{1}{\xi_3})\lambda^2 + (\frac{1}{\xi_1\xi_2} + \frac{1}{\xi_1\xi_3} + \frac{1}{\xi_2\xi_3} - \frac{a_{12}^2}{\xi_1\xi_2} - \frac{a_{23}^2}{\xi_2\xi_3} - \frac{a_{13}^2}{\xi_1\xi_3})\lambda$$
$$+\frac{1}{\xi_1\xi_2\xi_3}(a_{12}^2 + a_{23}^2 + a_{13}^2) - \frac{2a_{12}a_{23}a_{13}}{\xi_1\xi_2\xi_3} - \frac{1}{\xi_1\xi_2\xi_3} = 0 \tag{3-39}$$

Since for feasible region we have $p_i = \lambda\xi_i$, we have

$$p_1p_2p_3 - (p_2p_3 + p_1p_3 + p_1p_2) + (1 - a_{23}^2)p_1$$
$$+(1 - a_{13}^2)p_2 + (1 - a_{12}^2)p_3 + a_{12}^2 + a_{23}^2 + a_{13}^2 - 2a_{12}a_{22}a_{13} - 1 = 0 \tag{3-40}$$

If the gradient of the plane $\eta_1p_1 + \eta_2p_2 + \eta_3p_3$ be equal to the gradient of Eq. (3-40), we will have

$$\begin{cases} p_2p_3 - (p_2 + p_3) + 1 - a_{23}^2 = \Lambda\eta_1 \\ p_1p_3 - (p_1 + p_3) + 1 - a_{13}^2 = \Lambda\eta_2 \\ p_1p_2 - (p_1 + p_2) + 1 - a_{12}^2 = \Lambda\eta_3. \end{cases} \tag{3-41}$$

By solving the Eqs. (3-40) and (3-41), one can obtain

$$\begin{cases} \Lambda = 0, & \Lambda = \frac{a_{13}a_{23}}{\sqrt{\eta_1\eta_2}} - \frac{a_{12}(a_{13}\sqrt{\eta_1}+a_{23}\sqrt{\eta_2})}{\sqrt{\eta_1\eta_2\eta_3}} \\ \Lambda = -\frac{a_{23}^2}{\eta_1}, & \Lambda = \frac{a_{13}a_{23}}{\sqrt{\eta_1\eta_2}} + \frac{a_{12}(a_{13}\sqrt{\eta_1}+a_{23}\sqrt{\eta_2})}{\sqrt{\eta_1\eta_2\eta_3}} \\ \Lambda = -\frac{a_{13}^2}{\eta_2}, & \Lambda = -\frac{a_{13}a_{23}}{\sqrt{\eta_1\eta_2}} - \frac{a_{12}(a_{13}\sqrt{\eta_1}-a_{23}\sqrt{\eta_2})}{\sqrt{\eta_1\eta_2\eta_3}} \\ \Lambda = -\frac{a_{12}^2}{\eta_3}, & \Lambda = -\frac{a_{13}a_{23}}{\sqrt{\eta_1\eta_2}} + \frac{a_{12}(a_{13}\sqrt{\eta_1}-a_{23}\sqrt{\eta_2})}{\sqrt{\eta_1\eta_2\eta_3}}. \end{cases} \tag{3-42}$$

but, only $\Lambda = 0$ and $\Lambda = -\frac{a_{13}a_{23}}{\sqrt{\eta_1\eta_2}} + \frac{a_{12}(a_{13}\sqrt{\eta_1}-a_{23}\sqrt{\eta_2})}{\sqrt{\eta_1\eta_2\eta_3}}$ give the acceptable values for p_i. If the point of contact lies in the first octant and $0 \leq p_i \leq 1$, it gives the optimal solution. If not, then an optimal contact point occurs on one of the coordinate planes or even at one of the vertices.

Example 1: We assume that all of the prior probabilities are the same and equal to 1/3. If all of the overlaps are the same, i.e., $a_{12} = a_{13} = a_{13} = s$ where s a is real and positive number, then by using equations (3-35) and (3-36) we will obtain

$$\begin{cases} p_1 s^2 + p_3 s^2 + \Lambda/3(1-p_2) - 2s^2 + 2s^3 = 0 \\ p_1 s^2 + p_2 s^2 + \Lambda/3(1-p_3) - 2s^2 + 2s^3 = 0 \\ p_2 s^2 + p_3 s^2 + \Lambda/3(1-p_1) - 2s^2 + 2s^3 = 0 \end{cases} \tag{3-43}$$

In this case, $\Lambda = 0$ gives the optimal values $p_1 = p_2 = p_3 = 1 - s$ and $Q = s$.

We can generalize the above example by considering optimal distinguishability for WBE. That is, we consider equiangular tight frame WBE sequences (for example special Grassmanian frames)[29]. Let $\{\psi_i\}_{i=1}^N$ be an independent frame sequence such that

$$\langle \psi_i \mid \psi_j \rangle = s \quad \text{for} \quad i \neq j$$

$$\langle \psi_i \mid \psi_j \rangle = 1 \quad \text{for} \quad i = j \tag{3-44}$$

Thus, for optimal distinguishing between N independent vectors which are prepared with equal prior probabilities, we can prove (similar to example 1) that, optimal p_i's are given by

$$p_1 = ... = p_N = 1 - s.$$

One can also prove that, optimal distinguishability corresponds to equal measurement probabilities p that are equal to the minimum eigenvalue of frame operator. This can be proved by defining the frame operator as

$$S = \sum_{i=1}^{N} \mid \psi_i \rangle \langle \psi_i \mid, \tag{3-45}$$

such that

$$S_{kl} = \sum_{i=1}^{N} \langle k \mid \psi_i \rangle \langle \psi_i \mid l \rangle. \tag{3-46}$$

Then, we have $S = AA^\dagger$ where $A_{ki} = \langle k \mid \psi_i \rangle$. On the other hand we define Gramm matrix as follows

$$G = A^\dagger A. \tag{3-47}$$

One can show that S and G possess equal eigenvalues, thus we find eigenvalues of Gramm matrix instead of those of the frame operator. By using (3-45) and (3-48), one can easily see that for equiangular tight frame, the Gram matrix can be written as

$$G = I + s(C - 1), \tag{3-48}$$

where C is a matrix such that all of its matrix entries are equal to one, thus we have $C^2 = NC$. Therefore, eigenvalues of G are

$$1 + s(N - 1), \quad 1 - s, \tag{3-49}$$

with $s < 1$. Then, the minimum eigenvalue of the frame operator is equal to $1 - s$ and thus p is equal to $1 - s$.

Example 2: Consider $a_{12} = a_{13} = s_1$ and $a_{23} = s_2$, where both s_1 and s_2 are real and positive. Then by using equations (3-40) and (3-41) we will have

$$\left\{ \begin{array}{c} p_1 s_2^2 + p_3 s_1^2 + \Lambda/3(1 - p_2) - s_1^2 - s_2^2 + 2s_1^2 s_2 = 0 \\ p_1 s_2^2 + p_2 s_1^2 + \Lambda/3(1 - p_3) - s_1^2 - s_2^2 + 2s_1^2 s_2 = 0 \\ p_2 s_1^2 + p_3 s_1^2 + \Lambda/3(1 - p_1) - 2s_1^2 + 2s_1^2 s_2 = 0. \end{array} \right. \tag{3-50}$$

By solving the above equations we obtain:

$$\Lambda = 3s_1^2 - 6s_1s_2 \Rightarrow \begin{cases} p_1 = 1 - 2s_1 \\ p_2 = p_3 = s_2 - s_1 + 1 \end{cases} \Rightarrow Q = 2/3(2s_1 - s_2),$$

$$\Lambda = 0 \Rightarrow \begin{cases} p_1 = \frac{s_2 - s_1^2}{s_2} \\ p_2 = p_3 = 1 - s_2 \end{cases} \Rightarrow Q = 1/3(\frac{s_1^2}{s_2} + 2s_2). \quad (3\text{-}51)$$

One of the $\Lambda = 3s_1^2 - 6s_1s_2$ and $\Lambda = 0$ which gives smaller inconclusive answer, gives the optimal value provided that lies in the feasible region. If is not, then a contact point that it is optimal occurs on one of the coordinate plane or even at one of the vertices. Here we give two numerical examples such that, in one of them the contact point in equation (3-51) is optimal solution, whereas in another one it is not. First, consider the case in which the ensemble consists of three linearly independent states with equal prior probabilities 1/3, where

$$|\psi_1\rangle = [1, 0, 0]^T \quad |\psi_2\rangle = \tfrac{1}{3}[1, 2, 2]^T \quad |\psi_3\rangle = \tfrac{1}{3}[1, 2, -2]^T. \quad (3\text{-}52)$$

and $s_1 = \frac{1}{3}, s_2 = \frac{1}{9}$. Then, the optimal solution is given by

$$p_1 = \frac{1}{3}, \quad p_2 = p_3 = \frac{7}{9} \quad \text{and} \quad \mathrm{Q} = \frac{10}{27}. \quad (3\text{-}53)$$

In figure 3 we try to show the feasible region of this case which is a convex region.

As an another example, we consider the case in which the ensemble consists of three state vectors with equal probabilities 1/3, where

$$|\psi_1\rangle = \tfrac{1}{\sqrt{3}}[1, 1, 1]^T \quad |\psi_2\rangle = \tfrac{1}{\sqrt{2}}[1, 1, 0]^T \quad |\psi_3\rangle = \tfrac{1}{\sqrt{2}}[0, 1, 1]^T. \quad (3\text{-}54)$$

In this example the contact point occurs on one of the coordinate plane and optimal solution is given by:

$$p_1 = 0, \quad p_2 = p_3 = \frac{1}{6} \quad \text{and} \quad \mathrm{Q} = \frac{1}{9}. \quad (3\text{-}55)$$

Figure 4 shows the feasible region of this example.

3.1 Equal-probability measurement

A simple measurement that has been employed for unambiguous state discrimination is the measurement with $p_i = p$, for all $i = 1, 2, ..., N$. This measurement results in equal probability of correctly detecting each of the states and is called Equal-probability measurement (EPM). Using the feasible region, we are able to calculate the prior probabilities, so that EPM is optimal. Using equations (3-40) and (3-41), the prior probabilities in the optimal EPM measurement for unambitious discrimination of three states are given by

$$\eta_1 = \frac{(a_{12}^2 - 1)^2}{3 - 2a_{12}^2 - a_{13}^2 - a_{12}^2 a_{13}^2 - 2a_{23} + 2a_{23}a_{12}^2 + a_{12}^4}$$

$$\eta_2 = \frac{1 - a_{12}^2 - a_{13}^2 + a_{12}^2 a_{13}^2}{3 - 2a_{12}^2 - a_{13}^2 - a_{12}^2 a_{13}^2 - 2a_{23} + 2a_{23}a_{12}^2 + a_{12}^4}$$

$$\eta_3 = 1 - \eta_1 - \eta_2. \tag{3-56}$$

In this case the value of p is calculated from equation (3-40). If we consider more than three states, the functionality of the η_i in terms of $\{a_{ij}\}$ are too complicated to be written down, so it is not included here. However, if we consider geometrically uniform states, the problem will be easy. Let $S = \{|\psi_i\rangle = U_i|\psi\rangle, U_i \in \mathcal{G}\}$ be a set of geometrically uniform (GU) states generated by a finite group $\mathcal{G}$ of unitary matrices, where $|\psi\rangle$ is an arbitrary state. Now, let Φ be the matrix with columns $|\psi_i\rangle$. Then, the measurement which minimizes the probability of an inconclusive result could be reduced to an equal-probability measurement [27] and consists of the measurement operators

$$\Pi_i = p_i|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i\ |, \tag{3-57}$$

where, p_i is the inverse of the maximum eigenvalue of frame operator [27] and $|\ \tilde{\psi}_i\rangle = U_i\ |\ \tilde{\psi}\rangle, \quad U_i \in \mathcal{G}$ with

$$|\ \tilde{\psi}\rangle = (\Phi\Phi^*)^{-1}\ |\ \psi\rangle, \tag{3-58}$$

In this case, using the feasible region it is easy to show that for optimal EPM measurement all of prior probabilities are equal. In general, similar to the example 1, one can prove that for optimal EPM measurement all of p_i's are equal to the inverse of maximum eigenvalue of frame operator.

Example3: We now consider an example of a set of GU states. Consider the group $\mathcal{G}$ of 4 unitary matrices U_i, where

$$U_1 = I_4, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad U_4 = U_2U_3. \tag{3-59}$$

Now, let the set of GU states is given by $S = \{|\psi_i\rangle\} = U_i|\psi\rangle, 1 \leq i \leq 4\}$ with $|\psi\rangle = 1/(3\sqrt{2})[2,2,1,3]^T$. Then, we obtain

$$\Phi = \frac{1}{3\sqrt{2}} \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & -2 \\ 1 & 1 & -1 & -1 \\ 3 & -3 & -3 & 3 \end{pmatrix} \tag{3-60}$$

It should be noticed that, the reciprocal states $|\tilde{\psi}_i\rangle = U_i|\tilde{\psi}\rangle$ for $i = 1, ..., 4$ with

$$\mid \tilde{\psi}\rangle = (\Phi\Phi^*)^\dagger \mid \psi\rangle = \frac{1}{4\sqrt{2}} \begin{pmatrix} 3 \\ 3 \\ 6 \\ 2 \end{pmatrix}, \tag{3-61}$$

are also GU states with generating group $\mathcal{G}$. Therefore, we can provide the elements of POVM as $\Pi_i = p_i \mid \tilde{\psi}_i\rangle\langle\tilde{\psi}_i \mid$. Using feasible region it easy to show that the optimal contact point is given by

$$p_1 = p_2 = p_3 = p_4 = \frac{2}{9}, \tag{3-62}$$

then the equal probability measurement operators are given by

$$\Pi_i = \frac{2}{9} \mid \tilde{\psi}_j\rangle\langle\tilde{\psi}_i \mid, \quad i = 1, ..., 4 \tag{3-63}$$

where these results are in agreement with those of Ref.[27].

4 Lewnstein-Sanpera decomposition (LSD) as an optimal unambiguous discrimination

The idea of Refs. [30, 31] is based on the method of subtracting projections on product vectors from a given state, that is, for a given density matrix ρ and any set $V = \{|\tilde{\psi}_i\rangle\}$ of the states belonging to the range of ρ, one can subtract a density matrix $\rho' = \sum_i p_i \Pi_i$ (not necessarily normalized) with $p_i \geq 0$ such that $\delta\rho = \rho - \rho' \geq 0$, in the sense that $Tr(\rho') \leq 1$.

In the following we recall some important definitions and theorems about LSD.

Definition 1. A non-negative parameter p is called maximal with respect to a (not necessarily normalized) density matrix ρ and the projection operator $\Pi = | \tilde{\psi}\rangle\langle\tilde{\psi} |$ iff $\rho - p\Pi \geq 0$, and for every $\epsilon \geq 0$, the matrix $\rho - (p+\epsilon)\Pi$ is not positive definite. The maximal p thus determines the maximal contribution of Π that can be subtracted from ρ maintaining the non-negativity of the difference.

Lemma 1. p is maximal with respect to ρ and $\Pi = |\tilde{\psi}\rangle\langle\tilde{\psi}|$ iff:

1. if $|\tilde{\psi}\rangle \notin \mathcal{R}(\rho)$ then $p = 0$,

2. if $|\tilde{\psi}\rangle \in \mathcal{R}(\rho)$ then

$$0 < p = \frac{1}{\langle\tilde{\psi}|\frac{1}{\rho}|\tilde{\psi}\rangle}. \tag{4-64}$$

Definition 2. We say that a pair of non-negative (p_1, p_2) is *maximal* with respect to ρ and a pair of projection operators $\Pi_1 = |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|$, $\Pi_2 = |\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|$ iff $\rho - p_1\Pi_1 - p_2\Pi_2 \geq 0$, p_1 is maximal with respect to $\rho - p_2\Pi_2$ and to the projector Π_1, p_2 is maximal with respect to $\rho - p_1\Pi_1$ and to the projector Π_2, and the sum $p_1 + p_2$ is maximal.

Lemma 2. A pair (p_1, p_2) is maximal with respect to ρ and a pair of projectors (Π_1, Π_2) iff: (a) if $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle$ do not belong to $\mathcal{R}(\rho)$ then $p_1 = p_2 = 0$; (b) if $|\tilde{\psi}_1\rangle$ does not belong to $\mathcal{R}(\rho)$, while $|\tilde{\psi}_2\rangle \in \mathcal{R}(\rho)$ then $p_1 = 0$, $p_2 = \langle\tilde{\psi}_2|\frac{1}{\rho}|\tilde{\psi}_2\rangle^{-1}$; (c) if $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle \in \mathcal{R}(\rho)$ and $\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle = 0$ then $p_i = \langle\tilde{\psi}_i|\frac{1}{\rho}|\tilde{\psi}_i\rangle$, $i = 1, 2$; (d) finally, if $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle \in \mathcal{R}(\rho)$ and $\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle \neq 0$ then

$$p_1 = \frac{1}{D}\left(\langle\tilde{\psi}_2|\frac{1}{\rho}|\tilde{\psi}_2\rangle - |\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle|\right), \tag{4-65}$$

$$p_2 = \frac{1}{D}\left(\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_1\rangle - |\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle|\right), \tag{4-66}$$

where $D = \langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_1\rangle\langle\tilde{\psi}_2|\frac{1}{\rho}|\tilde{\psi}_2\rangle - |\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle|^2$.

Lemma 3. Let a hermitian density matrix ρ has a decomposition of the form $\rho = \rho' + (1-p)\delta\rho$, where ρ' is a part of density operator ρ which has the structure $\rho' = \sum_{i=1}^{n} p_i \Pi_i$, with Π_i being projection operator onto state $|\tilde{\psi}_i\rangle$ and $\sum_{i=1}^{n} p_i = 1$. Then the set of $\{p_i\}$, which are maximal with respect to the density matrix ρ and the set of the projection operators $\{\Pi_i\}$ form a manifold which generically has dimension $n-1$ and is determined by the following equation :

$$1 - \sum_i D_i p_i + \sum_{i<j} D_{ij} p_i p_j - \sum_{ijk} D_{ijk} p_i p_j p_k + ... + (-1)^n \sum_{i_1,...,i_n} p_{i_1} p_{i_2} ... p_{i_n} D_{i_1 i_2 ... i_n} = 0 \tag{4-67}$$

where the set of $\{D_{i_1,i_2...i_m}\}$ are the subdeterminants (minors) of matrix D defined by

$$D = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & ... & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & ... & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix}, \tag{4-68}$$

with $\tilde{a}_{ij} := \langle\tilde{\psi}_i \mid \frac{1}{\rho} \mid \tilde{\psi}_j\rangle$. Equation (4-67) determines feasible region via reciprocal states, that is it gives the domain of acceptable values of p_i. One way to drive the equation (4-67) is using semidefinite programming [15, 17].

In the rest of this section, we show that optimal unambiguous discrimination for N linearly independent states can be reduced to LSD method. Suppose a quantum system is prepared in a state secretly drawn from a known set $|\psi_1\rangle, ..., |\psi_N\rangle$ where each $|\psi_i\rangle$ is a pure state in the Hilbert space $\mathcal{H}$. In order to discriminate $|\psi_1\rangle, ..., |\psi_N\rangle$ unambiguously, one can construct a most general POVM consisting of $N+1$ elements $\Pi_0, \Pi_1, ..., \Pi_N$ such that

$$\Pi_i \geq 0 \quad , \quad i = 0, 1, ..., N, \quad \text{and} \quad \sum_{i=0}^{N} \Pi_i = I, \tag{4-69}$$

where I denotes the identity matrix in $\mathcal{H}$. Each element Π_i, $i = 1, ..., N$ of POVM corresponds to an identification of the corresponding state $|\psi_i\rangle$, while Π_0 corresponds to the inconclusive answer. For the sake of simplicity, we often specify only $\Pi_1, ..., \Pi_N$ for a given POVM since the left element Π_0 is uniquely determined by

$$\Pi_0 = I - \sum_{i=1}^{N} \Pi_i = I - \sum_i p_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|. \tag{4-70}$$

The goal of LSD is maximizing p_i's such that $\sum_{i=1}^{N} p_i$ is maximized. With assuming that density matrix in LSD method is equal to identity we obtain LSD as follows

$$\rho - \sum \Pi_i = I - \sum_i p_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|. \tag{4-71}$$

With comparison this relation and (4-70) it is clear that we can maximize success probability by using LSD method or minimize the inconclusive probability is minimized. Then we say that LSD is the same as OptUSDM and we use LSD in order to obtain the elements of the optimal POVM.

4.1 Analytical calculation of optimal POVM for unambiguous discrimination of quantum states via Lewnstein-Sanpera decomposition

In this section an analytical solution for unambiguous discrimination of two states by using Lewnstein-Sanpera decomposition is presented. For three linearly independent states, as the LSD method leads to a set of coupled equations which are in general difficult to solve, we embark on KKT (see Appendix I) method which makes the problem strongly easy. Since this condition is necessary and sufficient, then the answer will be exactly optimal for unambiguous discrimination.

4.1.1 Optimal unambiguous discrimination of two states

Suppose that, two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with arbitrary prior probabilities η_1 and η_2 are given. In order to obtain optimal POVM set for these two states by using LSD, we use the Lemma 2 of LSD for two states $|\tilde{\psi}_1\rangle$ and $|\tilde{\psi}_2\rangle$. Let corresponding density matrix in Hilbert space $\mathcal{H}$ is identity operator, i.e., $\rho = I$, and $\Pi_1' = \eta_1|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|$ and $\Pi_2' = \eta_2|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|$. Then, a pair (p_1, p_2) is maximal with respect to ρ and the pair of operators Π_1' and Π_2' iff $\rho - p_1\eta_1|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| - p_2\eta_2|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2| \geq 0$. Therefore, from Lemma 2 of LSD, the maximal pair (p_1, p_2) is given by

$$\begin{array}{c} p_1 = \frac{\tilde{a}_{22} - \sqrt{\frac{\eta_2}{\eta_1}}|\tilde{a}_{12}|}{\tilde{a}_{11}\tilde{a}_{22} - |\tilde{a}_{12}|^2} \\ p_2 = \frac{\tilde{a}_{11} - \sqrt{\frac{\eta_1}{\eta_2}}|\tilde{a}_{12}|}{\tilde{a}_{11}\tilde{a}_{22} - |\tilde{a}_{12}|^2}, \end{array} \tag{4-72}$$

where $\tilde{a}_{ij} = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle$. If the condition

$$\frac{|\tilde{a}_{12}|^2}{|\tilde{a}_{11}|^2 + |\tilde{a}_{12}|^2} \leq \eta_1 \leq \frac{|\tilde{a}_{11}|^2}{|\tilde{a}_{11}|^2 + |\tilde{a}_{12}|^2}, \tag{4-73}$$

is hold, then equation (4-72) is optimal solution.

If $\eta_1 \leq \eta_2$, then the optimal solution reads as

$$p_1 = 0, \quad p_2 = \frac{1}{\tilde{a}_{22}}. \tag{4-74}$$

If $\eta_2 \leq \eta_1$; then the optimal solution is given by

$$p_2 = 0, \quad p_1 = \frac{1}{\tilde{a}_{11}}. \tag{4-75}$$

Examples 4: For an example we consider following case: Alice gives Bob a qubit repaired in one of two states

$$|\psi_1\rangle = |\,0\rangle \quad , \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle + |\,1\rangle). \tag{4-76}$$

Since the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are non-orthogonal, there is no measurement that can distinguish them. In order to obtain optimal POVM, we define dual basis $|\tilde{\psi}_j\rangle$ as follows

$$|\tilde{\psi}_1\rangle = (|\,0\rangle - |\,1\rangle) \quad , \quad |\tilde{\psi}_2\rangle = \sqrt{2}\,|\,1\rangle. \tag{4-77}$$

Therefore, pairs p_1 and p_2 are given by

$$p_1 = p_2 = \frac{2-\sqrt{2}}{2},$$

and finally the elements of optimal POVM are obtained as

$$\Pi_1 = \frac{2-\sqrt{2}}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|), \quad \Pi_2 = (2-\sqrt{2})|1\rangle\langle 1|$$

$$\Pi_0 = 1 - \Pi_1 - \Pi_2.$$

4.1.2 Optimal unambiguous discrimination of three linearly independent states

Now, we consider three linearly independent normalized state vectors ψ_1 , ψ_2 and ψ_3 with arbitrary prior probabilities η_1, η_2 and η_3 in the three-dimensional complex vector space. The KKT conditions for unambiguous discrimination of three states are given by

$$\begin{cases} I - p_1|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| - p_2|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2| - p_3|\tilde{\psi}_3\rangle\langle\tilde{\psi}_3| \geq 0 \\ p_1 \geq 0, p_2 \geq 0, p_3 \geq 0 \\ (I - p_1|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| - p_2|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2| - p_3|\tilde{\psi}_3\rangle\langle\tilde{\psi}_3|)X = X(I - p_1|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| - p_2|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2| - p_3|\tilde{\psi}_3\rangle\langle\tilde{\psi}_3|) = 0 \\ z_1 p_1 = 0, \quad z_2 p_2 = 0, \quad z_3 p_3 = 0, \quad z_1 \geq 0, z_2 \geq 0, z_3 \geq 0 \\ Tr(X|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|) = z_1 + \eta_1, \quad Tr(X|\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|) = z_2 + \eta_2, Tr(X|\tilde{\psi}_3\rangle\langle\tilde{\psi}_3|) = z_3 + \eta_3 \quad \eta_1 \geq 0, \eta_2 \geq 0, \eta_3 \geq 0. \end{cases} \tag{4-78}$$

After some calculation, one can obtain

$$[(1-p_2\tilde{a}_{22})(1-p_3\tilde{a}_{33}) - p_2p_3|\tilde{a}_{23}|^2][(1-p_1\tilde{a}_{11}) - p_2\tilde{a}_{12}\sqrt{\frac{\eta_2}{\eta_1}} - p_3\tilde{a}_{13}\sqrt{\frac{\eta_3}{\eta_1}}] = 0$$

$$[(1-p_1\tilde{a}_{11})(1-p_3\tilde{a}_{33}) - p_1p_3|\tilde{a}_{13}|^2][(1-p_2\tilde{a}_{22}) - p_1\tilde{a}_{12}\sqrt{\frac{\eta_1}{\eta_2}} - p_3\tilde{a}_{23}\sqrt{\frac{\eta_3}{\eta_2}}] = 0$$

$$[(1-p_1\tilde{a}_{11})(1-p_2\tilde{a}_{22}) - p_1p_2|\tilde{a}_{12}|^2][(1-p_3\tilde{a}_{33}) - p_2\tilde{a}_{23}\sqrt{\frac{\eta_2}{\eta_3}} - p_1\tilde{a}_{13}\sqrt{\frac{\eta_1}{\eta_3}}] = 0. \tag{4-79}$$

With the following conditions

$$\begin{cases} (1-p_1\tilde{a}_{11}) \geq \frac{p_2|\tilde{a}_{12}|^2 + p_3|\tilde{a}_{13}|^2}{\tilde{a}_{11}} \\ (1-p_2\tilde{a}_{22}) \geq \frac{p_1|\tilde{a}_{12}|^2 + p_3|\tilde{a}_{23}|^2}{\tilde{a}_{22}} \\ (1-p_3\tilde{a}_{33}) \geq \frac{p_1|\tilde{a}_{13}|^2 + p_2|\tilde{a}_{23}|^2}{\tilde{a}_{33}}, \end{cases}, \qquad p_1 \geq 0, \quad p_2 \geq 0, \quad p_3 \geq 0 \tag{4-80}$$

the optimal answer is attained as

$$p_1 = \frac{(\tilde{a}_{22}\tilde{a}_{33} - |\tilde{a}_{23}|^2) - \sqrt{\frac{\eta_2}{\eta_1}}(\tilde{a}_{21}\tilde{a}_{33} - \tilde{a}_{31}\tilde{a}_{23}) + \sqrt{\frac{\eta_3}{\eta_1}}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{22}\tilde{a}_{31})}{\tilde{a}_{11}(\tilde{a}_{22}\tilde{a}_{33} - |\tilde{a}_{23}|^2) - a_{12}(\tilde{a}_{21}\tilde{a}_{33} - \tilde{a}_{31}\tilde{a}_{23}) + \tilde{a}_{13}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{22}\tilde{a}_{31})}$$
$$p_2 = \frac{(\tilde{a}_{11}\tilde{a}_{33} - |\tilde{a}_{13}|^2) - \sqrt{\frac{\eta_1}{\eta_2}}(\tilde{a}_{12}\tilde{a}_{33} - \tilde{a}_{32}\tilde{a}_{13}) + \sqrt{\frac{\eta_3}{\eta_2}}(\tilde{a}_{11}\tilde{a}_{32} - \tilde{a}_{31}\tilde{a}_{12})}{\tilde{a}_{11}(\tilde{a}_{22}\tilde{a}_{33} - |\tilde{a}_{23}|^2) - \tilde{a}_{12}(\tilde{a}_{21}\tilde{a}_{33} - \tilde{a}_{31}\tilde{a}_{23}) + \tilde{a}_{13}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{22}\tilde{a}_{31})}$$
$$p_3 = \frac{(\tilde{a}_{11}\tilde{a}_{22} - |\tilde{a}_{12}|^2) + \sqrt{\frac{\eta_1}{\eta_3}}(\tilde{a}_{12}\tilde{a}_{23} - \tilde{a}_{22}\tilde{a}_{13}) - \sqrt{\frac{\eta_2}{\eta_3}}(\tilde{a}_{11}\tilde{a}_{23} - \tilde{a}_{21}\tilde{a}_{13})}{\tilde{a}_{11}(\tilde{a}_{22}\tilde{a}_{33} - |\tilde{a}_{23}|^2) - \tilde{a}_{12}(\tilde{a}_{21}\tilde{a}_{33} - \tilde{a}_{31}\tilde{a}_{23}) + \tilde{a}_{13}(\tilde{a}_{21}\tilde{a}_{32} - \tilde{a}_{22}\tilde{a}_{31})}. \quad (4\text{-}81)$$

If the conditions (4-82) do not satisfied, then we search another optimal solution. If

$$\frac{|\tilde{a}_{23}|^2}{|\tilde{a}_{22}|^2 + |\tilde{a}_{23}|^2} \le \eta_2 \le \frac{|\tilde{a}_{22}|^2}{|\tilde{a}_{22}|^2 + |\tilde{a}_{23}|^2}, \quad (4\text{-}82)$$

is hold, the optimal solation is given by

$$\begin{cases} p_1 = 0 \\ p_2 = \frac{|\tilde{a}_{33}| - \sqrt{\frac{\eta_3}{\eta_2}}|\tilde{a}_{23}|}{|\tilde{a}_{22}||\tilde{a}_{33}| - |\tilde{a}_{23}|^2} \\ p_3 = \frac{|\tilde{a}_{22}| - \sqrt{\frac{\eta_2}{\eta_3}}|\tilde{a}_{23}|}{|\tilde{a}_{22}||\tilde{a}_{33}| - |\tilde{a}_{23}|^2}. \end{cases} \quad (4\text{-}83)$$

If

$$\frac{|\tilde{a}_{13}|^2}{|\tilde{a}_{11}|^2 + |\tilde{a}_{13}|^2} \le \eta_1 \le \frac{|\tilde{a}_{11}|^2}{|\tilde{a}_{11}|^2 + |\tilde{a}_{13}|^2}, \quad (4\text{-}84)$$

is hold, the optimal answer is given by

$$\begin{cases} p_1 = \frac{|\tilde{a}_{33}| - \sqrt{\frac{\eta_3}{\eta_1}}|\tilde{a}_{13}|}{|\tilde{a}_{11}||\tilde{a}_{33}| - |\tilde{a}_{13}|^2} \\ p_2 = 0 \\ p_3 = \frac{|\tilde{a}_{11}| - \sqrt{\frac{\eta_1}{\eta_3}}|\tilde{a}_{13}|}{|\tilde{a}_{11}||\tilde{a}_{33}| - |\tilde{a}_{13}|^2}. \end{cases} \quad (4\text{-}85)$$

If the condition

$$\frac{|\tilde{a}_{12}|^2}{|\tilde{a}_{11}|^2 + |\tilde{a}_{12}|^2} \le \eta_1 \le \frac{|\tilde{a}_{11}|^2}{|\tilde{a}_{11}|^2 + |\tilde{a}_{12}|^2}, \quad (4\text{-}86)$$

is satisfied, the optimal solution reads as

$$\begin{cases} p_1 = \frac{\tilde{a}_{22} - \sqrt{\frac{\eta_2}{\eta_1}}|\tilde{a}_{12}|}{\tilde{a}_{11}\tilde{a}_{22} - |\tilde{a}_{12}|^2} \\ p_2 = \frac{\tilde{a}_{11} - \sqrt{\frac{\eta_1}{\eta_2}}|\tilde{a}_{12}|}{\tilde{a}_{11}\tilde{a}_{22} - |\tilde{a}_{12}|^2}. \\ p_3 = 0 \end{cases} \quad (4\text{-}87)$$

If none of the above conditions is hold, in this case two of p_i vanish and the optimal POVM will be the same as Von Numan measurement.

Example 5: In this example we consider reciprocal independent states $\{\tilde{\psi}_i\}_{i=1}^{N}$ such that

$$\langle\tilde{\psi}_i \mid \tilde{\psi}_j\rangle = a \quad \text{for} \quad i \neq j$$

$$\langle\tilde{\psi}_i \mid \tilde{\psi}_j\rangle = 1 \quad \text{for} \quad i = j, \tag{4-88}$$

Thus, for optimal distinguishing of independent vectors that are prepared with equal probabilities, we minimize the inconclusive result Π_0 given by

$$\Pi_0 = I - \sum_{i=1}^{N} p_i \mid \tilde{\psi}_i\rangle\langle\tilde{\psi}_i \mid . \tag{4-89}$$

Now using LS's theorem and some analytical calculations we show that all of p_i's are equal, i.e., all of the elements of POVM possess equal probabilities, i.e.,

$$p_1 = p_2 = ... = p_N = \frac{1}{1 + a(N-1)}. \tag{4-90}$$

Similar to GU states, one can prove that the optimal distinguishability corresponds to equal measurement probabilities $p_i = p$ for all i such that p is equal to the inverse of maximum eigenvalue of the corresponding frame operator. In order to prove this fact, we define frame operator as

$$S = \sum_{i=1}^{N} \mid \tilde{\psi}_i\rangle\langle\tilde{\psi}_i \mid, \tag{4-91}$$

such that

$$S_{kl} = \sum_{i=1}^{N} \langle k \mid \tilde{\psi}_i\rangle\langle\tilde{\psi}_i \mid l\rangle. \tag{4-92}$$

Then, $S = AA^{\dagger}$ with $A_{ki} = \langle k \mid \tilde{\psi}_i\rangle$. On the other hand, the Gramm matrix is defined as follows

$$\tilde{G} = A^{\dagger}A. \tag{4-93}$$

Again, one can easily show that S and G have equal eigenvalues, thus we evaluate eigenvalues of Gramm matrix . The Gramm matrix $\tilde{G}$ can be written as

$$\tilde{G} = I + a(C - 1), \tag{4-94}$$

such that its eigenvalues are given by

$$\begin{cases} 1 + a(N-1) \\ \quad 1 - a, \end{cases} \tag{4-95}$$

where $a < 1$. Therefore, maximum eigenvalue of frame operator is equal to $1 + a(N-1)$ and thus p is given by $(1 + a(N-1))^{-1}$.

4.2 Discrimination of quantum states using approximated linear programming

As solving the problem analytically is so hard, approximated methods are useful for unambiguous discrimination of N linearly independent quantum states. The simplex method is the easiest way of solving it. The simplex algorithm is a common algorithm used to solve an optimization problem with a polytope feasible region, such as a linear programming problem. It is an improvement over the algorithm to test all feasible solution of the convex feasible region and then choose the optimal feasible solution. It does this by moving from one vertex to an adjacent vertex, such that the objective function is improved. This algorithm still guarantees that the optimal point will be discovered. In addition, only in the worst case scenario all vertices will be tested. Here, considering the scope of this paper, a complete treatment of the simplex algorithm is unnecessary; for a more complete treatment please refer to any LP text such as [32, 33]. In the following, we give some examples and compare the result with analytical solution. Suppose that, the quantum system is prepared in one of the two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with arbitrary prior probabilities η_1 and η_2. To unambiguously discrimination, one should calculate the optimal p_i's by solving equation $(3-27)$. According the equation $(3-27)$, the three extremal points

$$\begin{cases} \quad p_1 = 0 \\ p_2 = 1 - |\langle\psi_1|\psi_2\rangle|^2 \end{cases}, \quad \begin{cases} \quad p_2 = 0 \\ p_1 = 1 - |\langle\psi_1|\psi_2\rangle|^2 \end{cases}, \quad \Big\{ \; p_1 = p_2 = 1 - |\langle\psi_1|\psi_2\rangle|. \tag{4-96}$$

together with the origin ($p_1 = p_2 = 0$) form a polygon which surrounds the feasible region. Since the feasible region is not linear(see Figure 1) while the polygon is linear, so we use approximated simplex method to find optimal answer. For three linearly independent normalized state vectors $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ with arbitrary prior probabilities η_1, η_2 and η_3 in the three-dimensional complex vector space, there is seven extremal points. These points together with the origin form a polygon. The optimal solutions, resulting from the LP method, are consistent to that ones obtained analytically in sections 3 and 4 with high accuracy.

Since obtaining analytical solution for more than three states is difficult, the approximated LP method seems to be useful for optimal unambiguous discrimination of N linearly independent states. This method, not only enhance the speed of calculation but also is done with high precision.

5 Conclusion

Here in this work by reducing the theory of the semi-definite programming to a linear programming one with a feasible region of polygon type which can be solved via simplex method, we have been able to obtain optimal measurements to unambiguous discrimination of an arbitrary number of pure linearly independent quantum states and using the close connection between the Lewenstein-Sanpera decomposition and semi-definite programming, we have been able to obtain the optimal positive operator valued measure for some of the well known examples via Lewenstein-Sanpera decomposition method. Unambiguous discrimination of mixed states is under investigation.

Appendix I:

1 Karush-Kuhn-Tucker (KKT) theorem:

Assuming that functions g_i, h_i are differentiable and that strong duality holds, there exists vectors $\zeta \in R^k$, and $y \in R^m$, such that the gradient of dual Lagrangian $L(x^*, \zeta^*, y^*) = f(x^*) + \sum_i \zeta_i^* h_i(x^*) + \sum_i y_i^* g_i(x^*)$ over x vanishes at x^* :

$$h_i(x^*) = 0 \quad \text{(primal feasible)}$$

$$g_i(x^*) \leq 0 \quad \text{(primal feasible)}$$

$$y_i^* \geq 0 \quad \text{(dual feasible)}$$

$$y_i^* g_i(x^*) = 0$$

$$\bigtriangledown f(x^*) + \sum_i \zeta_i^* \bigtriangledown h_i(x^*) + \sum_i y_i^* \bigtriangledown g_i(x^*) = 0. \tag{I-1}$$

Then x^* and (ζ_i^*, y_i^*) are primal and dual optimal with zero duality gap. In summary, for any convex optimization problem with differentiable objective and constraint functions, the points which satisfy the KKT conditions are primal and dual optimal, and have zero duality gap. Necessary KKT conditions satisfied by any primal and dual optimal pair and for convex problems, KKT conditions are also suffcient. If a convex optimization problem with differentiable objective and constraint functions satisfies Slaters condition, then the KKT conditions provide necessary and suffcient conditions for optimality: Slaters condition implies that the optimal duality gap is zero and the dual optimum is attained, so x is optimal if and only if there are (ζ_i^*, y_i^*) such that, together with x satisfy the KKT conditions.

1.1 Slaters condition:

Suppose x^* solves

$$\text{minimize} \quad f(x) g_i(x) \geq b_i, i = 1, ..., m, \tag{I-2}$$

and the feasible set is non empty. Then there is a non-negative vector ζ such that for all x

$$L(x,\zeta) = f(x) + \zeta^T(b - g(x)) \leq f(x^*) = L(x^*,\zeta). \tag{I-3}$$

In addition, if $f(.), g_i(.), i = 1, ..., m$ are continuously differentiable, then

$$\frac{\partial f(x^*)}{\partial(x_j)} - \zeta\frac{\partial g(x^*)}{\partial(x)} = 0. \tag{I-4}$$

In the spatial case the vector x is a solution of the linear program

$$\text{minimize} \quad c^T x$$

$$\text{s.t} \quad Ax = bx \geq 0, \tag{I-5}$$

if and only if there exist vectors $\zeta \in R^k$, and $y \in R^m$ for which the following conditions hold for $(x,\zeta,y) = (x^*,\zeta^*,y^*)$

$$A^T\zeta + y = c \quad Ax = b \quad x_i \geq 0; y_i \geq 0;\ x_i y_i = 0, i = 1, ..., m. \tag{I-6}$$

A solution (x^*,ζ^*,y^*) is called strictly complementary, if $x^* + y^* > 0$, i. e., if there exists no index $i \in 1, ..., m$ such that $x_i^* = y_i^* = 0$.

References

[1] C.W. Helstrom, Quantum Detection and Estimation Theory (Academic Press, New York, (1976).

[2] I.D. Ivanovic, Phys. Lett. A 123, 257 (1987).

[3] D. Dieks, Phys. Lett. A 126, 303 (1988).

[4] A.Peres and D.R. Terno: J. phys.A 31, 7105(1998).

[5] G. Jaeger, A. Shimony, Phys. Lett A 197, 83(1995).

[6] L. M. Duan and G. C. Guo: Phy. Rev. Lett. 80, 4999(1998).

[7] Y. Sun, M. Hillery, and J. A. Bergou: Phys. Rev. A 64, 022311(2001).

[8] A. Chefles, Phys. Lett. A, vol. 239, pp. 339, (1998).

[9] M. X. Goemans and D. P. Williamson. Journal of the Association for Computing Machinery, 42(6): 1115, (1995).

[10] Z. Q. Luo. Mathematical Programming Series B, 97, 177 (2003).

[11] T. N. Davidson, Z.-Q. Luo, and K. M. Wong. IEEE Transactions on Signal Processing, 48(5), 1433 (2000).

[12] W. K. Ma, T. N. Davidson, K. M. Wong, Z.-Q. Luo, and P. C. Ching. IEEE Transactions on Signal Processing, 50, 912 (2002).

[13] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. Society for Industrial and Applied Mathematics, (1994).

[14] H. Barnum, M. Saks, and M. Szegedy. In Proceedings of the 18th IEEE Annual Conference on Computational Complexity, pages 179193 (2003).

[15] M. A. Jafarizadeh, M. Mirzaee, M. Rezaee. 2004. International Journal of quantum Information. Vol 2, no4,541 (2004)

[16] M. A. Jafarizadeh, M. Mirzaee, M. Rezaee. International Journal of quantum Information. Vol 3, no4, 511 (2005)

[17] M. A. Jafarizadeh, M. Mirzaee, M. Rezaee .Quantum Information Processing, Vol.4, No.3. 199 (2005).

[18] M. Mirzaee, M. Rezaee, M. A. Jafarizadeh, International Journal of Theoretical Physics, Vol. 46, No. 6, 1471(2007)

541-558.

[19] Y. C. Eldar, A. Megretski, and G. C. Verghese. IEEE Transactions on Information Theory, 49(4):10071012, (2003).

[20] Lawrence. Ip, Shors Algorithm is Optimal, University of California, Berkeley, USA MU Seminar, http://www.qcaustralia.org (2003).

[21] M. A. Jafarizadeh, M. Rezaee, and S. K. Seyed Yagoobi, Phys. Rev. A **72**, 062106 (2005).

[22] M. A. Jafarizadeh, M. Rezaee, and S. Ahadpour, Phys. Rev. A **74**, 042335 (2006).

[23] M. A. Jafarizadeh, G. Najarbashi, H. Habibian, Phys. Rev. A **75**, 052326 (2007).

[24] M. A. Jafarizadeh, R. Sufiani, e-print: quant-ph/0703106 (2007).

[25] M. A. Jafarizadeh, G. Najarbashi, Y. Akbari and H. Habibian, e-print: quant-ph/0704.2414(2007).

[26] K. Kraus. States, Effects, and Operations. Number 190 in Lecture Notes in Physics. Springer, Berlin, 1983.

[27] Y. C. Eldar, IEEE Trans. Inf. Theory 49, 446 (2003).

[28] G. H. Golub and C. F. Van Loan, Matrix Computations, Baltimore MD: Johns Hopkins Univ. Press, third edition, 1996.

[29] L.R. Welch. IEEE Trans. Info. Theory, **20**:397399, 1974.

[30] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. 80, 2261 (1998).

[31] S. Karnas and M. Lewenstein, J. Phys. A: Math. Gen. 34, 6919-6937, (2001)

[32] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, (2004).

[33] E. K. P. Chong and S. H. Żak, *An Introduction to Optimization*, John Wiley, NY (2001).

Figure Captions

Figure-1: Feasible region for unambiguous discrimination of two linearly independent states with $a_{12} =< \psi_1 \mid \psi_2 >$ is showed by shadow region and approximately feasible region is showed by a polygon.

Figure-2: Unambiguous discrimination of two pure states in Bloch sphere.

Figure-3: Feasible region for unambiguous discrimination of three linearly independent states with $a_{12} = a_{13} = \frac{1}{3} \quad and \quad a_{23} = \frac{1}{9}$.

Figure-4: Feasible region for unambiguous discrimination of three linearly independent with $a_{12} = a_{13} = \frac{2}{\sqrt{3}} \quad and \quad a_{23} = \frac{1}{2}$ states.