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Quantum tomography with wavelet transform in Banach space on Homogeneous space

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Abstract

The intimate connection between the Banach space wavelet reconstruction method on homogeneous spaces with both singular and nonsingular vacuum vectors, and some of well known quantum tomographies, such as: Moyal-representation for a spin, discrete phase space tomography, tomography of a free particle, Homodyne tomography, phase space tomography and SU(1,1) tomography is explained. Also both the atomic decomposition and banach frame nature of these quantum tomographic examples is explained in details.

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Finally the connection between the wavelet formalism on Banach space and Q-function is discussed.

1 INTRODUCTION

The mathematical theory of wavelet Transform finds nowadays an enormous success in various fields of science and technology, including treatment of large databases, data and image compression, signal processing, telecommunication and many other applications [1]. After the empirical discovery by Morlet [2], it was recognized from the very beginning by Grossmann, Morlet, Paul and daubechies[3] that wavelets are simply coherent states associated to affine group of the line (dilations and translations)[4, 5]. Thus, immediately the stage was set for a far reaching generalization[3, 6]. Unlike function which form orthogonal bases for space, Morlet wavelets are not orthogonal and form frames. Frames are the set of functions which are not necessarily orthogonal and which are not linearly independent. Actually, frames are a repeatable set of vectors in Hilbert space which produces each vectors in space with a natural representation.

Recently another concept called atomic decomposition have played a key role in further mathematical development of wavelet theory. Indeed atomic decomposition for any space of

function or distribution aims at representing any element in the form of a set of simple function which are called atoms[9]. As far as the Banach space is concerned, Feichtinger-Grocheing [10] provided a general and very flexible way to construct coherent atomic decompositions and Banach frames for certain Banach spaces, called coorbit spaces.

The concept of a quantum state represents one of the most fundamental pillars of the paradigm of quantum theory. Usually the quantum state is described either by state vector in Hilbert space, or density operator or a phase space probability density distribution (quasidistributions). The quantum states can be determined completely from the appropriated experimentally data by using the well known technic of quantum tomography or better to say tomographic transformation.

A general framework is already presented for the unification of the Hilbert space wavelets transformation on the one hand, and quasidistributions and tomographic transformation associated with a given pure quantum states on the other hand [11]. Here in this manuscript we are trying to present the intimate connection between the Banach space wavelet reconstruction method developed by Feichtinger-Grocheing [7, 10] and some of well known quantum tomographies associated with mixed states, such as: Moyal-representation for a spin [12], discrete phase space tomography [13], tomography of a free particle [14], Homodyne tomography [15, 16, 17, 18], phase space tomography [14, 19, 20] and SU(1,1) tomography [21], all which can be represented by density matrices. Since the density matrix can be presented through Banach space in quantum Physics [22]. Therefore, it is natural to do quantum tomography of each density matrix by using the wavelet transform and its inverse in Banach space on Homogeneous space corresponding to the associated density matrix. The quantum tomography used by this method for the mixed quantum states is completely consistent with other commonly used methods. Also both the atomic decomposition and banach frame nature of these quantum tomographic examples is explained in details.

The paper is organized as follows:

In section-2 we define wavelet transform and its inverse on homogeneous spaces with both singular and nonsingular vacuum vectors. In section -3 we obtain some typical quantum tomographic examples with nonsingular vacuum vectors, such as: Moyal-representation for a spin, discrete phase space tomography, then define its atomic decomposition and Banach frame bounds. In section -4 we obtain some typical quantum tomographic examples with singular vacuum vectors, such as: Homodyne tomography, phase space tomography, SU(1,1) tomography and tomography of a free particle and define its atomic decomposition and Banach frame bounds. Finally, the connection between the wavelet formalism on Banach space and Q-function is discussed. The paper is ended with a brief conclusion.

2 Wavelet transform, frame and atomic decomposition in Banach spaces on homogeneous space:

The following is a brief recapitulation of some aspects of the theory of wavelets, atomic decomposition and Banach frame on homogeneous space. We only mention those concepts that will be needed in the sequel, a more detailed treatment may be found for example in [7, 10]. Let G be locally compact group with left Haar measure $d\mu$ and H be a closed subgroup of G . Let U be a continuous representation of a group. The homogeneous space is meant by $X = G/H$. Since U is not directly defined on G/H , it is necessary to embed G/H in G . This can be realized by using the canonical fiber bundle structure of G with projection $\Pi : G \longrightarrow X$. Let $\sigma : X \longrightarrow G$ be a borel section of this fiber bundle i.e., $\Pi \circ \sigma(x) = x$ for all $x \in X$.

We could define a representation for homogeneous space $X \times X$ in the space $\mathcal{L}(\mathcal{B})$ of bounded linear operators $\mathcal{B} \rightarrow \mathcal{B}$:

$$T : X \times X \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{B})) : \hat{O} \rightarrow U(x_1)\hat{O}U(x_2^{-1}), \quad (2-1)$$

where if x_1 is equal to x_2 , the representation is called adjoint representation, and, if x_2 is equal

to identity operator, the representation is called left representation of homogeneous space.

Let $\mathcal{L}(\mathcal{B})$ be the space of bounded linear operator $\mathcal{B} \rightarrow \mathcal{B}$ in Banach space. We will say that $b_0 \in \mathcal{B}$ is a vacuum vector if for all $h \in H$ then $U(h)b_0 = \chi(h)b_0$ and also the set of vectors $b_x = U(x)b_0$ forms a family of coherent states, if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional) and a vector $b_0 \in \mathcal{B}$ (called vacuum vector) such that

$$C(b_0, b'_0) = \int_X \langle T(x^{-1})b_0, l_0 \rangle \langle T(x)b'_0, l'_0 \rangle d\mu(x), \quad (2-2)$$

is non-zero and finite, which is known as the admissibility relation.

If the subgroup H is non-trivial, one does not need to know wavelet transform on the whole group G , but it should be defined on only the homogeneous space G/H , then the reduced wavelet transform \mathcal{W} to a homogeneous space of function $F(X)$ is defined by a representation U of G on \mathcal{B} , a vacuum vector $b_0 \in \mathcal{B}$ and a test functional $l_0 \in \mathcal{B}^*$ such that[7]

$$\mathcal{W} : \mathcal{B} \rightarrow F(X) : \hat{O} \rightarrow \hat{O}(x) = [\mathcal{W}\hat{O}](x) = \langle U(x^{-1})\hat{O}, l_0 \rangle = \langle \hat{O}, \pi^*(x)l_0 \rangle \quad \forall x \in X. \quad (2-3)$$

The inverse wavelet transform \mathcal{M} from $F(X)$ to \mathcal{B} is given by the formula:

$$\mathcal{M} : F(X) \rightarrow \mathcal{B} : \hat{O}(x) \rightarrow \mathcal{M}[\hat{O}] = \int_X \hat{O}(x)b_x d\mu(x) = \int_X \hat{O}(x)U(x)b_0 d\mu(x). \quad (2-4)$$

The operator $P = \mathcal{M}\mathcal{W} : \mathcal{B} \rightarrow \mathcal{B}$ is a projection of \mathcal{B} into its linear subspace in which b_0 is cyclic (i.e., the set $\{T(x)b_0 | x \in X\}$ span Banach space \mathcal{B}), and $\mathcal{M}\mathcal{W}(\hat{O}) = P(\hat{O})$ in which the constant P is equal to $\frac{c(b_0, b'_0)}{\langle b_0, l'_0 \rangle}$. There are two different cases which correspond to different choices of vacuum vector:

a) **Non-singular cases:**

In this case, U is an irreducible representation, then the inverse wavelet transform \mathcal{M} is a left inverse operator on \mathcal{B} for the wavelet transform \mathcal{W} i.e., $\mathcal{M}\mathcal{W} = I$ for which admissibility relation (2-2) holds.

b) **Singular cases:**

In this case the representation U of G is neither square-integrable nor square-integrable modulo

a subgroup H . Therefore, the vacuum vector b_0 could not be selected within the original Banach space \mathcal{B} (representation space of U). Then, in the singular theorem, we assume that there is a topological linear space $\hat{\mathcal{B}}$ with \mathcal{B} as its subset such that:

1- \mathcal{B} is dense in $\hat{\mathcal{B}}$ and representation U could be uniquely extended to the continuous representation \hat{U} on $\hat{\mathcal{B}}$.

2- There exists $b_0 \in \hat{\mathcal{B}}$ such that the following relation holds for all $h \in H$

$$\hat{U}(h)b_0 = \chi(h)b_0, \quad \chi(h) \in \mathbb{C}.$$

3- There exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ such that $U(h)^*l_0 = \chi(h)l_0$

4- The following relation holds for a probe vector $p_0 \in \mathcal{B}$

$$C(b_0, p_0) = \left\langle \int_X \langle U(x^{-1})p_0, l_0 \rangle U(x)b_0 d\mu(x), l_0 \right\rangle, \quad (2-5)$$

where the integral converges in the weak topology of $\hat{\mathcal{B}}$.

5- The composition $\mathcal{MW} : \mathcal{B} \rightarrow \hat{\mathcal{B}}$ of the wavelet transform and the inverse wavelet transform map \mathcal{B} to \mathcal{B} .

The choice of probe vector is similar to regularization[27], which have been used in our calculations. According to the theory of distribution, the smoothness, regularity, and localization of a temper distributions can be improved by a function of the Schwartz class. Various regularizers can be used for numerical computations.

A good example is the Gaussian distribution :

$$R_\delta(x) = \exp\left(-\frac{x^2}{2\delta^2}\right),$$

where R_δ is a regularizer which has properties [27]

$$\lim_{\delta \rightarrow \infty} R_\delta(x) = 1, \quad R_\delta(0) = 1.$$

Frames can be seen as a generalization of basis in Hilbert or Banach space[28]. Banach frames and atomic decomposition are sequences that have basis-like properties but which need not to

be bases. Atomic decomposition has played a key role in the recent development of wavelet theory.

Now we define a decomposition of a Banach space on homogeneous space as follow:

Definition of Coorbit space: let \mathcal{B} be a Banach space and \mathcal{B}_d be an associated Banach space of scalar-valued sequences indexed by $N = \{1, 2, 3, \dots\}$, and let $\{y_i\}_{i \in N} \subset \mathcal{B}^*$ and $\{x_i\}_{i \in N} \subset \mathcal{B}$ be given. The coorbit space is the collecting of all functions for which wavelet transform is contained in \mathcal{B}_d . Similar to the definition of coorbit space in group, we can define coorbit spaces for $X=G/H$ by [26]:

$$M_P = \{\hat{O} \in \mathcal{B} : \mathcal{W}\hat{O} \in \mathcal{B}_d\} \quad \text{with} \quad 1 \leq d \leq \infty \quad \text{and} \quad \text{norm} \quad \|\hat{O}\|_{M_P} = \|\mathcal{W}\hat{O}\|_{\mathcal{B}_d}. \quad (2-6)$$

Definition of atomic decomposition: let M_P be a coorbit space and let \mathcal{B}_d be an associated Banach space of scalar-valued sequences indexed by $N = \{1, 2, 3, \dots\}$. Let $\{y_i = \pi(\sigma^{-1}(x_i))l_0\}_{i \in N} \subset \mathcal{B}^*$ and $\{\hat{O}_i = U(\sigma^{-1}(x_i))b_0\}_{i \in N} \subset M_P$ be given. If [26]:

- a) $\{\langle \hat{O}, y_i \rangle\} \in \mathcal{B}_d$ for each $\hat{O} \in M_P$,
- b) The norms $\|\hat{O}\|_{M_P}$ and $\|\{\langle \hat{O}, y_i \rangle\}\|_{\mathcal{B}_d}$ are equivalent,
- c) $\hat{O} = \sum_{i=1}^{\infty} \langle \hat{O}, y_i \rangle x_i$ for each $\hat{O} \in M_P$,

then $(\{y_i\}, \{x_i\})$ is an atomic decomposition of X with respect to \mathcal{B}_d and, if the norm equivalence is given by:

$$A\|\hat{O}\|_{M_P} \leq \|\{\langle \hat{O}, y_i \rangle\}\|_{\mathcal{B}_d} \leq B\|\hat{O}\|_{M_P}, \quad (2-7)$$

then A, B are a choice of atomic bounds for $(\{y_i\}, \{x_i\})$. If \mathbf{i} is a continuous index then $\sum_i \rightarrow \int d\mu(X)$.

Definition of Banach frame: let M_P be a coorbit space and let \mathcal{B}_d be an associated Banach space of scalar-valued sequences indexed by $N = \{1, 2, 3, \dots\}$. Let $\{y_i = \pi(\sigma^{-1}(x_i))l_0\}_{i \in N} \subset \mathcal{B}^*$ and $\{\hat{O}_i = U(\sigma^{-1}(x_i))b_0\}_{i \in N} \subset M_P$ and $S : \mathcal{B}_d \longrightarrow M_P$ be given. If [26]

- a) $\{\langle \hat{O}, y_i \rangle\} \in \mathcal{B}_d$ for each $\hat{O} \in M_P$,

b) The norms $\|\hat{O}\|_{M_P}$ and $\|\{\langle \hat{O}, y_i \rangle\}\|_{\mathcal{B}_d}$ are equivalent. so that,

$$A\|\hat{O}\|_{M_P} \leq \|\{\langle \hat{O}, y_i \rangle\}\|_{\mathcal{B}_d} \leq B\|\hat{O}\|_{M_P},$$

c) S is bounded and linear, and $S\{\langle \hat{O}, y_i \rangle\} = \hat{O}$ for each $\hat{O} \in M_P$.

Then $(\{y_i\}, S)$ is a Banach frame for M_P with respect to \mathcal{B}_d . The mapping S is a reconstruction operator. If the norm equivalence is given by $A\|\hat{O}\|_{M_P} \leq \|\{\langle \hat{O}, y_i \rangle\}\|_{\mathcal{B}_d} \leq B\|\hat{O}\|_{M_P}$, then A, B are a choice of frame bounds for $(\{y_i\}, S)$.

It is a remarkable fact that the admissibility condition is a relation analogous to frame. Again if \mathbf{i} is a continuous index then $\sum_i \rightarrow \int d\mu(X)$.

3 Quantum tomography with wavelet transform on homogeneous space (non-singular case)

3.1 Moyal-type representations for a spin

In Moyal's formulation of quantum mechanics, a quantum spin \mathbf{s} is described in terms of *continuous symbols* i.e., by smooth functions on a two-dimensional sphere. Such prescriptions to associate operators with Wigner functions, P - or Q -symbols, are conveniently expressed in terms of operator kernels satisfying the Stratonovich-Weyl postulates. Similar to this approach, a *discrete* Moyal formalism is defined on the basis of a modified set of postulates[12].

$$\hat{\Delta}_{\mathbf{n}} = \hat{U}_{\mathbf{n}} \hat{\Delta}_{\mathbf{n}_z} \hat{U}_{\mathbf{n}}^\dagger, \quad (3-1)$$

where $\hat{U}_{\mathbf{n}}$ represents a rotation which maps the vector \mathbf{n}_z to \mathbf{n} .

By defining the associated kernel as

$$\begin{aligned} \hat{\Delta}_{\mathbf{n}} &= |s, \mathbf{n}\rangle \langle s, \mathbf{n}| \equiv |\mathbf{n}\rangle \langle \mathbf{n}| \\ \hat{\Delta}^{\mathbf{n}} &= \sum_{m=-s}^s \sum_{l=0}^{2s} \frac{2l+1}{2s+1} \begin{pmatrix} s & l & s \\ s & 0 & s \end{pmatrix}^{-1} \begin{pmatrix} s & l & s \\ m & 0 & m \end{pmatrix} |m, \mathbf{n}\rangle \langle m, \mathbf{n}| \end{aligned} \quad (3-2)$$

$$= \sum_{m=-s}^s \Delta^m |m, \mathbf{n}\rangle \langle m, \mathbf{n}|. \quad (3-3)$$

The reconstruction relation can be written as

$$\hat{O} = \frac{(2s+1)}{4\pi} \int_{S^2} dn Tr[\hat{O} \hat{\Delta}_{\mathbf{n}}] \hat{\Delta}^{\mathbf{n}}. \quad (3-4)$$

In the wavelet notation, the Banach space is $(2s+1)^2$ -dimensional and group is $SU(2)$, the subgroup is $U(1)$ and measure is $d\mu(n) = \frac{2s+1}{4\pi} d(\mathbf{n})$ and the unitary irreducible representation of group is U_n which is the result of with adjoint representation on the any operators in Banach space:

$$\hat{T}(n) \hat{O} = \hat{U}_n \hat{O} \hat{U}_n^\dagger. \quad (3-5)$$

Then the wavelet transform in this Banach space with the test functional,

$$l_0(\hat{O}) = Tr(\hat{O} \sum_m \Delta^m |m, n_z\rangle \langle m, n_z|),$$

is given by:

$$\mathcal{W} : \hat{O} \longrightarrow \hat{O}(n) = \langle T(\hat{n})^\dagger \hat{O}, l_0 \rangle = Tr(\hat{U}_n^\dagger \hat{O} \hat{U}_n \sum_m \Delta^m |m, n_z\rangle \langle m, n_z|), \quad (3-6)$$

then we have:

$$\hat{O}(n) = Tr(\hat{O} \hat{U}_n \sum_m \Delta^m |m, n_z\rangle \langle m, n_z| \hat{U}_n^\dagger) = Tr(\hat{O} \hat{\Delta}^n).$$

If we choose vacuum vector $b_0 = |s, n_z\rangle \langle s, n_z|$, the inverse wavelet transform \mathcal{M} becomes left inverse operator of the wavelet transform \mathcal{W} :

$$\begin{aligned} \mathcal{M}\mathcal{W} = PI &\Rightarrow \mathcal{M} : \hat{O}(n) \longrightarrow \mathcal{M}(\hat{O}) = \int \langle \hat{T}^\dagger(n) \hat{O}, l_0 \rangle T(\hat{n}) b_0 \\ &= \int d\mu(n) Tr(\hat{O} \hat{\Delta}^n) \hat{U}_n |s, n_z\rangle \langle s, n_z| \hat{U}_n^\dagger \Rightarrow \hat{O} = \frac{1}{P} \left(\frac{2s+1}{4\pi} \int dn Tr(\hat{O} \hat{\Delta}^n) \hat{\Delta}^n \right). \end{aligned} \quad (3-7)$$

By using the relations:

$$\frac{2s+1}{4\pi} \int_{\mathcal{S}^2} d\mathbf{n} \operatorname{Tr} [\hat{\Delta}_{\mathbf{m}} \hat{\Delta}^{\mathbf{n}}] \hat{\Delta}_{\mathbf{n}} = \hat{\Delta}_{\mathbf{m}},$$

and

$$\operatorname{Tr} [\hat{\Delta}_{\mathbf{n}_z} \hat{\Delta}^{\mathbf{n}}] = \sum_{l=0}^{2s} \frac{2l+1}{2s+1} P_l(\cos \theta).$$

One can show that the constant on the left hand side of (2-2) is $C(b_0, b'_0) = 2s+1$ and the constant $P = \frac{C(b_0, b'_0)}{\langle b_0, l_0 \rangle} = 1$, and finally the reconstruction procedure of wavelet transform (operating the combination of wavelet transform and its inverse one, \mathcal{MW} on the operator \hat{O}) leads to the tomography relation (3-4).

By the same choice as above for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example. To do it, we need further to choose the set $\{\hat{T}(n)l_0\} \subset \mathcal{B}^*$ as the index sequence of functional which belongs to dual Banach space, then we can show the following conditions:

a) $\{\langle \hat{O}, \hat{T}(n)l_0 \rangle\} = \{Tr(\hat{T}^\dagger(n)\hat{O})\} \in \mathcal{B}_d$ for each $\hat{O} \in M_P$,

b) The norms $\|\hat{O}\|_{M_P}$ and $\|\{Tr(\hat{T}^\dagger(n)\hat{O})\}\| = [\int Tr(\hat{T}^\dagger(n)\hat{O}) \overline{Tr(\hat{T}^\dagger(n)\hat{O})} d\mu(n)]^{\frac{1}{2}}$ are equivalent such that they can satisfy the inequality (2-7) with the atomic bounds $A=B=1$, providing that we use the Hilbert-Schmidt norm for the operator \hat{O} and if we use the relation (3-4) we have:

c) $\hat{O} = \int Tr(\hat{T}^\dagger(n)\hat{O})\hat{T}(n)b_0 d\mu(n),$

Therefore, $\{\hat{T}(n)b_0, \hat{T}(n)l_0\}$ is an atomic decomposition of M_P of bounded operators acting on representation space with respect to \mathcal{B}_d with atomic bounds $A=B=1$.

Finally, by the same choice of vacuum vector, test functional and index sequence of functional as in the atomic decomposition case, yield the required conditions (a) and (b) for the existence of Banach frame as the atomic decomposition one, and in order to have the last condition for the existence of atomic decomposition, we can define the reconstruction operator S as follows:

c) $S\{Tr(\hat{T}^\dagger(n)\hat{O})\} = \int Tr(\hat{T}^\dagger(n)\hat{O})\hat{T}(n)d\mu(n) = \hat{O}$ for each $\hat{O} \in M_P$,

It is straightforward to show that the operator S as defined above is a linear bounded operator. Therefore, $\{T(\hat{n})l_0, S\}$ is Banach frame for M_p with respect to \mathcal{B}_d with frame bounds $A=B=1$.

3.2 Discrete phase space tomography

In ref [13] formalism was applied to represent the states and the evolution of a quantum system in phase space in finite dimensional Hilbert space and, finally, it was discussed how to perform direct measurement to determine the wigner function. This approach was based on the use of phase space point operator to define Wigner function. For discrete systems we can define finite translation operators \hat{Q} and \hat{V} , which respectively generate finite translation in position and momentum. The translation operator \hat{Q} generates cyclic shifts in the position basis and is diagonal in momentum basis:

$$\hat{Q}^m |n\rangle = |n+m\rangle, \quad \hat{Q}^m |k\rangle = \exp(-2\pi i m k / N) |k\rangle. \quad (3-8)$$

Similarly, the operator \hat{V} is a shift in the momentum basis and is diagonal in position basis :

$$\hat{V}^m |k\rangle = |k+m\rangle, \quad \hat{V}^m |n\rangle = \exp(2\pi i m n / N) |n\rangle. \quad (3-9)$$

Now by identifying the corresponding displacement operators, the discrete analogue of the phase space translation operator is given by:

$$\hat{U}(q, p) = \hat{Q}^q \hat{V}^p \exp(i\pi p q / N). \quad (3-10)$$

Here we can define the point operator as:

$$\hat{A}(q, p) = \frac{1}{(2N)^2} \sum_{n,m=0}^{2N-1} \hat{U}(m, k) \exp(-2\pi i \frac{(kq - mp)}{2N}), \quad (3-11)$$

or as:

$$\hat{A}(\alpha) = \frac{1}{2N} \hat{Q}^q \hat{R} \hat{V}^{-p} \exp(i\pi p q / N). \quad (3-12)$$

That \hat{R} is parity operator and it is worth noting that the phase space point operators have been defined on a lattice with $2N \times 2N$ points, but it has be shown that there are only N^2

independent phase space point operators on the set $G_N = \{\alpha = (q, p); 0 \leq q, p \leq N-1\}$. The tomography relation is given by:

$$\hat{\rho} = 1/N \sum_{\alpha \in G_N} Tr(\hat{\rho} \hat{U}^\dagger(\alpha)) \hat{U}(\alpha) = 4N \sum_{\alpha \in G_N} Tr(\hat{\rho} \hat{A}(\alpha)) \hat{A}(\alpha). \quad (3-13)$$

where $W(\alpha) = Tr(\hat{A}(\alpha) \hat{\rho})$ is Wigner function.

Now we try to obtain the tomography equation (3-13) via wavelets transform in Banach space. Obviously the group, subgroup and representation are finite Heisenberg, its center and $U(\alpha)$ respectively. Then the wavelet transform with the test functional

$$l_0(O) = Tr(O) \text{ for any operator } O$$

is given by

$$\mathcal{W} : \mathcal{B} \mapsto F(\alpha) : \hat{\rho} \longrightarrow \hat{\rho}(\alpha) = \langle \hat{\rho}, l_\alpha \rangle = \langle \hat{U}^\dagger(\alpha) \hat{\rho}, l_0 \rangle = Tr(\hat{U}^\dagger(\alpha) \hat{\rho}). \quad (3-14)$$

Since the representation is an irreducible representation, the inverse wavelet transform \mathcal{M} will be the left inverse operator of wavelet transform \mathcal{W} :

$$\mathcal{M} : F(\alpha) \mapsto \mathcal{B} : \hat{\rho}(\alpha) \longrightarrow \mathcal{M}[\hat{\rho}] = \hat{\rho} = \sum_{\alpha \in G_N} \langle \hat{\rho}, l_\alpha \rangle b_\alpha = \sum_{\alpha \in G_N} \langle \hat{U}^\dagger(\alpha) \hat{\rho}, l_0 \rangle \hat{U}(\alpha) b_0, \quad (3-15)$$

We can obtain tomography relation (3-13), for the admissible $b_0 = I/N$. By the same choice as above for vacuum vector and test functions, we can get the atomic decomposition and Banach frame for this example. To do it, we need just to choose the $\{U(\hat{\alpha})l_0\} \subset \mathcal{B}^*$, then we can show that:

$$\text{a) } \{\langle \hat{\rho}, U(\hat{\alpha})l_0 \rangle\} = \{Tr(\hat{\rho} \hat{U}^\dagger(\alpha))\} \in \mathcal{B}_d \text{ for each } \hat{\rho} \in M_P,$$

b) The norms $\|\hat{\rho}\|_{M_P}$ and $\|\{Tr(\hat{\rho} \hat{U}^\dagger(\alpha))\}\|$ are equivalent and in the sense that they satisfy the inequality (2-7) with the atomic bounds $A=B=1$, provided that we use the Hilbert-Schmidt norm for the operator \hat{O} and if we use the relation (3-12), we have,

$$\text{c) } \hat{\rho} = \sum_{\alpha} Tr(\hat{\rho} \hat{U}^\dagger(\alpha)) U(\alpha) b_0,$$

then $\{U(\hat{\alpha})b_0, U(\hat{\alpha})l_0\}$ is a linear atomic decomposition of M_P with respect to \mathcal{B}_d .

Finally by the same choice of vacuum vector, test functional and index sequence of functional as in the atomic decomposition case, we can show that the required conditions (a) and (b) for the existence of Banach frame as the atomic decomposition one, and in order to have the last condition for the existence of atomic decomposition, we can define the reconstruction operator S as follows

$$c) \ S\{Tr(\hat{\rho}\hat{U}^\dagger(\alpha))\} = \sum_{\alpha} Tr(\hat{\rho}\hat{U}^\dagger(\alpha))U(\hat{\alpha}) = \hat{\rho} \text{ for each } \hat{\rho} \in M_P,$$

then $\{U(\alpha)l_0, S\}$ is a Banach frame for M_P with respect to \mathcal{B}_d with frame bounds $A=B=1$.

4 Quantum tomography with wavelet transform on Homogeneous space singular case

4.1 Homodyne Tomography

The problem of measuring the density matrix $\hat{\rho}$ of radiation has been extensively considered both experimentally and theoretically[23]. Homodyne tomography is presently the only method that can be used to achieve such measurement. This method is based on the idea that the density matrix can be evaluated in optical Homodyne experiments from the collection of quadrature probability distribution for the radiation state. As shown in [24], the matrix can be obtained after calculating the Wigner function as the inverse Radon transform of such quadrature distributions [29]. Quantum homodyne tomography is used in quantum optic at the measurement of the quantum state of light. In this case, we get [15, 16, 17]:

$$\hat{\rho} = \int_{\mathcal{C}} \frac{d^2\alpha}{\pi} Tr[\hat{\rho}\hat{U}^\dagger(\alpha)]\hat{U}(\alpha), \quad (4-1)$$

where $\hat{U}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is a displacement operator. By Changing polar variable $\alpha = \frac{i}{2}ke^{i\phi}$ this formula becomes

$$\hat{\rho} = \int_0^{2\pi} \frac{d\phi}{\pi} \int_{-\infty}^{\infty} \frac{dk |k|}{4} Tr[\hat{\rho}e^{ikX_\phi}]e^{ikX_\phi}, \quad (4-2)$$

where $X_\phi = \frac{(a^\dagger e^{i\phi} + a e^{-i\phi})}{2}$ is field-quadrature operators that is measured by balance Homodyne [18].

Now we try to obtain the tomography equation (4-1) via wavelets transform in Banach space. Obviously the group is Heisenberg. Since the representation of H^R fails to be square-integrable, according to Stone-Von Neumann [30], we can factor out the center H^R and consider only the factor space.

For the vacuum vector and test functional, we need to choose the identity operator and $l_0(O) = Tr[O]$ for any operator O , respectively. Then the wavelet formula is given by:

$$\mathcal{W} : \mathcal{B} \mapsto F(\alpha) : \hat{O} \mapsto \hat{\rho}(\alpha) = \langle \hat{\rho}, l_\alpha \rangle = \langle \hat{\rho}, \hat{U}(\alpha) l_0 \rangle = \langle \hat{\rho} \hat{U}(\alpha)^\dagger, l_0 \rangle = Tr(\hat{\rho} \hat{U}(\alpha)^\dagger), \quad (4-3)$$

But above reference state is not admissible. Thus according the singular cases, we must select a probe vector $p_0 \in \mathcal{B}$ in which equation (2-5) is non-zero and finite. In this case, the probe vector is selected by:

$$p_0 = \int |\alpha\rangle \langle \alpha| e^{\left(\frac{-|\alpha|^2}{\Delta}\right)} \frac{d^2\alpha}{\pi}, \quad (4-4)$$

where Δ is non-zero and finite and $b_0 \in \hat{B}$ is identity. Since the representation is irreducible and $C(b_0, p_0) = \Delta$, then the inverse wavelet transform in \mathcal{M} is a left inverse operator on \mathcal{B} for the wavelet transform \mathcal{W} :

$$\mathcal{M}\mathcal{W} = I \Rightarrow \mathcal{M} : F(\alpha) \mapsto B : \hat{\rho}(\alpha) \mapsto \mathcal{M}[\hat{\rho}] = \mathcal{M}\mathcal{W}(\hat{\rho}), \quad (4-5)$$

then;

$$\hat{\rho} = \int d\mu(\alpha) \langle \hat{\rho}, l_\alpha \rangle b_\alpha = \int d\mu(\alpha) Tr(\hat{\rho} \hat{U}^\dagger(\alpha)) \hat{U}(\alpha) b_0, \quad (4-6)$$

where $d\mu(\alpha) = \frac{d^2\alpha}{\pi}$ is the invariant measure of the group of translation and group is unimodular. For b_0 is equal to I , the reconstruction procedure of wavelet transform (4-6) leads to the tomography relation (4-1).

In this relation $Tr(\hat{\rho} \hat{U}^\dagger(\alpha))$ is Wigner characteristic function. We also can obtain another quasidistribution characteristic functions with choosing different representations. For example, for P-function characteristic function [31], Q-function characteristic function [31], Husimi

characteristic function[32], Standard-ordered characteristic function [33] and Antistandard-ordered characteristic function [34], we need to choose the representations, $\hat{U}_{an}(\alpha) = e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}$, $\hat{U}_n(\alpha) = e^{-\alpha^*a}e^{\alpha a^\dagger}$, $\hat{U}_h(\nu) = e^{-\nu^*b}e^{\nu b^\dagger}$ ($\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger$ and $\mu^2 - \nu^2 = 1$), $\hat{U}_s(\xi.\eta) = e^{i\xi\hat{q}}e^{i\eta\hat{p}}$ and $\hat{U}_{as}(\xi.\eta) = e^{i\eta\hat{p}}e^{i\xi\hat{q}}$, respectively.

For the complex Fourier transform of the displacement operator \hat{U} [14]

$$\hat{\mathcal{U}}(\alpha) = \int \frac{d^2\xi}{\pi} \hat{U}(\xi) \exp(\alpha\xi^* - \alpha^*\xi), \quad (4-7)$$

the expansion of the operator in terms of the operator $\hat{\mathcal{U}}(\alpha)$ is given by

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} W(\alpha) \hat{\mathcal{U}}(\alpha), \quad (4-8)$$

where $W(\alpha)$ is Wigner function. Also by defining complex Fourier transform for each above representation, we can get its tomography relation for each quasidistribution. Now we will try to obtain the atomic decomposition and Banach frame for this example. Let M_P be a Banach space and let \mathcal{B}_d be an associated Banach space of scalar-valued sequences and let $\{\{\hat{U}(\alpha)l_0\} \subset \mathcal{B}^*\}$. Finally by the same choice of vacuum vector, test functional and index sequence, we can show that required conditions (a), (b) and (c) are satisfied by atomic bounds $A=B=1$. Therefore, $\{\hat{U}(\alpha)b_0, \hat{U}(\alpha)l_0\}$ is a linear atomic decomposition of M_p with respect to \mathcal{B}_d . Similarly, by using the relation (4-1) and definition S, $\{\hat{U}(\alpha)b_0, S\}$ is Banach frame for M_p with respect to \mathcal{B}_d with frame bounds $A=B=1$. We can generalize single mode Homodyne tomography to multimode state, too. In the wavelet notation, the irreducible representation is $\hat{U} = \hat{U}_0 \otimes \hat{U}_1 \otimes \dots \otimes \hat{U}_m$, which $\hat{U}_j = \exp(z_j\hat{a}_j^\dagger - z_j^*\hat{a}_j)$, and reduced wavelets formula with choose $b_0 = \hat{I} \otimes \hat{I} \otimes \dots \otimes \hat{I}$ is given by:

$$\begin{aligned} \mathcal{W} : B &\mapsto F(z_0, z_1, \dots, z_m) : \rho \mapsto \hat{\rho}(z_0, z_1, \dots, z_m) \\ &= \langle \hat{\rho}, l_{z_0, z_1, \dots, z_m} \rangle = \langle \hat{\rho}, \hat{U}(z_1, z_2, \dots, z_m)l_0 \rangle = \langle \hat{\rho}\hat{U}^\dagger(z_0, z_1, \dots, z_m), l_0 \rangle = \text{tr}(\hat{\rho}\hat{U}^\dagger(z_0, z_1, \dots, z_m)). \end{aligned} \quad (4-9)$$

But this reference state is not admissible. Thus according the singular cases, we must select a probe vector $p_0 \in \mathcal{B}$ in which that equation (2-5) is non-zero and finite. In this case, the probe vector is selected by:

$$p_0 = \int |z_0, z_1, \dots, z_m\rangle \langle z_0, z_1, \dots, z_m| e^{\left(\frac{-\sum_{j=0}^m |z_j|^2}{\Delta}\right)} d\mu(z_0, z_1, \dots, z_m), \quad (4-10)$$

where

$$|z_0, z_1, \dots, z_m\rangle = |z_0\rangle \otimes |z_1\rangle \otimes \dots \otimes |z_m\rangle, \quad (4-11)$$

and

$$d\mu(z_0, z_1, \dots, z_m) = \frac{d^2 z_0}{\pi} \frac{d^2 z_1}{\pi} \dots \frac{d^2 z_m}{\pi}, \quad (4-12)$$

where Δ is non-zero and finite, and $b_0 \in \hat{B}$ is identity. Since the representation is irreducible and $c(b_0, p_0) = \Delta^{m+1}$, the inverse wavelet transform in \mathcal{M} is a left inverse operator on \mathcal{B} for the wavelet transform \mathcal{W} :

$$\begin{aligned} \mathcal{M}\mathcal{W} = I &\Rightarrow \mathcal{M} : F(z_0, z_1, \dots, z_m) \mapsto B : \hat{\rho}(z_0, z_1, \dots, z_m) \mapsto \mathcal{M}[\hat{\rho}] \\ &= \mathcal{M}\mathcal{W}(\hat{\rho}) = \hat{\rho} = \int d\mu(z_0, z_1, \dots, z_m) \langle \hat{\rho}, l_{z_0, z_1, \dots, z_m} \rangle b_{z_0, z_1, \dots, z_m}, \end{aligned} \quad (4-13)$$

Then;

$$\hat{\rho} = \int_C \frac{d^2 z_0}{\pi} \int_C \frac{d^2 z_1}{\pi} \dots \int_C \frac{d^2 z_m}{\pi} Tr[\hat{\rho} \hat{U}^\dagger(z_0, z_1, \dots, z_m)] \hat{U}(z_0, z_1, \dots, z_m). \quad (4-14)$$

The atomic decomposition and Banach frame is similar to one mode Homodyne, and A , B are equal to identity.

4.2 Phase Space Tomography [14, 19, 20]:

Any marginal distribution is defined as the Fourier transform of the characteristic function $\mathcal{W}(X, \mu, \nu) = \int dk e^{-ikX} \langle e^{ik(\mu\hat{q} + \nu\hat{p})} \rangle$. This marginal distribution is related to the state of the quantum system which is expressed in terms of its Wigner function $W(q, p)$, as follows

$$\mathcal{W}(X, \mu, \nu) = \int dk e^{-ik(X - \mu\hat{q} - \nu\hat{p})} W(q, p) \frac{dk dq dp}{(2\pi)^2}. \quad (4-15)$$

It is possible to express the Wigner function in terms of the marginal distribution of homodyne outcomes through the tomographic formula. An invariant form connecting directly the marginal distribution $\mathcal{W}(X, \mu, \nu)$ and any operator was found

$$\hat{\rho} = \int dX d\mu d\nu \mathcal{W}(X, \mu, \nu) \hat{K}_{\mu\nu}, \quad (4-16)$$

where the kernel operator has the form:

$$\hat{K}_{\mu\nu} = \frac{1}{2\pi} e^{iX} e^{i\mu\nu} e^{-i\nu\hat{p}} e^{-i\mu\hat{q}}. \quad (4-17)$$

Now we can try to obtain the tomography equation (4-16) via wavelets transform in Banach space. Obviously the group is Heisenberg in phase space. For the vacuum vector and test functional we need to choose the identity operator and $l_0(O) = Tr[O]$ for any operator O , respectively. If we apply the induced wavelet transform for representation $\hat{U}(\mu, \nu) = e^{-i(\mu\hat{q} + \nu\hat{p})}$, we have:

$$\begin{aligned} \mathcal{W} : \mathcal{B} \mapsto F(\mu, \nu) : \hat{\rho} \mapsto \hat{\rho}(\mu, \nu) = \\ < \hat{\rho}, l_{(\mu, \nu)} > = < \hat{\rho}, \hat{U}(\mu, \nu) l_0 > = < \hat{\rho} \hat{U}^\dagger(\mu, \nu), l_0 > = Tr(\hat{\rho} \hat{U}^\dagger(\mu, \nu)). \end{aligned} \quad (4-18)$$

The vacuum vector $b_0 = \hat{I}$ is not admissible, then we choose a probe vector with the coherent state in the phase space [6] which is a translated Gaussian wave packet:

$$\eta_{\sigma(q,p)}(x) = (\pi^{-1/4}) \exp[-i(\frac{q}{2} - x)p] \exp[-\frac{(x - q)^2}{2}] \quad (4-19)$$

$$p_0 = \int | \eta_{\sigma(q,p)} > < \eta_{\sigma(q,p)} | \exp[\frac{-(q^2 + p^2)}{\Delta}] dq dp, \quad (4-20)$$

and the singularity condition gives $C(b_0, p_0) = \Delta$.

Since the representation is irreducible, the inverse wavelet transform \mathcal{M} is a left inverse operator on \mathcal{B} for the wavelet transform \mathcal{W} :

$$\begin{aligned} \mathcal{M}\mathcal{W} = I \Rightarrow \mathcal{M} : F(\mu, \nu) \mapsto B : \hat{\rho}(\mu, \nu) \mapsto \mathcal{M}[\hat{\rho}] = \mathcal{M}\mathcal{W}(\hat{\rho}) = \hat{\rho} \\ \hat{\rho} = \int d\mu d\nu < \hat{\rho}, l_{(\mu, \nu)} > b_{(\mu, \nu)} = \int d\mu d\nu Tr[\hat{\rho} \hat{U}^\dagger(\mu, \nu)] \hat{U}(\mu, \nu) b_0, \end{aligned} \quad (4-21)$$

Then for $b_0 = \hat{I}$, we have:

$$\hat{\rho} = \int d\mu d\nu \text{Tr}[\hat{\rho} \hat{R}^\dagger(\mu, \nu)] \hat{R}(\mu, \nu) = \int d\mu d\nu \text{Tr}[\hat{\rho} e^{i(\mu \hat{q} + \nu \hat{p})}] e^{-i(\mu \hat{q} + \nu \hat{p})}. \quad (4-22)$$

After simple calculation, we can obtain (4-16). The atomic decomposition and Banach frame are similar to one mode Homodyne, and A,B are equal to identity.

4.3 $SU(1, 1)$ Tomography:

The Lie algebra $su(1, 1)$ of the $SU(1, 1)$ group is spanned by the operators $\hat{K}_+, \hat{K}_-, \hat{K}_z$. The Casimir invariant operator that labels all the unitary irreducible representations of the group is given by $(\hat{K}_z)^2 - 1/2(\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+) = k(k+1)\hat{I}$, where the eigenvalue K is also called the Bargeman index.

Then the tomographic formula is given by:

$$\begin{aligned} \hat{\rho} = \frac{1}{\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \tanh(\theta) \text{Tr}[\{(-1)^{\hat{\mathbf{K}}_z} \mathbf{e}^{\theta(\mathbf{e}^{-i\phi} \hat{\mathbf{K}}_- - \mathbf{e}^{i\phi} \hat{\mathbf{K}}_+)}, \hat{\mathbf{K}}_z\}_+ \hat{\rho}] \times \\ e^{i\theta/2(e^{-i\phi} \hat{K}_+ + e^{i\phi} \hat{K}_-)} \hat{K}_z e^{-i\theta/2(e^{-i\phi} \hat{K}_+ + e^{i\phi} \hat{K}_-)}. \end{aligned} \quad (4-23)$$

In the following section, we will try to obtain the tomography equation (4-23) via wavelets transform in Banach space. Obviously the group is $SU(1, 1)$, and subgroup is $U(1)$ with reference state $b_0 = I$. By choosing

$$\hat{\pi}(x) = \hat{u}^\dagger(x) \hat{K}_z \hat{u}(x) \quad (4-24)$$

,

$$\hat{U}(x) = \{(-1)^{\hat{K}_z} e^{\theta(e^{i\phi} \hat{K}_+ - e^{-i\phi} \hat{K}_-)}, \hat{K}_z\}_+ \quad (4-25)$$

, where $\hat{u}(\theta, \phi) \equiv e^{-i\theta/2(e^{-i\phi} \hat{K}_+ + e^{i\phi} \hat{K}_-)}[21]$, the wavelet transform is given by:

$$\mathcal{W} : \mathcal{B} \rightarrow F(x) : \hat{\rho} \rightarrow \hat{\rho}(x) = [\mathcal{W}\hat{\rho}](x) = \langle \hat{U}(x^{-1})\hat{\rho}, l_0 \rangle = \langle \hat{\rho}, \pi^*(x)l_0 \rangle = \text{Tr}[\hat{U}^\dagger(x)\rho], \quad (4-26)$$

and inverse wavelet transform is given by

$$\mathcal{M} : F(x) \rightarrow \mathcal{B} : \hat{\rho}(x) \rightarrow \mathcal{M}[\hat{\rho}(x)] = \int_x \hat{\rho}(x) b_x d\mu(x) = \int_X \hat{\rho}(x) \pi(x) b_0 d\mu(x), \quad (4-27)$$

where $\hat{\pi}(x)$ is dual of $\hat{U}(x)$. The reference state is $b_0 = I$ but this reference state is not admissible. Thus according the singular cases, we must select a probe vector $p_0 \in \mathcal{B}$ in which equation (2-5) is non-zero and finite. In this case, the probe vector is selected by

$$p_0 = \sum_r b^r |r\rangle\langle r|, \quad (4-28)$$

where this probe vector is similar to thermal states described by the density operator ρ_T

$$\rho_T = \frac{1}{1 + \widetilde{N}} \sum_r \left(\frac{\widetilde{N}}{1 + \widetilde{N}} \right)^r |r\rangle\langle r|, \quad (4-29)$$

where $\widetilde{N} \equiv \langle \rho_T N \rangle = \frac{1}{\exp(\hbar\omega/KT) - 1}$, and $N = a^\dagger a$. In the high temperature this thermal state is proportional with identity. Since the representation is irreducible and $C(b_0, p_0) = \frac{1}{1-b}$, the inverse wavelet transform in \mathcal{M} is a left inverse operator on \mathcal{B} . Then the tomography formula for $SU(1, 1)$ group is given by the formula (4-23).

Now we will obtain atomic decomposition and Banach frame for this example. Let M_P be a coorbit space and let \mathcal{B}_d be an associated Banach space of scalar-valued sequences. Let $\{\hat{\pi}(x)l_0\} \subset \mathcal{B}^*$, then we can show that:

a) $\{\langle \hat{\rho}, \hat{\pi}(x)l_0 \rangle\} = \{Tr(\hat{\rho}\hat{U}^\dagger(x))\} \in \mathcal{B}_d$ for each $f \in M_P$,

b) The norms $\|\hat{\rho}\|_{M_P}$ and $\|\{Tr(\hat{\rho}\hat{B}^\dagger(x))\}\|$ are equivalent in the sense that they satisfy the inequality (2-7) with the atomic bounds $A=B=1$, provided that we use the Hilbert-Schmidt norm for the operator $\hat{\rho}$

$$\|Tr(\hat{\rho}\hat{U}^\dagger(x))\|^2 = \int d\mu(x) Tr(\hat{\rho}\hat{U}^\dagger(x)) \overline{Tr(\hat{\rho}\hat{\pi}(x))}, \quad (4-30)$$

Since the dual couple $\hat{U}(x)$ and $\hat{\pi}(x)$ satisfy the orthogonality relation [21]:

$$\delta_{mk}\delta_{nl} = \int d\mu(x) \langle m|B^\dagger(x)|n\rangle\langle l|C(x)\rangle,$$

then;

$$\|Tr(\hat{\rho}\hat{U}^\dagger(x))\|^2 = \int d\mu(x) \rho_{mn} U_{mn}^*(x) \rho_{kl}^* \pi_{kl}(x) = \|\hat{\rho}\|^2,$$

and if we use the relation (4-23), we have:

$$c) \hat{\rho} = \int d\mu(x) \text{Tr}(\hat{\rho} \hat{U}^\dagger(x)) \hat{\pi}(x),$$

Therefore, $\{\hat{\pi}(x)b_0, \hat{\pi}(x)l_0\}$ is an atomic decomposition of M_P with respect to \mathcal{B}_d with atomic bounds $A=B=1$. Similar to atomic decomposition, $\{\hat{\pi}(x)l_0, S\}$ is a Banach frame for coorbit space of operators with respect to \mathcal{B}_d with frame bounds A, B are equal to identity.

4.4 Tomography of a free particle

Here we will consider the tomography of a free particle. For simplicity we suppose a particle with unit mass and use normalized unit $\hbar/2 = 1$, so that the free Hamiltonian is given by $\hat{H}_F = \hat{p}^2$. The basis is constituted by the set of operator $\hat{R}(x, \tau) = e^{-i\hat{p}^2\tau}|x\rangle\langle x|e^{i\hat{p}^2\tau}$ [14]; then, a generic free particle density operator can be written as:

$$\hat{\rho} = \int_R \int_R dx d\tau p(x, \tau) \hat{R}(x, \tau), \quad (4-31)$$

where $p(x, \tau) = \text{Tr}[\hat{\rho} \hat{R}(x, \tau)]$ is the probability density of the particle to be at position x at time τ .

Now we try to obtain the tomography equation (4-31) via wavelets transform in Banach space. Obviously the group is $\{\hat{P}, \hat{X}, \hat{P}^2, I\}$ and subgroup is $\{\hat{X}, I\}$. The relevant representation for this example is adjoint representation:

$$\hat{T}(x, \tau)\hat{\rho} = \hat{U}(x, \tau)\hat{\rho}\hat{U}^{-1}(x, \tau) \quad \text{with} \quad \hat{U}(x, \tau) = e^{-i\hat{P}^2\tau}\hat{D}(x).$$

In this representation, $\hat{D}(x)$ is translation operator, so that $\hat{D}(x)|0\rangle = |x\rangle$, where $|x\rangle$ is eigenstate of position operator and \hat{P} is the momentum operator. On the other hand if we define:

$$\langle \hat{\rho}, l_0 \rangle = l_0(\hat{\rho}) = \text{Tr}(\hat{\rho} | 0 \rangle \langle 0 |).$$

the wavelet transform formula is given by:

$$\mathcal{W} : \mathcal{B} \mapsto F(x, \tau) : \hat{\rho} \mapsto \hat{\rho}(x, \tau) =$$

$$\begin{aligned}
& \langle \hat{\rho}, l_{(x,\tau)} \rangle = \langle \rho, \hat{T}(x, \tau) l_0 \rangle = \langle \hat{T}^\dagger(x, \tau) \hat{\rho}, l_0 \rangle = \\
& \text{tr}(\hat{T}^\dagger(x, \tau) \hat{\rho} | 0 \rangle \langle 0 |) = \text{Tr}(\hat{U}(x, \tau) | 0 \rangle \langle 0 | \hat{U}^\dagger(x, \tau) \hat{\rho}) = \text{Tr}(\hat{\rho} e^{-i\hat{P}^2 \tau} | x \rangle \langle x | e^{i\hat{P}^2 \tau}).
\end{aligned} \tag{4-32}$$

Also the inverse wavelet transform \mathcal{M} associated with wavelet transform \mathcal{W} is:

$$\begin{aligned}
& \mathcal{M}\mathcal{W} = PI \Rightarrow \mathcal{M} : F(x, \tau) \mapsto \mathcal{B} : \hat{\rho}(x, \tau) \mapsto \mathcal{M}[\hat{\rho}] = \\
& \int d\mu(x, \tau) \langle \hat{\rho}, l_{(x,\tau)} \rangle b_{(x,\tau)} = \int dx d\tau \text{Tr}[\hat{\rho} e^{-i\hat{P}^2 \tau} | x \rangle \langle x | e^{i\hat{P}^2 \tau}] T(\hat{x}, \tau) b_0,
\end{aligned} \tag{4-33}$$

The vacuum vector is $b_0 = |0 \rangle \langle 0|$, but this vacuum vector is not admissible. Thus according the singular cases, we must select a probe vector $p_0 \in \mathcal{B}$ in which equation (2-5) is non-zero and finite. In this case, the probe vector is selected by

$$p_0 = | D \rangle \langle D |, \tag{4-34}$$

where $\langle D | p \rangle = e^{-\frac{p^2}{D}}$. Its follows from bi-orthogonality and from the following relations [14](for $|j\rangle$, $j = p_1, p_2, p_3, p_4$)

$$\begin{aligned}
& \int_R \int_R dx d\tau \langle p_1 | \hat{R}(x, \tau) | p_2 \rangle \langle p_3 | \hat{R}(x, \tau) | p_4 \rangle \\
& = \int_R \int_R dx d\tau e^{-i\tau(p_2^2 - p_1^2 + p_3^2 - p_4^2)} \langle p_1 | x \rangle \langle x | p_2 \rangle \langle p_3 | x \rangle \langle x | p_4 \rangle \\
& = \int_R \int_R dx d\tau e^{-i\tau(p_2^2 - p_1^2 + p_3^2 - p_4^2)} e^{ix(p_1 - p_2 + p_3 - p_4)} \\
& = \delta(p_1 - p_3) \delta(p_2 - p_4).
\end{aligned} \tag{4-35}$$

we can show that the constant on left hand side of (2-5) is $C(b_0, p_0) = D/2\sqrt{\pi}$ and finally the reconstruction procedure of wavelet transform leads to the tomography relation (4-31). In order to obtain atomic decomposition and Banach frame for this example, let M_P be a coorbit space and let \mathcal{B}_d be an associated Banach space of scalar-valued sequences and $\{\{\hat{T}(x, \tau)l_0\} \subset \mathcal{B}^*\}$. Finally by the same choice of vacuum vector, test functional and index sequence, we can show that the required conditions (a), (b) and (c) are satisfied by atomic bounds $A=B=1$. Therefore,

$\{\hat{T}(x, \tau)b_0, \hat{T}(x, \tau)l_0\}$ is a linear atomic decomposition of M_p with respect to \mathcal{B}_d . Similarly, by using the relation (4-31) and definition S, $\{\hat{T}(x, \tau)b_0, S\}$ is Banach frame for M_p with respect to \mathcal{B}_d with frame bounds $A=B=1$.

4.5 Wavelet transform and Q-function:

Let $g \in \mathcal{L}^2(R)$ with $\|g\| = 1$ and the time-frequency translation of g be:

$$g^{[x_1, x_2]}(t) = e^{2\pi i t x_2} g(t + x_2) = U[x_1, x_2, 0]g(t), \quad (4-36)$$

where U is the unitary irreducible representation of the Heisenberg group H^R . To consider an arbitrary function $f \in L^2(R)$, we can compute the following inner product for pure state sampling [8]:

$$F(x_1, x_2) = \langle f, g^{[x_1, x_2]} \rangle, \quad (4-37)$$

where $g^{[x_1, x_2]} = U[x_1, x_2]g(t)$ is a coherent state. For the pure states, square of sampling is Q-function.

Now we will try to obtain Q-function via wavelet and we will show that the wavelet transform in the Banach space is Q-function. The group is Heisenberg and subgroup is identity and representation is adjoint. Then the wavelet transform is given by:

$$\mathcal{W} : \mathcal{B} \mapsto F(\alpha) :$$

$$\hat{\rho} \mapsto \hat{\rho}(\alpha) = \langle \hat{\rho}, l_\alpha \rangle = \langle \hat{\rho}, \hat{T}(\alpha)l_0 \rangle = \langle \hat{T}(\alpha)^\dagger \hat{\rho}, l_0 \rangle. \quad (4-38)$$

On the other hand if we choose:

$$\langle \hat{\rho}, l_0 \rangle = l_0(\hat{\rho}) = Tr[\hat{\rho}|0\rangle\langle 0|], \quad (4-39)$$

Then the wavelet transform for the adjoint representation is given by:

$$\begin{aligned} \mathcal{W} : \mathcal{B} \mapsto F(\alpha) : \hat{\rho} \mapsto \hat{\rho}(\alpha) &= Tr\{ \hat{T}(g)^\dagger \hat{\rho} |0\rangle\langle 0| \} \\ &= Tr\{ \hat{U}(\alpha)^\dagger(\hat{\rho})\hat{U}(\alpha)|0\rangle\langle 0| \} = \langle 0|\hat{U}(\alpha)^\dagger(\hat{\rho})\hat{U}(\alpha)|0\rangle = \langle \alpha|\hat{\rho}|\alpha\rangle = Q(\alpha). \end{aligned} \quad (4-40)$$

5 Conclusion

In this paper we have generalized wavelet transform and its inverse for tomography of density operator in Banach space on homogeneous space. Also we have explained some examples of the using the wavelet formalism in quantum tomography on homogeneous space and introduced frame and atomic decomposition for each of them. We have also presented the connection between the wavelet formalism on Banach space and Q-function.

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