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GROUP THEORETICAL APPROACH TO QUANTUM ENTANGLEMENT AND TOMOGRAPHY WITH WAVELET TRANSFORM IN BANACH SPACES

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The intimate connection between the Banach space wavelet reconstruction method for each unitary representation of a given group and some of well-known quantum tomographies, such as tomography of rotation group, spinor tomography and tomography of unitary group, is established. Also both the atomic decomposition and Banach frame nature of these quantum tomographic examples are revealed in detail. Finally, we consider separability criteria for any state with group theoretical wavelet transform on Banach spaces.

Keywords: Quantum tomography; wavelets, Banach space; group representations; frames; entanglement; separability criteria.

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1. Introduction

Entanglement is one of the most fascinating features of quantum mechanics. As Einstein, Podolsky and Rosen[1] pointed out, the quantum states of two physically-separated systems that interacted in the past can defy our intuitions about the outcome of local measurements. Moreover, it has recently been recognized that entanglement is a very important resource in quantum information processing.[2] A bipartite mixed state is said to be separable[3] (not entangled) if considered as a convex combination of pure product states.

A separability criterion is based on a simple property that can be shown to hold for every separable state. If some state does not satisfy this property, then it must be entangled. But the converse does not necessarily imply the state to be separable. One of the first and most widely-used related criteria is the positive partial transpose (PPT) criterion, introduced by Peres.[4] Furthermore, the necessary and sufficient condition for separability in $H_2 \otimes H_2$ and $H_2 \otimes H_3$ was shown by the Horodeckis,[5] which was based on a previous work by Woronowicz.[6] However, in higher dimensions, there are PPT states that are nonetheless entangled, as was first shown in Ref. 7, based on Ref. 6. These states are called *bound entangled states* because they have a peculiar property that no entanglement can be distilled from them by local operations.[8,10] Another approach to distinguish separable states from entangled ones involves the so-called *entanglement witness* (EW).[9–11] Some of entanglement measures and best separable state conditions using semidefinite programming method are given in Refs. 12–15. However, no investigation of the separability problem has been carried out there, as the work of Gu predates the seminal paper of Werner.[3] On the other side, the method of characteristic functions has already been successfully applied for studying other then entanglement genuine quantum features of quantum states in the works on non-classical states of quantum harmonic oscillator.[16] Choosing a compact group G and the set of its irreducible, unitary representation as the main ingredients of the mathematical representation of the state space, Korbicz and Lewenstein could define characteristic functions which are applied for testing states' entanglements. Although they do not present any new entanglement test, their results offer a new point of view on the separability problem. Moreover, they were able to translate the positivity of partial transpose (PPT) criterion[4] in the group theoretical language.[17]

Nowadays the mathematical theory of wavelet transform finds enormous success in various fields of science and technology, including the treatment of large databases, data and image compression, signal processing, telecommunication and many other applications.[18–20] After the empirical discovery by Morlet,[21] it was recognized from the very beginning by Grossmann, Morlet, Paul[22,23] and Daubechies[24] that wavelets are simply coherent states associated with the affine group of the line (dilations and translations). Thus, the stage was immediately set for a far-reaching generalization.[24–27] Unlike functions which form orthogonal bases for space, Morlet wavelets are not orthogonal and form frames. Frames are sets of functions which are not necessarily orthogonal and which are not linearly independent. Actually, frames are a redundant set of vectors in Hilbert space by which each vector in the space admits a natural representation.

Recently another concept called *atomic decomposition* has played a key role in the development of further mathematical wavelet theory. Indeed atomic decomposition for any space of function or distribution aims at representing any element in the form of a set of simple functions which are called *atoms*.[28] As far as the Banach space is concerned, Feichtinger–Grochenig[30] provided a general and very flexible way to construct coherent atomic decompositions and Banach frames for Banach spaces.

The concept of a quantum state represents one of the most fundamental pillars of the paradigm of the quantum theory. Usually the quantum state is described either by state vector in Hilbert space, by density operator or by a phase space probability density distribution (quasi-distributions). The quantum states can be completely determined from the appropriate experimental data by using the well-known technique of quantum tomography or tomographic transformation.

A general framework is already presented for the unification of the Hilbert space wavelets transformation on the one hand, and quasi-distributions and tomographic transformation associated with a given pure quantum state on the other hand.[29] In this manuscript we are trying to establish the intimate connection between the Banach space wavelet reconstruction method developed by Feichtinger–Grochenig,[30,31] Kisil[31] and some of the well-known quantum tomographies associated with mixed states, such as tomography of the rotation group,[32–35] spinor tomography,[36,37] discrete spin tomography[38] and tomography of the unitary group,[40,41] all of which can be represented by density matrices. For density matrices, we can define the norm as tr() which implies the absence of a scalar product in the density matrix space (so it is not a Hilbert space but a Banach one).[42,43] Therefore, it is natural to do quantum tomography of any density matrix by using the wavelet transform and its inverse in Banach space connected with the corresponding group representation associated with that density matrix. This obtained quantum tomography by Banach space wavelet method for density states is completely consistent with the quantum tomography obtained by other methods. Also both the atomic decomposition and Banach frame nature of these quantum tomographic examples are revealed in detail.

The paper is organized as follows. In Sec. 2, we define wavelet transform and its inverse for each unitary representation of a given group in Banach space and then define atomic decomposition and Banach frame in Banach space. In Sec. 3, we review the group theoretical approach to entanglement and obtain the quantum tomography associated with the unitary representation of a given group in Banach space and then define its atomic decomposition and Banach frame bounds. We finish this section with the derivation of some typical quantum tomographic and entanglement problem examples, such as rotation groups, spinor and unitary group, by using the Banach space wavelet reconstruction method. The paper ends with a brief conclusion.

2. Wavelet Transform, Frame and Atomic Decomposition on Banach Spaces

The following is a brief recapitulation of some aspects of the theory of wavelets, atomic decomposition, and Banach frames. We will only mention those concepts which are needed in the sequel, a more detailed treatment may be found in Refs. 28, 30, 44, 45.

Let G be locally compact group with left Haar measure $d\mu$ and let π be a continuous representation of a group G in a (complex) Banach space $\mathcal{B}$.

Now if $U(g)$ is a representation of a group G in a Banach space $\mathcal{B}$, a new representation for group $G \times G$ in the space $\mathcal{L}(\mathcal{B})$ of bounded linear operators acting on Banach space $\mathcal{B}$ is defined as

$$\pi : G \times G \to \mathcal{L}(\mathcal{L}(\mathcal{B})) : \hat{O} \to U(g_1)\hat{O}U(g_2^{-1}), \tag{1}$$

where $\hat{O} \in \mathcal{L}(\mathcal{B})$ and if g_1 is equal to g_2, the representation is called *adjoint representation*, and if g_2 is equal to the identity operator, the representation is called the *left representation of the group*. We will say that a set of vectors $b_{g_1,g_2} = \pi(g_1, g_2)b_0$ form a family of coherent states if there exist continuous non-zero linear functionals $l_0, l_0^{'} \in \mathcal{L}(\mathcal{B}^*)$ (called *test functional*) and vectors $b_0^{'}, b_0 \in \mathcal{L}(\mathcal{B})$ (called *vacuum vector*) such that

$$C(b_0, b_0^{'}) = \int_G \langle \pi(g_1^{-1}, g_2^{-1})b_0, l_0\rangle\langle \pi(g_1, g_2)b_0^{'}, l_0^{'}\rangle d\mu(g_1, g_2), \tag{2}$$

is non-zero and finite, which is known as the *admissibility relation*. For unitary representation in Hilbert spaces, the condition (2) is known as *square integrability*. Thus our definition describes an analog of square integrable representation for Banach space.

The wavelet transform $\mathcal{W}$ from Banach space $\mathcal{B}$ to a space of function $F(G\times G)$ is defined by a representation π of $G \times G$ on $\mathcal{B}$ and a test functional l_0, is given by the following formula

$$\begin{aligned} \mathcal{W} : \mathcal{B} \to F(G \times G) : \hat{O} \to \hat{O}(g_1, g_2) &= [\mathcal{W}\hat{O}](g_1, g_2) \\ &= \langle \pi(g_1^{-1}, g_2^{-1})\hat{O}, l_0\rangle = \langle \hat{O}, \pi^*(g_1, g_2)l_0\rangle \quad \forall g_1, \quad g_2 \in G \end{aligned} \tag{3}$$

where π^* is dual of π. The inverse wavelet transform $\mathcal{M}$ with a vacuum vector b_0 from $F(G \times G)$ to $\mathcal{B}$ is given by the formula

$$\begin{aligned} &\mathcal{M} : F(G \times G) \to \mathcal{B} : \hat{O}(g_1, g_2) \to \mathcal{M}[\mathcal{W}\hat{O}] \\ &= \int_G \hat{O}(g_1, g_2)b_{g_1,g_2} d\mu(g_1, g_2) = \int_G \hat{O}(g_1, g_2)\pi(g_1, g_2)b_0 d\mu(g_1, g_2). \end{aligned} \tag{4}$$

The operator $PI = \mathcal{M}\mathcal{W} : \mathcal{B} \longmapsto \mathcal{B}$ is a projection of $\mathcal{B}$ into its linear subspace in which b_0 is cyclic (i.e. the set $\{T(g)b_0|g \in G\}$ spans Banach space $\mathcal{B}$), and $\mathcal{M}\mathcal{W}(\hat{O}) = P(\hat{O})$ where the constant P is equal to $\frac{c(b_0,b_0^{'})}{\langle b_0,l_0^{'}\rangle}$. Especially, if the left representation U is an irreducible representation, then the inverse wavelet transform $\mathcal{M}$ is a left inverse operator of wavelet transform $\mathcal{W}$ on $\mathcal{B}$, i.e. $\mathcal{M}\mathcal{W} = I$.

Frames can be seen as a generalization of basis in Hilbert or Banach space.[28] Banach frames and atomic decomposition are sequences that have basis-like properties but which need not to be bases. Atomic decomposition has played a key role in the recent development of wavelet theory.

Now we define a decomposition of a Banach space as follows:

Definition of atomic decomposition. Let $\mathcal{B}$ be a Banach space, and $\mathcal{B}_2$ be an associated Banach space of scalar-valued sequences indexed by $N = \{1, 2, 3, \ldots\}$, and let $\{y_{i,j} = \pi^*(g_i, g_j)l_0\}_{i,j\in N} \subset \mathcal{B}^*$ where $\mathcal{B}^*$ is dual Banach space and $\{b_{i,j} = \pi(g_i, g_j)b_0\}_{i,j\in N} \subset \mathcal{B}$ be given. If

(i) $\{\langle \hat{O}, y_{i,j}\rangle\} \in \mathcal{B}_2$ for each $\hat{O} \in \mathcal{B}$,
(ii) the norms $\|\hat{O}\|_{\mathcal{B}}$ and $\|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{\mathcal{B}_2}$ are equivalent,
(iii) $\hat{O} = \sum_{i,j=1}^{\infty} \langle \hat{O}, y_{i,j}\rangle b_{i,j}$ for each $\hat{O} \in \mathcal{B}$,

then $(\{y_{i,j}\}, \{b_{i,j}\})$ is an atomic decomposition of $\mathcal{B}$ with respect to $\mathcal{B}_2$ and, if the norm equivalence is given by

$$A\|\hat{O}\|_{\mathcal{B}} \leq \|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{\mathcal{B}_2} \leq \mathcal{B}\|\hat{O}\|_{\mathcal{B}}, \tag{5}$$

then A, B are a choice of atomic bounds for $(\{y_{i,j}\}, \{b_{i,j}\})$.

Definition of Banach frame. Let $\mathcal{B}$ be a Banach space, and $\mathcal{B}_2$ be an associated Banach space of scalar-valued sequences indexed by $\mathcal{N}$ and let $\{y_{i,j} = \pi^*(g_i, g_j)l_0\}_{i\in N} \subset \mathcal{B}^*$ and $T : \mathcal{B}_2 \to \mathcal{B}$ be given. If

(i) $\{\langle \hat{O}, y_{i,j}\rangle\} \in \mathcal{B}_2$ for each $\mathcal{O} \in \mathcal{B}$,
(ii) the norms $\|x\|_{\mathcal{B}}$ and $\|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{\mathcal{B}_2}$ are equivalent, so that

$$A\|\hat{O}\|_{\mathcal{B}} \leq \|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{\mathcal{B}_2} \leq \mathcal{B}\|\hat{O}\|_{\mathcal{B}}, \tag{6}$$

(iii) T is bounded and linear, and $T\{\langle \hat{O}, y_{i,j}\rangle\} = \hat{O}$ for each $\hat{O} \in \mathcal{B}$, then $(\{y_{i,j}\}, T)$ is a Banach frame for $\mathcal{B}$ with respect to $\mathcal{B}_2$. The mapping T is called the *reconstruction operator* (inverse wavelet transform). If the norm equivalence is given by $A\|\hat{O}\|_{\mathcal{B}} \leq \|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{\mathcal{B}_2} \leq B\|\hat{O}\|_{\mathcal{B}}$, then A, B will be a choice of frame bounds for $(\{y_{i,j}\}, T)$.

Obviously one can show that the admissibility condition is the same as frame condition.

From the definition of Banach frame all states that a separated set $\{y_{i,j}, i, j \in N\}$ is a set of sampling for $\mathcal{B}^*$ if and only if the set of $\{y_{i,j}, i, j \in N\}$ is a frame for $\mathcal{B}^*$. It is well known from frame theory that there exists a dual frame $\{b_{i,j}, i, j \in N\} \subset \mathcal{B}$ that allows us to reconstruct functions $\{\hat{O}_{i,j} = \langle \hat{O}, y_{i,j}\rangle\} \subset \mathcal{B}$ explicitly as

$$\mathcal{MW}\hat{O} = \sum_{i,j} \langle \hat{O}, y_{i,j}\rangle b_{i,j}, \tag{7}$$

where $\mathcal{MW}$ is a map from $\mathcal{B}$ to $\mathcal{B}$ then $S = \mathcal{MW}$ is a frame operator similar to the definition of frame operator in Hilbert space.[28] The frame condition can be expressed in terms of the frame operator as

$$AI \leq S \leq BI. \tag{8}$$

Now according to the Schur's lemma a representation which commutes with all irreducible representations, must be a constant coefficient of the identity. This lemma is reduced to orthogonality relation and the resolution of identity for any coherent state. If b_0 is cyclic it satisfies the resolution of identity. Then if the representation is irreducible or b_0 is cyclic, $S = \mathcal{MW}$ is proportional to identity. Therefore, frame bounds A, B is equal to a constant (tight frame).[27]

3. Quantum Tomography and Entanglement via Group Theory with Wavelet Transform on Banach Space

Group tomography of a compact group G, with an irreducible unitary representation U acting on separable Hilbert space $\mathcal{H}$, means that every element of $\mathcal{B}(\mathcal{H})$, the Banach algebra of bounded linear operators acting on $\mathcal{H}$, can be constructed by the set $\{U(g), g \in G\}$ according to formula (10), where the set $\{U(g), g \in G\}$ is known as tomographic set and $\Phi(g) = \mathrm{Tr}[U^\dagger(g)\hat{O}]$ is the sampling set or tomogram set of a given operator $\hat{O}$.[46,47] When $\mathcal{H}$ is a finite-dimensional space, the hypothesis that $\{U(g)\}$ is a tomographic set is sufficient to reconstruct any given operator from the tomographic set by using (10), but the case of $\dim(\mathcal{H}) = \infty$ needs a further condition to make sure that every expression converges and that it can be attributed to a precise mathematical meaning. More explicitly, U needs to fulfil the following equality:

$$\begin{aligned}\int & d\mu(g)\langle f_1, U(g)f_2\rangle\langle f_3 U(g), f_4\rangle \\ &= \langle f_1, f_4\rangle\langle f_3, f_2\rangle \quad \forall\, |f_1\rangle, |f_2\rangle, |f_3\rangle, |f_4\rangle \in \mathcal{H},\end{aligned} \tag{9}$$

which is known as the square integrability of the representation $U(g)$. If $\hat{O}$ is a trace-class operator on $\mathcal{H}$ and $\{U(g)\}$ is a tomographic set and satisfies (9) then we have

$$\hat{O} = \int d\mu(g)\mathrm{Tr}[U^\dagger(g)\hat{O}]U(g). \tag{10}$$

Now we try to obtain the above explained tomography via wavelet transforms in Banach space. In order to do so, we need to choose the tomographic set $U(g)$ as a left continuous irreducible representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\begin{aligned}\mathcal{W} : \mathcal{B} \mapsto F(g) : \hat{O} \mapsto \hat{O}(g) &= \langle \hat{O}, l_g\rangle \\ &= \langle \hat{O}, U(g)l_0\rangle = \langle \hat{O}U(g)^\dagger, l_0\rangle = \mathrm{tr}(\hat{O}U(g)^\dagger).\end{aligned} \tag{11}$$

With those conditions, the inverse wavelet transform $\mathcal{M}$ becomes a left inverse operator of the wavelet transform $\mathcal{W}$:

$$\mathcal{M}\mathcal{W} = I \Rightarrow \mathcal{M} : F(g) \mapsto \mathcal{B} : \hat{O}(g) \mapsto \mathcal{M}[\hat{O}] = \mathcal{M}\mathcal{W}(\hat{O}) = \hat{O} = \int d\mu(g)\langle \hat{O}, l_g\rangle b_g. \tag{12}$$

Therefore, with the choice of $b_0 = I$ (identity operator), the tomography relation can be written as

$$\hat{O} = \int d\mu(g)\mathrm{Tr}(\hat{O}U(g)^\dagger)U(g). \tag{13}$$

By choosing $b_0 = I$, $b'_0 = |f_2\rangle\langle f_1|$, $l_0(\hat{O}) = \mathrm{Tr}(\hat{O}|f_4 =\rangle\langle f_3|)$ and $l'_0(\hat{O}) = \mathrm{Tr}(\hat{O})$ (with f_1, f_2, f_3, f_4 as arbitrary vectors in $\mathcal{H}$ and $\hat{O}$ as an arbitrary bounded operator), we can reduce the admissibility condition for wavelet transform on Banach space (2) to the square integrability for group theory tomography (9).

Similarly, by choosing the above for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example. Rather than all the above mentioned, to do so we also need to choose set $\{U(g)l_0\} \subset \mathcal{B}^*$ as the index sequence of functionals (with index set G) which belong to dual Banach space, then we can show that

(i) $\{\langle \hat{O}, U(g)l_0\rangle\} = \{\mathrm{Tr}(\hat{O}U^\dagger(g))\} \in \mathcal{B}_2$ for each $\hat{O} \in \mathcal{B}$,

(ii) the norms $||\hat{O}||_\mathcal{B}$ and $||\{\mathrm{Tr}(\hat{O}U^\dagger(g))\}|| = [\int \mathrm{Tr}(\hat{O}U^\dagger(g))\overline{\mathrm{Tr}(\hat{O}U^\dagger(g))}d\mu(g)]^{\frac{1}{2}}$ are equivalent since

$$\begin{aligned}\|\{\mathrm{Tr}(\hat{O}U^\dagger(g))\}\| &= \left[\int \mathrm{Tr}(\hat{O}U^\dagger(g))\overline{\mathrm{Tr}(\hat{O}U^\dagger(g))}d\mu(g)\right]^{1/2} \\ &= \left|\int \mathrm{tr}(U^\dagger(g)\hat{O})\mathrm{tr}(U(g)\hat{O}^\dagger)d\mu(g)\right|^{1/2} \\ &= \left|\mathrm{tr}\left(\int \mathrm{tr}(U^\dagger(g)\hat{O})U(g)d\mu(g)\hat{O}^\dagger\right)\right|^{1/2},\end{aligned}$$

if we use the tomography relation (13), we obtain

$$\|\{\mathrm{Tr}(\hat{U}^\dagger(g)\hat{O})\}\| = |\mathrm{tr}(\hat{O}\hat{O}^\dagger)|^{1/2} = \|\hat{O}\|,$$

such that they can saturate the inequality (5) with the atomic bounds $A = B = 1$, provided that we use the Hilbert–Schmidt norm for the operator $\hat{O}$ and if we use the relation (10), we have

(iii) $\hat{O} = \int \mathrm{Tr}(\hat{O}U^\dagger(g))U(g)d\mu(g)$.

Therefore, $\{U(g)b_0, U(g)l_0\}$ is an atomic decomposition of Banach space of bounded operators acting on representation space with respect to $\mathcal{B}_2$ with atomic bounds $A = B = 1$.

Finally by the same choice of vacuum vector, test functional and index sequence of functional as in the atomic decomposition case, we can show that the required conditions (i), (ii) for the existence of Banach frame is the same as the atomic decomposition one, and in order to have the last condition for the existence of the atomic decomposition, we can define the reconstruction operator T as follows:

(iv) $T\{\mathrm{Tr}(\hat{O}U^\dagger(g))\} = \int \mathrm{Tr}(\hat{O}U^\dagger(g))U(g)d\mu(g) = \hat{O}$ for each $\hat{O} \in \mathcal{B}$.

It is straightforward to show that the operator T as defined above is a linear bounded operator.

Therefore $\{U(g)l_0, T\}$ is a Banach frame for $\mathcal{B}$ with respect to $\mathcal{B}_2$ with frame bounds $A = B = 1$.

Korbicz and Lewenstein proceeded to the reformulation of the separability problem in terms of the group theoretical language.[17] For clarification, let us assume that ρ is separable, i.e. there exists a decomposition of type $\rho = \sum_i p_i|u_i\rangle\langle u_i| \otimes |v_i\rangle\langle v_i|$.

By definition of the characteristic function[17] or the sampling function $\Phi_\rho(g_1, g_2)$ from wavelet transformation in Banach space with the above density matrix for irreducible representation $U(g) := U_1(g_1) \otimes U_2(g_2)$, we obtain that[17]

$$\Phi_\rho(g_1, g_2) = \mathrm{tr}(\rho U(g)) = \sum_i p_i K_i(g_1)\eta_i(g_2), \tag{14}$$

where $K_i(g_1) = \langle u_i \mid U(g_1)u_i\rangle$, $\eta_i(g_2) = \langle v_i \mid U(g_2)v_i\rangle$.

The theories we present here are standard and they are mentioned and proved in Ref. 17.

Theorem 1. *Let G be a compact dynamical group and π, τ irreducible representations. A state ρ is separable iff its characteristic function can be written in the form $\Phi_\rho(g_1, g_2) = \sum_i p_i K_i(g_1)\eta_i(g_2)$, where $K_i, \eta_i \in \mathcal{P}_1(G)$ (where $\mathcal{P}_1(G)$ is the space of all normalized positive definite functions on G) and the equality holds almost everywhere w.r.t. the Haar measure dg on $G \times G$.*

Theorem 2. *Let G be a compact dynamical group, π, τ irreducible representations at G, and ρ an arbitrary state in $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$: the condition ($\rho$ is separable) $\Rightarrow \tilde{\Phi}_\rho \in \mathcal{P}(G \otimes G)$ either leads to PPT criterion for ρ when $\pi \sim \bar{\pi}$ or is empty otherwise,*[17] *where $\bar{\pi}(g) := \pi(g^{-1})$ and $\tilde{\Phi}(g_1, g_2) := \Phi(g_1^{-1}, g_2)$.*

3.1. *Quantum tomography and entanglement for rotation group*

The most general (unnormalized) spin density matrix $\hat{\rho}$ is a $(2s+1) \times (2s+1)$ hermitian matrix with $(2s+1)^2$ real parameters.[39] Various methods have been proposed to determine $\hat{\rho}$. The expectations of $(2s+1)^2 - 1$ linearly independent spin multipoles do fix a unique (normalized) density operator.[48,49] In order to establish a down-to-earth approach, it is natural to restrict the measurements to those performed with a standard Stern–Gerlach apparatus, the quantization axis of which can be arbitrarily oriented in space. Therefore, here we deal with Hilbert space spin tomography i.e. angular momentum tomography[32,33,35,38] with Hilbert space $H = C^{2s+1}$, s being the spin of the particle, and the corresponding group being $SU(2)$.

Obviously, the diagonal elements of the density matrix are non-negative and their sum is equal to unity, their physical meaning being the probabilities of observing the value of spin projection on fixed axis in space. Obviously the relevant representation of $SU(2)$ group is irreducible unitary representation $U(\Omega) = U(\alpha, \beta, \gamma)$, where α, β, γ are Euler angles which are also our tomographic set at the same time. The haar invariant measure for $SU(2)$ is given by:[50]

$$dg(\alpha, \beta, \gamma) = \frac{2s+1}{4\pi^2} \sin(\beta) d\alpha d\beta d\gamma, \tag{15}$$

and finally spin tomography can be written as

$$\rho = \frac{2s+1}{4\pi^2} \int_0^{2\pi} \sin(\beta) d\alpha d\beta d\gamma \mathrm{Tr}[\rho U^\dagger(\alpha, \beta, \gamma)] U(\alpha, \beta, \gamma). \tag{16}$$

Now we try to obtain the above explained tomography via wavelets transform in Banach space. In order to do so, we need to choose the tomographic set $U(\alpha,\beta,\gamma)$ as a left continuous representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\begin{aligned}\mathcal{W} &: \mathcal{B} \mapsto F(G) : \hat{\rho} \mapsto \hat{\rho}(\alpha,\beta,\gamma)\\ &= \langle \hat{\rho}, l_{(\alpha,\beta,\gamma)}\rangle = \langle \hat{\rho}, U(\alpha,\beta,\gamma) l_0\rangle = \langle \hat{\rho} U^{\dagger}(\alpha,\beta,\gamma), l_0\rangle = \mathrm{Tr}(\hat{\rho} U^{\dagger}(\alpha,\beta,\gamma)). \end{aligned} \tag{17}$$

With this condition, inverse wavelet transform in $\mathcal{M}$ is a left inverse operator on B for the wavelet transform $\mathcal{W}$:

$$\begin{aligned}\mathcal{M} : F(G) \mapsto \mathcal{B} : \hat{\rho}(\alpha,\beta,\gamma) \mapsto \mathcal{M}[\hat{\rho}] = \mathcal{M}(\hat{\rho})\\ \mathcal{M}\mathcal{W}(\hat{\rho}) = \int d\mu(\alpha,\beta,\gamma)\langle \hat{\rho}, l_{(\alpha,\beta,\gamma)}\rangle b_{(\alpha,\beta,\gamma)}. \end{aligned} \tag{18}$$

The constant on the left-hand side of (2) also becomes proportional to the dimension of unitary representation, that is, $C(b_0, b_0^{'}) = 2s+1 = d$, where d is dimensional of representation. Finally the constant P becomes equal to one, i.e. $P = \frac{c(b_0,b_0^{'})}{\langle b_0, l_0^{'}\rangle} = 1$, and the reconstruction procedure of wavelet transform (operation of the combination of wavelet transform and its inverse one, $\mathcal{M}\mathcal{W}$ on the density operator $\hat{\rho}$) leads to the tomography relation (16).

By the same choice as above for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example. Therefore, $\{U(\alpha,\beta,\gamma)b_0, U(\alpha,\beta,\gamma)l_0\}$ is an atomic decomposition of Banach space of bounded operators acting on spin s representation space with respect to $\mathcal{B}_2$ with atomic bounds $A = B = 1$.

Finally, we define the reconstruction operator T as follows:

$$\begin{aligned} &T\{\mathrm{Tr}(\hat{\rho} U^{\dagger}(\alpha,\beta,\gamma))\}\\ &\quad = \int \mathrm{Tr}(\hat{\rho} U^{\dagger}(\alpha,\beta,\gamma)) U(\alpha,\beta,\gamma) d\mu(\alpha,\beta,\gamma) = \hat{\rho} \quad \text{for each } \hat{\rho} \in \mathcal{B}. \end{aligned}$$

Therefore, $\{U(\alpha,\beta,\gamma)l_0, T\}$ is a Banach frame for Banach space of operators acting on spin representation space with respect to $\mathcal{B}_2$ with frame bounds $A = B = 1$.

Let us first introduce $(2s+1)^2$ irreducible multipole tensor operators T_{LM}[51]:

$$\widehat{T}^{s}_{LM} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-s}^{s} |sm\rangle\langle sm'| C^{sm}_{sm'LM} = \mathrm{Tr}(\hat{\rho} U^{\dagger}(g_1, g_2)). \tag{19}$$

By substituting the following identity:

$$\widehat{U}^{s}(\alpha,\beta,\gamma) = \sum_{L,M,m,m'} \sqrt{\frac{2l+1}{2s+1}} C^{sm}_{sm'\,LM} U^{s}{}_{mm'}(\Omega) \widehat{T}^{s}_{LM}. \tag{20}$$

in Eq. (16) where $C^{sm}_{sm'LM}$ are Clebsch–Gordan coefficients, and by using the sums involving product of two Clebsch–Gordan coefficients, we get

$$\hat{\rho} = \sum_{LM} \mathrm{Tr}[\hat{\rho}\widehat{T}^{s\dagger}_{LM}]\widehat{T}^{s}_{LM}. \tag{21}$$

On the other hand, we know that these operators are the most convenient basis for $\mathcal{B}$, Banach space of operators acting on $(2s+1)$-dimensional Hilbert space associated with the representation of rotation group, in the sense that any linear bounded operator acting in the $(2s+1)$-dimensional spin state space can be expanded in terms of these multipole tensor operators.[34]

In the wavelet notation for the two partite rotation system, the irreducible representation $SU(2) \times SU(2)$ is $U(g_1, g_2) = U_1(g_1) \otimes U_2(g_2)$, with $U_j = D(\alpha_j, \beta_j, \gamma_j)$, and the characteristic function is defined as

$$\begin{aligned} \mathcal{W} : \mathcal{B} \mapsto F(G) : \Phi(g_1, g_2) &= \langle \hat{\rho}, U(g_1, g_2) l_0 \rangle \\ &= \langle \hat{\rho} U^\dagger(g_1, g_2), l_0 \rangle = \mathrm{Tr}(\hat{\rho} U^\dagger(g_1, g_2)). \end{aligned} \tag{22}$$

Using inverse wavelet transform with vacuum vector $b_0 = I \otimes I$, we give the the following formula to the two partite spin tomography:

$$\begin{aligned} \rho = \left(\frac{2s+1}{4\pi^2}\right)^2 \int_0^{2\pi} dg_1 \int_0^{2\pi} dg_2 \mathrm{Tr}[\rho D^\dagger(\alpha_1, \beta_1, \gamma_1) D^\dagger(\alpha_2, \beta_2, \gamma_2)] \\ \times D(\alpha_1, \beta_1, \gamma_1) D(\alpha_2, \beta_2, \gamma_2). \end{aligned} \tag{23}$$

From Theorems 1 and 2, ρ is separable iff the characteristic function is written as (14).

As an example we consider $3 \otimes 3$ representation of $SU(2) \otimes SU(2)$ group. Three-dimensional representation of $SU(2)$ group is defined in Ref. 17 as

$$U_1(g_1) = \begin{pmatrix} \alpha_1^2 & \alpha_1\bar{\beta}_1 & \bar{\beta_1}^2 \\ 2\alpha_1\beta_1 & |\alpha_1|^2 - |\beta_1|^2 & -2\bar{\alpha}_1\bar{\beta}_1 \\ \beta_1^2 & \beta_1\bar{\alpha}_1 & \bar{\alpha_1}^2 \end{pmatrix}$$

$$U_2(g_2) = \begin{pmatrix} \alpha_2^2 & \alpha_2\bar{\beta}_2 & \bar{\beta_2}^2 \\ 2\alpha_2\beta_2 & |\alpha_2|^2 - |\beta_2|^2 & -2\bar{\alpha}_2\bar{\beta}_2 \\ \beta_2^2 & \beta_2\bar{\alpha}_2 & \bar{\alpha_2}^2 \end{pmatrix}, \tag{24}$$

where α_i and $\beta_i, i = 1, 2$ are group parameters. The $3 \otimes 3$ un-normalized Bell diagonal separable states[10,11] are defined as

$$\begin{aligned} \rho_m &= \sum_k |\psi_{km}\rangle\langle\psi_{km}| = \sum_l |l\rangle\langle l| \otimes |l+m\rangle\langle l+m|, \\ \rho'_m &= \sum_k |\psi_{mk}\rangle\langle\psi_{mk}| = \sum_{l,l',k} \omega^{m(l-l')} |l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|, \\ \rho''_n &= \sum_k |\psi_{nk,k}\rangle\langle\psi_{nk,k}| = \sum_{l,l',k} \omega^{nk(l-l')} |l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|, \end{aligned} \tag{25}$$

where $n = 0, 1, 2, m = 0, 1, 2$ and $\omega = e^{\frac{2\pi i}{3}}$. One can show that any convex sum of these states is separable.[10] We can obtain the characteristic function $\Phi(g_1, g_2)$ for any convex sum of the abovementioned separable states but we consider a particular simple case ρ_0. Then the characteristic function $\Phi(g_1, g_2)$ for ρ_0 is obtained as

$$\Phi(g_1, g_2) = \mathrm{Tr}\big(\rho_0 U_1^\dagger(g_1) \otimes U_2^\dagger(g_2)\big). \tag{26}$$

By some calculations we have

$$\Phi(g_1, g_2) = \alpha_1^2\alpha_2^2 + \big(\mid \alpha_1 \mid^2 - \mid \beta_1 \mid^2\big)\big(\mid \alpha_2 \mid^2 - \mid \beta_2 \mid^2\big) + \bar{\alpha_1}^2\bar{\alpha_2}^2. \tag{27}$$

By definition of $K_i(g_1) = \langle i|U_1(g_1)|i\rangle$ and $\eta_i(g_2) = \langle i|U_1(g_1)|i\rangle$ we have

$$\begin{aligned} &K_1(g_1) = \alpha_1^2 \quad K_2(g_1) = \mid \alpha_1 \mid^2 - \mid \beta_1 \mid^2 \quad\ K_3(g_1) = \bar{\alpha_1}^2 \\ &\eta_1(g_2) = \alpha_1^2 \quad \eta_2(g_2) = \mid \alpha_1 \mid^2 - \mid \beta_1 \mid^2 \quad\ \eta_3(g_2) = \bar{\alpha_1}^2. \end{aligned} \tag{28}$$

Therefore, the characteristic function is rewritten as

$$\Phi(g_1, g_2) = \sum_{i=1}^{3} K_i(g_1)\eta_i(g_2), \tag{29}$$

which agrees with Theorem 1 and we show that state ρ_0 is separable. Similarly, one can show that $\{\rho_1, \rho_2\}$, $\{\rho_i'\}$ and $\{\rho_i''\}$ for all $i = 0, 1, 2$ are separable states.

At the end of this section we try to introduce some possible bound entangled states. For this we first introduce partial transpose (PPT) operators states. We consider the following $3 \otimes 3$ density matrix defined as

$$\rho = p(|\psi_{00}\rangle\langle\psi_{00}|) + \frac{1-p}{3}(\mu_1\rho_1 + \mu_2\rho_2), \tag{30}$$

where it is positive for $0 \leq p, \mu_1, \mu_2 \leq 1$ and $\sum_{i=1}^{2} \mu_i = 1$. Now in order to make the partial transpose of density matrix (30) positive, i.e. to obtain PPT density matrix, we need to use the relations $\rho_i^{T_A} = \rho_i$ and substitute the partial transpose of Bell state projection operator $|\psi_{00}\rangle\langle\psi_{00}|$

$$(|\psi_{00}\rangle\langle\psi_{00}|)^{T_A} = \frac{1}{3}\sum_{m,l=0}^{2} \omega^{ml}|\psi_{m,l}\rangle\langle\psi_{m,3-l}|, \tag{31}$$

in partial transpose of Eq. (30). Now, the positivity of partial transpose of density matrix (30) implies that

$$\rho^{T_A} \geq 0 \Rightarrow p \leq \max\left\{\frac{\sqrt{\mu_1\mu_2}}{1+\sqrt{\mu_1\mu_2}}\right\}. \tag{32}$$

For $\mu_1 = \mu_2 = \frac{1}{2}$ the parameter p is optimal and equal to $p = \frac{1}{3}$. Then $3 \otimes 3$ PPT density matrix is reduced to

$$\rho_{\mathrm{PPT}} = p(|\psi_{01}\rangle\langle\psi_{01}|) + \frac{1-p}{3}(\rho_1\mu_1 + \rho_2\mu_1), \quad p \leq \max\left\{\frac{\sqrt{\mu_1\mu_2}}{1+\sqrt{\mu_1\mu_2}}\right\}. \tag{33}$$

This PPT state is a bound entangled state if there exists a non-decomposable entanglement witness W when $\mathrm{Tr}(\rho_{\mathrm{PPT}}W) < 0$. Let us consider a $3 \otimes 3$ non-decomposable Choi entanglement witness[11] as

$$W = \frac{1}{6}\left(2\sum_{k=0}^{2}|\psi_{k0}\rangle\langle\psi_{k0}| + \sum_{k=0}^{2}|\psi_{k2}\rangle\langle\psi_{k2}| - 3|\psi_{00}\rangle\langle\psi_{00}|\right), \tag{34}$$

where satisfying $\mathrm{Tr}[W\rho_{\mathrm{PPT}}] = \mu_2 - p < 0$. Then ρ_{PPT} is entangled if $\mu_2 < p < \frac{\sqrt{\mu_1\mu_2}}{1+\sqrt{\mu_1\mu_2}}$.

Now, we obtain the characteristic function $\Phi(g_1, g_2)$ for this bound entangled state as

$$\begin{aligned}
\Phi(g_1, g_2) &= \mathrm{Tr}[\rho U(g_1, g_2)] \\
&= \frac{1-p}{3}\big(\alpha_1^2((\mid\alpha_2\mid^2 - \mid\beta_2\mid^2)\mu_1 + \bar{\alpha_2}^2\mu_2) \\
&\quad + (\mid\alpha_1\mid^2 - \mid\beta_1\mid^2)(\bar{\alpha_2}^2\mu_1 + \alpha_2{}^2\mu_2) \\
&\quad + \bar{\alpha_1}^2(\alpha_2^2 + (\mid\alpha_2\mid^2 - \mid\beta_2\mid^2)\mu_2) \\
&\quad + \frac{p}{9}[\alpha_1^2\alpha_2^2 + (\mid\alpha_2\mid^2 - \mid\beta_2\mid^2)(\mid\alpha_1\mid^2 - \mid\beta_1\mid^2) \\
&\quad + \bar{\alpha_1}^2\bar{\alpha_2}^2 + 4\alpha_1\beta_1\alpha_2\beta_2 + \beta_1^2\beta_2{}^2 + \alpha_1\bar{\beta}_1\alpha_2\bar{\beta}_2 \\
&\quad + \beta_1\bar{\alpha_1}\beta_2\bar{\alpha_2} + \bar{\beta}_1{}^2\bar{\beta}_2{}^2 + 4\bar{\alpha_1}\bar{\beta}_1\bar{\alpha_2}\bar{\beta}_2].
\end{aligned} \tag{35}$$

Through Theorem 1, one can obtain a general characteristic function for general separable density matrix as a convex mixture of products of positive definite functions $K_i(g_1) = \langle u_i \mid U(g_1)u_i\rangle$ and $\eta_i(g_2) = \langle v_i \mid U(g_2)v_i\rangle$ where by using the generally normalized vector for u_i and v_i, we obtained the separable characteristic function. Comparing this separable characteristic function with Eq. (35), we show that the function (35) cannot be represented as a convex mixture of products of positive definite functions $K_i(g_1) = \langle u_i \mid U(g_1)u_i\rangle$ and $\eta_i(g_2) = \langle v_i \mid U(g_2)v_i\rangle$, depending on parameters (α_1, β_1) and (α_2, β_2) respectively.

3.2. *Tomography of quantum spinor states*

The family of the probability distribution functions of the 1/2-spin projection is parameterized by the point's coordinates θ and φ on the sphere of unity radius. This parameterization coincides with the physical meaning of the marginal distribution in the sense that the distribution function $w(m, \Omega)$ is the probability to observe the spin projection m if we measure this spin projection on the quantization axis which is parallel to the vector normal to the surface of the sphere of the unit radius at the point with the coordinates θ and φ. If we know the positive and normalized marginal distribution $w(m, \Omega)$, then, as it was shown in Refs. 36, 37 and 52, the matrix elements $\hat{\rho}_{ik}^{(j)}$ can be calculated with the help of the measurable marginal distribution $w(m, \Omega)$ of the particle with an arbitrary spin j and the values of indices $m = -j, -j+1, \ldots, j$ by means of the relation:

$$(-1)^k \hat{\rho}_{ik}^{(j)} = \sum_{J'=0}^{2j} \sum_{M=-J'}^{J'} (2J'+1)^2 \sum_{m=-j}^{j} (-1)^m \otimes \int w(m,\Omega) D_{0M}^{J'}(\Omega) \frac{d\Omega}{8\pi^2} \begin{pmatrix} j & j & J' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} j & j & J' \\ i & -k & M \end{pmatrix}, \quad (36)$$

where $i, k = -j, -j+1, \ldots, j$ and the integration has been operationalized according to the rotation angles φ, θ, ψ

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta. \quad (37)$$

Since in this example adjoint representation is used, the ψ parameter is removed and the representation is limited to homogeneous space, but the representation which we use is group representation. Therefore, the spin tomography can be obtained via wavelet transform in Banach space on Homogeneous space $S^2 = SU(2)/U(1)$.

Then definition of the wavelet transforms for adjoint representation is given by

$$\hat{\rho}(\Omega) = \langle \hat{\rho}, l_\Omega \rangle = \langle \pi(\Omega)\hat{\rho}, l_0 \rangle = \langle U(\Omega)\hat{\rho}(\Omega)U^\dagger(\Omega), l_0 \rangle, \quad (38)$$

where U is irreducible representation of $SU(2)$ group for spin J. In this case, if we choose the test functional as $l_0(\hat{\rho}) = \mathrm{Tr}[\hat{\rho} \mid j, m_1\rangle\langle j, m_2 \mid]$, the corresponding wavelet transform becomes

$$\begin{aligned} \hat{\rho}(\Omega) &= \langle U(\Omega) \mid j, m_1\rangle\langle j, m_2 \mid U^\dagger(\Omega), \hat{\rho}\rangle \\ &= \langle j, m_1 \mid U(\Omega)\hat{\rho}U^\dagger(\Omega)\mathrm{mid}\ j, m_2\rangle = \omega(\Omega, m_1, m_2). \end{aligned} \quad (39)$$

The inverse wavelet transform is

$$\mathcal{M}(\hat{\rho}) = \int d\Omega \ \ \omega(\Omega, m_1, m_2)U^\dagger(\Omega)b_0U(\Omega), \quad (40)$$

if we choose $b_0 = \mid j, m_1\rangle\langle j, m_2 \mid$, the inverse wavelet transform becomes

$$\mathcal{M}(\hat{\rho}) = \int d\Omega\omega(\Omega, m_1, m_2)U^\dagger(\Omega) \mid j, m_1\rangle\langle j, m_2 \mid U(\Omega). \quad (41)$$

Therefore, the matrix elements $\mathcal{M}(\hat{\rho})_{ik}$ take the following forms:

$$\mathcal{M}(\hat{\rho})_{ik} = \int d\Omega\omega(\Omega, m_1, m_2)\langle j, i \mid U^\dagger(\Omega) \mid j, m_1\rangle\langle j, m_2 \mid U(\Omega) \mid j, k\rangle. \quad (42)$$

By putting $m_1 = m_2 = m$, and taking the average sum over m, we get

$$\mathcal{M}(\hat{\rho})_{ik} = \left(\frac{1}{2j+1}\right) \sum_m \int d\Omega\omega(\Omega, m)(-1)^{m-j} D_{im}^j(\Omega) D_{-k-m}^j(\Omega). \quad (43)$$

Now, if we expand the product of D functions in terms of D function (by using addition rule of D functions), after some algebra we get

$$\mathcal{MW}\hat{\rho}_{ik}^{(j)} = \left(\frac{1}{2j+1}\right) \sum_{J'=0}^{2j} \sum_{M=-J'}^{J'} (2J'+1)^2 \sum_{m=-j}^{j} (-1)^{m-k}$$

$$\otimes \int w(m,\Omega)\, D_{0M}^{J'}(\Omega)\, \frac{d\Omega}{8\pi^2} \begin{pmatrix} j & j & J' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} j & j & J' \\ i & -k & M \end{pmatrix}. \tag{44}$$

After some routine calculation, we can also show that the constant on left-hand side of (2) is $C(b_0, b_0^{'}) = 2j+1$, the constant $P = \frac{c(b_0,b_0^{'})}{\langle b_0, l_0^{'} \rangle}$ is equal to $\frac{1}{2j+1}$, and finally the reconstruction procedure of wavelet transform (operation of the combination of wavelet transform and its inverse one, $\mathcal{MW}$ on the density operator $\hat{\rho}$) leads to the tomography relation (36).

By the above choice of vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example with the atomic and frame bounds $A = B = \frac{1}{2j+1}$.

In the wavelet notation, the irreducible representation for the two partite spin j group is $\hat{U}(\Omega_1, \Omega_2) = \hat{U}_{\Omega_1} \otimes \hat{U}_{\Omega_2}$, and tomography formula with vacuum vector $b_0 = |j, m_1\rangle\langle j, m_1| \otimes |j, m_2\rangle\langle j, m_2|$ is given by

$$\rho = \int d\Omega_1 d\Omega_2 \omega(\Omega, m_1, m_2) U^\dagger(\Omega_1, \Omega_2) b_0 U(\Omega_1, \Omega_2), \tag{45}$$

where $\omega(\Omega, m_1, m_2) = \langle j.m_1|\langle j.m_2|U(\Omega_1,\Omega_2)\rho U^\dagger(\Omega_1,\Omega_2)|j.m_1\rangle|j.m_2\rangle$ is the characteristic function which requires the considerations of separability criterion.

3.3. *Discrete spin tomography*

It is easy to characterize a class of discrete representations whose reconstruction is similar to the continuous representation. The most general form of a density matrix of a single qubit can be written as

$$\rho(\vec{n}) = \frac{1}{2}(1 + a\vec{n}.\vec{\sigma}),$$

where $0 \le a \le 1$ and the unit vector in Bloch sphere $\vec{n}$ is the direction along which the spin is pointing up, and it satisfies the following conditions[53]:

$$\Sigma_{\alpha=1}^{K} \vec{n_\alpha} = 0, \tag{46}$$

$$\frac{1}{3}\delta_{jk} = \frac{1}{K}\Sigma_\alpha (n_\alpha)_j (n_\alpha)_k.$$

In the dihedral and tetrahedral subgroups of $SU(2)$, these conditions are satisfied by unit vectors that point to the vertices of any regular polyhedron inscribed within the Bloch sphere i.e. a tetrahedron, octahedron, cube, icosahedron, or dodecahedron. The four projectors for a tetrahedron are linearly independent.

An octahedron gives rise to the six cardinal-direction representations on the Bloch sphere. The regular polyhedral by no means exhausts the possibility of representations of this sort. Simultaneously, we can use the vertices forms of any number of polyhedral.

For spin $s = 1$, it is possible to find a finite group instead of $SU(2)$. In fact,we consider the 12 element tetrahedric group composed of the $\pm\frac{2\pi}{3}$ rotations around the versos:

$$\Big\{\vec{n}_1 = \frac{1}{\sqrt{3}}(1,1,1),\ \vec{n}_2 = \frac{1}{\sqrt{3}}(1,-1,-1),$$
$$\vec{n}_3 = \frac{1}{\sqrt{3}}(-1,1,-1),\ \vec{n}_4 = \frac{1}{\sqrt{3}}(-1,-1,1)\Big\},$$

of the π rotations around

$$\{\vec{n}_5 = (1,0,0), \vec{n}_6 = (0,1,0), \vec{n}_7 = (0,0,1)\},$$

and of the identity. It induces a unitary irreducible representation on the space C^3, given by 3×3 rotation matrices. Hence, the tomography for $s = 1$ can be written as [35,38]:

$$\hat{\rho} = \frac{1}{4}\sum_{m=-1}^{1}\sum_{j=1}^{7} P(\vec{n}_j, m)K_j(m - \vec{s}.\vec{n}_j) + \frac{1}{4}I, \tag{47}$$

where $P(\vec{n}_j, m)$ is the probability of having outcome $\mathbf{m}$ which is the result of measuring the operator $\vec{s}.\vec{n}$ and $K_j(m-\vec{s}.\vec{n}_j)$ is a kernel function and $R_n(\Psi) = e^{i\Psi(\vec{s}.\vec{n})}$ is a unitary representation. So if we choose the identity operator as the vacuum vector, the corresponding wavelet transform or characteristic function becomes

$$\begin{aligned}\mathcal{W} : \mathcal{B} \mapsto F(n,\Psi) : \hat{\rho} \mapsto \hat{\rho}(n,\Psi)\\ = \langle \hat{\rho}, l_{(n,\Psi)}\rangle = \langle \hat{\rho}, R_n(\Psi)l_0\rangle = \langle R_n^{\dagger}(\Psi)\hat{\rho}, l_0\rangle = \mathrm{tr}(R_n^{\dagger}(\Psi)\hat{\rho}).\end{aligned} \tag{48}$$

With the choice of the identity operator as a vacuum vector and the test functional $l_0(\hat{\rho}) = \mathrm{Tr}[\hat{\rho}]$, the inverse wavelet transform $\mathcal{M}$, (4) becomes left inverse operator of the wavelet transform $\mathcal{W}$:

$$\begin{aligned}\mathcal{M}\mathcal{W} = PI \Rightarrow \mathcal{M} : F(n,\Psi) \mapsto \mathcal{B} : \hat{\rho}(n,\Psi)\\ \mapsto \mathcal{M}[\hat{\rho}] = \mathcal{M}\mathcal{W}(\hat{\rho}) = \sum_{\Psi,n}\frac{1}{P}\langle \hat{\rho}, l_{(n,\Psi)}\rangle b_{(n,\Psi)}.\end{aligned} \tag{49}$$

Evaluating the trace over the complete set of vectors $|\overrightarrow{n}, m\rangle$, which are the eigenstates of $\overrightarrow{s}.\overrightarrow{n}$, with eigenvalues m, and by taking into account $\langle \overrightarrow{n}, m|\hat{\rho}|\overrightarrow{n}, m\rangle = p(\overrightarrow{n}, m)$, we get

$$\mathcal{M}\mathcal{W}(\hat{\rho}) = \frac{1}{P}\sum_{m=-1}^{1}\sum_{j=1}^{7} P(\vec{n}_j, m)K_j(m - \vec{s}.\vec{n}_j) + \frac{1}{P}I. \tag{50}$$

We can show that the constant, on the left-hand side of (2), $C(b_0, b_0^{'}) = 12$, the constant $P = \frac{c(b_0,b_0^{'})}{\langle b_0,l_0^{'}\rangle} = 4$, and finally the reconstruction procedure of wavelet transform (operating the combination of wavelet transform and its inverse one, $\mathcal{MW}$ on the operator $\hat{\rho}$) lead to the tomography relation (47).

By the above choice for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example similar to rotation group with bound $A = B = 4$.

In the wavelet notation, the irreducible representation for two partite spin system is $R_n\hat{(}\psi) = R_n\hat{(}\psi_1) \otimes R_n\hat{(}\psi_2)$, and then the wavelet transform or characteristic function is defined by

$$F(n, \Psi) := \langle \hat{\rho}, l_{(n,\Psi)} \rangle = \mathrm{tr}(R_n^{\dagger}(\Psi)\hat{\rho}). \tag{51}$$

The tomography formula with vacuum vector $b_0 = I \otimes I$ is given by

$$\rho = \left(\frac{1}{4}\right)^2 \sum_{m_1=-1}^{1} \sum_{m_2=-1}^{1} \sum_{j=1}^{7} R_n\hat{(}\psi), \tag{52}$$

where $P(\vec{n}_j, m_1, m_2) = \langle \overrightarrow{n}, m_1 | \langle \overrightarrow{n}, m_2 | \hat{\rho} | \overrightarrow{n}, m_2 \rangle | \overrightarrow{n}, m_1 = \rangle$.

3.4. *Unitary group tomography*

Here in this case we consider a state with d levels. To do so we prepare the generators for $SU(d)$ systems and thereby construct the density matrices for a qudit system. When we choose an irreducible square integrable representation of $SU(d)$ group as $U(\Omega) = e^{i\hat{J}.\hat{n}\psi}$, the tomography relation is given by

$$\rho = \int d\mu(\Omega) \mathrm{Tr}[U^{\dagger}(\Omega)\rho] U(\Omega). \tag{53}$$

Now we try to obtain the above explained tomography via wavelets transform in Banach space. The wavelet transform $\mathcal{W}$ from Banach space $\mathcal{B}$ to a space of function $F(G)$ is defined by a representation $U(\Omega) = e^{i\hat{J}.\hat{n}\psi}$ of G on $\mathcal{B}$, with the selection of a vacuum vector b_0 which is equal to identity, and a test functional $l_0(\hat{\rho}) = \mathrm{Tr}(\hat{\rho})$ is given by the following formula:

$$\mathcal{W} : \mathcal{B} \mapsto F(\Omega) : \rho \mapsto \widehat{\rho}(\Omega) = \langle \rho, l_{\Omega} \rangle = \langle U^{\dagger}(\Omega)\rho, l_0 \rangle = \mathrm{Tr}[U^{\dagger}(\Omega)\rho]. \tag{54}$$

Also the constant on the left-hand side of (2) becomes proportional to the dimension of the unitary representation, that is, $C(b_0, b_0^{'}) = d$, where d is dimensional of representation. Finally the constant P becomes equal to one, i.e. $P = \frac{c(b_0, b_0^{'})}{\langle b_0, l_0^{'} \rangle} = 1$, and the reconstruction procedure of wavelet transform (operating the combination of wavelet transform and its inverse one, $\mathcal{MW}$ on the density operator $\hat{\rho}$) leads to the tomography relation (53).

We know that we can expand the $SU(d)$ representation in terms of the generators J as

$$U(\Omega) = \sum_{i=0}^{d^2-1} a_i(\Omega) J_i, \quad a_i(\Omega) = \frac{1}{d} \mathrm{Tr}[U(\Omega) J_i], \tag{55}$$

and using the relation $\mathrm{Tr}[J_i J_j] = d\delta_{ij}$, the tomography relation can be written as

$$\rho = \int d\mu(\Omega) \sum_{i,j} \mathrm{Tr}[J_i \rho] J_j a_i^*(\Omega) a_j(\Omega). \tag{56}$$

Now using the identity

$$\int d\mu(\Omega) a_i^*(\Omega) a_j(\Omega) = \frac{1}{d}\delta_{ij}, \tag{57}$$

the tomography relation reduces to

$$\rho = \frac{1}{d} \sum_j \mathrm{Tr}[J_i \rho] J_i. \tag{58}$$

The generators of $SU(d)$ group may be conveniently constructed by the elementary matrices of d dimensions, $\{e_j^k | k, j = 1, \ldots, d\}$.

$$(e_j^k)_{\mu\nu} = \delta_{\nu j}\delta_{\mu k}, \quad 1 \leq \nu, \quad \mu \leq d. \tag{59}$$

There are $d(d-1)$ traceless matrices,

$$\Theta_j^k = e_j^k + e_k^j \tag{60}$$

$$\beta_j^k = -i(e_j^k - e_k^j), \quad 1 \leq k < j \leq d, \tag{61}$$

which are the off-diagonal generators of the $SU(d)$ group. We add the $d-1$ traceless matrices

$$\eta_r^r = \sqrt{\frac{2}{r(r+1)}} \left[\sum_{j=1}^{r} e_j^j - r e_{r+1}^{r+1} \right], \tag{62}$$

as the diagonal generators and obtain a total of $d^2 - 1$ generators. $SU(2)$ generators are, for instance, given as $\{X = \Theta_2^1 = e_2^1 + e_1^2\ ,\ Y = \beta_2^1 = -i(e_2^1 - e_1^2)\ ,\ Z = \eta_1^1 = e_1^1 - e_2^2\}$.

We now define the λ-matrices, which are similar to Pauli matrices in $SU(2)$ case:

$$\lambda_{(j-1)^2+2(k-1)} = \Theta_j^k \tag{63}$$

$$\lambda_{(j-1)^2+2k-1} = \beta_j^k \tag{64}$$

$$\lambda_{j^2-1} = \eta_{j-1}^{j-1}. \tag{65}$$

In conjunction with a scaled d-dimensional identity operator they form a complete hermitian operator basis.[40] If we replace J_i with $\lambda_i/2$, unitary group tomography relation (58), is reduced to

$$\hat{\rho}_d = \frac{1}{d} \sum_{j=0}^{d^2-1} r_j \hat{\lambda}_j, \tag{66}$$

in which $\widehat{\rho}_d$ is a density matrix of dimension d, a qudit, and $\mathrm{Tr}[\widehat{\rho}_d]=1$ implies that the coefficient r_0 is one. The condition $\mathrm{Tr}\left[\rho_d^2\right]\leq 1$ requires $\sum_{j=1}^{d^2-1} r_j^2 \leq d(d-1)/2$.

At the end we can show that $\{U(\Omega)b_0, U(\Omega)l_0\}$ is atomic decomposition and $(\{U(\Omega)l_0\}, T)$ is Banach frame with atomic bounds $A=B=1$.

We can extend these results to 2-qudits. It is shown that for multiple qudits one needs only to consider a space of operators defined by the tensor product of the generators, $SU(d)\otimes SU(d)$, and the representation is the tensor product of the representations, where this representation is irreducible. If we choose the vacuum vector as $b_0=I_d\otimes I_d$ for admissibility condition we will have $C(b_0,b_0')=d^n$. Then 2-qudit tomography can be written as

$$\hat{\rho}_{nd}=\frac{1}{d^n}\sum_{j_1\cdots j_n=0}^{d^2-1} r_{j_1\cdots j_n}\hat{\lambda}_{j_1}\otimes\cdots\otimes\hat{\lambda}_{j_n}, \tag{67}$$

where $r_{00\cdots 0}=1$ and $r_{j_1,j_2,\ldots,j_n}=\frac{d^n}{2^n}\mathrm{Tr}[\lambda_{j_1}\otimes\lambda_{j_2}\cdots\otimes\lambda_{j_n}\rho](j_1,j_2,\ldots,j_n=1,2,\ldots,d)$. On the other hand in the wavelet notation, the characteristic function for separability consideration is defined as

$$\Phi(\Omega)=\langle\rho,l_\Omega\rangle=\langle U^\dagger(\Omega)\rho,l_0\rangle=\mathrm{Tr}[U^\dagger(\Omega)\rho], \tag{68}$$

where $U(\Omega)=U(\omega_1)\otimes U(\omega_2)$. As an example just like the $SU(2)$ group we can consider un-normalized $d\otimes d$ separable states[11] as

$$\begin{aligned}
\rho_m &= \sum_k |\psi_{km}\rangle\langle\psi_{km}| = \sum_l |l\rangle\langle l|\otimes|l+m\rangle\langle l+m|,\\
\rho'_m &= \sum_k |\psi_{mk}\rangle\langle\psi_{mk}| = \sum_{l,l',k} \omega^{m(l-l')}|l\rangle\langle l'|\otimes|l+k\rangle\langle l'+k|,\\
\rho''_n &= \sum_k |\psi_{nk,k}\rangle\langle\psi_{nk,k}| = \sum_{l,l',k} \omega^{nk(l-l')}|l\rangle\langle l'|\otimes|l+k\rangle\langle l'+k|,
\end{aligned} \tag{69}$$

where $n=0,1,\ldots,d-1, m=0,1,\ldots,d-1$ and $\omega=e^{\frac{2\pi i}{d}}$. Corresponding characteristic function for ρ_0 state is written by

$$\Phi(g_1,g_2)=\mathrm{Tr}\big(\rho_0 U_1^\dagger(g_1)\otimes U_2^\dagger(g_2)\big). \tag{70}$$

By some calculation and using $SU(d)$ representation[17] we have

$$\Phi(g_1,g_2)=\sum_{i=1}^{d} K_i(g_1)\eta_i(g_2), \tag{71}$$

where following Theorem 1, we show that state ρ_0 is separable and we can also show that $\{\rho_1,\ldots,\rho_{d-1}\},\ldots,\{\rho_0^{'},\ldots,\rho_{d-1}^{'}\}$ and $\{\rho_0'',\ldots,\rho_{d-1}''\}$ are separable.

4. Conclusions

The Banach space wavelets transformation nature of quantum tomography of mixed quantum states has been revealed. Also by considering various well-known examples of quantum tomography, we show that the quantum tomography of mixed quantum states is almost the same as the Banach space wavelets reconstruction formalism associated with some unitary representation of finite or infinite group. Utilizing this fact, we can also explain the frame and atomic decomposition nature of quantum tomography of mixed states. Finally, considering the definition of the characteristic function, we have found the separability criterion by group theoretical approach for any mixed state. Homogeneous space and projective representation quantum tomography and quantum entanglement via wavelet transform in Banach space are under investigation.

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