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Quantum tomography with wavelet transform in Banach space on homogeneous space

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Abstract. In this study the intimate connection is established between the Banach space wavelet reconstruction method on homogeneous spaces with both singular and nonsingular vacuum vectors, and some of the well known quantum tomographies, such as: Moyal-representation for a spin, discrete phase space tomography, tomography of a free particle, Homodyne tomography, phase space tomography and SU(1,1) tomography. And both the atomic decomposition and the Banach frame nature of these quantum tomographic examples are also revealed in details. Finally the connection between the wavelet formalism on Banach space and Q-function is discussed.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.)

1 Introduction

The mathematical theory of wavelet transform has recently gained considerable success in various fields of science and technology, including the treatment of large databases, data and image compression, signal processing, telecommunications and many other applications [1–3]. After Morlet's empirical discovery [4], Grossmann et al. [5,6] and Daubechies [7] recognized from the very beginning that wavelets are simply coherent states associated with affine group of the line (dilations and translations). Thus, immediately the stage was set for a far reaching generalization [7,8].

Unlike the Haar functions which form an orthogonal basis, Morlet wavelets are not orthogonal and form frames. Frames are sets of functions which are not necessarily orthogonal and which are not linearly independent. Actually, a frame for a Hilbert space is a redundant set of vectors (overcomplete) which yield, in a stable way, a representation for each vector in the space [9].

Recently another concept called the atomic decomposition has played a key role in the further mathematical development of the wavelet theory. Indeed the atomic decomposition for any space of function or distribution aims to represent any element in the form of a set of simple functions which are called atoms [10]. As far as the Banach space is concerned, Feichtinger-Grochenig [11] provided a general and very flexible way to construct coherent atomic decompositions and the Banach frames for certain Banach spaces, called coorbit spaces.

The concept of a quantum state represents one of the most fundamental pillars of the paradigm of quantum theory. Usually the quantum state is described either by the state vector in Hilbert space, the density operator, or a phase space probability density distribution (quasi-distributions). The quantum states can be determined completely from the appropriate experimental data by using the well-known technique of the quantum tomography or better to say the tomographic transformation.

A general framework is already presented for the unification of the Hilbert space wavelets transformation on one hand, and the quasi-distributions and the tomographic transformation associated with a given pure quantum states on the other hand [12]. Here in this manuscript it has been attempted to establish the intimate connection between the Banach space wavelet method developed by Feichtinger-Grochenig [11] and Kisil [13] and some of the well-known quantum tomographies associated with the mixed states.

The density matrix can be presented in the Banach space in quantum physics [14]. Therefore, it is natural to do quantum tomography of each density matrix by using the wavelet transform and its inverse in the Banach space

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on homogeneous space associated with the corresponding density matrix. This obtained quantum tomography by the Banach space wavelet method for density states is completely consistent with the quantum tomography obtained by other methods. Both the atomic decomposition and the Banach frame nature of these quantum tomographic examples are also revealed in detail.

The organization of the paper is as follows: First the definition of the wavelet transform and its inverse on homogeneous spaces with both singular and nonsingular vacuum vectors are given. Then we obtain some typical quantum tomographic examples with nonsingular vacuum vectors, such as: Moyal-representation for a spin, discrete phase space tomography, and as result we present their atomic decomposition and their Banach frame bounds. After that some typical quantum tomographic examples with singular vacuum vectors, such as: Homodyne tomography, phase space tomography, SU(1,1) tomography and tomography of a free particle are obtained. And later its atomic decomposition and Banach frame bounds is presented. Finally, the connection between the wavelet formalism on the Banach space and Q-function is discussed. The paper ends with a brief conclusion.

2 Wavelet transform, frame and atomic decomposition in Banach spaces on homogeneous space

The following is a brief recapitulation of some aspects of the theory of wavelets, the atomic decomposition and the Banach frame on homogeneous space. We have only mentioned the concepts which are needed in the sequel, a more detailed treatment can be found in [11,13]. Let G be a locally compact group and H be its closed normal subgroup. Let $X = G/H$ be the corresponding homogeneous space with an invariant measure $d\mu$ and since in general the representation is not directly defined on X, $x \in G/H$ can be defined only after an embedding $\sigma : G/H \rightarrow G$ which is fixed in the principal bundle $G \rightarrow G/H$.

In this part of our discussions for simplicity, we write $U(x)$ instead of $U(\sigma(x))$, for a fixed section $\sigma : G/H \rightarrow G$. Given a Banach space $\mathcal{B}$, we could define a new representation π for embedding $X \times X$ in the space $\mathcal{L}(\mathcal{B})$ of bounded linear operators $\mathcal{B} \rightarrow \mathcal{B}$, let U be a continuous representation of a group:

$$\pi : X \times X \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{B})) : \hat{O} \rightarrow U(x_1)\hat{O}U(x_2^{-1}), \qquad (2.1)$$

where if x_1 is equal to x_2, the representation is called an adjoint representation, and, if x_2 is equal to the identity operator, the representation is called a left representation of the homogeneous space.

Let $\mathcal{L}(\mathcal{B})$ be the space of the bounded linear operators $\mathcal{B} \rightarrow \mathcal{B}$ in the Banach space, we will say that $b_0 \in \mathcal{B}$ is a vacuum vector if for all $h \in H_1 \times H_2$ with normal subgroups H_1 and H_2 we have $\pi(h)b_0 = \chi(h)b_0$ where χ is a character of the subgroup H and also the set of vectors $b_{x_1,x_2} = \pi(x_1,x_2)b_0$ forms a family of coherent states, if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional where $\mathcal{B}^*$ is dual of $\mathcal{B}$) such that

$$C(b_0, b_0^{'}) = \int_X \int_X \langle \pi(x_1^{-1}, x_2^{-1})b_0, l_0 \rangle \langle \pi(x_1, x_2)b_0^{'}, l_0^{'} \rangle d\mu(x_1, x_2), \qquad (2.2)$$

is non-zero and finite, which is known as the admissibility relation.

If the subgroup H is non-trivial, one does not need to know the wavelet transform on the whole group G, but it should be defined on only the homogeneous space G/H, then the reduced wavelet transform $\mathcal{W}$ to a homogeneous space of the function L_2 is defined by a representation π of $X \times X$ on $\mathcal{B}$ and a test functional $l_0 \in \mathcal{B}^*$ such that [13]

$$\begin{aligned}\mathcal{W} : \mathcal{B} \rightarrow L_2(X \times X) : \hat{O} \rightarrow \hat{O}(x_1, x_2) = [\mathcal{W}\hat{O}](x_1, x_2) \\ = \langle \pi(x_1^{-1}, x_2^{-1})\hat{O}, l_0 \rangle = \langle \hat{O}, \pi^*(x_1, x_2)l_0 \rangle \quad \forall x_1, x_2 \in X,\end{aligned} \qquad (2.3)$$

where π^* is dual of π. The inverse wavelet transform $\mathcal{M}$ with a vacuum vector $b_0 \in \mathcal{B}$ from $L_2(X \times X)$ to $\mathcal{B}$ is given by the formula:

$$\mathcal{M} : L_2(X \times X) \rightarrow \mathcal{B} : \hat{O}(x_1, x_2) \rightarrow \mathcal{M}[\hat{O}]$$

$$\begin{aligned}\mathcal{M}\mathcal{W}(\hat{O}) &= \int_X \int_X \hat{O}(x_1, x_2) b_{x_1,x_2} d\mu(x_1, x_2) \\ &= \int_X \int_X \hat{O}(x_1, x_2)\pi(x_1, x_2)b_0 d\mu(x_1, x_2).\end{aligned} \qquad (2.4)$$

The operator $PI = \mathcal{M}\mathcal{W} : \mathcal{B} \longmapsto \mathcal{B}$ is a projection of $\mathcal{B}$ into the linear subspace in which b_0 is cyclic (i.e., the set $\{\pi(x_1, x_2)b_0 | x_1, x_2 \in X\}$ spans the Banach space $\mathcal{B}$), and $\mathcal{M}\mathcal{W}(\hat{O}) = P(\hat{O})$ in which the constant P is equal to $\frac{C(b_0,b_0^{'})}{\langle b_0,l_0^{'} \rangle}$. There are two different cases which correspond to different choices of the vacuum vector:

a) **Non-singular cases**

In this case, π is an irreducible representation, then the inverse wavelet transform $\mathcal{M}$ is a left inverse operator on $\mathcal{B}$ for the wavelet transform $\mathcal{W}$ i.e., $\mathcal{M}\mathcal{W}$=I for which admissibility relation (2.2) holds and the vector b_0 can be found in $\mathcal{B}$.

b) **Singular cases**

In this case the representation π of G is neither a square-integrable nor a square-integrable modulo of a subgroup H. Therefore, the vacuum vector b_0 could not be selected within the original Banach space $\mathcal{B}$ (the representation space of U). Then, in the singular case, we assume that there is a probe vector $p_0 \in \mathcal{B}$ where the following integral converges:

$$C(b_0, p_0) = \left\langle \int_X \int_X \langle \pi(x_1^{-1}, x_2^{-1})p_0, l_0 \rangle \pi(x_1, x_2)b_0 d\mu(x_1, x_2), l_0 \right\rangle. \qquad (2.5)$$

The choice of a probe vector is similar to regularization [25], which has been used in our calculations. According to the theory of distributions, the smoothness, regularity, and localization of a tempered distribution can be improved by a function of the Schwartz class. Various regulators can be used for numerical computations.

The frames can be seen as a generalization of bases in the Hilbert or the Banach space [10]. The Banach frames and the atomic decomposition are sequences that have basis-like properties but which need not to be bases. The atomic decomposition has played a key role in the recent development of the wavelet theory.

Now we define a decomposition of a Banach space on homogeneous space as follow:

The definition of the coorbit space. Let $\mathcal{B}$ be a Banach space. The coorbit space is the collection of all functions for which the wavelet transform is contained in L_2. Similar to the definition of the coorbit space in the group, we can define the coorbit spaces for $X = G/H$ as [24]:

$$M_2 = \{\hat{O} \in \mathcal{B} : \mathcal{W}\hat{O} \in L_2\} \quad with \quad norm \quad ||\hat{O}||_{M_2} = ||\mathcal{W}\hat{O}||_{L_2}. \quad (2.6)$$

The definition of the atomic decomposition. Let M_2 be a coorbit space and let L_2 be an associated Banach space. Let $\{y_{i,j} = \pi^*(x_i, x_j)l_0\}$ and $\{\hat{b}_{i,j} = \pi(x_i, x_j)b_0\}$ be given. The set [24] $(\{y_{i,j}\}, \{b_{i,j}\})$ is an atomic decomposition of $\mathcal{B}$ with respect to L_2 and, the norm equivalence is given by:

$$A\|\hat{O}\|_{M_2} \leq \|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{L_2} \leq B\|\hat{O}\|_{M_2}, \quad (2.7)$$

then A, B are a choice of the atomic bounds for $(\{y_{i,j}\}, \{b_{i,j}\})$. And $\hat{O} = \sum_{i,j=1}^{\infty} \langle O, y_{ij}\rangle b_{ij}$ for each $\hat{O}$.

The definition of the Banach frame. Let T is a bounded linear operator such that $T\{\langle \hat{O}, y_{i,j}\rangle\} = \hat{O}$ for each $\hat{O} \in M_2$.

Then $(\{y_{i,j}\}, T)$ is a Banach frame for M_2 with respect to L_2. The mapping T is a reconstruction operator (inverse wavelet transform). If the norm equivalence is given by $A\|\hat{O}\|_{M_2} \leq \|\{\langle \hat{O}, y_{i,j}\rangle\}\|_{L_2} \leq B\|\hat{O}\|_{M_2}$, then A, B are a choice of the frame bounds for $(\{y_{i,j}\}, T)$.

It is a remarkable fact that the admissibility condition is an analogous relation to the existence of frame bounds.

From the definition of the Banach frame all of the states that define a separated set $\{y_{i,j}\}$ are a set of sampling for L_2 if and only if the set of $\{y_{i,j}\}$ is a frame for L_2. It is inferred from frame theory that there exists a dual frame $\{b_{i,j}\} \subset L_2$ that allows us to reconstruct functions $\{\hat{O}_{i,j} = \langle \hat{O}, y_{i,j}\rangle\} \subset L_2$ explicitly as

$$\mathcal{MW}\hat{O} = \sum_{i,j} \langle \hat{O}, y_{i,j}\rangle b_{i,j}, \quad (2.8)$$

where $\mathcal{MW}$ is a map from M_2 to M_2 then $S = \mathcal{MW}$ is a frame operator similar to the definition of the frame operator in the Hilbert space [10]. The frame condition can be expressed in terms of the frame operator as:

$$AI \leq S \leq BI. \quad (2.9)$$

Now according to the Schur's lemma a representation that commutes with all irreducible representations must be a constant multiple of the identity. This lemma reduces to an orthogonality relation and the resolution of the identity for any coherent state. In other cases if b_0 is cyclic it satisfies the resolution of the identity. Then if the representation be irreducible or b_0 is cyclic, $S = \mathcal{MW}$ is proportional to the identity. Therefore, frame bounds A, B are constant (tight frame) [8].

3 Quantum tomography with wavelet transform on homogeneous space (non-singular case)

3.1 Moyal-type representations for a spin

For a spin s, in [15] a 'Stratonovich-Weyl' correspondence as a rule which maps each operator $\widehat{O}$ on the $(2s+1)$-dimensional Hilbert space $\mathcal{H}_s$ to a function on the phase space of the classical spin, $\mathcal{S}^2$ is defined. A *discrete* Moyal formalism is defined as [16].

$$\hat{\Delta}_{\mathbf{ns}} = \widehat{U}_{\mathbf{n}} \hat{\Delta}_{\mathbf{n_z s}} \hat{U}_{\mathbf{n}}^{\dagger}, \quad (3.1)$$

where $\hat{U}_{\mathbf{n}}$ represents a rotation which maps the vector $\mathbf{n}_z$ to $\mathbf{n}$.

The associated kernels are defined as

$$\hat{\Delta}_{\mathbf{ns}} = |s, \mathbf{n}\rangle\langle s, \mathbf{n}| \equiv |\mathbf{n}\rangle\langle \mathbf{n}|, \quad (3.2)$$

$$\hat{\Delta}^{\mathbf{ns}} = \sum_{m=-s}^{s} \Delta^m |m, \mathbf{n}\rangle\langle m, \mathbf{n}|. \quad (3.3)$$

In the wavelet notation, the Banach space is $(2s+1)^2$-dimensional and the group is $SU(2)$, the subgroup is $U(1)$ and the measure is $d\mu(n) = \frac{2s+1}{4\pi} d(\mathbf{n})$ and the unitary irreducible representation of the group is U_n which is the result of the adjoint representation on any operators in the Banach space:

$$\hat{\pi}(n)\hat{O} = \hat{U_n}\hat{O}\hat{U}_n^{\dagger}. \quad (3.4)$$

Then the wavelet transform in this Banach space with the test functional,

$$l_0(\hat{O}) = Tr\left(\hat{O} \sum_m \Delta^m |m, n_z\rangle\langle m, n_z|\right), \quad (3.5)$$

is given by:

$$\begin{aligned} \mathcal{W}\hat{\rho} = \hat{\rho}(n) &= \langle \hat{\pi(n)}^{\dagger} \hat{O}, l_0\rangle \\ &= Tr\left(\hat{U_n}^{\dagger} \hat{O} \hat{U_n} \sum_m \Delta^m \mid m, n_z\rangle\langle m, n_z \mid\right), \quad (3.6) \end{aligned}$$

then we have:

$$\hat{O}(n) = Tr\left(\hat{O}\hat{U_n} \sum_m \Delta^m \mid m, n_z\rangle\langle m, n_z \mid \hat{U_n}^{\dagger}) = Tr(\hat{O}\hat{\Delta}^{ns}\right).$$

If we choose the vacuum vector $b_0 = |\ s, n_z\rangle\langle s, n_z\ |$, the inverse wavelet transform $\mathcal{M}$ becomes the left inverse operator of the wavelet transform $\mathcal{W}$:

$$\begin{aligned}\mathcal{M}\mathcal{W}\hat{\rho} &= \int \langle \hat{\pi}^\dagger(n)\hat{O}, l_0\rangle \hat{\pi(n)} b_0 \\ &= \int d\mu(n) Tr(\hat{O}\hat{\Delta}^{ns})\hat{U_n}\,|s, n_z\rangle\langle s, n_z|\,\hat{U_n}^\dagger \\ &\Rightarrow \hat{O} = \frac{1}{P}\left(\frac{2s+1}{4\pi}\int dn Tr(\hat{O}\hat{\Delta^{ns}})\hat{\Delta_{ns}}\right).\end{aligned} \quad (3.7)$$

By using the relations:

$$\frac{2s+1}{4\pi}\int_{\mathcal{S}^2} d\mathbf{n}\ \mathrm{Tr}\ \left[\hat{\Delta}_{\mathbf{ms}}\hat{\Delta}^{\mathbf{ns}}\right]\hat{\Delta}_{\mathbf{ns}} = \hat{\Delta}_{\mathbf{ms}},$$

and

$$\mathrm{Tr}\ \left[\hat{\Delta}_{\mathbf{n}_z s}\hat{\Delta}^{\mathbf{ns}}\right] = \sum_{l=0}^{2s}\frac{2l+1}{2s+1}P_l(\cos\theta),$$

one can show that the constant on the left hand side of (2.2) is

$$C(b_0, b_0') = \int_X \langle \hat{\pi}^\dagger|s, n_z\rangle\langle s, n_z|, l_0\rangle\langle\hat{\pi}|s, n_z\rangle\langle s, n_z|, l_0\rangle d\mu(n),$$

now using definitions (3.4) and (3.5) we have

$$\begin{aligned}C(b_0, b_0') &= \\ &\int_X \langle \hat{U_n}^\dagger|s, n_z\rangle\langle s, n_z|\hat{U_n}, l_0\rangle\langle\hat{U_n}|s, n_z\rangle\langle s, n_z|\hat{U_n}^\dagger, l_0\rangle d\mu(n) \\ &= \int tr(\hat{\Delta}_{n_z s}\hat{\Delta}^{ns})tr(\hat{\Delta}^{n_z s}\hat{\Delta}_{ns})d\mu(n) \\ &= tr(\hat{\Delta}^{n_z s}\hat{\Delta}_{n_z s}) = 2s+1,\end{aligned}$$

then the constant $P = \frac{C(b_0, b_0')}{\langle b_0, l_0\rangle} = 1$, and finally the reconstruction procedure of the wavelet transform (operating the combination of the wavelet transform and its inverse one, $\mathcal{M}\mathcal{W}$ on the operator $\hat{O}$) leads to the following tomography relation:

$$\hat{O} = \frac{(2s+1)}{4\pi}\int_{S^2} dn Tr[\hat{O}\hat{\Delta}_{\mathbf{ns}}]\hat{\Delta}^{\mathbf{ns}}. \quad (3.8)$$

By the same choice as above for the vacuum vectors and test functions, we can get the atomic decomposition and the Banach frame for this example. Then we can show the following conditions:

The norms $||\hat{O}||_{M_2}$ and $||\{Tr(\hat{\pi}^\dagger(n)\hat{O})\}|| = [\int Tr(\hat{\pi}^\dagger(n)\hat{O})\overline{Tr(\hat{\pi}^\dagger(n)\hat{O})}d\mu(n)]^{\frac{1}{2}}$ are equivalent since

$$\begin{aligned}||\{Tr(\hat{\pi}^\dagger(n)\hat{O})\}|| &= \left[\int Tr(\hat{\pi}^\dagger(n)\hat{O})\overline{Tr(\hat{\pi}^\dagger(n)\hat{O})}d\mu(n)\right]^{\frac{1}{2}} \\ &= |\int tr(\pi^\dagger(n)\hat{O})tr(\pi(n)\hat{O}^\dagger)d\mu(n)|^{1/2} \\ &= \left|tr\left(\int tr(\pi^\dagger(n)\hat{O})\pi(n)d\mu(n)\hat{O}^\dagger\right)\right|^{1/2},\end{aligned}$$

if we use the tomography relation (3.8) we obtain

$$||\{Tr(\hat{\pi}^\dagger(n)\hat{O})\}|| = |tr(\hat{O}\hat{O}^\dagger)|^{1/2} = ||\hat{O}||,$$

such that they can saturate the inequality (2.7) with the atomic bounds $A = B = 1$, provided that we use the the Hilbert-Schmidt norm for the operator $\hat{O}$ and if we use the relation (3.8) we have:

$$\hat{O} = \int Tr(\hat{\pi}^\dagger(n)\hat{O})\hat{\pi}(n)b_0 d\mu(n)$$

Therefore, $\{\hat{\pi}(n)b_0, \hat{\pi}(n)l_0\}$ is an atomic decomposition of M_2 of bounded operators acting on representation space with respect to L_2 with atomic bounds $A = B = 1$.

Finally, for the Banach frame we use the same choice of the vacuum vector and the test functional as in the atomic decomposition case. In order to have the Banach frame condition for the existence of the atomic decomposition, we can define the reconstruction operator T as follows:

$$T\{Tr(\hat{\pi^\dagger}(n)\hat{O})\} = \int Tr(\hat{\pi^\dagger}(n)\hat{O})\hat{\pi}(n)d\mu(n) = \hat{O}$$

$$\text{for each} \quad \hat{O} \in M_2.$$

It is straightforward to show that the operator T as defined above is a linear bounded operator. Therefore, $\{\hat{\pi}(n)l_0, T\}$ is a Banach frame for M_2 with respect to L_2 with frame bounds $A = B = 1$.

3.2 Discrete phase space tomography

In [17] the formalism was applied to represent the states and the evolution of a quantum system in the phase space in the finite dimensional Hilbert space. For discrete systems we can define the finite translation operators $\hat{Q}$ and $\hat{V}$, which respectively generate the finite translation in position and momentum. Now by identifying the corresponding displacement operators, the discrete analogue of the phase space translation operator is given by:

$$\hat{U}(q, p) = \hat{Q}^q\hat{V}^p\exp(i\pi pq/N). \quad (3.9)$$

Here we can define the point operator as:

$$\hat{A}(q, p) = \frac{1}{(2N)^2}\sum_{n,m=0}^{2N-1}\hat{U}(m, k)\exp\left(-2\pi i\frac{(kq - mp)}{2N}\right), \quad (3.10)$$

or:

$$\hat{A}(\alpha) = \frac{1}{2N}\hat{Q}^q\hat{R}\hat{V}^{-p}\exp(i\pi pq/N), \quad (3.11)$$

where $\hat{R}$ is a parity operator and it is worth noting that the phase space point operators have been defined on a lattice with $2N \times 2N$ points, but it has been shown that there are only N^2 independent phase space point operators on the set $G_N = \{\alpha = (q, p); 0 \leq q, p \leq N - 1\}$.

Now in order to obtain the tomography equation via the wavelets transform in Banach space we can define: the group, the subgroup and the representation are respectively the finite Heisenberg group, its center and $U(\alpha)$. Then the wavelet transform with the test functional

$$l_0(O) = Tr(O) \text{ for any operator } O,$$

is given by

$$\mathcal{W}\hat{\rho} = \hat{\rho}(\alpha) = \langle \hat{\rho}, l_\alpha \rangle = Tr(\hat{U}^\dagger(\alpha)\hat{\rho}). \tag{3.12}$$

Since the representation is an irreducible representation, the inverse wavelet transform $\mathcal{M}$ will be the left inverse operator of wavelet transform $\mathcal{W}$:

$$\mathcal{M}\mathcal{W}\hat{\rho} = \sum_{\alpha \in G_N} \langle \hat{\rho}, l_\alpha \rangle b_\alpha = \sum_{\alpha \in G_N} \langle \hat{U}^\dagger(\alpha)\hat{\rho}, l_0 \rangle \hat{U}(\alpha) b_0. \tag{3.13}$$

We can obtain the tomography relation as follows:

$$\begin{aligned} \hat{\rho} &= 1/N \sum_{\alpha \in G_N} Tr(\hat{\rho}\hat{U}^\dagger(\alpha))\hat{U}(\alpha) \\ &= 4N \sum_{\alpha \in G_N} Tr(\hat{\rho}\hat{A}(\alpha))\hat{A}(\alpha), \end{aligned} \tag{3.14}$$

where $W(\alpha) = Tr(\hat{A}(\alpha)\hat{\rho})$ is a Wigner function, for the admissible $b_0 = I/N$.

Through the same choice as above for the vacuum vector and the test functions, we can get the atomic decomposition and the Banach frame for this example. Then we can show that:

The norms $||\hat{\rho}||_{M_2}$ and $||\{Tr(\hat{\rho}\hat{U}^\dagger(\alpha))\}||$ are equivalent since

$$\begin{aligned} ||\{Tr(\hat{U}^\dagger(\alpha)\hat{O})\}|| &= \left[\int Tr(\hat{U}^\dagger(\alpha)\hat{O})\overline{Tr(\hat{U}^\dagger(\alpha)\hat{O})} d\mu(\alpha)\right]^{\frac{1}{2}} \\ &= \left|\int tr(U^\dagger(\alpha)\hat{O}) tr(U(\alpha)\hat{O}^\dagger) d\mu(\alpha)\right|^{1/2} \\ &= \left|tr\left(\int tr(U^\dagger(\alpha)\hat{O}) U(\alpha) d\mu(\alpha)\hat{O}^\dagger\right)\right|^{1/2}, \end{aligned}$$

if we use the tomography relation (3.14) we obtain

$$||\{Tr(\hat{U}^\dagger(\alpha)\hat{O})\}|| = |tr(N\hat{O}\hat{O}^\dagger)|^{1/2} = \sqrt{N}||\hat{O}||.$$

Therefore they saturate the inequality (2.7) with the atomic bounds $A = B = \sqrt{N}$, provided that we use the Hilbert-Schmidt norm for the operator $\hat{O}$. Also if we use the relation (3.11), we have,

$$\hat{\rho} = \sum_\alpha Tr(\hat{\rho}\hat{U}^\dagger(\alpha))U\hat{(\alpha)} b_0,$$

hence $\{U\hat{(\alpha)} b_0, U\hat{(\alpha)} l_0\}$ is a linear atomic decomposition of M_2 with respect to L_2.

Finally, we can define the reconstruction operator T as follows:

$$T\{Tr(\hat{\rho}\hat{U}^\dagger(\alpha))\} = \sum_\alpha Tr(\hat{\rho}\hat{U}^\dagger(\alpha))U\hat{(\alpha)} = \hat{\rho}$$

for each $\hat{\rho} \in M_2$ therefore, $\{U\hat{(\alpha)} l_0, T\}$ is a Banach frame for M_2 with respect to L_2 with the frame bounds $A = B = \sqrt{N}$.

4 Quantum tomography with wavelet transform on homogeneous space (singular case)

4.1 Homodyne tomography

The problem of measuring the density matrix $\hat{\rho}$ of radiation has been extensively considered both experimentally and theoretically [22]. The Homodyne tomography is rather the only method that can be used to achieve such measurement [19]. This method is based on the idea that the density matrix for the radiation states can be evaluated in the optical Homodyne experiments by using a collection of the quadrature probability distribution.

Now we wish to obtain the tomography equation via the wavelets transform in the Banach space. Obviously the group is the Heisenberg group H^R. Since the representation of H^R fails to be square-integrable, according to Stone-Von Neumann [23], we can factor out the center H^R and consider only the factor space.

For the vacuum vector and the test functional, we need to choose the identity operator and $l_0(O) = Tr[O]$ for any operator O, respectively. Then the wavelet formula for the left translation is given by:

$$\mathcal{W}\hat{\rho} = \hat{\rho}(\alpha) = \langle \hat{\rho}, l_\alpha \rangle = Tr(\hat{\rho}\hat{U}(\alpha)^\dagger), \tag{4.1}$$

where $\hat{U}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is a displacement operator. But the above reference state is not admissible. Thus according to the prescription of singular cases, we have to select a probe vector $p_0 \in \mathcal{B}$ in which the equation (2.5) is non-zero and finite. In this case, the probe vector can be selected as:

$$p_0 = \int \mid \alpha \rangle \langle \alpha \mid e^{(\frac{-|\alpha|^2}{\Delta})} \frac{d^2\alpha}{\pi}, \tag{4.2}$$

where Δ is non-zero and finite and $b_0 \in \mathcal{B}$ is an identity operator. Since the representation is irreducible and

$$\begin{aligned} &C(b_0, p_0) = \\ &\left\langle \int_\alpha \left\langle U(\alpha)^\dagger \int_{\alpha'} |\alpha'\rangle \langle \alpha'| e^{\left(\frac{-|\alpha'|^2}{\Delta}\right)} \frac{d^2\alpha'}{\pi}, l_0 \right\rangle U(\alpha) \frac{d^2\alpha}{\pi}, l_0 \right\rangle \\ &= \int_\alpha \int_{\alpha'} tr\left(U^\dagger(\alpha) \mid \alpha'\rangle\langle\alpha' \mid e^{\left(\frac{-|\alpha'|^2}{\Delta}\right)} \frac{d^2\alpha'}{\pi}\right) \\ &\qquad \times tr(U(\alpha)) \frac{d^2(\alpha)}{\pi} = \Delta, \end{aligned}$$

where $tr(U(\alpha)) = \pi\delta^2(\alpha)$ then the inverse wavelet transform in $\mathcal{M}$ is a left inverse operator on $\mathcal{B}$ for the wavelet transform $\mathcal{W}$:

$$\mathcal{M}\mathcal{W}\hat{\rho} = \int d\mu(\alpha)\langle \hat{\rho}, l_\alpha \rangle b_\alpha = \int d\mu(\alpha) Tr(\hat{\rho}\hat{U}^\dagger(\alpha))\hat{U}(\alpha) b_0, \tag{4.3}$$

where $\mathrm{d}\mu(\alpha) = \frac{d^2\alpha}{\pi}$ is an invariant measure. For $b_0 = I$, the reconstruction procedure of wavelet transform (4.3)

Table 1. The correspondence between different characteristic functions and representations.

charasteristic function	representation	quasi distribution function
$Tr(\hat{\rho}\hat{U}_{an}(\alpha))$	$\hat{U}_{an}(\alpha) = e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}$	P-function
$Tr(\hat{\rho}\hat{U}_n(\alpha))$	$\hat{U}_n(\alpha) = e^{-\alpha^* a}e^{\alpha a^\dagger}$	Q-function
$Tr(\hat{\rho}\hat{U}_h(\alpha))$	$\hat{U}_h(\nu) = e^{-\nu^* b}e^{\nu b^\dagger}$	Husimi function
	$(\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger$ and $\mu^2 - \nu^2 = 1)$	
$Tr(\hat{\rho}\hat{U}_s(\xi.\eta))$	$\hat{U}_s(\xi.\eta) = e^{i\xi\hat{q}}e^{i\eta\hat{p}}$	Standard-ordered function
$Tr(\hat{\rho}\hat{U}_{as}(\xi.\eta))$	$\hat{U}_{as}(\xi.\eta) = e^{i\eta\hat{p}}e^{i\xi\hat{q}}$	Antistandard-ordered function

leads to the tomography relation:

$$\hat{\rho} = \int_{\mathcal{C}} \frac{d^2\alpha}{\pi} Tr[\hat{\rho}\hat{U}^\dagger(\alpha)]\hat{U}(\alpha). \tag{4.4}$$

In tomography relation (4.4), the expression $Tr(\hat{\rho}\hat{U}^\dagger(\alpha))$ is a Wigner characteristic function. We can also obtain other quasi-distribution characteristic functions by choosing different representations according to Table 1.

For the complex Fourier transform of the displacement operator $\hat{U}$ [18]

$$\hat{\mathcal{U}}(\alpha) = \int \frac{d^2\xi}{\pi}\hat{U}(\xi)\exp(\alpha\xi^* - \alpha^*\xi), \tag{4.5}$$

the expansion of the operator in terms of the operator $\hat{\mathcal{U}}(\alpha)$ is given by

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} W(\alpha)\hat{\mathcal{U}}(\alpha), \tag{4.6}$$

where $W(\alpha)$ is a Wigner function. Also by defining the complex Fourier transform for each of the above representations, we can get the tomography relation for each quasi-distribution. Now we try to obtain the atomic decomposition and the Banach frame for this example. Applying the same choice of the vacuum vector and the test functional we can show that the required atomic decomposition conditions are satisfied by the atomic bounds $A = B = 1$, since we have

$$\begin{aligned} ||\{Tr(\hat{U}^\dagger(\alpha)\hat{O})\}|| &= \left[\int Tr(\hat{U}^\dagger(\alpha)\hat{O})\overline{Tr(\hat{U}^\dagger(\alpha)\hat{O})}d\mu(\alpha)\right]^{\frac{1}{2}} \\ &= \left|\int tr(U^\dagger(\alpha)\hat{O})tr(U(\alpha)\hat{O}^\dagger)d\mu(\alpha)\right|^{1/2} \\ &= \left|tr\left(\int tr(U^\dagger(\alpha)\hat{O})U(\alpha)d\mu(\alpha)\hat{O}^\dagger\right)\right|^{1/2}, \end{aligned}$$

if we use the tomography relation (4.4) we obtain

$$||\{Tr(\hat{U}^\dagger(\alpha)\hat{O})\}|| = |tr(\hat{O}\hat{O}^\dagger)|^{1/2} = ||\hat{O}||.$$

Therefore, $\{\hat{U}(\alpha)b_0, \hat{U}(\alpha)l_0\}$ is a linear atomic decomposition of M_2 with respect to L_2. Similarly, by using the relation (4.4) and definition T, $\{\hat{U}(\alpha)b_0, T\}$ is a Banach frame for M_2 with respect to L_2 with frame bounds $A = B = 1$. We can also generalize the single mode Homodyne tomography to a multimode state. In the wavelet notation, the irreducible representation is $\hat{U} = \hat{U}_0 \bigotimes \hat{U}_1 \bigotimes ... \bigotimes \hat{U}_m$, with $\hat{U}_j = \exp(z_j\hat{a}_j^\dagger - z_j^*\hat{a}_j)$, and the tomography formula with the vacuum vector $b_0 = \hat{I} \bigotimes \hat{I} \bigotimes ... \bigotimes \hat{I}$ is given by

$$\hat{\rho} = \int_{\mathcal{C}} \frac{d^2z_0}{\pi}\int_{\mathcal{C}} \frac{d^2z_1}{\pi}\cdots\int_{\mathcal{C}} \frac{d^2z_m}{\pi} \times Tr[\hat{\rho}\hat{U}^\dagger(z_0, z_1, ..., z_m)]\hat{U}(z_0, z_1, ..., z_m). \tag{4.7}$$

The atomic decomposition and the Banach frame is similar to a one mode Homodyne, and the frame bounds A, B are equal to the identity.

4.2 Phase space tomography

Any marginal distribution is defined as the Fourier transform of the characteristic function $\mathcal{W}(X, \mu, \nu) = \int dk e^{-ikX}\langle e^{ik(\mu\hat{q}+\nu\hat{p})}\rangle$. This marginal distribution is related to the state of the quantum system which is expressed in terms of its Wigner function $W(q, p)$ [20], as follows

$$\mathcal{W}(X, \mu, \nu) = \int dk e^{-ik(X-\mu\hat{q}-\nu\hat{p})}W(q, p)\frac{dkdqdp}{(2\pi)^2}. \tag{4.8}$$

Now we can obtain the tomography equation via wavelets transform in the Banach space. Obviously the group is the Heisenberg group in phase space. For the vacuum vector and the test functional we need to choose the identity operator and $l_0(O) = Tr[O]$ for any operator O, respectively. If we apply the reduced wavelet transform for the left translation $\hat{U}(\mu, \nu) = e^{-i(\mu\hat{q}+\nu\hat{p})}$, we have:

$$\mathcal{W}\hat{\rho} = \hat{\rho}(\mu, \nu) = \langle\hat{\rho}, l_{(\mu,\nu)}\rangle = Tr(\hat{\rho}\hat{U}^\dagger(\mu, \nu)). \tag{4.9}$$

The vacuum vector $b_0 = \hat{I}$ is not admissible, then we choose a probe vector with coherent states in the phase space [8] which is a shifted Gaussian wave packet:

$$\eta_{\sigma(q,p)}(x) = (\pi^{-1/4})\exp\left[-i\left(\frac{q}{2} - x\right)p\right]\exp\left[-\frac{(x-q)^2}{2}\right] \tag{4.10}$$

$$p_0 = \int \mid \eta_{\sigma(q,p)}\rangle\langle\eta_{\sigma(q,p)} \mid \exp\left[\frac{-(q^2+p^2)}{\Delta}\right]dqdp, \tag{4.11}$$

and the singularity condition gives $C(b_0, p_0) = \Delta$.

Since the representation is irreducible, the inverse wavelet transform $\mathcal{M}$ is a left inverse operator on $\mathcal{B}$ for the wavelet transform $\mathcal{W}$:

$$\mathcal{M}\mathcal{W}\hat{\rho} = \int d\mu(\mu,\nu)\langle\hat{\rho}, l_{(\mu,\nu)}\rangle b_{(\mu,\nu)} = \int d\mu d\nu Tr[\hat{\rho}\hat{U}^{\dagger}(\mu,\nu)]\hat{U}(\mu,\nu)b_0, \quad (4.12)$$

Then for $b_0 = \hat{I}$, we have:

$$\hat{\rho} = \int d\mu d\nu Tr[\hat{\rho}\hat{R}^{\dagger}(\mu,\nu)]\hat{R}(\mu,\nu) = \int d\mu\nu Tr[\hat{\rho}e^{i(\mu\hat{q}+\nu\hat{p})}]e^{-i(\mu\hat{q}+\nu\hat{p})}. \quad (4.13)$$

With a simple calculation, we can obtain the tomography relation as follows:

$$\hat{\rho} = \int dX d\mu d\nu \mathcal{W}(X,\mu,\nu)\hat{K}_{\mu\nu}, \quad (4.14)$$

where the kernel operator has the form:

$$\hat{K}_{\mu\nu} = \frac{1}{2\pi}e^{iX}e^{i\mu\nu}e^{-i\nu\hat{p}}e^{-i\mu\hat{q}}. \quad (4.15)$$

The atomic decomposition and the Banach frame are similar to a one mode Homodyne, and the frame bounds A, B are equal to the identity.

4.3 SU(1,1) tomography

The Lie algebra $su(1,1)$ of the $SU(1,1)$ group is spanned by the operators $\hat{K}_+, \hat{K}_-, \hat{K}_z$. The Casimir invariant operator that labels all the unitary irreducible representations of the group is given by $(\hat{K}_z)^2 - 1/2(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+) = k(k+1)\hat{I}$, where the eigenvalue k is also called the Bargmann index [21].

Now we obtain the tomography equation via the wavelets transform in the Banach space. Obviously the group is $SU(1,1)$, and the subgroup is U(1) with the reference state $b_o = I$. By choosing

$$\hat{\pi}^*(x) = \hat{u}^{\dagger}(x)\hat{K}_z\hat{u}(x), \quad (4.16)$$

$$\hat{U}(x) = \{(-1)^{\hat{K}_z}e^{\theta(e^{i\phi}\hat{K}_+ - e^{-i\phi}\hat{K}_-)}, \hat{K}_z\}_+, \quad (4.17)$$

where $\hat{u}(\theta,\phi) \equiv e^{-i\theta/2(e^{-i\phi}\hat{K}_+ + e^{i\phi}\hat{K}_-)}$[21], the wavelet transform is :

$$\mathcal{W}\hat{\rho} = \hat{\rho}(x) = \langle\hat{U}(x^{-1})\hat{\rho}, l_0\rangle = \langle\hat{\rho}, \pi^*(x)l_0\rangle = Tr[\hat{U}^{\dagger}(x)\rho], \quad (4.18)$$

and the inverse wavelet transform is shown by

$$\mathcal{M}\mathcal{W}[\hat{\rho}] = \int_X \hat{\rho}(x)b_x d\mu(x) = \int_X \hat{\rho}(x)\pi(x)b_0 d\mu(x), \quad (4.19)$$

where $\hat{\pi}^*(x)$ is dual of $\hat{U}(x)$. The reference state is $b_0 = I$ but this reference state is not admissible, therefore, again according to the prescription of singular cases, we have to select a probe vector $p_0 \in \mathcal{B}$ in which equation (2.5) is non-zero and finite. In this case, the probe vector can be selected as

$$p_0 = \sum_r b^r \mid r\rangle\langle r \mid, \quad (4.20)$$

where this probe vector is similar to the thermal states described by the density operator ρ_T

$$\rho_T = \frac{1}{1+\widetilde{N}}\sum_r \left(\frac{\widetilde{N}}{1+\widetilde{N}}\right)^r \mid r\rangle\langle r \mid, \quad (4.21)$$

with $\widetilde{N} \equiv \langle\rho_T N\rangle = \frac{1}{\exp(\hbar\omega/KT)-1}$, and $N = a^{\dagger}a$. At high temperatures the thermal state is proportional with the identity operator. Since the representation is irreducible and $C(b_0, p_0) = \frac{1}{1-b}$, the inverse wavelet transform in $\mathcal{M}$ is a left inverse operator on $\mathcal{B}$. Then the tomography formula for $SU(1,1)$ group is given by

$$\hat{\rho} = \frac{1}{\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \tanh(\theta) \times Tr[\{(-\mathbf{1})^{\hat{\mathbf{K}}_{\mathbf{z}}}e^{\theta(e^{-i\phi}\hat{\mathbf{K}}_- - e^{i\phi}\hat{\mathbf{K}}_+)}, \hat{\mathbf{K}}_{\mathbf{z}}\}_+\hat{\rho}] \times e^{i\theta/2(e^{-i\phi}\hat{K}_+ + e^{i\phi}\hat{K}_-)}\hat{K}_z e^{-i\theta/2(e^{-i\phi}\hat{K}_+ + e^{i\phi}\hat{K}_-)}. \quad (4.22)$$

Here we obtain the atomic decomposition and the Banach frame for this example. The norms $||\hat{\rho}||_{M_2}$ and $||\{Tr(\hat{\rho}\hat{B}^{\dagger}(x))\}||$ are equivalent in the sense that they satisfy the inequality (2.7) with the atomic bounds $A = B = 1$, provided that we use the Hilbert-Schmidt norm for the operator $\hat{\rho}$

$$||Tr(\hat{\rho}\hat{U}^{\dagger}(x))||^2 = \int d\mu(x)Tr(\hat{\rho}\hat{U}^{\dagger}(x))\overline{Tr(\hat{\rho}\hat{\pi}^*(x))}, \quad (4.23)$$

Since the dual couple $\hat{U}(x)$ and $\hat{\pi}^*(x)$ satisfy the orthogonality relation [21]:

$$\delta_{mk}\delta_{nl} = \int d\mu(x)\langle m|B^{\dagger}(x)|n\rangle\langle l|C(x)|k\rangle,$$

then;

$$||Tr(\hat{\rho}\hat{U}^{\dagger}(x))||^2 = \int d\mu(x)\rho_{mn}U^*_{mn}(x)\rho^*_{kl}\pi_{kl}(x) = ||\hat{\rho}||^2,$$

and if we use the relation (4.22), we have:

$$\hat{\rho} = \int d\mu(x)Tr(\hat{\rho}\hat{U}^{\dagger}(x))\hat{\pi}^*(x),$$

Therefore, $\{\hat{\pi}^*(x)b_0, \hat{\pi}^*(x)l_0\}$ is an atomic decomposition of M_2 with respect to L_2 with the atomic bounds $A = B = 1$. Similar to the atomic decomposition, $\{\hat{\pi}^*(x)l_0, T\}$ is a Banach frame for the coorbit space of the operators with respect to L_2 with frame bounds $A = B = 1$.

4.4 Tomography of a free particle

In this section we intend to consider the tomography of a free particle. For simplicity we suppose a particle with a unit mass and use the normalized unit $\hbar/2 = 1$, so the free Hamiltonian is given by $\hat{H}_F = \hat{p}^2$. The base is constituted by the set of operators $\hat{R}(x,\tau) = e^{-i\hat{p}^2\tau}|x\rangle\langle x|e^{i\hat{p}^2\tau}$ [18].

Again we try to obtain the tomography equation via the wavelets transform in the Banach space. Obviously the group is $\{\hat{P}, \hat{X}, \hat{P}^2, I\}$ and the subgroup is $\{\hat{X}, I\}$. The relevant representation for this example is an adjoint representation:

$$\hat{\pi}(x,\tau)\hat{\rho} = \hat{U}(x,\tau)\hat{\rho}\hat{U}^{-1}(x,\tau) \ \ with\ \hat{U}(x,\tau) = e^{-i\hat{P}^2\tau}\hat{D}(x).$$

In this representation, $\hat{D}(x)$ is a translation operator, such that $\hat{D}(x)|0\rangle = |x\rangle$, where $|x\rangle$ is eigenstate of the position operator and $\hat{P}$ is the momentum operator. On the other hand if we define:

$$\langle\hat{\rho}, l_0\rangle = l_0(\hat{\rho}) = Tr(\hat{\rho} \mid 0\rangle\langle 0 \mid),$$

the wavelet transform formula is given by:

$$\mathcal{W}\hat{\rho} = \hat{\rho}(x,\tau) = \langle\hat{\rho}, l_{(x,\tau)}\rangle = Tr(\hat{\rho}e^{-i\hat{P}^2\tau} \mid x >< x \mid e^{i\hat{P}^2\tau}). \tag{4.24}$$

Also the inverse wavelet transform $\mathcal{M}$ associated with the wavelet transform $\mathcal{W}$ is:

$$\begin{aligned}\mathcal{M}\mathcal{W}[\hat{\rho}] &= \int d\mu(x,\tau)\langle\hat{\rho}, l_{(x,\tau)} > b_{(x,\tau)} \\ &= \int dx d\tau Tr[\hat{\rho}e^{-i\hat{P}^2\tau} \mid x\rangle\langle x \mid e^{i\hat{P}^2\tau}]\pi(\hat{x,\tau})b_0.\end{aligned} \tag{4.25}$$

The vacuum vector is defined as $b_0 = |0\rangle\langle 0|$, but it is not admissible. Thus according to the prescription of the singular cases, we have to select a probe vector $p_0 \in \mathcal{B}$ in which equation (2.5) is non-zero and finite. In this case, the probe vector can be selected as

$$p_0 = \mid D\rangle\langle D \mid, \tag{4.26}$$

where $\langle D \mid p\rangle = e^{-\frac{p^2}{\delta}}$ such that δ is a real and nonzero parameter. Using the bi-orthogonality and the following relations [18] (for $|j\rangle$, $j = p_1, p_2, p_3, p_4$)

$$\begin{aligned}&\int_R\int_R dx\, d\tau\, \langle p_1|\hat{R}(x,\tau)|p_2\rangle\, \langle p_3|\hat{R}(x,\tau)|p_4\rangle \\ &= \int_R\int_R dx\, d\tau\, e^{-i\tau(p_2^2-p_1^2+p_3^2-p_4^2)}\, \langle p_1|x\rangle\langle x|p_2\rangle\, \langle p_3|x\rangle\langle x|p_4\rangle \\ &\quad = \int_R\int_R dx\, d\tau\, e^{-i\tau(p_2^2-p_1^2+p_3^2-p_4^2)}\, e^{ix(p_1-p_2+p_3-p_4)} \\ &\qquad = \delta(p_1-p_3)\,\delta(p_2-p_4)\end{aligned} \tag{4.27}$$

we can show that the constant coefficient on the left of (2.5) is $C(b_0,p_0) = \delta/2\sqrt{\pi}$ and finally the reconstruction procedure of the wavelet transform leads to the tomography relation as follow:

$$\hat{\rho} = \int_R\int_R dx\, d\tau\, p(x,\tau)\, \hat{R}(x,\tau), \tag{4.28}$$

where $p(x,\tau) = \mathrm{Tr}[\hat{\rho}\, \hat{R}(x,\tau)]$ is the probability density of the particle to be at position x at time τ. In order to obtain the atomic decomposition and the Banach frame for this example by the same choice of the vacuum vector and the test functional, we can show that the required atomic decomposition conditions are satisfied by atomic bounds $A = B = 1$. Therefore, $\{\hat{\pi}(x,\tau)b_0, \hat{\pi}(x,\tau)l_0\}$ is a linear atomic decomposition of M_2 with respect to L_2. Similarly, by using the relation (4.28) and the definition T, $\{\hat{\pi}(x,\tau)b_0, T\}$ is a Banach frame for M_2 with respect to L_2 with frame bounds $A = B = 1$.

4.5 Wavelet transform and Q-function

Let $g \in \mathcal{L}^2(R)$ with $\parallel g \parallel = 1$ and the time-frequency translation of g be:

$$g^{[x_1,x_2]}(t) = e^{2\pi i t x_1}g(t+x_2) = U[x_1,x_2,0]g(t), \tag{4.29}$$

where U is the unitary irreducible representation of the Heisenberg group H^R. For an arbitrary function $f \in L^2(R)$, the pure state sampling [30] associated with it, can be given in terms of inner product as:

$$F(x_1,x_2) = \langle f, g^{[x_1,x_2]}\rangle, \tag{4.30}$$

where $g^{[x_1,x_2]} = U[x_1,x_2]g(t)$ is a coherent state. In pure states, the square of sampling is Q-function.

Now we try to obtain Q-function via wavelet and show that the wavelet transform in the Banach space is a Q-function. The group is the Heisenberg group and the subgroup is the identity and the representation is an adjoint. Then the wavelet transform is given by:

$$\mathcal{W}\hat{\rho} = \hat{\rho}(\alpha) = \langle\hat{\rho}, l_\alpha\rangle = \langle\hat{\pi}(\alpha)^\dagger\hat{\rho}, l_0\rangle. \tag{4.31}$$

On the other hand if we choose:

$$\langle\hat{\rho}, l_0\rangle = l_0(\hat{\rho}) = Tr[\hat{\rho}|0\rangle\langle 0|], \tag{4.32}$$

then the wavelet transform for the adjoint representation is given by:

$$\begin{aligned}\mathcal{W}\hat{\rho} &= \hat{\rho}(\alpha) = Tr\{\hat{\pi}(g)^\dagger\hat{\rho}|0\rangle\langle 0|\} \\ &= \langle 0|\hat{U}(\alpha)^\dagger(\hat{\rho})\hat{U}(\alpha)|0\rangle = \langle\alpha|\hat{\rho}|\alpha\rangle = Q(\alpha).\end{aligned} \tag{4.33}$$

5 Conclusion

The intimate connection between the generalized wavelets and its inverse in Banach space connected with the homogeneous manifolds, and quantum tomography of the density operators is revealed. This connection is shown in some well known physical quantum tomographic examples in details, further we introduce the Banach frame and the atomic decomposition for each. We have also a established a connection between the wavelet formalism on the Banach space and Q-function.

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