International Journal of Quantum Information
Vol. 7, No. 2 (2009) 517–528

OPTIMAL UNAMBIGUOUS DISCRIMINATION OF TWO FINITE-DIMENSIONAL COHERENT STATES

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Received 23 December 2008

An exact analytical solution to the optimal unambiguous state discrimination involving two finite-dimensional coherent states that occur with given prior probabilities is presented using the Lewenstein–Sanpera decomposition method. Furthermore, a numerical method is advised for efficient solving of the unambiguous state discrimination of the two finite-dimensional coherent states with the same dimensions. In this manner, it is shown that the maximum success rate for the unambiguous states discrimination of the two finite-dimensional coherent states with the arbitrary prior probability is decreased by increasing the dimensionality of the finite-dimensional coherent states. Also, the success rate for the unambiguous states discrimination of the two coherent states satisfies the upper bound proportional to the fidelity of the states for a given prior probability.

Keywords: Optimal unambiguous discrimination; finite-dimensional coherent states; Lewenstein–Sanpera decomposition.

PACS Number(s): 03.65.Ud

1. Introduction

The discrimination of quantum states is an essential task in quantum information and quantum communications.[1–3] Many novel protocols, especially in quantum communication and quantum cryptography, have been proposed based on the fact that non-orthogonal states cannot be discriminated determinately. One possible discrimination strategy is the so-called Unambiguous State Discrimination (USD) where the states are successfully identified with non-unit probability,[4–10] but without error. It was first considered for pure states in the literatures.[4–7,10]

Recently, the mixed states have attracted a lot of attention in the quantum information community.[11–16] Since the perfect discrimination between non-orthogonal quantum states is impossible, measurement strategies for state discrimination have been developed that are optimized with respect to various criteria. Optimal USD is an extreme case in that we are looking for a measurement that either identifies a state uniquely (conclusive result) or fails to identify it (inconclusive result). The goal is to minimize the fraction of the inconclusive results.

Unambiguous state discrimination between m states has $m+1$ outcomes. There are m possible conclusive results, and the inconclusive result. Since no projective measurement in an n-dimensional Hilbert space can have more than m outcomes, generalized measurements are required. Generalized measurements (or positive-operator valued measures, POVMs) provide the most general means of transforming the state of a quantum system.[17,18]

Coherent states are widespread in very different fields of physics such as quantum optics, quantum information, photonics etc. Then in the theoretical and experimental view of the point, the study of optimal measurement and discrimination of these states is very significant. For this reason, we are interested in introducing an analytical and numerical method to discriminate between the two coherent states with the same and finite-dimensionalities.

In this paper, we have considered the problem of unambiguous discrimination between the two finite-dimensional coherent states (FDCS). In this manner, we used the Lewenstein–Sanpera decomposition (LSD) method in order to presentation an analytical solution to the problem. Also, we provided a numerical method for efficient solving of the unambiguously discriminating of FDCS. By using this method, we showed that the maximum success rate for the USD of the two FDCS with the arbitrary prior probability would be decreased by increasing the FDCS dimensionality. Furthermore, we pointed out that the success rate for the USD of the two FDCS satisfied the upper bound proportional to the fidelity of the states for a given prior probability.

The paper is organized as follows. Following this introduction in Sec. 1. In Sec. 2, the mathematical background of the USD is presented. In Sec. 3, the LSD method, an important definition and a lemma about LSD are recalled. In Sec. 4, a brief discussion on the generalization of the standard infinite-dimensional coherent states (IDCS) to comprise the finite-dimensional case is proposed. In Sec. 5, analytical calculations of the optimal POVM for the USD of FDS via LSD are introduced. Section 6 involves our numerical method to calculate the optimal USD. Section 7 is devoted to the conclusion.

2. Unambiguous Quantum State Discrimination

Suppose that a quantum system is described by a mixed quantum state ρ, drawn from a collection $\{\rho_1, \rho_2, \ldots, \rho_m\}$ of mixed quantum states on an n-dimensional complex Hilbert space $\mathcal{H}$, with prior probabilities $\eta_1, \eta_2, \ldots, \eta_m$ respectively, where

$m \le n$. Without loss of generality, assume that all eigenvectors of ρ_i span Hilbert space $\mathcal{H}$. To unambiguously discriminate ρ, one can construct POVMs comprising $m+1$ elements $E_0, E_1, \ldots, E_m$ such that,

$$E_i \ge 0, \quad i = 0, 1, \ldots, m, \quad \text{and} \quad \sum_{i=0}^{m} E_i = I, \tag{1}$$

where I denotes the identity matrix in Hilbert space $\mathcal{H}$. These operators are defined in such a way that $\mathrm{Tr}(\rho E_k)$ is the probability which a system prepared in a state ρ is inferred to be in the state ρ_k, while $\mathrm{Tr}(\rho E_0)$ is the probability that the measurement fails to give a definite answer. The fact that no errors are made in the identification of the states requires

$$\mathrm{Tr}(\rho_i E_j) = \delta_{ij}. \tag{2}$$

Therefore, the probability P of detecting the systems state correctly is as follows

$$P = \sum_{j=1}^{m} \mathrm{Tr}(\rho_j E_j), \tag{3}$$

and the failure probability can be displayed as

$$Q = 1 - P = 1 - \sum_{j=1}^{m} \mathrm{Tr}(\rho_j E_j). \tag{4}$$

The main objective is to design an optimum measurement where the failure probability is to be made as small as possible.

In the above discussion, we have given a general description on USD among m mixed input states. To be more specific, in the following, we will focus on the USD between the two mixed states as a particular case. In the simple case of unambiguous discrimination between the two mixed states ρ_1 and ρ_2 with prior probabilities η_1 and η_2, Eqs. (2) and (4) have the following forms:

$$\mathrm{Tr}(\rho_1 E_2) = \mathrm{Tr}(\rho_2 E_1) = 0, \tag{5}$$

$$Q = \eta_1 \mathrm{Tr}(\rho_1 E_0) + \eta_2 \mathrm{Tr}(\rho_2 E_0) = 1 - \eta_1 \mathrm{Tr}(\rho_1 E_1) - \eta_2 \mathrm{Tr}(\rho_2 E_2). \tag{6}$$

In order to find the optimum measurement, it is necessary to find the explicit expressions for the detection operators. To do this based on Herzog and his collaborators' works,[19] one need the explicit form of the spectral representation of the density operators,

$$\rho_1 = \sum_{l=1}^{d_1} r_l |r_l\rangle\langle r_l|, \quad \rho_2 = \sum_{i=1}^{d_2} s_i |s_i\rangle\langle s_i|,$$

where $r_l, s_i \neq 0$ and $\langle r_l|r_i\rangle = \langle s_l|s_i\rangle = \delta_{li}$. Furthermore, the projection operators are introduced as

$$P_1 = \sum_{l=1}^{d_1} |r_l\rangle\langle r_l|, \quad P_2 = \sum_{i=1}^{d_2} |s_i\rangle\langle s_i|,$$

where d_1 and d_2 are the number of non-vanishing eigenvalues. The non-normalized state vectors $|r_l^{\perp}\rangle$, $(l = 1, \ldots, d_l)$ are constructed as follows

$$|r_l^{\perp}\rangle = |r_l\rangle - P_2|r_l\rangle.$$

It means that each eigenstate of ρ_1 is decomposed into a component lying within the support of ρ_2 and a component being perpendicular to it. A complete orthonormal basis $|v_i\rangle$ is constructed using Gram–Schmidt orthogonalization procedure in the subspace spanned by the non-normalized state vectors $|r_l^{\perp}\rangle$ and the projection onto this subspace is denoted by $P_1^{\perp}$. Similarly, the projection $P_2^{\perp}$ is determined. These projections have the following forms

$$P_1^{\perp} = I - P_2 = \sum_{i=1}^{d_{1\perp}} |v_i\rangle\langle v_i|, \quad P_2^{\perp} = I - P_1 = \sum_{i=1}^{d_{2\perp}} |u_i\rangle\langle u_i|,$$

where $d_{1\perp} = n - d_2$ and $d_{2\perp} = n - d_1$ are the dimensionalities of the corresponding bases.

Now one can specify the general structure of all detection operators, E_1 and E_2, which describes unambiguous discrimination, i.e. it satisfies Eqs. (1) and (2),

$$E_1 = \sum_{i,j=1}^{d_{1\perp}} \alpha_{ij}|v_i\rangle\langle v_j|, \quad E_2 = \sum_{i,j=1}^{d_{2\perp}} \beta_{ij}|u_i\rangle\langle u_j|. \tag{7}$$

It is clear that the USD of a pair of mixed states is possible when they have distinct supports. In fact, the necessary condition for the existence of the non-zero probability of error-free discrimination shows that each of the density operators has a non-zero overlap with the intersection of the kernels of the others. In this manner, for distinguishing between the two mixed quantum states, general upper and lower bounds have been derived for the maximum probability of success.[13] To determine the particular measurement that is optimally suited for unambiguous discrimination of the given states, one has to insert the general *ansatz* for E_1 and E_2 into the second line of Eqs. (1)–(6) and determine the parameters α_{ij} and β_{ij} in such a way which minimizes the failure probability Q under the constraint where the operator $E_0 = I - E_1 - E_2$ is positive. Raynal and Lütkenhaus in Ref. 15 have presented another method using the partial reduction theorem which represents an analytical solutions for the unambiguous discrimination of the two mixed states.

3. Lewenstein–Sanpera Decomposition (LSD) as an Optimal Unambiguous Discrimination

The idea of Refs. 20–23 is based on the method of subtracting projections on product vectors from a given state, that is, for a given density matrix ρ and any set of the states $V = \{|\tilde{\psi}_i\rangle\}$ belonging to the range of ρ, one can subtract a density matrix $\rho' = \sum_i p_i \Pi_i$ (not necessarily normalized) with $p_i \geq 0$ such that $\delta\rho = \rho - \rho' \geq 0$, in the sense that $\mathrm{Tr}(\rho') \leq 1$.

Below, an important definition and lemma about LSD are recalled:

Definition 1. We say that a pair of non-negative (p_1, p_2) is *maximal* with respect to ρ and a pair of projection operators $\Pi_1 = |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|$, $\Pi_2 = |\tilde{\psi}_2\rangle\langle\tilde{\psi}_2|$ if

$$\rho - p_1\Pi_1 - p_2\Pi_2 \geq 0. \tag{8}$$

p_1 is maximal with respect to $\rho - p_2\Pi_2$ and to the projector Π_1, p_2 is maximal with respect to $\rho - p_1\Pi_1$ and to the projector Π_2, and the sum $p_1 + p_2$ is maximal.

Lemma 1. *A pair (p_1, p_2) is maximal with respect to ρ and a pair of projectors (Π_1, Π_2).* (i) *If $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle$ do not belong to $\mathcal{R}(\rho)$ (range of ρ) then $p_1 = p_2 = 0$.* (ii) *If $|\tilde{\psi}_1\rangle$ does not belong to $\mathcal{R}(\rho)$, while $|\tilde{\psi}_2\rangle \in \mathcal{R}(\rho)$, then $p_1 = 0$, $p_2 = \langle\tilde{\psi}_2|\frac{1}{\rho}|\tilde{\psi}_2\rangle^{-1}$.* (iii) *If $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle \in \mathcal{R}(\rho)$ and $\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle = 0$ then $p_i = \langle\tilde{\psi}_i|\frac{1}{\rho}|\tilde{\psi}_i\rangle$, $i = 1, 2$.* (iv) *Finally, if $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle \in \mathcal{R}(\rho)$ and $\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle \neq 0$, then*

$$p_1 = \frac{1}{D}\left(\langle\tilde{\psi}_2|\frac{1}{\rho}|\tilde{\psi}_2\rangle - |\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle|\right), \tag{9}$$

$$p_2 = \frac{1}{D}\left(\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_1\rangle - |\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle|\right), \tag{10}$$

where $D = \langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_1\rangle\langle\tilde{\psi}_2|\frac{1}{\rho}|\tilde{\psi}_2\rangle - |\langle\tilde{\psi}_1|\frac{1}{\rho}|\tilde{\psi}_2\rangle|^2$.

The rest of this section shows that the optimal unambiguous discrimination for m states can be reduced to LSD method. Suppose a quantum system is prepared in a state secretly drawn from a known set $|\psi_1\rangle, \ldots, |\psi_m\rangle$ where each $|\psi_i\rangle$ is a pure state in the Hilbert space $\mathcal{H}$. In order to discriminate $|\psi_1\rangle, \ldots, |\psi_m\rangle$ unambiguously, one can construct a more general POVM consisting of $m+1$ elements $E_0, E_1, \ldots, E_m$ such that

$$E_i \geq 0, \quad i = 0, 1, \ldots, m, \quad \text{and} \quad \sum_{i=0}^{m} E_i = I, \tag{11}$$

where I denotes the identity matrix in $\mathcal{H}$. Each element E_i, $i = 1, \ldots, m$ of POVM corresponds to an identification of the corresponding state $|\psi_i\rangle$, while E_0 corresponds to the inconclusive answer. For the sake of simplicity, we often specify only $E_1, \ldots, E_m$ for a given POVM since the left element E_0 is uniquely determined by

$$E_0 = I - \sum_{i=1}^{m} E_i = I - \sum_i p_i\eta_i\Pi_i = I - \sum_i p_i\eta_i|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|, \tag{12}$$

where the vectors $|\tilde{\psi}_i\rangle$ are the reciprocal states associated with the states $|\psi_i\rangle$, i.e. there are unique vectors such that

$$\langle\tilde{\psi}_i|\psi_k\rangle = \delta_{ik}, \quad 1 \leq i, k \leq N. \tag{13}$$

When the reciprocal states $|\tilde{\psi}_i\rangle$ are normalized, $\Pi_i = |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$ is the projection operator.

The goal of LSD is maximizing p_i''s such that $\sum_{i=1}^m p_i'$ is maximized. Assuming that density matrix in LSD method is equal to identity, we obtain LSD as follows

$$\rho - \sum p_i'\Pi_i = I - \sum_i p_i'|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|. \tag{14}$$

Through the comparison of this relation and Eq. (12) with $p_i' = p_i\eta_i$, it becomes clear that we can maximize success probability by using LSD method or minimize the inconclusive probability. Then we say that LSD is the same as the optimal USD and we use LSD in order to obtain the elements of the optimal POVM.

4. Finite-Dimensional Coherent States

There are some methods for generalizing the standard infinite-dimensional coherent states to comprise the finite-dimensional (FD) case. Two ways are common:

(i) The "truncated coherent" states which are produced by the truncation of the standard infinite-dimensional coherent states expansion.[24–26]
(ii) The generalization of the displacement operator acting on vacuum,[26–30] the states which are referred to as the generalized coherent states.

In the second method, the coherent state is constructed in infinite dimension Hilbert space using the displacement operator $D(\alpha,\alpha^*) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ acting on vacuum state $|0\rangle$. In order to construct the generalized coherent states in the FD Hilbert space[29]

$$|\alpha\rangle_s = \hat{D}_s(\alpha,\alpha^*)|0\rangle, \tag{15}$$

we can use displacement operator as

$$\hat{D}_s(\alpha,\alpha^*) = e^{(\alpha\hat{a}_s^\dagger - \alpha^*\hat{a}_s)}, \tag{16}$$

where the $\hat{a}_s$ and $\hat{a}_s^\dagger$ are the FD annihilation and creation operators. The Fock expansion of the generalized coherent states has the following form

$$|\alpha\rangle_s = \sum_{n=0}^{s} e^{in\phi} b_n^s |n\rangle, \tag{17}$$

where

$$b_n^s = \frac{s!}{s+1}\frac{(-i)^n}{\sqrt{n!}}\sum_{k=0}^{s} e^{ix_k|\alpha|}\frac{\mathrm{He}_n(x_k)}{\mathrm{He}_s^2(x_k)}. \tag{18}$$

Here, $x_k \equiv x_k^{s+1}$ are the roots, $He_{s+1}(x_k) = 0$, of the Hermite polynomial[26]

$$\mathrm{He}_n(x) \equiv 2^{(-n)/2} H_n(x/\sqrt{2}). \tag{19}$$

The generalized CS for spacial cases $s = 1, 2$ have the following forms

$$|\alpha\rangle_1 = \cos|\alpha||0\rangle + e^{i\phi}\sin|\alpha||1\rangle, \tag{20}$$

$$\begin{aligned} |\alpha\rangle_2 &= \frac{1}{3}[\cos(\sqrt{3}|\alpha|) + 2]|0\rangle \\ &\quad + \frac{1}{\sqrt{3}}e^{i\phi}\sin(\sqrt{3}|\alpha|))|1\rangle \\ &\quad + \frac{\sqrt{2}}{3}e^{2i\phi}[1 - \cos(\sqrt{3}|\alpha|)]|2\rangle.. \end{aligned} \tag{21}$$

5. Analytical Calculation of Optimal POVM for Unambiguous Discrimination of the FD Coherent States via Lewenstein–Sanpera Decomposition

In this section, an analytical solution for unambiguous discrimination of FDCS, Eq. (17), by using the Lewenstein–Sanpera decomposition is presented. Suppose that, two FDCS $\rho_1 = |\alpha\rangle_s\langle\alpha|$ and $\rho_2 = |\beta\rangle_s\langle\beta|$ with arbitrary prior probabilities η_1 and $\eta_2 = 1-\eta_1$ have been given. In order to obtain the optimal POVM set for these two states, the projection operators perpendicular to ρ_1 and ρ_2 are introduced as

$$\begin{aligned} P_1^{\perp} &= I - |\beta\rangle_s\langle\beta| = |v\rangle\langle v|, \\ P_2^{\perp} &= I - |\alpha\rangle_s\langle\alpha| = |u\rangle\langle u|, \end{aligned} \tag{22}$$

where

$$|u\rangle = \frac{1}{\sqrt{\sum_{i=0}^{s}|\tilde{\alpha}_i|^2}}\sum_{i=0}^{s}(-1)^i\bar{\tilde{\alpha}}_i|i\rangle, \tag{23}$$

and

$$|v\rangle = \frac{1}{\sqrt{\sum_{i=0}^{s}|\tilde{\beta}_i|^2}}\sum_{i=0}^{s}(-1)^i\bar{\tilde{\beta}}_i|i\rangle \tag{24}$$

are the normalized states,

$$\tilde{\alpha}_i = \sum_{k=1}^{s}(-1)^{k+1}\alpha_{i+k} \quad \text{with} \quad \begin{cases} \alpha_{(i+k)} \Rightarrow (-1)^s\alpha_{(i+k)\bmod (s+1)} \\ \text{for any } (i+k) > s \end{cases}, \tag{25}$$

and $\tilde{\beta}_i$ is defined like $\tilde{\alpha}_i$. The most general *ansatz* for the detection operators E_1 and E_2 can be written as

$$\begin{aligned} E_1 &= p_1\sqrt{\eta_1}|v\rangle\langle v|\sqrt{\eta_1} = p_1\eta_1\Pi_1, \\ E_2 &= p_2\sqrt{\eta_2}|u\rangle\langle u|\sqrt{\eta_2} = p_2\eta_2\Pi_2, \end{aligned} \tag{26}$$

where $\Pi_1 = |v\rangle\langle v|$ and $\Pi_2 = |u\rangle\langle u|$ are projection operators. Let the density matrix ρ introduced in Eq. (8) be the identity operator in Hilbert space $\mathcal{H}$, i.e. $\rho = I$.

Then, a pair (p'_1, p'_2) is maximal with respect to ρ and the pair of operators Π_1 and Π_2 if

$$E_0 = I - p'_1|v\rangle\langle v| - p'_2|u\rangle\langle u| \geq 0,$$

where $p'_i = p_i\eta_i$.

Now if the condition

$$|\langle u|v\rangle| \leq \sqrt{\frac{\eta_1}{\eta_2}} \leq \frac{1}{|\langle u|v\rangle|}, \tag{27}$$

is held, considering the LSD lemma, the maximal pair (p_1, p_2) is obtained by

$$\begin{aligned} p_1 &= \frac{1 - \sqrt{\frac{\eta_1}{\eta_2}}|\langle u|v\rangle|}{\eta_1(1 - |\langle u|v\rangle|^2)}, \\ p_2 &= \frac{1 - \sqrt{\frac{\eta_2}{\eta_1}}|\langle u|v\rangle|}{\eta_2(1 - |\langle u|v\rangle|^2)}, \end{aligned} \tag{28}$$

and the success probability is presented by

$$P = \eta_1 \mathrm{Tr}[E_1\rho_1] + \eta_2 \mathrm{Tr}[E_2\rho_2]. \tag{29}$$

If $\sqrt{\eta_1/\eta_2} \geq 1/|\langle u|v\rangle|$, then the optimal solution is shown as

$$p_1 = 0, \quad p_2 = 1, \tag{30}$$

and the success probability is obtained by

$$P = \eta_2 \mathrm{Tr}[E_2\rho_2]. \tag{31}$$

If $\sqrt{\eta_1/\eta_2} \geq |\langle u|v\rangle|$, then the optimal solution is revealed by

$$p_2 = 0, \quad p_1 = 1, \tag{32}$$

and finally the success probability is

$$P = \eta_1 \mathrm{Tr}[E_1\rho_1]. \tag{33}$$

Below, we study a well-known example by using our USD coherent state method.

5.1. *Unambiguous discrimination of the two-dimensional coherent states*

The two-dimensional coherent states in the Bloch sphere is defined as

$$|\alpha\rangle_1 = \sum_{i=0}^{1} \alpha_i|i\rangle, \tag{34}$$

where $|\alpha_0|^2 + |\alpha_1|^2 = 1$ and $\alpha_0 = \cos(|\alpha|), \alpha_1 = e^{i\phi}\sin(|\alpha|)$. Suppose that the two finite-dimensional density matrix $\rho_1 = |\alpha\rangle_1\langle\alpha|$ and $\rho_2 = |\beta\rangle_1\langle\beta|$ correspond to coherent states $|\alpha\rangle_1$ and $|\beta\rangle_1$ with the arbitrary prior probabilities η_1 and $\eta_2 = 1-\eta_1$

are given. The normalized dual vectors (23) and (24) for the above coherent states are reduced to

$$|u\rangle = \bar{\tilde{\alpha}}_1|0\rangle - \bar{\tilde{\alpha}}_0|1\rangle, \tag{35}$$

$$|v\rangle = \bar{\tilde{\beta}}_1|0\rangle - \bar{\tilde{\beta}}_0|1\rangle. \tag{36}$$

Then the most general *ansatz* for the detection operators (E_1 and E_2) can be written as follows

$$E_1 = p_1\eta_1|v\rangle\langle v|, \quad E_2 = p_2\eta_2|u\rangle\langle u|. \tag{37}$$

By the definition of the fidelity $F = \mathrm{Tr}\left[\sqrt{\rho_2^{\frac{1}{2}}\rho_1\rho_2^{\frac{1}{2}}}\right]$ and considering Eqs. (35) and (36), it is easy to show that $F = |\langle u|v\rangle|$. Using the LSD lemma for the two states $|u\rangle$ and $|v\rangle$ in the case of

$$F \leq \sqrt{\frac{\eta_1}{\eta_2}} \leq \frac{1}{F}, \tag{38}$$

we have

$$\begin{aligned} p_1 &= \frac{1-\sqrt{\frac{\eta_1}{\eta_2}}F}{\eta_1(1-F^2)}, \\ p_2 &= \frac{1-\sqrt{\frac{\eta_2}{\eta_1}}F}{\eta_2(1-F^2)}, \end{aligned} \tag{39}$$

so Eq. (39) will be the optimal solution and the success probability is obtained by

$$P = 1 - 2\sqrt{\eta_1\eta_2}F. \tag{40}$$

The relations (38) and (40) are in agreement with Ref. 15 when one considers its results for the optimal unambiguous discrimination of the two pure states.

If $\sqrt{\eta_1/\eta_2} \geq 1/F$, the optimal solution will be

$$p_1 = 0, \quad p_2 = 1, \tag{41}$$

and the success probability is

$$P = \eta_2\mathrm{Tr}[E_2\rho_2] = \eta_2\sin^2(|\alpha| - |\beta|). \tag{42}$$

Finally, if $\sqrt{\eta_1/\eta_2} \geq F$, then the optimal solution is obtained by

$$p_2 = 0, \quad p_1 = 1, \tag{43}$$

and the success probability is

$$P = \eta_1\mathrm{Tr}[E_1\rho_1] = \eta_1\sin^2(|\alpha| - |\beta|). \tag{44}$$

Comparing these exact solutions for the optimal USD of the two pure coherent states with the results of Ref. 15, one can observe the agreement between these two methods.

6. Numerical Method to Calculate the Optimal State Discrimination

We have carried out extensive tests on our analytical algorithm of the unambiguous state discriminations involving the two FDCS in the Hilbert space using a numerical method.

As introduced in Sec. 3, the success rate for the unambiguous coherent state discrimination can be maximized with respect to the conditions imposed on prior probabilities η_1, η_2. Using Eqs. (17) and (18), the FDCS $|\alpha\rangle_s$ and $|\beta\rangle_s$ with $s = \{1,\ldots,7\}$ are defined up to the phase factors ϕ_α and ϕ_β (depending on the given coherent states). The phase factors ϕ_α and ϕ_β, the amplitudes of α and β and the prior probabilities of the two FDCS (η_1, η_2), are defined randomly in the Monte Carlo method. For the pair of the FDCS with the same dimension, this numerical algorithm can be used to find the optimal success rate for any arbitrary dimensions. For an example, we have used this algorithm for the FDCS with the same dimensions in the dimension range between $\{1,\ldots,7\}$.

It is revealed that the maximum success rates are decreased with the increasing the dimensionality of the FDCS. Figure 1 clearly shows the results. As expected, the maximum success rate generally decreases proportionally to the increasing of the dimensionality of the FDCS. We have also found that the maximum success rates satisfy the upper bound $P \leq 1 - 2\sqrt{\eta_1\eta_2}F$, which is proportional to the fidelity of the states for any given prior probabilities. For an example, the case $\eta_1 = \eta_2 = 1/2$ is shown in Fig. 2.

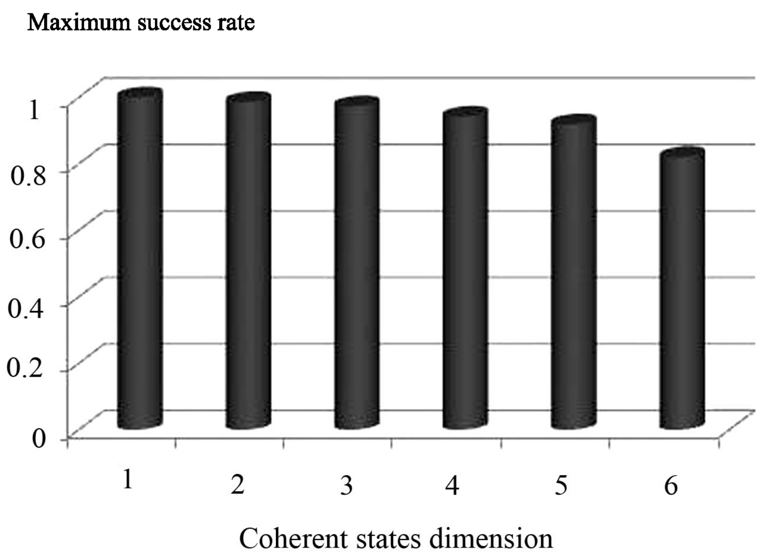

Fig. 1. The success rate for the unambiguous states discrimination of the two FD coherent states is decreased by increasing the dimensionality of the FDCS.

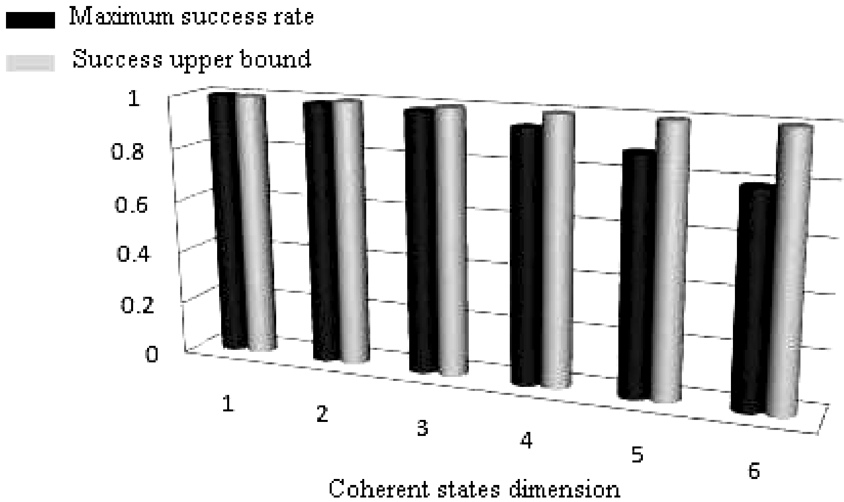

Fig. 2. The success rate for the unambiguous states discrimination of the two coherent states satisfies the definite upper-bound proportional to the fidelity of the states for given $\eta_1 = \eta_2 = 1/2$.

7. Conclusion

An exact analytical solution has been presented to the optimal USD involving the two FDCS with the given prior probabilities using the LSD optimization method. The numerical method for the USD of the two FDCS with the same dimension has also been proposed. It is shown that the maximum success rate for the USD of the two FDCS with the arbitrary prior probability is decreased by increasing the dimensionality of these states. Hence, the success rate for the USD of the two FDCS and the upper bound proportionality to the fidelity of the states for a given prior probability has been compared. The analytical and numerical solutions to the optimal USD involving an arbitrary FDCS is under investigation as well.

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