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Bell states diagonal entanglement witnesses

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Abstract

It has been shown that finding generic Bell states diagonal entanglement witnesses (BDEW) for $d_1 \otimes d_2 \otimes \dots \otimes d_n$ systems exactly reduces to a linear programming if the feasible region be a polygon by itself and approximately obtains via linear programming if the feasible region is not a polygon. Since solving linear programming for generic case is difficult, the multi-qubits, $2 \otimes N$ and $3 \otimes 3$ systems for the special case of generic BDEW for some particular choice of their parameters have been considered. In the rest of this paper we obtain the optimal non decomposable entanglement witness for $3 \otimes 3$ system for some particular choice of its parameters. By proving the optimality of the well known reduction map and combining it with the optimal and non-decomposable $3 \otimes 3$ BDEW (named critical entanglement witnesses) the family of optimal and non-decomposable $3 \otimes 3$ BDEW have also been obtained. Using the approximately critical entanglement witnesses, some $3 \otimes 3$ bound entangled states are so detected. So the well known Choi map as a particular case of the positive map in connection with this witness via Jamiołkowski isomorphism has been considered which approximately is obtained via linear programming.

Keywords: Entanglement witness, Bell decomposable state, non decomposable entanglement witness, Optimal entanglement witness, Choi map.

PACs Index: 03.65.Ud

1 Introduction

Entanglement is one of the most fascinating features of quantum mechanics. As Einstein, Podolsky and Rosen [1] pointed out, the quantum states of two physically separated systems that interacted in the past can defy our intuitions about the outcome of local measurements. Moreover, it has recently been recognized that entanglement is a very important resource in quantum information processing[2]. A bipartite mixed state is said to be separable [3] (not entangled) if it can be written as a convex combination of pure product states.

A separability criterion is based on a simple property that can be shown to hold for every separable state. If some state does not satisfy this property, then it must be entangled. But the converse does not necessarily imply the state to be separable. One of the first and most widely used related criterion is the Positive Partial Transpose (PPT) criterion, introduced by Peres [4]. Furthermore, the necessary and sufficient condition for separability in $H_2 \otimes H_2$ and $H_2 \otimes H_3$ was shown by Horodeckis [5], which was based on a previous work by Woronowicz [6]. However, in higher dimensions, there are PPT states that are nonetheless entangled, as was first shown in [7], based on [6]. These states are called bound entangled states because they have the peculiar property that no entanglement can be distilled from them by local operations [8].

Another approach to distinguish separable states from entangled states involves the so called entanglement witness (EW) [9]. An EW for a given entangled state ρ is an observable W whose expectation value is nonnegative on any separable state, but strictly negative on an entangled state ρ .

There is a correspondence relating entanglement witnesses to linear positive (but not completely positive) maps from the operators on H_A to the operators on H_B via Jamiolkowski isomorphism, or vice versa[10].

There has been much work on the separability problem, particularly from the Innsbruck-

Hannover group, as reviewed in [11, 12], that emphasizes convexity and proceeds by characterizing entanglement witnesses in terms of their extreme points, the so-called optimal entanglement witnesses[13], and PPT entangled states in terms of their extreme points, the edge PPT entangled states [14, 15].

Having constructed the EW , one can decompose it into a sum of local measurements, then the expectation value can be measured with simple method. This decomposition has to be optimized in a certain way since we want to use the smallest number of measurements possible[18, 19, 20, 21].

In this paper, we show that finding generic Bell states diagonal entanglement witnesses (BDEW) for $d_1 \otimes d_2 \otimes \dots \otimes d_n$ systems reduces to a convex optimization problem. If the feasible region for this optimization problem constructs a polygon by itself, the corresponding boundary points of the convex hull will minimize exactly optimization problem. This problem is called linear programming , and the simplex method is the easiest way of solving it. If the feasible region is not a polygon, with the help of tangent planes in this region at points which are determined either analytically or numerically one can define new convex hull which is a polygon and has encircled the feasible region. The points on the boundary of the polygon can approximately determine the minimum value of optimization problem. Thus approximated value is obtained via linear programming. In general it is difficult to find this region and solve the optimization problem, thus it is difficult to find any generic multipartite EW. In the following sections we consider some simple but important examples which are solved exactly or approximately by linear programming method. Then we consider the multi-qubits and $2 \otimes N$ with exactly minimum value by linear programming and $3 \otimes 3$ systems with approximately minimum value by linear programming and then establish $3 \otimes 3$ optimality condition together with non-decomposability properties for some particular choice of its parameters. Then we combine the optimal well known reduction map, and the optimal as well as the non-decomposable $3 \otimes 3$ BDEW (i.e., the critical entanglement witnesses) to obtain further family of optimal and

non-decomposable $3 \otimes 3$ BDEW. Finally, using the critical entanglement witnesses some $3 \otimes 3$ bound entangled states are detected and we consider the well known Choi map as a particular case of the positive map in connection with this witness via Jamiołkowski isomorphism which approximately is obtained via linear programming.

The paper is organized as follows:

In section 2 we give a brief review of entanglement witness. In section 3 we show that finding generic Bell states diagonal entanglement witnesses for $d_1 \otimes d_2 \otimes \dots \otimes d_n$ systems reduces to a linear programming problem. In section 4, we consider BDEW for multi-qubit system. In section 5, we provide BDEW for $2 \otimes N$. In section 6, we provide BDEW for $3 \otimes 3$ systems. Section 7 is devoted to prove the n-d of critical EW and introduce a new family of optimal nd-EW via combining critical EW with the well known reduction maps. In section 8, using the critical EW, we will be able to detect a bound BD entangled state. In section 9, we consider the well known Choi map as a particular case of the positive map connect with this witness via Jamiołkowski isomorphism. Finally in section 10 using the optimal EW, we show that some separable Bell states diagonal lies at the boundary of separable region. The paper is ended with a brief conclusion together with three appendices devoted to the proof of A) the optimization of product distributions B)optimality of critical, reduction map C)simplex method for solving linear programming problem.

2 Entanglement witness

Here we mention briefly those concepts and definitions of EW that will be needed in the sequel, a more detailed treatment may be found for example in [6, 10, 17].

Let S be a convex compact set in a finite dimensional Banach space. Let ρ be a point in the space with ρ which is not in S . Then there exists a hyperplane[17] that separates ρ from S .

A hermitian operator (an observable) W is called an entanglement witness (EW) iff

$$\exists \rho \text{ such that } Tr(\rho W) < 0 \quad (2-1)$$

$$\forall \rho' \in S \quad Tr(\rho' \hat{W}) \geq 0. \quad (2-2)$$

Definition 1: An EW is decomposable iff there exists operators P, Q such that

$$W = P + Q^{T_A} \quad P, Q \geq 0. \quad (2-3)$$

Decomposable EW can not detect PPT entangled states[6].

Definition 2: An EW is called non-decomposable entanglement witness (nd-EW) iff there exists at least one PPT entangled state which the witness detects[6].

Definition 3: The (decomposable) entanglement witness is tangent to $S(P)$ iff there exists a $\sigma \in S(P)$ with $Tr(W\sigma) = 0$ ($Tr(W\rho) = 0$).

Using these definitions we can restate the consequences of the Hahn-Banach theorem [17] in several ways:

Theorem:

- 1- ρ is entangled iff there exists a witness W such that $Tr(\rho W) < 0$.
- 2- ρ is a PPT entangled state iff there exists an non-decomposable entanglement witness W such that $Tr(\rho W) < 0$.
- 3- σ is separable iff for all EW $Tr(W\sigma) \geq 0$.

From theoretical point of view this theorem is quite powerful. However, it is not useful to construct witnesses that detect a given state ρ .

We know that a strong relation was developed between entanglement witnesses and positive maps[6, 10]. Notice that an entanglement witness only gives one condition (namely $Tr(W\rho) < 0$) while for the map $(I_A \otimes \phi)\rho$ to be positive definite, there are many conditions that have to be satisfied. Thus the map is much stronger, while the witnesses are much weaker in detecting entanglement. It is shown that this concept is able to provide a more detailed classification of entangled states.

3 Bell states diagonal entanglement witnesses

As we know, one can expand any trace class observable in the Bell basis as

$$W = \sum_{i_1 i_2 \dots i_n} W_{i_1 i_2 \dots i_n} |\psi_{i_1 i_2 \dots i_n}\rangle \langle \psi_{i_1 i_2 \dots i_n}| \quad (3-4)$$

where $|\psi_{i_1 i_2 \dots i_n}\rangle (0 \leq i_1 \leq d_1, 0 \leq i_2 \leq d_2, \dots, 0 \leq i_n \leq d_n, \text{ and } d_1 \leq d_2 \leq \dots \leq d_n)$ stands for the orthonormal states for a $d_1 \otimes d_2 \dots \otimes d_n$ Bell state defined as

$$|\psi_{i_1 i_2 \dots i_n}\rangle = (\Omega)^{i_1} \otimes (S)^{i_2} \otimes \dots \otimes (S)^{i_n} |\psi_{00 \dots 0}\rangle \quad (3-5)$$

where Ω and S are phase modules and shift operators for a $d_1 \otimes d_2 \otimes \dots \otimes d_n$ defined as

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{d-1} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (3-6)$$

with $\omega = \exp(\frac{2\pi i}{d})$ and

$$|\psi_{00 \dots 0}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d_1-1} |i\rangle_1 |i\rangle_2 \dots |i\rangle_n. \quad (3-7)$$

W is a trace one observable i.e., $Tr(W) = 1$ and we have $\sum_{i_1 i_2 \dots i_n} W_{i_1 i_2 \dots i_n} = 1$.

Let us split the observable W into its positive and negative spectra as:

$$W = \sum_{k=1}^{n^+} \lambda_k^+ |\phi_k^+\rangle \langle \phi_k^+| - \sum_{k=1}^{n^-} |\lambda_k^-| |\phi_k^-\rangle \langle \phi_k^-|, \quad (3-8)$$

where $\lambda_k^+ (\lambda_k^-)$ are the positive (negative) eigenvalues $|\phi_k^+\rangle (|\phi_k^-\rangle)$, and we have $n^+ + n^- = d^n$.

Denoting $\sum |\lambda_k^-| = s > 0$ we can write (3-8) as:

$$W = (1 + s)\rho^+ - s\rho^-, \quad (3-9)$$

where ρ^\pm are two normalized positive operators or density matrices defined as

$$\rho^+ = \frac{1}{1+s} \sum_{k=1}^{n^+} (\lambda_k^+ |\phi_k^+\rangle \langle \phi_k^+|) \quad , \quad \rho^- = \frac{1}{s} \sum_{k=1}^{n^-} (\lambda_k^- |\phi_k^-\rangle \langle \phi_k^-|). \quad (3-10)$$

Now, using the Lewenstein-Sanpera technique [25, 26, 27] the identity operator $\frac{I_{d_1 d_2 \dots d_n}}{d_1 d_2 \dots d_n}$ can be written in terms of ρ^- and the other positive states as

$$\frac{I_{d_1 d_2 \dots d_n}}{d_1 d_2 \dots d_n} = \lambda \rho^- + (1 - \lambda) \rho'^- , \quad 0 < \lambda < 1. \quad (3-11)$$

By using the above equation we can replace ρ^- in Eq.(3-9) in terms of the identity operator. So, Eq.(3-9) is written as a sum of the identity and positive operators. Thus we have

$$W = \mathbf{r} \frac{I_{d_1 d_2 \dots d_n}}{d_1 d_2 \dots d_n} + (1 - \mathbf{r}) \rho, \quad (3-12)$$

where

$$\rho = \frac{(1 + s)\lambda}{\lambda + s} \rho^+ + s \left(\frac{1 - \lambda}{s + \lambda} \right) \rho'^-, \quad (3-13)$$

and $\mathbf{r} = -\frac{s}{\lambda} < 0$.

In this paper we have considered only trace one observables which are diagonal in the Bell states. Hence we restrict ourselves to the Bell states diagonal ρ defined as

$$\rho = \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} |\psi_{i_1 i_2 \dots i_n}\rangle \langle \psi_{i_1 i_2 \dots i_n}| , \quad q_{i_1 i_2 \dots i_n} > 0 \quad \text{and} \quad \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} = 1. \quad (3-14)$$

Finally, by substituting (3-14) in (3-12) the trace one Bell states diagonal W observables are

$$W = \mathbf{r} \frac{I_{d_1 d_2 \dots d_n}}{d_1 d_2 \dots d_n} + (1 - \mathbf{r}) \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} |\psi_{i_1 i_2 \dots i_n}\rangle \langle \psi_{i_1 i_2 \dots i_n}|. \quad (3-15)$$

The observable given by (3-15) is not a positive operator and can not be an EW provided that its expectation value on any pure product state is positive. For a given product state $|\gamma\rangle = |\alpha\rangle_1 |\alpha\rangle_2 \dots |\alpha\rangle_n$ the non negativity of

$$\text{Tr}(W|\gamma\rangle\langle\gamma|) \geq 0 \quad (3-16)$$

implies that

$$\frac{-d_1 d_2 \dots d_n \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n}}{1 - d_1 d_2 \dots d_n \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n}} \leq \mathbf{r} \leq 0, \quad (3-17)$$

where $P_{i_1 i_2 \dots i_n} = |\langle \gamma | \psi_{i_1 i_2 \dots i_n} \rangle|^2$.

Denoting the summation in the numerator and the dominator in (3-17) by $C(\gamma) = d_1 d_2 \dots d_n \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n}$ we see that the least possible $\mathbf{r}_0 = -\frac{C(\gamma)}{1-C(\gamma)}$ is the decreasing function of $C(\gamma)$ for $C(\gamma) < 1$ (obviously for $C(\gamma) > 1$ all \mathbf{r} while being positive provide positive expectation value). Therefore, for given parameters $q_{i_1 i_2 \dots i_n} > 0$, with $\sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} = 1$, the least allowed value of the parameter \mathbf{r} , called the critical parameter (denoted by \mathbf{r}_c) is obtained from the product state γ which minimizes $C_\gamma = \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n}$, with $0 \leq P_{i_1 i_2 \dots i_n} \leq 1$ and the constraint $\sum_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n} = 1$. As for the completeness of the Bell state $\sum_{i_1 i_2 \dots i_n} |\psi_{i_1 i_2 \dots i_n}\rangle \langle \psi_{i_1 i_2 \dots i_n}| = 1$, the determination of \mathbf{r}_c reduces to the following optimization problem[24]

$$\begin{aligned} \text{minimize } C_\gamma &= \sum_{i_1 i_2 \dots i_n} q_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n}(\gamma) \\ 0 &\leq P_{i_1 i_2 \dots i_n}(\gamma) \leq \frac{1}{d_1} \\ \sum_{i_1 i_2 \dots i_n} P_{i_1 i_2 \dots i_n}(\gamma) &= 1. \end{aligned} \tag{3-18}$$

Always the distribution $P_{i_1 i_2 \dots i_n}$ satisfies $0 \leq P_{i_1 i_2 \dots i_n}(\gamma) \leq \frac{1}{d_1}$ for all pure product states (the proof is given in the Appendix A). One can calculate the distributions $P_{i_1 i_2 \dots i_n}(\gamma)$, consistent with the aforementioned optimization problem, from the information about the boundary of feasible region. To achieve the feasible region we obtain the extreme points corresponding to the product distributions $P_{i_1 i_2 \dots i_n}(\gamma)$ for every given product states by applying the special conditions on $q_{i_1 i_2 \dots i_n}$'s parameters. C_γ themselves are functions of the product distributions, and they are in turn are functions of γ . They are not real variables of γ but the product states will be multiplicative. If this feasible region constructs a polygon by itself, the corresponding boundary points of the convex hull will minimize exactly C_γ in Eq. (3-18). This problem is called linear programming, and the simplex method is the easiest way of solving it. If the feasible region is not a polygon, with the help of tangent planes in this region at points which are determined either analytically or numerically one can define new convex hull which is a polygon and has encircled the feasible region. The points on the boundary of the polygon can approximately determine the minimum value C_γ from Eq.(3-18), thus the problem is that of a

linear programming again. In general it is difficult to find this region and solve the optimization problem, thus it is difficult to find any generic multipartite EW. In the following sections we consider some simple but important examples which are solved as linear programming problem.

4 Bell states diagonal entanglement witnesses for multi-qubit system

Here we provide a multi-qubit entanglement witness. From the previous section one can show that the Bell states diagonal observable W for multi qubit system is defined by

$$W = \mathbf{r} \frac{I_{2^n}}{2^n} + (1 - \mathbf{r}) \sum_{i_1, \dots, i_n=0}^1 q_{i_1, i_2, \dots, i_n} |\psi_{i_1, i_2, \dots, i_n}\rangle \langle \psi_{i_1, i_2, \dots, i_n}|, \quad (4-19)$$

where $|\psi_{i_1, i_2, \dots, i_n}\rangle$ is a Bell state:

$$|\psi_{i_1, i_2, \dots, i_n}\rangle = (\sigma_z)^{i_1} \otimes (\sigma_x)^{i_2} \otimes \dots \otimes (\sigma_x)^{i_n} |\psi_{0,0,\dots,0}\rangle, \quad (4-20)$$

with

$$|\psi_{0,0,\dots,0}\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^1 |i\rangle_1 |i\rangle_2 \dots |i\rangle_n, \quad (4-21)$$

and σ_z and σ_x are the Pauli operators. This observable is not a positive operator and can not be an EW provided that its expectation value on any product state $|\gamma\rangle = |\alpha\rangle_1 |\alpha\rangle_2 \dots |\alpha\rangle_n$ is positive.

We consider an easy case $q_{00\dots 00} = 0$, $q_{10\dots 00} = x$ with all the other q 's being equal, i.e., $q_{i_1, i_2, \dots, i_n} = \frac{1-x}{2(2^{n-1}-1)}$ except for $i_1 = i_2 = \dots = i_n = 0$ and $i_2 = i_3 = \dots = i_n = 0, i_1 = 1$. Then the observable W reduces to the following form

$$W = \mathbf{r} \frac{I_{2^n}}{2^n} + \frac{(1 - \mathbf{r})}{2(2^{n-1} - 1)} ((1-x)I_{2^n} - (1-x)|\psi_{0,0,\dots,0}\rangle \langle \psi_{0,0,\dots,0}| + ((2^n - 1)x - 1)|\psi_{1,0,\dots,0}\rangle \langle \psi_{1,0,\dots,0}|). \quad (4-22)$$

We can calculate C_γ from the non negativity of $Tr(W|\gamma\rangle \langle \gamma|)$ for a given product state $|\gamma\rangle$

$$C_\gamma = \frac{1}{2(2^{n-1} - 1)} ((1-x) - (1-x)P_{00\dots 00} + ((2^n - 1)x - 1)P_{10\dots 00}). \quad (4-23)$$

According to the definition of product distributions, we have

$$\begin{aligned} P_{00\dots 0} &= \frac{1}{2} | \alpha_1 \alpha_2 \dots \alpha_n + \beta_1 \beta_2 \dots \beta_n |^2 \\ P_{10\dots 0} &= \frac{1}{2} | \alpha_1 \alpha_2 \dots \alpha_n - \beta_1 \beta_2 \dots \beta_n |^2, \end{aligned} \quad (4-24)$$

where

$$|\alpha\rangle_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \quad i = 1, 2, \dots, n. \quad (4-25)$$

For the most general given product states γ , we determine the extreme allowed values of these product distributions. Let

$$|\alpha\rangle_1 = |\alpha\rangle_2 = \dots = |\alpha\rangle_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (4-26)$$

then the product distributions are

$$\begin{cases} P_{00\dots 00} = \frac{1}{2} \\ P_{10\dots 00} = \frac{1}{2} \end{cases}. \quad (4-27)$$

Another choice will make $P_{10\dots 0}$ minimal

$$|\alpha\rangle_1 = |\alpha\rangle_2 = \dots = |\alpha\rangle_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (4-28)$$

and the product distributions will become

$$\begin{cases} P_{10\dots 0} = \frac{1}{2^{n-1}} \\ P_{10\dots 00} = 0 \end{cases}. \quad (4-29)$$

Similarly we can also make $P_{00\dots 0}$ minimal

$$|\alpha\rangle_2 = |\alpha\rangle_3 = \dots = |\alpha\rangle_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\alpha\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} P_{00\dots 00} = 0 \\ P_{10\dots 00} = \frac{1}{2^{n-1}} \end{cases}. \quad (4-30)$$

From the definition of the convex function [24] we can show that the convex combination of these distributions provide a convex region called the feasible region, where all points in the

interior of this region satisfy the positivity constraint of $Tr(W|\gamma\rangle\langle\gamma|)$. Then we have

$$\left\{ \begin{array}{l} \text{maximize} \quad -C_\gamma = \frac{1}{2(2^{n-1}-1)}(-(1-x) + (1-x)P_{00\dots 00} - ((2^n-1)x-1)P_{10\dots 00}) \\ \text{subject to} \quad \quad \quad 2P_{00\dots 00} - 2P_{10\dots 00}(1 - \frac{1}{2^{n-1}}) \leq \frac{1}{2^{n-1}} \\ \quad \quad \quad 2P_{10\dots 00} - 2P_{00\dots 00}(1 - \frac{1}{2^{n-1}}) \leq \frac{1}{2^{n-1}} \\ \quad \quad \quad P_{00\dots 00} \geq 0, P_{10\dots 00} \geq 0. \end{array} \right. \quad (4-31)$$

Now we must prove that the feasible region, constructed from the convex of these points, is a polygon. Let $P_+ = P_{00\dots 0}$ and $P_- = P_{10\dots 0}$. The equation of the line passing through $(P_+ = \frac{1}{2^{n-1}}, P_- = 0)$ and $(P_+ = \frac{1}{2}, P_- = \frac{1}{2})$ is

$$P_- = (\frac{2^{n-1}}{2^{n-1}-2})P_+ - \frac{1}{2^{n-1}-2}. \quad (4-32)$$

Let us further assume $P_- = \lambda P_+$. By intersecting this equation with the one above we get

$$P_+ = \frac{1}{2^{n-1} - \lambda(2^{n-1} - 2)}. \quad (4-33)$$

Now if we assume $\lambda = 0$ we arrive at the point $(P_+ = \frac{1}{2^{n-1}}, P_- = 0)$, for $\lambda = 1$ we conclude $(P_+ = \frac{1}{2}, P_- = \frac{1}{2})$. One can rewrite Eq.(4-24) as

$$\begin{aligned} P_\pm = \frac{1}{2}(&|\alpha_1|^2 |\alpha_2|^2 \dots |\alpha_n|^2 + (1-|\alpha_1|^2)(1-|\alpha_2|^2)\dots(1-|\alpha_n|^2) \\ &\pm 2|\alpha_1|(1-|\alpha_1|^2)|\alpha_2|(1-|\alpha_2|^2)\dots|\alpha_n|(1-|\alpha_n|^2)\cos(\phi)). \end{aligned} \quad (4-34)$$

Thus we write the Lagrangian as

$$\mathcal{L} = P_+ + \mu(P_- - \lambda P_+), \quad (4-35)$$

where μ is the Lagrange multiplier. With $|\alpha_i| = \cos\theta_i$ we maximize \mathcal{L} with respect to θ_i 's and ϕ

$$\left\{ \begin{array}{l} \theta_1 = \theta_2 = \dots = \theta_n \Rightarrow |\alpha_1| = |\alpha_2| = \dots = |\alpha_n| \\ \phi = 0 \end{array} \right., \quad (4-36)$$

such that

$$\tan^n \theta_i = \frac{1 - \sqrt{\lambda}}{1 + \sqrt{\lambda}}, \quad (4-37)$$

so that

$$P_+ = \left(\frac{2}{1 + \sqrt{\lambda}}\right) \frac{1}{\left(1 + \left(\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^{\frac{2}{n}}\right)^n}. \quad (4-38)$$

As we see the equation for P_+ is less than the one in (4-33), moreover, this relation indicates the correctness of the result (4-26)-(4-30). Thus the convex hull is a polygon and the optimization problem will be converted into the linear programming one.

There is no simple analytical formula for solving a linear programming, however there are a variety of very effective methods, including the simplex method to solve them. So, minimization solutions of C_γ is obtained by the simplex method[24] and we have (see Appendix C):

I) For $0 \leq x \leq \frac{1}{2^{n-1}+1}$ the extreme points of the feasible region are $P_{00\dots 00} = P_{10\dots 00} = \frac{1}{2}$ and the minimum value of C_γ is defined by $(C_\gamma)_{min} = \frac{x}{2}$. By substituting these values in (3-17) we have

$$-\frac{2^{n-1}x}{1 - 2^{n-1}x} \leq \mathbf{r} \leq 0 \Rightarrow \mathbf{r}_c = -\frac{2^{n-1}x}{1 - 2^{n-1}x}, \quad (4-39)$$

where \mathbf{r}_c is called the critical \mathbf{r} . By substituting \mathbf{r}_c in (4-19) this observable has positive expectation value under any product state, thus it will be an EW called critical EW equal to

$$W_c(x) = \frac{1}{2(2^{n-1} - 1)} \left(I_{2^n} - \frac{1-x}{1 - 2^{n-1}x} |\psi_{0,0,\dots,0}\rangle \langle \psi_{0,0,\dots,0}| + \frac{(2^n - 1)x - 1}{1 - 2^{n-1}x} |\psi_{1,0,\dots,0}\rangle \langle \psi_{1,0,\dots,0}| \right), \quad (4-40)$$

which in the special case where $x = \frac{1}{2^{n-1}}$ the $W_c(x)$ reduces to

$$W_{red} = \frac{1}{2(2^{n-1} - 1)} (I_{2^n} - 2 |\psi_{0,0,\dots,0}\rangle \langle \psi_{0,0,\dots,0}|), \quad (4-41)$$

which is the well known reduction map.

II) For $\frac{1}{2^{n-1}+1} \leq x \leq 1$ the extreme points of the feasible region are $P_{00\dots 00} = \frac{1}{2^{n-1}}$ and $P_{10\dots 00} = 0$ respectively. Therefore, from the simplex method we get $(C_\gamma)_{min} = \frac{1-x}{2^n}$, hence $\mathbf{r}_c = -\frac{1-x}{x}$ and the critical EW is calculated to be

$$W_c(x) = \frac{1}{(2^{n-1} - 1)} \left(\frac{1-x}{x} \frac{I_{2^n}}{2^n} - \frac{1-x}{2x} |\psi_{0,0,\dots,0}\rangle \langle \psi_{0,0,\dots,0}| + \frac{(2^n - 1)x - 1}{2x} |\psi_{1,0,\dots,0}\rangle \langle \psi_{1,0,\dots,0}| \right). \quad (4-42)$$

Note that this choice of q is not the only way of defining a BDEW for multi-qubit system in the one parameter representation. Let us consider the alternative definition for the one parameter BDEW by studying the following example. Assume $q_{00\dots 01} = x$ and set all the other q 's to be equal. Thus we have

$$W = \mathbf{r} \frac{I_{2^n}}{2^n} + \frac{(1 - \mathbf{r})}{2(2^{n-1} - 1)} ((1 - x)I_{2^n} - (1 - x)|\psi_{0,0,\dots,0}\rangle\langle\psi_{0,0,\dots,0}| + ((2^n - 1)x - 1)|\psi_{0,0,\dots,1}\rangle\langle\psi_{0,0,\dots,1}|). \quad (4-43)$$

Similarly we can find the extreme points of $P_{00\dots 00}$ and $P_{00\dots 01}$ as

$$|\alpha\rangle_1 = |\alpha\rangle_2 = \dots = |\alpha\rangle_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00\dots 00} = \frac{1}{2} \\ P_{00\dots 01} = 0 \end{array} \right. . \quad (4-44)$$

$$|\alpha\rangle_1 = |\alpha\rangle_2 = \dots = |\alpha\rangle_{n-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\alpha\rangle_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00\dots 00} = 0 \\ P_{00\dots 01} = \frac{1}{2} \end{array} \right. . \quad (4-45)$$

Also we know that the convex combination of $P_{00\dots 00}$ and $P_{00\dots 01}$ provides a convex region or a feasible region. Then we have an optimization problem as follows:

$$\left\{ \begin{array}{ll} \text{minimize} & C_\gamma = \frac{1}{2(2^{n-1}-1)}((1-x) - (1-x)P_{00\dots 00} + ((2^n-1)x-1)P_{00\dots 01}) \\ \text{subject to} & \frac{1}{2} - P_{00\dots 00} - P_{00\dots 01} \geq 0 \\ & P_{00\dots 00}, P_{00\dots 01} \geq 0 \end{array} \right. . \quad (4-46)$$

Here the optimization is converted to the linear programming problem. To prove, we must show that the feasible region is a polygon. Let us suppose $P_+ = P_{00\dots 00}$ and $P_- = P_{00\dots 01}$ with

$$\left\{ \begin{array}{l} P_+ = \frac{1}{2} |\alpha_1\alpha_2\dots\alpha_n + \beta_1\beta_2\dots\beta_n|^2 \\ P_- = \frac{1}{2} |\beta_1\alpha_2\dots\alpha_n + \alpha_1\beta_2\dots\beta_n|^2, \end{array} \right. \quad (4-47)$$

where as before we have $P_- = \lambda P_+$ and $|\alpha_i| = \cos \theta_i$. By maximizing the Lagrangian we get $\theta_2 = \theta_3 = \dots = \theta_n$. Thus we have

$$P_+ + P_- = \frac{1}{2}(\cos^{2n-2} \theta_2 + \sin^{2n-2} \theta_2) \leq \frac{1}{2}. \quad (4-48)$$

The line passing through the points $(P_+ = \frac{1}{2}, P_- = 0)$ and $(P_+ = 0, P_- = \frac{1}{2})$ is $P_- = -P_+ + \frac{1}{2}$, which is always located above the curve obtained in the Eq.(4-48). Therefore, all the points are within the feasible region and this region constructs a polygon. Thus the above optimization problem reduces to a linear programming one. This minimization is exactly solved in the same way as mentioned above, and the critical EW is obtained as

$$\mathbf{r}_c = \frac{-2^n(1-x)}{2(2^{n-1}(x+1)-2)} \Rightarrow \quad (4-49)$$

$$W_c = \frac{1}{(2^{n-1}(x+1)-2)}((1-x)I_{2^n} - 2(1-x)|\psi_{00\dots 00}\rangle\langle\psi_{00\dots 00}| + 2((2^n-1)x-1)|\psi_{00\dots 01}\rangle\langle\psi_{00\dots 01}|).$$

5 Bell states diagonal entanglement witnesses for $2 \otimes N$ system

Here, we will find a $2 \otimes N$ entanglement witness. From the previous discussions we can define the Bell states diagonal observable W as

$$W = \mathbf{r} \frac{I_{2N}}{2N} + (1-\mathbf{r}) \sum_{i=0}^{N-1} \sum_{\alpha=0}^1 q_{i\alpha} |\psi_{i\alpha}\rangle\langle\psi_{i\alpha}|, \quad (5-50)$$

where $|\psi_{i\alpha}\rangle = I_2 \otimes (S)^i (\Omega)^\alpha |\psi_{00}\rangle$, with $|\psi_{00}\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^1 |k\rangle|k\rangle$ and

$$\omega = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \quad (5-51)$$

Similar to multi-qubit let $q_{00} = 0$ and $q_{10} = x$ and let all the other q 's be equal to $\frac{1-x}{2N-2}$. Then by obtaining the expectation value of W on the product states and finding the product

distributions we have

$$|\alpha\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\alpha\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00} = \frac{1}{2} \\ P_{10} = \frac{1}{2} \end{array} \right. \quad (5-52)$$

$$|\alpha\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\alpha\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00} = \frac{1}{2} \\ P_{10} = 0 \end{array} \right. \quad (5-53)$$

$$|\alpha\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad |\alpha\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00} = 0 \\ P_{10} = \frac{1}{2} \end{array} \right. \quad (5-54)$$

The feasible region is a rectangular and the optimization problem reduces to linear programming. Therefore, by using simplex method for $0 \leq x \leq \frac{1}{N+1}$ we find the minimum value of $C_\gamma = \frac{x}{2}$ and critical \mathbf{r} as $\mathbf{r}_c = \frac{-Nx}{1-Nx}$, and the critical EW is defined as

$$W_c = \frac{1}{2(N-1)} (I_{2N} - \frac{1-x}{1-Nx} |\psi_{00}\rangle\langle\psi_{00}| + \frac{(2N-1)x-1}{1-Nx} |\psi_{10}\rangle\langle\psi_{10}|). \quad (5-55)$$

For the critical \mathbf{r} we find $\mathbf{r}_c = -\frac{1-x}{x}$ in the region $\frac{1}{N+1} \leq x \leq 1$ and the critical EW has the following form

$$W_c(x) = \frac{1}{2} \left(\frac{1-x}{x} \frac{I_{2N}}{2N} - \frac{1-x}{2x} |\psi_{00}\rangle\langle\psi_{00}| + \frac{(2N-1)x-1}{2x} |\psi_{10}\rangle\langle\psi_{10}| \right). \quad (5-56)$$

In another one parameter EW example we assume that $q_{01} = x$ and set all the other q 's to

be equal so that we have

$$W = \mathbf{r} \frac{I_{2N}}{2N} + \frac{(1 - \mathbf{r})}{2(N - 1)} ((1 - x)I_{2N} - (1 - x)|\psi_{00}\rangle\langle\psi_{00}| + ((2N - 1)x - 1)|\psi_{01}\rangle\langle\psi_{01}|). \quad (5-57)$$

Similarly we can find the extreme points of P_{00} and P_{01} as

$$|\alpha\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\alpha\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00} = \frac{1}{2} \\ P_{01} = 0 \end{array} \right., \quad (5-58)$$

$$|\alpha\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\alpha\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \left\{ \begin{array}{l} P_{00} = 0 \\ P_{01} = \frac{1}{2} \end{array} \right.. \quad (5-59)$$

Then the critical EW is defined as

$$\mathbf{r}_c = \frac{-2N(1 - x)}{2(N(x + 1) - 2)} \Rightarrow \quad (5-60)$$

$$W_c = \frac{1}{(N(x + 1) - 2)} ((1 - x)I_{2N} - (1 - x)|\psi_{00}\rangle\langle\psi_{00}| + ((2N - 1)x - 1)|\psi_{01}\rangle\langle\psi_{01}|).$$

6 Bell states diagonal entanglement witnesses for $3 \otimes 3$ system

Here we provide a $3 \otimes 3$ Bell diagonal entanglement witness. One can show that the Eq. (3-15) for a $3 \otimes 3$ system reads as

$$W = \mathbf{r} \frac{I_9}{9} + (1 - \mathbf{r}) \sum_{i_1, i_2=0}^2 q_{i_1, i_2} |\psi_{i_1, i_2}\rangle\langle\psi_{i_1, i_2}|. \quad (6-61)$$

It is difficult to prove whether or not the EW for a $3 \otimes 3$ system is optimal. Also it is difficult to see for which value of the allowed \mathbf{r} , EW are (or are not) decomposable. Therefore, to investigate the optimality and non decomposability of these EW we restrict ourselves below to some particular choice of q_{ij} :

Because the distributions $0 \leq P_{ij} \leq \frac{1}{3}$ and the minimum value of C_γ are dependent on the coefficients q_{ij} , we consider a special case for the coefficients q_{ij} defined by

$$q_{01} = q_{02} = q_{11} = q_{22} = q_{12} = q_{21} = \frac{1}{8}, \quad q_{10} = x \quad \text{and} \quad q_{20} = \frac{1}{4} - x, \quad 0 \leq x \leq \frac{1}{4}. \quad (6-62)$$

By substituting these values in (3-15) we get

$$W(x) = \mathbf{r} \frac{I_9}{9} + (1 - \mathbf{r}) \left(\frac{I_9}{8} - \frac{1}{8} |\psi_{00}\rangle\langle\psi_{00}| - \frac{8x-1}{8} (|\psi_{10}\rangle\langle\psi_{10}| - |\psi_{20}\rangle\langle\psi_{20}|) \right). \quad (6-63)$$

By using (3-16) for non-negativity of the observable W we find the distributions P_{ij} as a function of x . The minimum value of C_γ is obtained from the boundary of the feasible region, i.e., we have

$$C_\gamma = \frac{1}{8} (1 - P_{00} - (8x - 1)(P_{10} - P_{20})). \quad (6-64)$$

For given product states $|\gamma\rangle = |\alpha\rangle_1 |\alpha\rangle_2$ one can obtain the extreme points of the product distributions as

$$\left\{ \begin{array}{l} |\alpha\rangle_1 = |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow (P_{00} = \frac{1}{3}, P_{10} = 0, P_{20} = 0) \\ \\ |\alpha\rangle_1 = |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ \bar{\omega} \end{pmatrix} \rightarrow (P_{00} = 0, P_{10} = \frac{1}{3}, P_{20} = 0) \\ \\ |\alpha\rangle_1 = |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \bar{\omega} \\ \omega \end{pmatrix} \rightarrow (P_{00} = 0, P_{10} = 0, P_{20} = \frac{1}{3}) \end{array} \right.$$

$$\left\{ \begin{array}{l} |\alpha\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \rightarrow (P_{00} = 0, P_{10} = \frac{1}{4}, P_{20} = \frac{1}{4}) \\ |\alpha\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \bar{\omega} \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -\bar{\omega} \end{pmatrix} \rightarrow (P_{00} = \frac{1}{4}, P_{10} = 0, P_{20} = \frac{1}{4}) \\ |\alpha\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \omega \\ 0 \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\omega \\ 0 \end{pmatrix} \rightarrow (P_{00} = \frac{1}{4}, P_{10} = \frac{1}{4}, P_{20} = 0) \end{array} \right. \quad (6-65)$$

and

$$|\alpha\rangle_1 = |\alpha\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow (P_{00} = \frac{1}{3}, P_{10} = \frac{1}{3}, P_{20} = \frac{1}{3}) . \quad (6-66)$$

By convex combination of these points we obtain the possible region. Thus we have an optimization problem as

$$\left\{ \begin{array}{ll} \text{minimize} & C_\gamma = \frac{1}{8}(1 - P_{00} - (8x - 1)(P_{10} - P_{20})) \\ \text{subject to} & 1 - 3P_{00} - P_{10} + P_{20} \geq 0 \\ & 1 + P_{00} - 3P_{10} - P_{20} \geq 0 \\ & 1 - P_{00} + P_{10} - 3P_{20} \geq 0 \\ & P_{00}, P_{10}, P_{20} \geq 0. \end{array} \right. \quad (6-67)$$

One can prove analytically that the region can be encircled with a polygon and the optimization problem is reduced to a linear programming. To prove, we begin from the definition of the product distributions

$$\begin{aligned} P_{00} &= \frac{1}{3} | \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 |^2 \\ P_{10} &= \frac{1}{3} | \alpha_1\beta_1 + \alpha_2\beta_2\omega + \alpha_3\beta_3\bar{\omega} |^2 \\ P_{20} &= \frac{1}{3} | \alpha_1\beta_1 + \alpha_2\beta_2\bar{\omega} + \alpha_3\beta_3\omega |^2 . \end{aligned} \quad (6-68)$$

Without loss of generality, one can assume that

$$|\alpha_1\beta_1| = x_1, \quad |\alpha_2\beta_2| = x_2, \quad |\alpha_3\beta_3| = x_3. \quad (6-69)$$

The Schwartz inequality yields

$$x_1 + x_2 + x_3 \leq 1. \quad (6-70)$$

Since we are looking the extreme points we will choose the maximum value in the inequality (6-70), that is $x_1 + x_2 + x_3 = 1$. Thus the product distributions are written as

$$\begin{aligned} P_{00} &= \frac{1}{3} |x_1 + x_2 e^{i\phi_2} + x_3 e^{i\phi_3}|^2 \\ P_{10} &= \frac{1}{3} |x_1 + x_2 e^{i\phi_2} \omega + x_3 e^{i\phi_3} \bar{\omega}|^2 \\ P_{20} &= \frac{1}{3} |x_1 + x_2 e^{i\phi_2} \bar{\omega} + x_3 e^{i\phi_3} \omega|^2. \end{aligned} \quad (6-71)$$

Now, supposing that P_{00} and P_{10} are fixed values, we conclude

$$\phi_2 = \phi_3, \quad x_2 = x_3. \quad (6-72)$$

Thus

$$\begin{aligned} P_{00} &= \frac{1}{3} |1 + 2x_2(1 - \cos \phi_2)|^2 \\ P_{10} &= \frac{1}{3} |1 + 2x_2(1 - \cos \phi_2 + \frac{2\pi}{3})|^2 \\ P_{20} &= \frac{1}{3} |1 + 2x_2(1 - \cos \phi_2 - \frac{2\pi}{3})|^2. \end{aligned} \quad (6-73)$$

We write down the equation for the planes passing through the obtained extreme points for P_{ij} and maximize it with respect to the variables ϕ_2 and x_2 . Where the obtained values for ϕ_2 and x_2 are indicative of the violation from the equations of planes, that is there are points which are located out of these planes, or in other word the planes have become convex. Thus,

the equations of the planes (Fig 1) and their maximum violation(D) are obtained as follows

$$\left\{ \begin{array}{lll} 1) 3P_{10} - P_{20} + P_{00} - 1 = 0 & x_2 = \frac{7}{61}, \cos \phi_2 = \frac{-1}{7} & D = \frac{2}{61} \\ 2) 3P_{10} + P_{20} - P_{00} - 1 = 0 & x_2 = \frac{7}{61}, \cos \phi_2 = \frac{-11}{14} & D = \frac{2}{61} \\ 3) 3P_{20} + P_{10} - P_{00} - 1 = 0 & x_2 = \frac{7}{61}, \cos \phi_2 = \frac{-11}{14} & D = \frac{2}{61} \\ 4) 3P_{20} - P_{10} + P_{00} - 1 = 0 & x_2 = \frac{7}{61}, \cos \phi_2 = \frac{-1}{7} & D = \frac{2}{61} \\ 5) 3P_{00} + P_{20} - P_{10} - 1 = 0 & x_2 = \frac{7}{61}, \cos \phi_2 = \frac{13}{14} & D = \frac{2}{61} \\ 6) 3P_{00} - P_{20} + P_{10} - 1 = 0 & x_2 = \frac{7}{61}, \cos \phi_2 = \frac{13}{14} & D = \frac{2}{61}. \end{array} \right. \quad (6-74)$$

It is seen that the points thus obtained are located out of the considered plane. Thus, the equations of the planes passing through the new extreme points which are parallel to the above plane are obtained. For example, the equation of plane parallel to $3p_{00} + P_{10} + p_{20} = 1$ is $3p_{00} + P_{10} + p_{20} = 1 + \frac{2}{61}$ which under permutation (P_{00}, P_{10}, P_{20}) will act similarly. Tack arbitrary any three of planes passing through the new extreme, $(P_{00} = \frac{1}{3}, P_{10} = \frac{1}{3}, P_{20} = \frac{1}{3})$, $(P_{10} = 0, P_{20} = 0, P_{00} = 0)$ and $P_{10} + P_{20} + P_{00} - 1 = 0$ and intersecting with each other. Hence new extreme points will be produced. Thus we have encircled a polygon by its feasible region and the optimization problem will be reduced to that of a linear programming. The vertices of this polygon are the solutions of the problem provided that they obey $(P_{00} \leq \frac{1}{3}, P_{10} \leq \frac{1}{3}, P_{20} \leq \frac{1}{3})$ and by substituting them into equation C_γ one can determine its minimum value. Thus, for $\frac{67}{756} \leq x \leq \frac{61}{378}$, the extreme point is defined by $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and finally $(C_\gamma)_{min} = (\frac{1}{12})$. Having found the critical \mathbf{r} we substitute it in (3-15) and obtain a family of EW (called critical EW). Thus we have

$$p_c = -3, \quad W_c(x) = \frac{1}{2} \left(\frac{1}{3} I_9 - |\psi_{00}\rangle\langle\psi_{00}| + (8x - 1)(|\psi_{10}\rangle\langle\psi_{10}| - |\psi_{20}\rangle\langle\psi_{20}|) \right), \quad (6-75)$$

where $W_c(x)$ reduces to the following well known reduced EW at $x = \frac{1}{8}$:

$$W_{red} = \frac{I_9 - 3|\psi_{00}\rangle\langle\psi_{00}|}{6}, \quad (6-76)$$

In the Appendix B it is shown that the above EW is optimal in contrast to the conclusion that it is a decomposable EW Ref.[28].

In the Appendix B, we discuss the possible choice of x consistent with $C_{mn} = \frac{1}{12}$ and the optimality of the corresponding $W_c(x)$.

Also we prove in the following section that W_c is nd-EW for all values of $\frac{67}{756} \leq x \leq \frac{61}{378}$, except for $x = \frac{1}{8}$. Besides taking a convex combination of W_c and W_{red} , i.e.,

$$W_\Lambda = \Lambda W_c + (1 - \Lambda)W_{red}, \quad (6-77)$$

we obtain a new EW which is optimal (see Appendix B) and is also an nd-EW for certain value of the parameter Λ as will be shown in section 7.

However we can consider other values for q_{ij} in (3-15), e.g., $q_{20} = q_{02} = q_{11} = q_{22} = q_{12} = q_{21} = \frac{1}{8}$, $q_{10} = x$ and $q_{01} = \frac{1}{4} - x$, $0 \leq x \leq \frac{1}{4}$ then define the observable W by substituting the above condition in (3-15) as follows

$$W(x) = \mathbf{r} \frac{I_9}{9} + (1 - \mathbf{r}) \left(\frac{I_9}{8} - \frac{1}{8} |\psi_{00}\rangle\langle\psi_{00}| - \frac{8x-1}{8} (|\psi_{10}\rangle\langle\psi_{10}| - |\psi_{01}\rangle\langle\psi_{01}|) \right). \quad (6-78)$$

By using (3-16) for non-negativity of the observable W we find the distributions P_{ij} as a function of x . The minimum value of C_γ is obtained from the boundary of the feasible region, i.e., we have

$$C_\gamma = \frac{1}{8} (1 - P_{00} - (8x - 1)(P_{10} - P_{01})). \quad (6-79)$$

For given product states $|\gamma\rangle = |\alpha\rangle_1 |\alpha\rangle_2$ one can obtain the extreme points of the product

distributions as

$$\left\{ \begin{array}{l}
 |\alpha\rangle_1 = |\alpha\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow (P_{00} = \frac{1}{3}, P_{10} = \frac{1}{3}, P_{01} = 0) \\
 |\alpha\rangle_1 = |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow (P_{00} = \frac{1}{3}, P_{10} = 0, P_{01} = \frac{1}{3}) \\
 |\alpha\rangle_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega \\ \omega \\ \bar{\omega} \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} \bar{\omega} \\ \omega \\ \bar{\omega} \end{pmatrix} \rightarrow (P_{00} = 0, P_{10} = \frac{1}{3}, P_{01} = \frac{1}{3}) \\
 \\
 |\alpha\rangle_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ \omega \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ \bar{\omega} \end{pmatrix} \rightarrow (P_{00} = \frac{1}{3}, P_{10} = 0, P_{01} = 0) \\
 |\alpha\rangle_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ \bar{\omega} \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ \bar{\omega} \end{pmatrix} \rightarrow (P_{00} = 0, P_{10} = \frac{1}{3}, P_{01} = 0) \quad . \quad (6-80) \\
 |\alpha\rangle_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega \\ 1 \end{pmatrix}, |\alpha\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ \bar{\omega} \end{pmatrix} \rightarrow (P_{00} = 0, P_{10} = 0, P_{01} = \frac{1}{3})
 \end{array} \right.$$

Similar to (P_{00}, P_{10}, P_{20}) we obtain the extreme points. Then by convex combination of these points we obtain the feasible region (see Fig 2). The optimization problem is in the following form

$$\left\{ \begin{array}{ll}
 \text{minimize} & C_\gamma = \frac{1}{8}(1 - P_{00} - (8x - 1)(P_{10} - P_{01})) \\
 \text{subject to} & \frac{2}{3} - P_{00} - P_{10} - P_{01} \geq 0 \\
 & P_{00}, P_{10}, P_{01} \geq 0.
 \end{array} \right. \quad (6-81)$$

We have been able to find analytically the extreme points which at the same time don't

violate the plane ($\frac{2}{3} - P_{00} - P_{10} - P_{01} = 0$). But we have failed to prove in general that no point lies out of the plane. Therefore, we have proved numerically that there is no violation from the plane. Thus, this feasible region is a convex hull or a polygon itself and reduces the optimization problem to a linear programming. So the vertices of the polygon are the solutions of the problem which by substituting them into equation C_γ one can determine its minimum value as $(C_\gamma)_{min} = (\frac{2-(8x-1)}{24})$. We can find the critical r and by substituting the critical r in (3-15) we obtain a family of EW (called critical EW) resulting in

$$\mathbf{r}_c = \frac{-3 + 24x}{-1 + 24x}$$

$$W_c(x) = \frac{1}{3(-1 + 24x)}((8x - 1)I_9 - 3|\psi_{00}\rangle\langle\psi_{00}| - 3(8x - 1)(|\psi_{10}\rangle\langle\psi_{10}| - |\psi_{01}\rangle\langle\psi_{01}|)). \quad (6-82)$$

The obtained EW for two sets of triplet P , namely (P_{00}, P_{20}, P_{20}) and (P_{00}, P_{20}, P_{01}) can produce the most general form EW corresponding to combination of another triplet P_{ij} 's. Since under Fourier transform one can transform all the shifts in to the modulation. Moreover, the shift and modulation operators themselves can affect on the corresponding EW too, and so produce new combination of triplet.

7 Non-decomposable $3 \otimes 3$ Bell states diagonal entanglement witnesses

By calculating the partial transpose of $W_c(\frac{67}{756} \leq x \leq \frac{61}{378})$ (for $\{P_{00}, P_{10}, P_{20}$ case) we prove that it is an nd-EW. The necessary and sufficient condition for non-decomposability of W_c reduces to the negativity of its partial transpose. Using the following relation

$$(|\psi_{j'k'}\rangle\langle\psi_{jk}|)^{T_A} = \frac{1}{3} \sum_{l,m} \omega^{ml} |\psi_{m+j',l+k'}\rangle\langle\psi_{m+j,3-(l-k)}|, \quad (7-83)$$

one can show that $(W_c)^{T_A}$ is a block diagonal, i.e., we have

$$(W_c)^{T_A} = \sum_{j,k,k'} (O_j)_{kk'} |\psi_{j'k'}\rangle\langle\psi_{jk}|,$$

with the matrices O_j calculated as

$$O_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{6} & C_j \\ 0 & \bar{C}_j & \frac{1}{6} \end{pmatrix}, \quad (7-84)$$

with

$$C_j = \frac{4}{3}(x\omega + (\frac{1}{4} - x)\bar{\omega})\bar{\omega}^j, \quad j = 0, 1, 2. \quad (7-85)$$

Using the fact that $|C_2| = |C_1| = |C_0|$ one can show that the matrices O_j have the same eigenvalues

$$\begin{cases} \lambda = 0 \\ \lambda_{\pm} = \lambda_{\pm}^j = \frac{1}{6} \pm \frac{1}{6}\sqrt{4 + 48x(4x - 1)}. \end{cases} \quad (7-86)$$

The above equation indicates that λ_- is negative except for the particular case in which $x = \frac{1}{8}$, i.e., W_{red} . Then different eigenvectors are so obtained

$$\begin{cases} \lambda = 0 & \rightarrow |\phi_j^0\rangle = |\psi_{j0}\rangle, \\ \lambda = \lambda_{\pm} & \rightarrow |\phi_j^{\pm}\rangle = \frac{1}{\sqrt{|\beta_{\pm}^j|^2 + 1}}(\beta_{\pm}^j |\psi_{j1}\rangle + |\psi_{j2}\rangle), \end{cases}, \quad (7-87)$$

where

$$\beta_{\pm}^j = \frac{C_j \lambda_{\pm}}{\lambda_{\pm}^2 - \lambda_{\pm} B}. \quad (7-88)$$

So we conclude that W_c^{TA} has three eigenvalues, namely λ_0, λ_{\pm} , each with degeneracy 3, and the following projection operators

$$\begin{cases} Q_+ = \sum_{j=0}^2 |\phi_j^+\rangle \langle \phi_j^+| \\ Q_- = \sum_{j=0}^2 |\phi_j^-\rangle \langle \phi_j^-| \\ Q_0 = \sum_{j=0}^2 |\phi_j^0\rangle \langle \phi_j^0|. \end{cases} \quad (7-89)$$

Here we have

$$W_c^{TA} = \lambda_+ Q_+ + \lambda_- Q_-. \quad (7-90)$$

The equation indicates that W_c^{TA} is not a positive definite operator except for the particular case W_{red} , hence it is non-decomposable entanglement witness.

We are interested in the n-d of EW given in (3-15) for the allowed values of p. Therefore, we write Eq.(3-15) as

$$W = \varepsilon I_9/9 + (1 - \varepsilon)W_c, \quad (7-91)$$

with

$$\varepsilon = \frac{\mathbf{r} + 3}{4}. \quad (7-92)$$

Now, expanding $I_9/9$ in terms of the projection operator (7-89) as

$$I_9 = Q_0^{T_A} + Q_-^{T_A} + Q_+^{T_A}, \quad (7-93)$$

the EW given by (3-15) can be written as

$$W = \varepsilon/9 Q_0^{T_A} + \left(\frac{\varepsilon}{9} + (1 - \varepsilon)\lambda_+\right) Q_+^{T_A} + \left(\frac{\varepsilon}{9} - (1 - \varepsilon)|\lambda_-|\right) Q_-^{T_A}. \quad (7-94)$$

The above form of EW indicates that its partial transpose W^{T_A} is positive, i.e., it is decomposable EW if we have

$$W^{T_A} \geq 0 \Rightarrow \left(\frac{\varepsilon}{9} - (1 - \varepsilon)|\lambda_-|\right) \geq 0 \rightarrow \mathbf{r} \geq \frac{-3 + 9|\lambda_-|}{1 + 9|\lambda_-|}, \quad (7-95)$$

for $\frac{-3+9|\lambda_-|}{1+9|\lambda_-|} \leq \mathbf{r} \leq -3$. It is not easy to tell where the EW is or is not decomposable. In the next section using some bound entangled state we will investigate their non-decomposability.

Now, in the remaining part of this section we try to obtain some nd-EW by taking the convex combination $W_c(x)$ for all $\frac{67}{756} \leq x \leq \frac{61}{378}$ and W_{red} (6-76) as

$$W_\Lambda(x) = \Lambda W_c(x) + (1 - \Lambda)W_{red}, \quad \Lambda \in [0, 1]. \quad (7-96)$$

In order to test the positivity of $W_\Lambda^{T_A}(x)$ we must first expand W_c and W_{red} in terms of the positive diagonal operators. Thus at first we write the projection operators defined in (7-89) in the following form

$$Q_\pm = \sum_{k=0}^2 |\chi_k^\pm\rangle \langle \chi_k^\pm|, \quad Q_0 = \sum_{k=0}^2 |\psi_{k0}\rangle \langle \psi_{k0}| \quad (7-97)$$

with

$$| \chi_k^\pm \rangle = (| \psi_{k1} \rangle \pm \omega^{-k} | \psi_{k2} \rangle). \quad (7-98)$$

Now writing $I_9/9$ in terms of the projection operator (7-97) and using the fact that

$$(| \psi_{00} \rangle \langle \psi_{00} |)^{TA} = \frac{1}{3} \left(\sum_{k=0}^2 | \psi_{k0} \rangle \langle \psi_{k0} | + \sum_{k=0}^2 | \chi_k^+ \rangle \langle \chi_k^+ | - \sum_{k=0}^2 | \chi_k^- \rangle \langle \chi_k^- | \right)$$

and

$$W_c^{TA}(x) = \lambda_+ \sum_{k=0}^2 | \chi_k^+ \rangle \langle \chi_k^+ | - | \lambda_- | \sum_{k=0}^2 | \chi_k^- \rangle \langle \chi_k^- |, \quad (7-99)$$

we get for the partial transpose $W_\Lambda(x)$ in Eq.(7-96)

$$W_\Lambda^{TA}(x) = \Lambda(\lambda_+) \sum_{k=0}^2 | \chi_k^+ \rangle \langle \chi_k^+ | + (-\Lambda | \lambda_- | + \frac{(1-\Lambda)}{3}) | \chi_k^- \rangle \langle \chi_k^- |. \quad (7-100)$$

This expression implies that $W_\Lambda^{TA}(x)$ is positive, since

$$\Lambda \leq \frac{1}{1 + 3 | \lambda_- |}. \quad (7-101)$$

Again, for $\frac{1}{1+3|\lambda_-|} \leq \Lambda \leq 1$, it is not easy to talk about decomposable or non-decomposable $W_\Lambda(x)$, and one needs to find some bound entangled states to show their non-decomposability, this will be done in the following section.

8 Detection of bound entangled state with Bell states diagonal entanglement witnesses

Now if we succeed to find any bound entangled state [6, 5] so that BDEW is able to detect this bound state corresponding to BDEW, from definition 2 in section 1 EW will be an nd-EW.

Let a bound entangled Bell decomposable state be written as

$$\rho = \mu Q_0^{TA} + \eta Q_+^{TA} + \zeta Q_-^{TA}, \quad \rho^{TA} \geq 0 \Rightarrow \{\mu, \eta, \zeta\} \geq 0. \quad (8-102)$$

Optimal BDEW must detect this bound state, i.e.,

$$Tr[W_c \rho] < 0 \Rightarrow \eta \lambda_+ < \zeta | \lambda_- |. \quad (8-103)$$

On the other hand this bound state must be positive. For simplicity we use the operator W_c and the identity operator I_9 in the bound state definition

$$Q_+^{T_A} = \frac{W_c + |\lambda_-| (I_9 - Q_0^{T_A})}{|\lambda_-| + \lambda_+}, \quad Q_-^{T_A} = \frac{-W_c + \lambda_+ (I_9 - Q_0^{T_A})}{|\lambda_-| + \lambda_+}, \quad (8-104)$$

so that the bound state reduces to the following form

$$\rho = (\mu - \frac{\eta |\lambda_-| + \zeta \lambda_+}{|\lambda_-| + \lambda_+}) Q_0^{T_A} + (\frac{\eta |\lambda_-| + \zeta \lambda_+}{|\lambda_-| + \lambda_+}) I_9 + (\frac{\eta - \zeta}{|\lambda_-| + \lambda_+}) W_c. \quad (8-105)$$

In this case $Q_0 = |\psi_{00}\rangle\langle\psi_{00}| + |\psi_{10}\rangle\langle\psi_{10}| + |\psi_{20}\rangle\langle\psi_{20}|$ and by substituting this result in the Eq.(8-105) we get

$$\begin{aligned} \rho = & (\mu - \frac{\eta - \zeta}{3(|\lambda_-| + \lambda_+)}) |\psi_{00}\rangle\langle\psi_{00}| + (\mu + (12x - 1) \frac{\eta - \zeta}{3(|\lambda_-| + \lambda_+)}) |\psi_{10}\rangle\langle\psi_{10}| \\ & + (\mu - (12x - 2) \frac{\eta - \zeta}{3(|\lambda_-| + \lambda_+)}) |\psi_{20}\rangle\langle\psi_{20}| + \frac{\eta(|\lambda_-| - \frac{1}{6}) - \zeta(|\lambda_+| + \frac{1}{6})}{3(|\lambda_-| + \lambda_+)} (|\psi_{01}\rangle\langle\psi_{01}| \\ & + |\psi_{02}\rangle\langle\psi_{02}| + |\psi_{11}\rangle\langle\psi_{11}| + |\psi_{22}\rangle\langle\psi_{22}| + |\psi_{12}\rangle\langle\psi_{12}| + |\psi_{21}\rangle\langle\psi_{21}|). \end{aligned} \quad (8-106)$$

The positivity of ρ requires that all the Bell states diagonal operator coefficients to be positive, and that this condition be imposed on the coefficient μ only. So we get

$$\begin{cases} x \geq \frac{1}{8} & \mu \geq \frac{(12x-1)(\frac{1}{3}-2\eta)}{(12x-1)+3(|\lambda_-|+\lambda_+)} \\ x \leq \frac{1}{8} & \mu \geq \frac{(2-12x)(\frac{1}{3}-2\eta)}{(2-12x)+3(|\lambda_-|+\lambda_+)}, \end{cases} \quad (8-107)$$

which, in this case means Q_0 is on the boundary. Now by using this bound entangled BD state we can find n-d condition for BDEW. We know EW will be an nd-EW if this EW is able to detect any bound state. Then by using the equations (7-94) and (8-102) we have

$$Tr(W\rho) = (\frac{\varepsilon\mu}{3} + 3(\frac{\varepsilon}{9\lambda_+} + (1-\varepsilon)\eta\lambda_+ + 3(\frac{\varepsilon}{9\lambda_-} - (1-\varepsilon)\zeta)|\lambda_-| < 0. \quad (8-108)$$

Now by substituting ε from Eq.(7-92) we obtain

$$\mathbf{r} < \frac{-3 + 27(\zeta|\lambda_-| - \eta\lambda_+)}{1 + 27(\zeta|\lambda_-| - \eta\lambda_+)}, \quad (8-109)$$

where the calculated \mathbf{r} is greater than the represented \mathbf{r} for EW in Eq.(7-95). Therefore, we can find one of the p's corresponding to EW which is an nd-EW. Non-decomposable generalized EW for a general case is under investigation.

9 Choi map

Choi positive map [16] $\phi(a, b, c) : M^3 \rightarrow M^3$ is defined as

$$\phi_{a,b,c}(\rho) = \begin{pmatrix} a\rho_{11} + b\rho_{22} + c\rho_{33} & 0 & 0 \\ 0 & a\rho_{22} + b\rho_{33} + c\rho_{11} & 0 \\ 0 & 0 & a\rho_{33} + b\rho_{11} + c\rho_{22} \end{pmatrix} - \rho, \quad (9-110)$$

where $\rho \in M^3$. It was shown that $\phi(a, b, c)$ is positive iff

$$a \geq 1, \quad a + b + c \geq 3, \quad 1 \leq a \leq 3. \quad (9-111)$$

Using Jamiołkowski [10] isomorphism between the positive map and the operators we obtain the following $3 \otimes 3$ EW corresponding to Choi map

$$W_{Choi} = \frac{1}{3(a+b+c-1)} \left(a \sum_{k=0}^2 |\psi_{k0}\rangle\langle\psi_{k0}| + b \sum_{k=0}^2 |\psi_{k2}\rangle\langle\psi_{k2}| + c \sum_{k=0}^2 |\psi_{k1}\rangle\langle\psi_{k1}| - 3|\psi_{00}\rangle\langle\psi_{00}| \right). \quad (9-112)$$

Similar to BDEW we expand $|\psi_{00}\rangle\langle\psi_{00}|$ using the identity operator and the other Bell diagonal states:

$$|\psi_{00}\rangle\langle\psi_{00}| = I_9 - \sum_{i \neq j=0}^2 |\psi_{ij}\rangle\langle\psi_{ij}|. \quad (9-113)$$

Then we reduce EW to the following form

$$\begin{aligned} W_{Choi} = \frac{1}{3(a+b+c-1)} & \left(-(3-a)I_9 + 3 \sum_{k=1}^2 |\psi_{k0}\rangle\langle\psi_{k0}| \right. \\ & \left. + (b+3-a) \sum_{k=0}^2 |\psi_{k2}\rangle\langle\psi_{k2}| + (c+3-a) \sum_{k=0}^2 |\psi_{k1}\rangle\langle\psi_{k1}| \right). \end{aligned} \quad (9-114)$$

Comparing with BDEW (3-15) we have

$$\mathbf{r} = -\frac{3(3-a)}{(a+b+c-1)}, \quad (9-115)$$

and the EW operator is defined as

$$W_{Choi} = \mathbf{r}I_9/9 + (1-\mathbf{r}) \left(\frac{1}{(8-2a+b+c)} \sum_{k=1}^2 |\psi_{k0}\rangle\langle\psi_{k0}| + \frac{(b+3-a)}{3(8-2a+b+c)} \sum_{k=0}^2 |\psi_{k2}\rangle\langle\psi_{k2}| \right)$$

$$+\frac{(c+3-a)}{3(8-2a+b+c)}\sum_{k=0}^2|\psi_{k1}\rangle\langle\psi_{k1}|). \quad (9-116)$$

By comparing (9-114) with (3-15) we obtain the coefficients q_{ij}

$$q_{10} = q_{20} = \frac{1}{(8-2a+b+c)}, \quad q_{02} = q_{12} = q_{22} = \frac{b+3-a}{3(8-2a+b+c)},$$

$$q_{01} = q_{11} = q_{21} = \frac{c+3-a}{3(8-2a+b+c)}. \quad (9-117)$$

Note that if r is negative, as introduced in EW above, this operator will be positive, but not a completely positive map. For $\mathbf{r} \leq 0$ we have $1 \leq a \leq 3$. By assuming $a \geq b \geq c$, the minimum negative eigenvalue of choi EW (9-116) is given by

$$\frac{\mathbf{r}}{9} + (1-\mathbf{r})\frac{c+3-a}{3(8-2a+b+c)} < 0, \quad (9-118)$$

where upon substituting \mathbf{r} from Eq.(9-117) we get $1 \leq a \leq 2$. This is equal to the introduced positivity condition of Choi map in [16].

By using (3-16) for non-negativity of the observable W_{choi} we find the distributions P_{ij} as a function of q_{ij} . The minimum value of C_γ is obtained from the boundary of the feasible region, i.e., we have

$$(C_\gamma) = \frac{1}{(8-2a+b+c)}\mathcal{P}_1 + \frac{(b+3-a)}{3(8-2a+b+c)}\mathcal{P}_2 + \frac{(c+3-a)}{3(8-2a+b+c)}\mathcal{P}_3, \quad (9-119)$$

where $\mathcal{P}_1 = \sum_{k=1}^2 P_{k0}$, $\mathcal{P}_2 = \sum_{k=0}^2 P_{k2}$ and $\mathcal{P}_3 = \sum_{k=0}^2 P_{k1}$. We can find the extreme value of $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ which is obtained under the product states $|\gamma\rangle = |\alpha\rangle_1 |\alpha\rangle_2$ as

$$\left\{ \begin{array}{l} \mathcal{P}_1 = |\alpha_1|^2 |\beta_1|^2 + |\alpha_2|^2 |\beta_2|^2 + |\alpha_3|^2 |\beta_3|^2 \\ -\frac{1}{3} ||\alpha_1|| |\beta_1| + |\alpha_2|| |\beta_2| e^{i\phi_2} + |\alpha_3|| |\beta_3| e^{i\phi_3} |^2 \\ \mathcal{P}_2 = |\alpha_1|^2 |\beta_1|^2 + |\alpha_3|^2 |\beta_1|^2 + |\alpha_3|^2 |\beta_2|^2 \\ \mathcal{P}_3 = |\alpha_1|^2 |\beta_2|^2 + |\alpha_2|^2 |\beta_3|^2 + |\alpha_3|^2 |\beta_1|^2 \end{array} \right., \quad (9-120)$$

where $|\alpha\rangle_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ and $|\alpha\rangle_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$. One can obtain the extreme points of the

$(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ as

$$\left\{ \begin{array}{l} |\alpha\rangle_1 = |\alpha\rangle_2 = \begin{pmatrix} 1 \\ \omega \\ \bar{\omega} \end{pmatrix} \rightarrow (\mathcal{P}_1 = \frac{1}{3}, \mathcal{P}_2 = \frac{1}{3}, \mathcal{P}_3 = \frac{1}{3}) \\ |\alpha\rangle_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |\alpha\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow (\mathcal{P}_1 = 0, \mathcal{P}_2 = 1, \mathcal{P}_3 = 0) \\ |\alpha\rangle_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |\alpha\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow (\mathcal{P}_1 = 0, \mathcal{P}_2 = 0, \mathcal{P}_3 = 1) \\ |\alpha\rangle_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |\alpha\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow (\mathcal{P}_1 = \frac{2}{3}, \mathcal{P}_2 = 0, \mathcal{P}_3 = 0) \end{array} \right. . \quad (9-121)$$

The convex combination of all extreme points provide a convex or a feasible region (Fig-3), then we have the following optimization problem

$$\left\{ \begin{array}{l} \text{minimize} \quad (C_\gamma) = (\frac{1}{(8-2a+b+c)}\mathcal{P}_1 + \frac{(b+3-a)}{3(8-2a+b+c)}\mathcal{P}_2 + \frac{(c+3-a)}{3(8-2a+b+c)}\mathcal{P}_3 \\ \text{subject to} \quad 1 - \frac{3}{2}\mathcal{P}_1 - \mathcal{P}_2 - \frac{1}{2}\mathcal{P}_3 \leq 0 \\ \quad \quad \quad 1 - \frac{3}{2}\mathcal{P}_1 - \frac{1}{2}\mathcal{P}_2 - \mathcal{P}_3 \leq 0 \\ \quad \quad \quad 1 - \mathcal{P}_1 - \mathcal{P}_2 - \mathcal{P}_3 \geq 0 \\ \quad \quad \quad \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \geq 0. \end{array} \right. \quad (9-122)$$

Whether analytically we have been able to show that we will have violation only from the two planes

$$2 - 3\mathcal{P}_1 - 2\mathcal{P}_2 - \mathcal{P}_3 = 0$$

$$2 - 3\mathcal{P}_1 - 2\mathcal{P}_3 - \mathcal{P}_2 = 0.$$

Now let us assume that the maximum value of the violation from the planes is $\Delta < 1$. Thus, the equation of the plane passing through the new extreme points, parallel to the above plane,

is obtained. Next we derive the intersection of the following adjacent planes

$$\left\{ \begin{array}{ll} 1) & 3\mathcal{P}_1 + \mathcal{P}_2 + 2\mathcal{P}_3 - (2 + \Delta) = 0 \\ 2) & 3\mathcal{P}_1 + 2\mathcal{P}_2 + \mathcal{P}_3 - (2 + \Delta) = 0 \\ 3) & \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 - 1 = 0 \\ 4) & \mathcal{P}_1 = 0 \\ 5) & \mathcal{P}_2 = 0 \\ 6) & \mathcal{P}_3 = 0 \\ 7) & \mathcal{P}_1 = \frac{2}{3} \\ 8) & \mathcal{P}_2 = 1 \\ 9) & \mathcal{P}_3 = 1 \end{array} \right. , \quad (9-123)$$

where new extreme points are obtained from intersecting the above planes. Next we calculate C_γ for all the newly obtained extreme points and compare them with each other. Some easy calculations gives the minimum value of the parameter C_γ which is independent from Δ

$$(C_\gamma)_{min} = \frac{6 + 2(c - a)}{9(8 - 2a + b + c)}, \quad (9-124)$$

then the critical value of the parameter r is obtained as

$$r_c = \frac{-6 + 2(a - c)}{2 + b - c}. \quad (9-125)$$

For $a = b = c = 1$ the parameter r reduces to $r_c = -3$ corresponding to the well known reduction map. On the other hand, EW (9-116) must have positive trace under any product state $|\gamma\rangle\langle\gamma|$. Thus the introduced r in EW must satisfy

$$r \geq r_c \Rightarrow \frac{-3(3 - a)}{(a + b + c - 1)} \geq \frac{-6 + 2(a - c)}{2 + b - c}, \quad (9-126)$$

where the inequality is satisfied for all value of $0 \leq a \leq 2$ and $a \geq b \geq c$.

10 Some separable states at the boundary of separable region

Here we introduce some set of separable states as

$$\begin{aligned}
 \rho_m &= \sum_k |\psi_{km}\rangle\langle\psi_{km}| = \sum_l |l\rangle\langle l| \otimes |l+m\rangle\langle l+m|, \\
 \rho'_m &= \sum_k |\psi_{mk}\rangle\langle\psi_{mk}| = \sum_{l,l',k} \omega^{m(l-l')} |l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|, \\
 \rho''_n &= \sum_k |\psi_{nk,k}\rangle\langle\psi_{nk,k}| = \sum_{l,l',k} \omega^{nk(l-l')} |l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|,
 \end{aligned} \tag{10-127}$$

where $n = 0, 1, 2$, $m = 0, 1, 2$. One can show that the convex sum of ρ_0 , $\rho''_0 = \rho'_0$, i.e, $\rho_\mu^S = \mu\rho_0 + (1 - \mu)\rho'_0$, is orthogonal to the optimal $W_\Lambda = \Lambda W_c + (1 - \Lambda)W_{red}$, i.e., we have $Tr(W_\Lambda \rho_\mu^S) = 0$.

Hence, ρ_μ^S lie at the boundary of the separable region [28]. On the other hand, one can show that by acting the local unitary operation U_{ij} over W_Λ as $(W_\Lambda)_{ij} = U_{ij}(W_\Lambda)U_{ij}^\dagger$ he obtains a new set of optimal EW, $(W_\Lambda)_{ij}$, the application of which is not only to get a new set of bound entangled states by acting local unitary operation, but also to obtain some separable states $(\rho_\mu^S)_{ij} = U_{ij}\rho_\mu^S U_{ij}^\dagger$ as such which are the convex sum of separable states (10-127) at the boundary of separable states.

11 Conclusion

We have shown that finding generic Bell states diagonal entanglement witnesses (BDEW) for $d_1 \otimes d_2 \otimes \dots \otimes d_n$ systems has reduced to a linear programming problem. Since solving linear programming for generic case is difficult we have considered the following special cases. Also we have considered BDEW for multi-qubit, $2 \otimes N$ and $3 \otimes 3$ systems and then have considered optimality condition for $3 \otimes 3$ EW. Also, we have considered an n-d condition over $3 \otimes 3$ BDEW and have obtained this condition for some special cases exactly. We have defined

extensive group of nd-BDEW by combining critical EW and the reduction map (each with special coefficients). Then we have defined the Bell decomposable bound entangled state and have considered detection of this state with optimal BDEW and a general BDEW. Finally, we have considered Choi map as an example of BDEW. Optimality and non-decomposability of EW for multi-qubit and $2 \otimes N$ as well as EW for generic bipartite $d_1 \otimes d_2$ systems and multipartite $d_1 \otimes d_2 \otimes \dots \otimes d_n$ are under investigation. As a physical implementation of EW we know that the optimization of decomposition of EW to find the smallest number of measurements possible for local measurement on a system can be used. Therefore to make use of this implementation of EW for the obtained EW's is currently under investigation.

APPENDIX A

Minimization of the product distributions:

In Eq.(3-5) the Bell orthonormal states for a $d_1 \otimes d_2 \otimes \dots \otimes d_n$ ($d_1 \leq d_2 \leq \dots \leq d_n$) have been introduced by applying local unitary operation on $|\psi_{00}\rangle$. Let us further consider a pure product state $|\gamma\rangle = |\alpha\rangle_1 |\alpha\rangle_2 \dots |\alpha\rangle_n$. Then the product distributions can be written as

$$P_{i_1, i_2, \dots, i_n}(\gamma) = |\langle \gamma | \psi_{i_1, i_2, \dots, i_n} \rangle|^2. \quad (\text{A-i})$$

It easily follows that

$$0 \leq P_{i_1, i_2, \dots, i_n}(\gamma) \leq \frac{1}{d_1}. \quad (\text{A-ii})$$

On the other hand, from the completeness of Bell states:

$$\sum_{i_1, i_2, \dots, i_n} |\psi_{i_1, i_2, \dots, i_n}\rangle \langle \psi_{i_1, i_2, \dots, i_n}| = I_{d_1} \otimes I_{d_2} \otimes \dots \otimes I_{d_n}, \quad (\text{A-iii})$$

we have $\sum_{i_1, i_2, \dots, i_n} P_{i_1, i_2, \dots, i_n}(\gamma) = 1$, which leads to

$$\sum_{i_1, i_2, \dots, i_n} |\langle \gamma | \psi_{i_1, i_2, \dots, i_n} \rangle|^2 = d_1. \quad (\text{A-iv})$$

The above equation indicates that if we can show that for a particular choice of $|\alpha\rangle_i$'s, the

d_1 -number of $|\langle \gamma | \psi_{i_1, i_2, \dots, i_n} \rangle|^2 = P_{i_1, i_2, \dots, i_n}$ can have their maximum value equal to $\frac{1}{d_1}$, then the remaining ones will be zero.

To minimize the summation $C = \sum_{ij} q_{ij} P_{ij}$ for a $3 \otimes 3$ system, assuming that $q_{00} = 0$, let us first suppose that $|\alpha\rangle = |\beta\rangle$ so that $P_{00} = \frac{1}{3}$. Then we find the set $|\langle \alpha | U_{ij} | \beta \rangle|^2 = 1$ for different possible choices of $|\alpha\rangle$ and U_{ij} :

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega \\ \bar{\omega} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \bar{\omega} \\ \omega \end{pmatrix}, \quad |\psi_{01}\rangle, |\psi_{02}\rangle, \quad \min(\sum_{ij} q_{ij}) = q_{01} + q_{02},$$

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\psi_{10}\rangle, |\psi_{20}\rangle, \quad \min(\sum_{ij} q_{ij}) = q_{10} + q_{20},$$

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 1 \\ \omega \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \omega \\ 1 \\ 1 \end{pmatrix}, \quad |\psi_{11}\rangle, |\psi_{22}\rangle, \quad \min(\sum_{ij} q_{ij}) = q_{11} + q_{22},$$

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 1 \\ \bar{\omega} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \bar{\omega} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \bar{\omega} \\ 1 \\ 1 \end{pmatrix}, \quad |\psi_{12}\rangle, |\psi_{21}\rangle, \quad \min(\sum_{ij} q_{ij}) = q_{12} + q_{21}.$$

The above relations imply that $C_{mn} = \frac{1}{3}(q_1 + q_2)$, where q_1 and q_2 correspond to two of q_{ij} appearing in the same row.

APPENDIX B

Critical entanglement witness is optimal:

According to the References [14, 15], an EW will be optimal if for all positive operator P and $\varepsilon > 0$, the operator

$$W' = (1 + \varepsilon)W_c - \varepsilon P \tag{B-i}$$

is not an EW. In order to prove the critical EW given in (6-75) is optimal, we first show that

$$\text{Tr}(W_c|\alpha\rangle\langle\alpha| \otimes |\alpha^*\rangle\langle\alpha^*|) = 0. \quad (\text{B-ii})$$

It just suffices to check that for the product distribution $P_{ij} = \langle \psi_{ij} | \alpha \rangle \langle \alpha | \otimes |\alpha^*\rangle \langle \alpha^* | \psi_{ij} \rangle$, we have $P_{00} = \frac{1}{3}$, $P_{01} = P_{02}$, $P_{11} = P_{22}$, $P_{12} = P_{21}$.

Substituting P_{ij} given above in (B-ii), it is easy to see that $\text{Tr}(W_c|\alpha\rangle\langle\alpha| \otimes |\alpha^*\rangle\langle\alpha^*|) = 0$. Also it is straightforward to see that there exists no positive operator P with the constraint

$$\text{Tr}(P|\alpha\rangle\langle\alpha| \otimes |\alpha^*\rangle\langle\alpha^*|) = 0, \quad \forall |\alpha\rangle$$

. Therefore, there exist no positive operator \mathbf{r} to satisfy (B-i). Hence W_c , and in particular W_{red} , are optimal.

APPENDIX C

Simplex method for solving multi-qubit minimization problem

We know that simplex method is an elegant way for solving linear programming problems. As an example we obtain the $P_{00\dots 00}$ and $P_{10\dots 00}$ constraints in Eq.(4-31), thus we have two slack variables which are defined as

$$\omega_1 = \frac{1}{2^{n-1}} - 2P_{00\dots 00} + 2P_{10\dots 00}\left(1 - \frac{1}{2^{n-1}}\right), \quad \omega_2 = \frac{1}{2^{n-1}} - 2P_{10\dots 00} + 2P_{00\dots 00}\left(1 - \frac{1}{2^{n-1}}\right). \quad (\text{B-i})$$

We carry out this procedure to transform the inequality constraints (4-31) into equality

$$\left\{ \begin{array}{ll} \text{maximize} & -C_\gamma = \frac{1}{2(2^{n-1}-1)}(-(1-x) + (1-x)P_{00\dots 00} + ((2^n-1)x-1)P_{10\dots 00}) \\ \text{subject to} & \omega_1 = \frac{1}{2^{n-1}} - 2P_{00\dots 00} + 2P_{10\dots 00}\left(1 - \frac{1}{2^{n-1}}\right) \\ & \omega_2 = \frac{1}{2^{n-1}} - 2P_{10\dots 00} + 2P_{00\dots 00}\left(1 - \frac{1}{2^{n-1}}\right) \\ & P_{00\dots 00}, P_{10\dots 00}, \omega_1, \omega_2 \geq 0. \end{array} \right. \quad (\text{B-ii})$$

Now we rewrite the first equation in (B-ii) in terms of ω_1 and ω_2 , making use of the slack

variables:

$$-C_\gamma = \frac{1}{2(2^{n-1} - 1)}(-(1-x) + \frac{(1-x)a - (2^n - 1)x + 1}{2(a^2 - 1)}\omega_1 + \frac{(1-x) - ((2^n - 1)x - 1)a}{2(a^2 - 1)}\omega_2), \quad (\text{B-iii})$$

where $a = 1 - \frac{1}{2^{n-2}}$. For $0 \leq x \leq \frac{1}{2^{n-1}+1}$ the coefficients ω_1 and ω_2 are both negative. Now from the simplex method we conclude $\omega_1 = \omega_2 = 0$, i.e., $P_{00\dots 00} = P_{10\dots 00} = \frac{1}{2}$. Thus the minimum value of $C_\gamma = \frac{x}{2}$. For $\frac{1}{2^{n-1}+1} \leq x \leq 1$, from (B-ii), we see that the coefficient $P_{10\dots 00}$ is negative, so that $P_{10\dots 00} = 0$, hence $P_{00\dots 00} = \frac{1}{2^{n-1}}$. Therefore, we find the minimum value of C_γ as $(C_\gamma)_{\min} = \frac{1-x}{2^n}$.

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Figure Captions

Figure-1: Feasible region for $3 \otimes 3$ systems for particular choice of $q_{00} = 0$, $q_{10} = x$, $q_{20} = \frac{1}{4} - x$ and others q 's being equal, i.e., when the linear programming variables are P_{00} , P_{10} and P_{20} .

Figure-2: Feasible region for $3 \otimes 3$ systems for particular choice of $q_{00} = 0$, $q_{10} = x$, $q_{01} = \frac{1}{4} - x$ and others q 's being equal, i.e., when the linear programming variables are P_{00} , P_{10} and P_{01} .

Figure-3: Feasible region for $3 \otimes 3$ Choi map for particular choice of $a \geq b \geq c$, i.e., when the linear programming variables are $\mathcal{P}_1 \sum_{k=1}^2 P_{k0}$, $\mathcal{P}_2 \sum_{k=0}^2 P_{k2}$ and $\mathcal{P}_1 \sum_{k=0}^2 P_{k1}$.