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Exact calculation of robustness of entanglement via convex semi-definite programming

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February 1, 2008

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Abstract

In general the calculation of robustness of entanglement for the mixed entangled quantum states is rather difficult to handle analytically. Using the the convex semi-definite programming method, the robustness of entanglement of some mixed entangled quantum states such as: $2 \otimes 2$ Bell decomposable (BD) states, a generic two qubit state in Wootters basis, iso-concurrence decomposable states, $2 \otimes 3$ Bell decomposable states, $d \otimes d$ Werner and isotropic states, a one parameter $3 \otimes 3$ state and finally multi partite isotropic state, is calculated exactly, where thus obtained results are in agreement with those of :$2 \otimes 2$ density matrices, already calculated by one of the authors in [1, 2]. Also an analytic expression is given for separable states that wipe out all entanglement and it is further shown that they are on the boundary of separable states as pointed out in [3]. **Keywords: Robustness of entanglement, Semi-definite programming, Bell decomposable states, Werner and isotropic states.**

PACs Index: 03.65.Ud

1 INTRODUCTION

Quantum entanglement has recently been attracted much attention as a potential resource for communication and information processing [4, 5]. Entanglement usually arises from quantum correlations between separated subsystems which can not be created by local actions on each subsystem. By definition, a mixed state ρ of a bipartite system is said to be separable (non entangled) if it can be written as a convex combination of product states

$$\rho = \sum_i w_i \, \rho_i^{(1)} \otimes \rho_i^{(2)}, \qquad w_i \geq 0, \quad \sum_i w_i = 1,$$

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ are states of subsystems 1 and 2, respectively. Although, in the case of pure states of bipartite systems it is easy to check whether a given state is, or is not entangled, the question is yet an open problem in the case of mixed states. Therefore having a measure to quantify entanglement of mixed states is likely to be valuable and several measures of entanglement have been proposed [6, 7, 8, 9].

One useful quantity introduced in [10] as a measure of entanglement is robustness of entanglement. It corresponds to the minimal amount of mixing with locally prepared states which washes out all entanglement. An analytical expression for pure states of bipartite systems by using partial transpose has been given in [10]. Meanwhile the robustness of entanglement has been calculated for a Werner states. Moreover, in [11] Vidal and Werner have computed the robustness of entanglement for density operators with symmetry. In [12] Rudolph, using cross norm has clarified the relationship of the greatest cross norm with the robustness of entanglement and has determined the value of the greatest cross norm for Bell diagonal states. A geometrical interpretation of robustness is given in [3] and it is pointed out that two corresponding separable states needed to wipe out all entanglement are necessarily on the boundary of separable set. On the other hand, the robustness of entanglement of few mixed quantum states such as: $2 \otimes 2$ Bell decomposable (BD) states and a generic two qubit state in Wootters basis is already calculated by one of the authors in[1, 2]. In Ref [13] has characterized the

robustness of entanglement, and its relation to the permutation symmetries, for the basic set of eight entangled three particle states of spin-1/2 objects. Authors in [14], have studied the robustness of multi-party entanglement under local decoherence, modeled by partially depolarizing channels acting independently on each subsystem. Unfortunately, in general, the above mentioned quantity as the most proposed measures of entanglement involves exteremization which is difficult to handel analytically.

On the other hand, over the past years, semidefinite programming (SDP) has been recognized as valuable numerical tools for control system analysis and design. In (SDP) one minimizes a linear function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. SDP, has been studied (under various names) as far back as the 1940s. Subsequent research in semidefinite programming during the 1990s was driven by applications in combinatorial optimization[15], communications and signal processing [16, 17, 18], and other areas of engineering[19]. Although semidefinite programming is designed to be applied in numerical methods it can be used for analytic computations, too. Some authors try to use the SDP to construct an explicit entanglement witness [20, 21]. Kitaev used semidefinite programming duality to prove the impossibility of quantum coin flipping [22], and Rains gave bounds on distillable entanglement using semidefinite programming [23]. In the context of quantum computation, Barnum, Saks and Szegedy reformulated quantum query complexity in terms of a semidefinite program [24]. The problem of finding the optimal measurement to distinguish between a set of quantum states was first formulated as a semidefinite program in 1972 by Holevo4, who gave optimality conditions equivalent to the complementary slackness conditions [25]. Recently, Eldar, Megretski and Verghese showed that the optimal measurements can be found efficiently by solving the dual followed by the use of linear programming [26]. Also in [27] used semidefinite programming to show that the standard algorithm implements the optimal set of measurements. All of the above mentioned applications indicate that the method of SDP is very useful.

Here in this paper, by using the the convex semi-definite programming method, the robustness of entanglement of some mixed entangled quantum states such as: $2 \otimes 2$ Bell decomposable (BD) states, a generic two qubit state in Wootters basis, iso-concurrence decomposable states, $2 \otimes 3$ Bell decomposable states, $d \otimes d$ Werner and isotropic states, a one parameter $3 \otimes 3$ state and finally multi partite isotropic state, is calculated exactly, where thus obtained results are in agreement with those of: $2 \otimes 2$ density matrices, already calculated by one of the authors in[1, 2]. Also an analytic expression is given for separable states that wipe out all entanglement and it is further shown that they are on the boundary of separable states as pointed out in [3].

The paper is organized as follows:

In sections 2 and 3 we give brief review of semidefinite programming and robustness of entanglement, respectively. In section 4, by using the semi-definite programing method we calculate the robustness of entanglement of some mixed entangled quantum states, such as: $2 \otimes 2$ Bell decomposable (BD) states, a generic two qubit state in Wootters basis, iso-concurrence decomposable states, $2 \otimes 3$ Bell decomposable states, $d \otimes d$ Werner and isotropic states, a one parameter $3 \otimes 3$ state and finally multi partite isotropic state. The paper is ended with a brief conclusion.

2   Semi-definite programming

A semidefinite programming(SDP) is a particular type of convex optimization problem [28]. A semidefinite programming problem requires minimizing a linear function subject to a linear matrix inequality (LMI) constraint [29]:

$$\begin{aligned} \text{minimize} \quad & \mathcal{P} = c^T x \\ \text{subject to} \quad & F(x) \geq 0, \end{aligned}$$

(2-1)

where c is a given vector, $x = (x_1, ..., x_n)$, and $F(x) = F_0 + \sum_i x_i F_i$, for some fixed hermitian matrices F_i. The inequality sign in $F(x) \geq 0$ means that $F(x)$ is positive semidefinite.

This problem is called the primal problem. Vectors x that satisfy the constraint $F(x) \leq 0$ are called primal feasible points, and if they satisfy $F(x) > 0$ they are called strictly feasible points. The minimal objective value $c^T x$ is by convention denoted as $\mathcal{P}^*$ and is called the primal optimal value.

The minimization is performed over the vector x, whose component are the variables of the problem. The vector x which satisfies the LMI, is called a feasible solution, and the set of all feasible solutions, is called the feasible set.

A very important property of a (SDP) is its convexity, since the feasible set defined by the above constraints is convex. For this reason, semidefinite programming has a nice duality structure, with, the associated dual program being:

$$\text{maximize} \quad -Tr[F_0 Z]$$
$$Z \geq 0 \tag{2-2}$$
$$Tr[F_i Z] = c_i.$$

Here the variable is the real symmetric (or Hermitean) matrix Z, and the data c, F_i are the same as in the primal problem. Correspondingly, matrices Z satisfying the constraints are called dual feasible (or strictly dual feasible if $Z > 0$). The maximal objective value $-Tr F_0 Z$, the dual optimal value, is denoted as d^*.

The objective value of a primal feasible point is an upper bound on $\mathcal{P}^*$, and the objective value of a dual feasible point is a lower bound on d^*. The main reason why one is interested in the dual problem is that one can prove that, under relatively mild assumptions, $\mathcal{P}^* = d^*$. This holds, for example, if either the primal problem or the dual problem are strictly feasible, i.e. there either exist strictly primal feasible points or strictly dual feasible points. If this or other conditions are not fulfilled, we still have that $d^* \leq \mathcal{P}^*$. Furthermore, when both the primal and dual problem are strictly feasible, one proves the following optimality condition on x:

A primal feasible x and a dual feasible Z are optimal which is denoted by $\hat{x}$ and $\hat{Z}$ if and

only if

$$F(\hat{x})\hat{Z} = \hat{Z}F(\hat{x}) = 0. \tag{2-3}$$

This latter condition is called the complementary slackness condition.

In one way or another, numerical methods for solving SDP problems always exploit the inequality $d \le d^* \le \mathcal{P}^* \le \mathcal{P}$, where d and $\mathcal{P}$ are the objective values for any dual feasible point and primal feasible point, respectively. The difference

$$\mathcal{P} - d = c^T x + Tr[F_0 Z] = Tr[F(x)Z] \ge 0 \tag{2-4}$$

is called the duality gap, and the optimal value $\mathcal{P}^*$ is always bracketed inside the interval $[d, \mathcal{P}]$. These numerical methods try to minimize the duality gap by subsequently choosing better feasible points. Under the requirements of the above-mentioned theorem, the duality gap can be made arbitrarily small (as far as numerical precision allows).

Equation (2-3) together with (2-2) and (2-1) constitute a set of necessary and sufficient conditions for $\hat{x}$ to be an optimal solution to the problem of (2-1), when both the primal and the dual are strictly feasible.

3   Robustness of entanglement

According to [10] for a given entangled state ρ and separable state ρ'', a new density matrix ρ'_s) can be constructed as,

$$\rho'_s = \frac{1}{s+1}(\rho + s\rho''), \quad s \ge 0, \tag{3-5}$$

where it can be either entangled or separable. It was pointed that there always exits the minimal **s** corresponding to one ρ'' such that ρ'_s is separable. This minimal s is called the robustness of ρ relative to ρ'', denoted by $R(\rho||\rho'')$. The absolute robustness of ρ is defined as the quantity,

$$R(\rho||S) \equiv min R(\rho||\rho''), \quad \rho'' \in S, \tag{3-6}$$

where S is the set of separable states.

Du et al. in [3] have given a geometrical interpretation of robustness and pointed that if **s** in Eq. (3-5) is minimal among all separable states ρ'', i.e. **s** is the absolute robustness of ρ, then ρ'' and ρ'_s in Eq. (3-5) are necessarily on the boundary of the separable states.

4 Robustness of entanglement via semi-definite programming

Unfortunately, the above mentioned quantity as the most proposed measure of entanglement involves extremization which are difficult to handle analytically. One of authors have given analytical expression for the robustness of entanglement of some $2 \otimes 2$ density matrices in [1, 2], here in this section we try to obtain robustness of entanglement for many categories of states, namely, $2 \otimes 2$ Bell decomposable (BD) states, a generic two qubit state in Wootters basis, iso-concurrence decomposable states, $2 \otimes 3$ Bell decomposable states, $d \otimes d$ Werner and isotropic states, a one parameter $3 \otimes 3$ state and finally multi partite isotropic state, via semi-definite programming method. As we will see in the following, besides the elegance of semi-definite programming in the calculation of the robustness of entanglement, also we do not need to define any kind of norm for mixed quantum states in order to calculate their robustness of entanglement as it is done in Ref. [1, 2].

4.1 Robustness of entanglement for Bell-decomposable state

A Bell decomposable (BD) state is defined by:

$$\rho = \sum_{i=1}^{4} p_i \left| \psi_i \right\rangle \left\langle \psi_i \right|, \qquad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{4} p_i = 1, \tag{4-7}$$

where $\left| \psi_i \right\rangle$ is Bell state, given by:

$$\left| \psi_1 \right\rangle = \left| \phi^+ \right\rangle = \frac{1}{\sqrt{2}} (\left| \uparrow\uparrow \right\rangle + \left| \downarrow\downarrow \right\rangle), \tag{4-8}$$

$$|\psi_2\rangle = \left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle), \tag{4-9}$$

$$|\psi_3\rangle = \left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \tag{4-10}$$

$$|\psi_4\rangle = \left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{4-11}$$

In terms of Pauli's matrices, ρ can be written as,

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} t_i\sigma_i \otimes \sigma_i), \tag{4-12}$$

where

$$\begin{aligned}
t_1 &= p_1 - p_2 + p_3 - p_4, \\
t_2 &= -p_1 + p_2 + p_3 - p_4, \\
t_3 &= p_1 + p_2 - p_3 - p_4.
\end{aligned} \tag{4-13}$$

From the positivity of ρ we get

$$\begin{aligned}
1 + t_1 - t_2 + t_3 &\geq 0, \\
1 - t_1 + t_2 + t_3 &\geq 0, \\
1 + t_1 + t_2 - t_3 &\geq 0, \\
1 - t_1 - t_2 - t_3 &\geq 0.
\end{aligned} \tag{4-14}$$

These equations form a tetrahedral with its vertices located at $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, -1, -1)$ [30]. In fact these vertices denote the Bell states given in Eqs. (4-8) to (4-11), respectively.

On the other hand ρ given in Eq. (4-12) is separable if and only if t_i satisfy Eq. (4-14) and,

$$\begin{aligned}
1 + t_1 + t_2 + t_3 &\geq 0, \\
1 - t_1 - t_2 + t_3 &\geq 0, \\
1 + t_1 - t_2 - t_3 &\geq 0, \\
1 - t_1 + t_2 - t_3 &\geq 0.
\end{aligned} \tag{4-15}$$

Inequalities (4-14) and (4-15) form an octahedral with its vertices located at $O_1^\pm = (\pm 1, 0, 0)$, $O_2^\pm = (0, \pm 1, 0)$ and $O_3^\pm = (0, 0, \pm 1)$. So, tetrahedral is divided into five regions. Central regions, defined by octahedral, are separable states $(p_k \leq \frac{1}{2})$. There are also four smaller equivalent tetrahedral corresponding to entangled states$(p_k > \frac{1}{2}$ for only one of $k = 1, ..., 4)$, where $p_k = \frac{1}{2}$ denote to boundary between separable and entangled region. Each tetrahedral takes one Bell state as one of its vertices. Three other vertices of each tetrahedral form a triangle which is its common face with octahedral (See Fig. 1).

Here in this section we evaluate robustness of entanglement for all BD-states with semi-definite programming method, and we give an explicit form the corresponding ρ_s' and ρ'' which are on the boundary of the separable states.

Now for a given BD density matrix

$$\rho = \sum_{i=1}^4 p_i |\psi_i\rangle\langle\psi_i|, \quad p_1 > \frac{1}{2}, \quad \sum_{i=2}^4 p_i < \frac{1}{2} \tag{4-16}$$

and arbitrary separable density matrix

$$\rho_s' = \sum_{i=1}^4 p_i' |\psi_i\rangle\langle\psi_i|, \quad \text{with } p_1' \leq \frac{1}{2}, \tag{4-17}$$

according to the SDP method explained in section (2), we have to optimize $\mathcal{P} = c^T x = -Tr(\Lambda\rho)$ with

$$F(x) = F_0 + \Lambda F_1 = \rho_s' + \frac{1}{1+s}(-\rho) \geq 0, \tag{4-18}$$

Therefore, we have

$$\mathcal{P} = -\Lambda, \text{ and } \Lambda = x = \frac{1}{1+s} \tag{4-19}$$

Now using the complementary slackness equation (2-3) with a optimal feasible pair$(\hat{Z}, \hat{\Lambda})$, we have

$$\hat{Z}(\rho_s' - \hat{\Lambda}\rho) = 0, \tag{4-20}$$

or

$$\hat{Z}(I - \hat{\Lambda}\rho\rho_s'^\dagger) = 0, \tag{4-21}$$

where $\rho_s^{'\dagger}$ is pseudo inverse of $\rho_s^{'}$. By substituting (4-16) and (4-17) into (4-21) and considering the positivity of $\rho_s^{'} - \hat{\Lambda}\rho$ and after some elementary algebra we arrive at the following results,

$$\hat{\Lambda} = min\{\frac{p_1^{'}}{p_1}, \frac{p_2^{'}}{p_2}, \frac{p_3^{'}}{p_3}, \frac{p_4^{'}}{p_4}\}. \tag{4-22}$$

Now defining

$$s\rho'' = (1+s)\rho_s^{'} - \rho, \tag{4-23}$$

according to equation (3-5) we get

$$sp_i'' + p_i = (1+s)p_i^{'}, \quad i = 1, 2, 3, 4 \tag{4-24}$$

Hence using the above equation we get the following result for the parameter

$$s = \frac{p_1 - p_1^{'}}{p_1^{'} - p''_1} \tag{4-25}$$

The choice $\hat{\Lambda} = \frac{p_1^{'}}{p_1}$ leads to the rank three density matrix ρ'' with $p_1'' = 0$ and the parameter

$$s_1 = \frac{p_1 - p_1^{'}}{p_1^{'}} \tag{4-26}$$

but other choices of $\hat{\Lambda} = \frac{p_i^{'}}{p_i}$, $i = 2, 3, 4$ leads to rank three matrices with $p_1'' \neq 0$ and $s_i = \frac{p_i - p_i^{'}}{p_1 - p_1''}$. Comparing $s_i, i = 1, 2, 3, 4$, one can show that s_1 is smaller than the others, hence for given separable density matrix $\rho_s^{'}$, the choice $\hat{\Lambda} = \frac{p_1^{'}}{p_1}$ yields the minimum parameter s_1.

We see that the parameter $s_1 = \frac{p_1}{p_1^{'}} - 1$ is a monotonic decreasing function of $p_1^{'}$ in the separable region $0 \leq p_1^{'} \leq 1/2$ and its minimum value can be obtained for $p_1^{'} = 1/2$ states which lies at the boundary of separable region. Therefore the robustness of entanglement, that is, the minimum of s

$$s = 2p_1 - 1 \tag{4-27}$$

corresponds to separable states ρ' and ρ'' lying at the corresponding boundaries $p_1^{'} = \frac{1}{2}$ and $p_1'' = 0$ of separable region, in agreement with the results of references [31, 1, 3].

So far using the SDP optimization method we have proved that for a given entangled density matrix ρ minimum s in formula (3-5) is achived for separable states $\rho_s^{'}$ and ρ'' lying

at the boundary of separable region. One we choose ρ'' at that part of boundary of separable region far from ρ, one can determine ρ'_s simply from the intersection of a straight line drawn from ρ to ρ'' and the segment of the boundary of separable region near to ρ. As it is shown in Fig (2) the boundary $\mathcal{S}'_1 = P_1 P_2 P_3$ is near ρ and others $\mathcal{S}_2 = P_1 P_3 O_2^- O_3^-$, $\mathcal{S}_3 = P_1 P_2 O_1^- O_2^-$, $\mathcal{S}_4 = P_2 P_3 O_1^- O_3^-$ are far from it. Therefore, first we have to choose ρ'' at one of $\mathcal{S}'_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, but we see that only the choice of $\mathcal{S}'_1$ leads to $p''_1 = 0$, that is minimum s_1 and other choices leads to greater value of parameter s. But the choice of ρ'' at any point of $\mathcal{S}'_1$=boundary will yields the same minimum robustness of entanglement, which result in the choice of ρ'_s belong to a triangle defined by

$$
\begin{aligned}
A &= \{p'_1 = \tfrac{1}{2}, \quad p'_2 = \tfrac{1}{2} - \tfrac{1-2p_2}{4p_1}, \quad p'_3 = \tfrac{1}{2} - \tfrac{1-2p_3}{4p_1}, \quad p'_4 = \tfrac{1-p_2-p_3}{2p_1} - \tfrac{1}{2}\}, \\
B &= \{p'_1 = \tfrac{1}{2}, \quad p'_2 = \tfrac{1}{2} - \tfrac{1-2p_2}{4p_1}, \quad p'_3 = \tfrac{1-p_2-p_4}{2p_1} - \tfrac{1}{2}, \quad p'_4 = \tfrac{1}{2} - \tfrac{1-2p_4}{4p_1}\}, \\
C &= \{p'_1 = \tfrac{1}{2}, \quad p'_2 = \tfrac{1-p_3-p_4}{2p_1} - \tfrac{1}{2}, \quad p'_3 = \tfrac{1}{2} - \tfrac{1-2p_3}{4p_1}, \quad p'_4 = \tfrac{1}{2} - \tfrac{1-2p_4}{4p_1}\}.
\end{aligned}
\tag{4-28}
$$

lying at boundary $\mathcal{S}'_1$.

4.2 Robustness of entanglement for 2×2 density matrix in Wootters's basis

Here, we find robustness of a generic two qubit density matrix. To this aim we first review Wootters's basis as presented by Wootters in [9]. Wootters in [9, **?**] has shown that for any two qubit density matrix ρ there always exist a decomposition

$$
\rho = \sum_i |x_i\rangle \langle x_i|,
\tag{4-29}
$$

called Wootters's basis, such that

$$
\langle x_i | \tilde{x}_j \rangle = \lambda_i \delta_{ij},
\tag{4-30}
$$

where λ_i are square roots of eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho\tilde{\rho}$ and

$$
\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),
\tag{4-31}
$$

where ρ^* is the complex conjugate of ρ when it is expressed in a standard basis such as

$\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}, \{|\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ and σ_y represent Pauli matrix in local basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Based on this,

the concurrence of the mixed state ρ is defined by $\max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ [9] to see the

explicit form of the wootters basis of the generic $2 * 2$ density matrix see ref. [33].

Now let us define states $|x_i'\rangle$ as

$$|x_i'\rangle = \frac{|x_i\rangle}{\sqrt{\lambda_i}}, \qquad \text{for } i = 1, 2, 3, 4. \tag{4-32}$$

Then ρ can be expanded as

$$\rho = \sum_i \lambda_i |x_i'\rangle \langle x_i'|, \tag{4-33}$$

and Eq. (4-30) takes the following form

$$\left\langle x_i' | \tilde{x}_j' \right\rangle = \delta_{ij}. \tag{4-34}$$

Here in this section we obtain the robustness for a generic two qubit density matrix with

SDP method. Our method of evaluation of robustness is based on the decomposition of

density matrix given by Wootters in [9]. By defining $P_i = \lambda_i K_i$, where $K_i = \langle x_i' | x_i' \rangle$, then

normalization condition of ρ leads to

$$Tr(\rho) = \sum_{i=1}^{4} P_i = 1, \qquad P_i > 0. \tag{4-35}$$

This means that with respect to coordinates P_i, the space of density matrices forms a tetra-

hedral. With respect to this representation separability condition $\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \leq 0$ takes

the following form

$$\frac{P_1}{K_1} - \frac{P_2}{K_2} - \frac{P_3}{K_3} - \frac{P_4}{K_4} \leq 0. \tag{4-36}$$

The states that saturate inequality (4-36) form a plane called $\mathcal{S}_1$ (see Fig. 3). All states

violating inequality (4-36) are entangled states for which $\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4$. These states

form an entangled region with $\mathcal{S}_1$ as its separable boundary. There exist, however, three other

entangled regions corresponding to the dominating λ_j ($j = 2, 3, 4$), respectively. These regions

also define separable planes $\mathcal{S}_j$. Four planes $\mathcal{S}_i$ together with four planes $\mathcal{S}'_i$, corresponding to $\lambda_i = 0$, form an irregular octahedral corresponding to the separable states. This geometry is similar to that of Bell decomposable states but here we have an irregular octahedral associated to separable states [32]. Figure 3 shows a perspective of this geometry, where two separable planes $\mathcal{S}_1$ and $\mathcal{S}'_1$ are shown explicitly.

Now in order to obtain the robustness of ρ, suppose that a ray from ρ is drawn such that intersects the boundary planes of separable region at points ρ'_s and ρ'', respectively. Although ρ'_s is necessarily on the plane $\mathcal{S}_1$, but ρ'' is allowed to lie on any plane $\mathcal{S}'_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ or $\mathcal{S}_4$, where we evaluate robustness for each case separately.

Again for a given generic entangled density matrix ρ in wootters basis

$$\rho = \sum \lambda_i |x'_i\rangle\langle x'_i|, \quad \lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 \tag{4-37}$$

and an arbitrary separable density matrix in the same wootters basis

$$\rho'_s = \sum \lambda'_i |x'_i\rangle\langle x'_i|, \quad \lambda'_1 < \lambda'_2 + \lambda'_3 + \lambda'_4 \tag{4-38}$$

the SDP optimization of $-Tr(\Lambda\rho)$ with respect to $\rho'_s - \Lambda\rho > 0$ yields

$$\hat{\Lambda} = min\{\frac{\lambda'_1}{\lambda_1}, \frac{\lambda'_2}{\lambda_2}, \frac{\lambda'_3}{\lambda_3}, \frac{\lambda'_4}{\lambda_4}\}. \tag{4-39}$$

In this case ρ'' can be written as a convex sum of three vertices of the plane

$$\rho'' = \sum_i \lambda''_i |x'_i\rangle\langle x'_i| = a_2\sigma_2 + a_3\sigma_3 + a_4\sigma_4, \qquad a_2 + a_3 + a_4 = 1, \tag{4-40}$$

where σ_i are separable states that can be written as a convex sum of two corresponding vertices of tetrahedral as

$$\sigma_2 = \frac{1}{K_3 + K_4} |x'_3\rangle\langle x'_3| + \frac{1}{K_3 + K_4} |x'_4\rangle\langle x'_4|, \tag{4-41}$$

$$\sigma_3 = \frac{1}{K_2 + K_4} |x'_2\rangle\langle x'_2| + \frac{1}{K_2 + K_4} |x'_4\rangle\langle x'_4|, \tag{4-42}$$

$$\sigma_4 = \frac{1}{K_2 + K_3} |x'_2\rangle\langle x'_2| + \frac{1}{K_2 + K_3} |x'_3\rangle\langle x'_3|, \tag{4-43}$$

and λ_i'' are

$$\lambda_1'' = 0, \tag{4-44}$$

$$\lambda_2'' = \frac{a_3}{K_2 + K_4} + \frac{a_4}{K_2 + K_3} = \frac{\lambda_1}{\lambda_1 + \lambda_1'}(\lambda_2' - \frac{\lambda_1}{\lambda_1'}\lambda_2), \tag{4-45}$$

$$\lambda_3'' = \frac{a_2}{K_3 + K_4} + \frac{a_4}{K_2 + K_3} = \frac{\lambda_1}{\lambda_1 + \lambda_1'}(\lambda_3' - \frac{\lambda_1}{\lambda_1'}\lambda_3), \tag{4-46}$$

$$\lambda_4'' = \frac{a_2}{K_3 + K_4} + \frac{a_3}{K_2 + K_4} = \frac{\lambda_1}{\lambda_1 + \lambda_1'}(\lambda_4' - \frac{\lambda_1}{\lambda_1'}\lambda_4). \tag{4-47}$$

By expanding ρ_s' as convex sum of ρ and ρ'' and also using the fact that the coordinates of ρ_s' satisfy the equation

$$\frac{P_1'}{K_1} - \frac{P_2'}{K_2} - \frac{P_3'}{K_3} - \frac{P_4'}{K_4} = 0, \tag{4-48}$$

after some algebra, coordinates λ_i' of ρ_s' can be written as

$$\lambda_1' = \frac{\left(\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3}\right)\lambda_1}{\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3} + \frac{C}{2}}, \tag{4-49}$$

$$\lambda_2' = \frac{\left(\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3}\right)\lambda_2 + \frac{C}{2}\left(\frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3}\right)}{\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3} + \frac{C}{2}}, \tag{4-50}$$

$$\lambda_3' = \frac{\left(\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3}\right)\lambda_3 + \frac{C}{2}\left(\frac{a_2}{K_3+K_4} + \frac{a_4}{K_2+K_3}\right)}{\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3} + \frac{C}{2}}, \tag{4-51}$$

$$\lambda_4' = \frac{\left(\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3}\right)\lambda_4 + \frac{C}{2}\left(\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4}\right)}{\frac{a_2}{K_3+K_4} + \frac{a_3}{K_2+K_4} + \frac{a_4}{K_2+K_3} + \frac{C}{2}}, \tag{4-52}$$

where $C = \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$ is the concurrence of ρ. By using the above result one can evaluate robustness of ρ relative to ρ'' as

$$s_1 = \frac{1}{\hat{\Lambda}} - 1 = \frac{C}{\frac{2a_2}{K_3+K_4} + \frac{2a_3}{K_2+K_4} + \frac{2a_4}{K_2+K_3}}. \tag{4-53}$$

Analogue to the above method one can evaluate robustness of ρ for the case that ρ'' lies on the plane $\mathcal{S}_2$ i.e. $\hat{\Lambda} = \frac{\lambda_2'}{\lambda_2}$. In this case ρ_s'' can be expanded as convex sum of three vertices of the plane

$$\rho'' = b_1\sigma_1 + b_3\sigma_3 + b_4\sigma_4, \qquad b_1 + b_3 + b_4 = 1, \tag{4-54}$$

where

$$\sigma_1 = \frac{1}{K_1 + K_2} |x_1'\rangle \langle x_1'| + \frac{1}{K_1 + K_2} |x_2'\rangle \langle x_2'|, \tag{4-55}$$

and σ_3 and σ_4 are defined in Eqs. (4-42) and (4-43). Then after some algebra we obtain the corresponding robustness as

$$s_2 = \frac{C}{\frac{2b_3}{K_2+K_4} + \frac{2b_4}{K_2+K_3}}. \tag{4-56}$$

Similarly in cases that separable state ρ_s'' are on the planes $\mathcal{S}_3$ and $\mathcal{S}_4$ we obtain relative robustness of ρ as

$$s_3 = \frac{C}{\frac{2c_2}{K_3+K_4} + \frac{2c_3}{K_2+K_4}}, \tag{4-57}$$

and

$$s_4 = \frac{C}{\frac{2d_3}{K_2+K_4} + \frac{2d_4}{K_2+K_3}}, \tag{4-58}$$

respectively. Equations. (4-53), (4-56), (4-57) and (4-58) show that in order to achieve the minimum robustness it is enough to consider the case that separable state ρ_s'' lies on the plane $\mathcal{S}_1'$. Hence the choice $\hat{\Lambda} = \frac{\lambda_1'}{\lambda_1}$ corresponds to smaller value of parameter s_1 than others. Therefore for the separable density matrix ρ_s' given in (4-38), the minimum value of parameter s_1, is given by

$$s_1 = \frac{\lambda_1}{\lambda_1'} - 1 = \frac{p_1}{p_1'} - 1 \tag{4-59}$$

Obviously s_1 is a decreasing function of parameter p_1'. Again the minimum value of s_1 can be obtain from maximum possible value p_1', since s_1 is a monotonic decreasing function of p_1', and its maximum value corresponds to the separable states with

$$\lambda_1' = \lambda_2' + \lambda_3' + \lambda_4', \tag{4-60}$$

i.e. , the separable state ρ_s' lying on it boundary of separable region in agreement with [31, 1, 3]. With this consideration we are now allowed to choose coefficients a_i in such a way that Eq. (4-53) becomes minimum. It is easy to see that this happens as long as the coefficient a_k

corresponding to the term $\min(K_i + K_j)$ becomes one. Therefore the robustness of ρ relative to ρ_s'' is

$$s = \frac{\min(K_i + K_j)}{2} C \tag{4-61}$$

which is one of the main results of this work. Here the minimum is taken over all combination of $K_i + K_j$ for $i, j = 2, 3, 4$. Equation (4-61) implies that for two qubit systems robustness is proportional to the concurrence. We see that the minimum robustness given in Eq. (2-3) corresponds to $a_i = \delta_{ik}$, therefore, by using Eq. (4-40) we get the following result for ρ''

$$\rho_s'' = \sigma_k. \tag{4-62}$$

Also by using $a_i = \delta_{ik}$ in Eqs. (4-49) to (4-52) one can easily obtain the coordinates λ_i' of separable state ρ_s'.

As we will show in the next section the Bell decomposable states correspond to the $K_i = 1$ for $i = 1, 2, 3, 4$, therefore in Bell decomposable states Eq. (4-61) implies that the robustness is equal to the concurrence.

One can show that thus obtained robustness is minimum over all separable states.

4.3   Iso-concurrence decomposable states

In this section we define iso-concurrence decomposable (ICD) states, then we give their separability condition and evaluate robustness of entanglement. The iso-concurrence states are defined by

$$|\phi_1\rangle = \cos\theta \, |00\rangle + \sin\theta \, |11\rangle, \qquad |\phi_2\rangle = \sin\theta \, |00\rangle - \cos\theta \, |11\rangle, \tag{4-63}$$

$$|\phi_3\rangle = \cos\theta \, |01\rangle + \sin\theta \, |10\rangle, \qquad |\phi_4\rangle = \sin\theta \, |01\rangle - \cos\theta \, |10\rangle. \tag{4-64}$$

It is quite easy to see that the above states are orthogonal, thus span the Hilbert space of $2 \otimes 2$ systems. Also by choosing $\theta = \frac{\pi}{4}$ the above states reduce to Bell states. Now we can define

ICD states as

$$\rho = \sum_{i=1}^{4} p_i \, |\phi_i\rangle \, \langle\phi_i| \, , \qquad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{4} p_i = 1. \tag{4-65}$$

These states form a four simplex (tetrahedral) with its vertices defined by $p_1 = 1$, $p_2 = 1$, $p_3 = 1$ and $p_4 = 1$, respectively.

Peres-Horodeckis criterion [34, 35] for separability implies that the state given in Eq. (4-65) is separable if and only if the following inequalities are satisfied

$$(p_1 - p_2) \leq \sqrt{4p_3 p_4 / \sin^2 2\theta + (p_3 - p_4)^2}, \tag{4-66}$$

$$(p_2 - p_1) \leq \sqrt{4p_3 p_4 / \sin^2 2\theta + (p_3 - p_4)^2}, \tag{4-67}$$

$$(p_3 - p_4) \leq \sqrt{4p_1 p_2 \sin^2 2\theta + (p_1 - p_2)^2}, \tag{4-68}$$

$$(p_4 - p_3) \leq \sqrt{4p_1 p_2 / \sin^2 2\theta + (p_1 - p_2)^2}. \tag{4-69}$$

Inequalities (4-66) to (4-69) divide tetrahedral of density matrices to five regions. Central regions, defined by the above inequalities, form a deformed octahedral and are separable states. In four other regions one of the above inequality will not hold, therefore they represent entangled states. Bellow we consider entangled states corresponding to the violation of inequality (4-66) i.e. the states which satisfy the following inequality

$$(p_1 - p_2) > \sqrt{4p_3 p_4 / \sin^2 2\theta + (p_3 - p_4)^2}. \tag{4-70}$$

In order to obtain the robustness of ICD states, we have to follow the method presented by Wootters in [9]. Starting from the spectral decomposition for ICD states, and defining subnormalized orthogonal eigenvectors, the wootters basis of ICD states can be defined as

$$|x_1\rangle = -i\alpha_1\sqrt{p_1}|\phi_1\rangle + i\alpha_2\sqrt{p_2}|\phi_2\rangle,$$

$$|x_2\rangle = \quad \alpha_2\sqrt{p_1}|\phi_1\rangle + \quad \alpha_1\sqrt{p_2}|\phi_2\rangle,$$

$$|x_3\rangle = \quad \alpha_3\sqrt{p_3}|\phi_3\rangle - \quad \alpha_4\sqrt{p_4}|\phi_4\rangle, \tag{4-71}$$

$$|x_1\rangle = -i\alpha_4\sqrt{p_3}|\phi_3\rangle - i\alpha_3\sqrt{p_4}|\phi_4\rangle,$$

where

$$\alpha_1 = \frac{\left((p_1 + p_2) \sin 2\theta + \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right)}{\sqrt{2} \left(4p_1p_2 \cos^2 2\theta + (p_1 + p_2)^2 \sin^2 2\theta + (p_1 + p_2) \sin 2\theta \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right)},$$

$$\alpha_2 = \frac{\sqrt{2p_1p_2} \cos 2\theta}{\left(4p_1p_2 \cos^2 2\theta + (p_1 + p_2)^2 \sin^2 2\theta + (p_1 + p_2) \sin 2\theta \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right)},$$

(4-72)

$$\alpha_3 = \frac{\left((p_3 + p_4) \sin 2\theta + \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right)}{\sqrt{2} \left(4p_3p_4 \cos^2 2\theta + (p_3 + p_4)^2 \sin^2 2\theta + (p_3 + p_4) \sin 2\theta \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right)},$$

$$\alpha_4 = \frac{\sqrt{2p_3p_4} \cos 2\theta}{\left(4p_3p_4 \cos^2 2\theta + (p_3 + p_4)^2 \sin^2 2\theta + (p_3 + p_4) \sin 2\theta \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right)}.$$

Now it is easy to evaluate λ_i which yields

$$\lambda_1 = \tfrac{1}{2} \left((p_1 - p_2) \sin 2\theta + \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right),$$

$$\lambda_2 = \tfrac{1}{2} \left((p_2 - p_1) \sin 2\theta + \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right),$$

$$\lambda_3 = \tfrac{1}{2} \left((p_3 - p_4) \sin 2\theta + \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right),$$

$$\lambda_4 = \tfrac{1}{2} \left((p_4 - p_3) \sin 2\theta + \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right).$$

(4-73)

Therefore, coefficients $K_i = \frac{p_i}{\lambda_i}$ are:

$$K_1 = \frac{p_1}{\tfrac{1}{2} \left((p_1 - p_2) \sin 2\theta + \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right)},$$

$$K_2 = \frac{p_2}{\tfrac{1}{2} \left((p_2 - p_1) \sin 2\theta + \sqrt{4p_1p_2 + (p_1 - p_2)^2 \sin^2 2\theta} \right)},$$

(4-74)

$$K_3 = \frac{p_3}{\tfrac{1}{2} \left((p_3 - p_4) \sin 2\theta + \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right)},$$

$$K_4 = \frac{p_4}{\tfrac{1}{2} \left((p_4 - p_3) \sin 2\theta + \sqrt{4p_3p_4 + (p_3 - p_4)^2 \sin^2 2\theta} \right)}.$$

Writing the ICD state in wootters basis, we evaluate its robustness of entanglement with respect to the set separable state, diagonal in ICD basis, simply by chossing the separable states ρ'_s and ρ''_s on the corresponding boundaries, as follows,

$$\rho'_s = \frac{1}{2} \left((p'_1 - p'_2) \sin 2\theta + \sqrt{4p'_1p'_2 + (p'_1 - p'_2)^2 \sin^2 2\theta} \right) \langle x'_1 | x'_1 \rangle$$

$$+\frac{1}{2}\left((p_2'-p_1')\sin 2\theta+\sqrt{4p_1'p_2'+(p_1'-p_2')^2\sin^2 2\theta}\right)\langle x_2'|x_2'\rangle$$

$$+\frac{1}{2}\left((p_3'-p_4')\sin 2\theta+\sqrt{4p_3'p_4'+(p_3'-p_4')^2\sin^2 2\theta}\right)\langle x_3'|x_3'\rangle$$

$$+\frac{1}{2}\left((p_4'-p_3')\sin 2\theta+\sqrt{4p_3'p_4'+(p_3'-p_4')^2\sin^2 2\theta}\right)\langle x_4'|x_4'\rangle,$$

$$\rho_s''=\frac{1}{2}\left((p_1''-p_2'')\sin 2\theta+\sqrt{4p_1''p_2''+(p_1''-p_2'')^2\sin^2 2\theta}\right)\langle x_1'|x_1'\rangle \qquad (4\text{-}75)$$

$$+\frac{1}{2}\left((p_2''-p_1'')\sin 2\theta+\sqrt{4p_1''p_2''+(p_1''-p_2'')^2\sin^2 2\theta}\right)\langle x_2'|x_2'\rangle$$

$$+\frac{1}{2}\left((p_3''-p_4'')\sin 2\theta+\sqrt{4p_3''p_4''+(p_3''-p_4'')^2\sin^2 2\theta}\right)\langle x_3'|x_3'\rangle$$

$$+\frac{1}{2}\left((p_4''-p_3'')\sin 2\theta+\sqrt{4p_3''p_4''+(p_3''-p_4'')^2\sin^2 2\theta}\right)\langle x_4'|x_4'\rangle,$$

with the corresponding robustness of entanglement as:

$$s=\frac{\min(K_i+K_j)}{2}C \qquad (4\text{-}76)$$

where K_i are given in (4-74).

It is obvious that, BD state correspond to particular case of $\theta=\frac{\pi}{4}$, ICD ones with $K_i=1$ for $i=1,2,3,4$. Therefore, the robustness of entanglement given in (4-76) become $s=C$ which is agreement with (4-27)

4.4 $2\otimes 3$ Bell decomposable state

In this subsection we obtain robustness of entanglement for Bell decomposable states of $2\otimes 3$ quantum systems. A Bell decomposable density matrix acting on $2\otimes 3$ Hilbert space can be defined by

$$\rho=\sum_{i=1}^{6}p_i|\psi_i\rangle\langle\psi_i|, \qquad 0\le p_i\le 1, \quad \sum_{i=1}^{6}p_i=1, \qquad (4\text{-}77)$$

where $|\psi_i\rangle$ are Bell states in $H^6\cong H^2\otimes H^3$ Hilbert space, defined by:

$$|\psi_1\rangle=\frac{1}{\sqrt{2}}(|11\rangle+|22\rangle), \qquad |\psi_2\rangle=\frac{1}{\sqrt{2}}(|11\rangle-|22\rangle),$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |23\rangle), \qquad |\psi_4\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |23\rangle), \tag{4-78}$$

$$|\psi_5\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |21\rangle), \qquad |\psi_6\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |21\rangle).$$

It is quite easy to see that the above states are orthogonal and hence it can span the Hilbert space of $2 \otimes 3$ systems. From Peres-Horodeckis [34, 35] criterion for separability we deduce that the state given in Eq. (4-77) is separable if and only if the following inequalities are satisfied

$$(p_1 - p_2)^2 \leq (p_3 + p_4)(p_5 + p_6), \tag{4-79}$$

$$(p_3 - p_4)^2 \leq (p_5 + p_6)(p_1 + p_2), \tag{4-80}$$

$$(p_5 - p_6)^2 \leq (p_1 + p_2)(p_3 + p_4). \tag{4-81}$$

In the following we always assume without loss of generality that $p_1 \geq p_2$, $p_3 \geq p_4$ and $p_5 \geq p_6$. Now in order to obtain robustness of entanglement for BD state given in Eq. (4-77) we choose $\rho'_s = \sum_i p'_i |\psi_i\rangle \langle\psi_i|$ and $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. We also assume without loss of generality that ρ_s lies on the separable-entangled boundary defined by (all other cases where ρ_s lies on other surfaces can be treated similarly)

$$p'_1 - p'_2 = \sqrt{(p'_3 + p'_4)(p'_5 + p'_6)}. \tag{4-82}$$

Moreover ρ_s must satisfies the other two separability conditions (4-80) and (4-81). This means that entangled state ρ violates separability condition (4-79), i.e. we have

$$p_1 \geq p_2 + \sqrt{(p_3 + p_4)(p_5 + p_6)}. \tag{4-83}$$

Here the boundary of the separable states is given by

$$\left((p_1 - p_2)^2 - (p_3 + p_4)(p_5 + p_6)\right) = 0. \tag{4-84}$$

States that saturate inequality (4-79),(4-80),(4-81)) form a plane called $\mathcal{S}_1$. All states violating these three inequality are entangled states. These states form an entangled region

with $\mathcal{S}_1$ as its separable boundary. There exist, however, other entangled regions correspond to the dominating p_j $(j = 2, .., 6)$, respectively. These regions also define separable planes $\mathcal{S}_j$. The Planes $\mathcal{S}_i$ together with the planes $\mathcal{S}'_i$, correspond to $p_i = 0$.

Below in the rest of this subsection we will use Eqs. (3-5) and (4-21) to calculate robustness of entanglement for $2 \otimes 3$ entangled Bell decomposable density matrix

$$\rho = \sum_{i=1}^{6} p_i |\psi_i\rangle\langle\psi_i| \ \ with \ \ p_1 > p_2 + \sqrt{(p_3 + p_4)(p_5 + p_6)}, \tag{4-85}$$

and an arbitrary separable density matrix

$$\rho'_s = \sum_{i=1}^{6} p'_i |\psi_i\rangle\langle\psi_i|, \qquad p'_1 \leq p'_2 + \sqrt{(p'_3 + p'_4)(p'_5 + p'_6)} \tag{4-86}$$

The SDP optimization of $-Tr(\Lambda\rho)$ with respect to $\rho'_s - \Lambda\rho > 0$ yields

$$\hat{\Lambda} = min\{\frac{p'_1}{p_1}, \frac{p'_2}{p_2}, \frac{p'_3}{p_3}, \frac{p'_4}{p_4}, \frac{p'_5}{p_5}, \frac{p'_6}{p_6}\}, \tag{4-87}$$

The choice of $\hat{\Lambda} = \frac{p'_i}{p_i}$, $i = 1, ..., 6$ consistent with positivity of $\rho - s\rho'_s$ implies that ρ'' lies at the boundary $\mathcal{S}_i$, $i = 1, ..., 6$. Numerical calculation indicates that the minimum s is achieved with the choice of $\hat{\Lambda} = \frac{p'_1}{p_1}$ (for more details refer to ref[36]). Therefore the robustness of entanglement becomes

$$s_1 = \frac{p_1 - p'_1}{p'_1}, \tag{4-88}$$

we see that s is a monotonic decreasing function of p'_1 and its minimum is achieved for $p'_1 = p'_2 + \sqrt{(p'_3 + p'_4)(p'_5 + p'_6)}$ which implies that ρ'_s lies at the boundary as separable state, too, in agreement with those results of ref[36].

Now in order to obtain minimum robustness of entanglement with respect to the set of separable states diagonal in 2×3, BD states, all we need to draw a line from ρ to interest the separable boundary

$$p'_1 = p'_2 + \sqrt{(p'_3 + p'_4)(p'_5 + p'_6)}, \tag{4-89}$$

at ρ'_s and the other boundary

$$p''_1 = p''_2 + \sqrt{(p''_3 + p''_4)(p''_5 + p''_6)}. \tag{4-90}$$

at ρ'', respectively.

Therefore, from the relation $\rho = (1+s)\rho' - s\rho''$ we have

$$p_i = (1+s)p_i' - sp_i''. \tag{4-91}$$

Hence the robustness of entanglement s_1 becomes

$$s_1 = \frac{(p_1 - p_2)^2 - (p_3 + p_4)(p_5 + p_6)}{-2(p_1 - p_2)(p_1'' - p_2'') + (p_3 + p_4)(p_5'' + p_6'') + (p_5 + p_6)(p_3'' + p_4'')}. \tag{4-92}$$

where the maximization of the denominator, by using the Lagrange multipliers method due to existence of constrains (4-90) and normalization of $Tr(\rho'') = 1$, leads to the following results for the minimums robustness of entanglement

$$s_1 = \frac{3((p_1 - p_2)^2 - (p_3 + p_4)(p_5 + p_6))}{2(\sqrt{(2p_1 - 1)^2 + 3((p_1 - p_2)^2 - (p_3 + p_4)(p_5 + p_6))} - (2p_2 - 1))}. \tag{4-93}$$

4.5 Werner states

Werner states are the only states that are invariant under local unitary operations and for $d \otimes d$ systems the Werner states are defined by [37]

$$\rho = \frac{1}{d^3 - d}((d - f)I + (df - 1)\mathcal{F}), \qquad -1 \leq f \leq 1, \tag{4-94}$$

where I stands for identity operator and $\mathcal{F} = \sum_{i,j} |ij\rangle \langle ji|$. It is shown that Werner state is separable iff $0 \leq f \leq 1$.

Now to obtain the optimal robustness of entanglement with respect to accessible separable region, that is, density matrices of werner type with $0 \leq f' \leq 1$, all we need is to choose an arbitrary point ρ_s' in separable region $0 \leq f' \leq 1$. Then the SDP method of optimization of $-Tr(\rho\Lambda)$ with respect to $\rho_s' - \Lambda\rho > 0$ yields

$$\hat{\Lambda} = min\{\frac{f'+1}{f+1}, \frac{1-f'}{1-f}\} = \frac{1-f'}{1-f}, \tag{4-95}$$

where the second equality follows from the fact that parameters f and f' are restricted to the regions $f \in (-1, 0)$ and $f' \in (0, 1)$, respectively.

Therefore, for the corresponding parameter s we get

$$s = \frac{f' - f}{-f' + 1} = -1 + \frac{1 - f}{1 - f'}, \tag{4-96}$$

which is a monotonic decreasing function of f'. Hence, the optimal robustness of entanglement is $s = -f$ which corresponds to the choice of werner type separability matrix with $f' = 0$. On the other hand we have

$$\rho'' = \rho - (s+1)\rho'_s = \frac{1}{d(d+1)} (I + \mathcal{F}), \tag{4-97}$$

which corresponds to the separable states with $f = 1$ in Eq.(4-94).

Again both separable states ρ'_s and ρ'' are at the boundary of separable region in agreement with [1, 31, 3].

4.6 Isotropic states

The $d \otimes d$ bipartite isotropic states are the only ones that are invariant under $U \otimes U^*$ operations, where * denotes complex conjugation. The isotropic states of $d \otimes d$ systems are defined by [38]

$$\rho = \frac{1 - F}{d^2 - 1} \left(I - \left| \psi^+ \right\rangle \left\langle \psi^+ \right| \right) + F \left| \psi^+ \right\rangle \left\langle \psi^+ \right|, \qquad 0 \le F \le 1, \tag{4-98}$$

where $\left| \psi^+ \right\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ is maximally entangled state. It is shown that isotropic state is separable when $0 \le F \le \frac{1}{d}$ [38].

Again to obtain the optimal robustness of entanglement with respect to accessible separable region, that is, density matrices of Isotropic type with $0 \le F' \le \frac{1}{d}$, all we need is to choose an arbitrary point ρ'_s in separable region $0 \le F' \le \frac{1}{d}$. Then the SDP method of optimization of $-Tr(\rho\Lambda)$ with respect to $\rho'_s - \Lambda\rho > 0$ yields

$$\hat{\Lambda} = min\{\frac{1 - F'}{1 - F}, \frac{F'}{F}\} = \frac{F'}{F}, \tag{4-99}$$

where the second equality follows from the fact that parameters F and F' are restricted to the regions $F \in (1/d, 1)$ and $F' \in (0, 1/d)$, respectively.

Therefore, for the corresponding parameter s we get

$$s = \frac{F}{F'} - 1, \tag{4-100}$$

which is a monotonic decreasing function of F'. Hence, the optimal robustness of entanglement is $s = dF - 1$ which corresponds to the choice of Isotropic type separability matrix with $F' = \frac{1}{d}$. On the other hand we have

$$\rho'' = \rho - (s+1)\rho'_s = \frac{1}{d^2 - 1} \left(I - \left| \psi^+ \right\rangle \left\langle \psi^+ \right| \right), \tag{4-101}$$

which corresponds to separable states with $F = 0$ in Eq.(4-98).

Again both separable states ρ'_s and ρ'' are at the boundary of separable region in agreement with [1, 31, 3].

4.7 One parameter $3 \otimes 3$ state

Finally let us consider a one parameter state acting on $H^9 \cong H^3 \otimes H^3$ Hilbert space as [30]

$$\rho = \frac{2}{7} \left| \psi^+ \right\rangle \left\langle \psi^+ \right| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \qquad 2 \leq \alpha \leq 5, \tag{4-102}$$

where

$$\left| \psi^+ \right\rangle = \tfrac{1}{\sqrt{3}} \left(|11\rangle + |22\rangle + |33\rangle \right),$$
$$\sigma_+ = \tfrac{1}{3} \left(|12\rangle \langle 12| \, |23\rangle \langle 23| + |31\rangle \langle 31| \right), \tag{4-103}$$
$$\sigma_- = \tfrac{1}{3} \left(|21\rangle \langle 21| \, |32\rangle \langle 32| + |13\rangle \langle 13| \right).$$

ρ is separable iff $2 \leq \alpha \leq 3$, it is bound entangled iff $3 \leq \alpha \leq 4$ and it is distillable entangled state iff $4 \leq \alpha \leq 5$ [30].

Similarly in order to obtain the optimal robustness of entanglement with respect to accessible separable region, that is, density matrices of $3 \otimes 3$ type with $2 \leq \alpha' \leq 3$, all we need is to choose and arbitrary point ρ'_s in separable region $2 \leq \alpha' \leq 3$. Then the SDP method of optimization of $-Tr(\rho \Lambda)$ with respect to $\rho'_s - \Lambda \rho > 0$ yields

$$\hat{\Lambda} = \frac{\alpha'}{\alpha}, \tag{4-104}$$

where the second equality follows from the fact that parameters α and α' are restricted to the regions $\alpha \in (3,5)$ and $\alpha' \in (2,3)$, respectively.

Therefore, for the corresponding parameter s we get

$$s = \frac{\alpha}{\alpha'} - 1, \tag{4-105}$$

which is a monotonic decreasing function of α'. Hence, the optimal robustness of entanglement is $s = \frac{\alpha}{3} - 1$ which corresponds to the choice of $3 \otimes 3$ type separability matrix with $\alpha' = 3$. On the other hand we have

$$\rho'' = \rho - (s+1)\rho'_s = \frac{2}{7}\left|\psi^+\right\rangle\left\langle\psi^+\right| + \frac{5}{7}\sigma_-, \tag{4-106}$$

which corresponds to the separable states with $\alpha = 0$ in Eq.(4-102).

Again both separable states ρ'_s and ρ'' are at the boundary of separable region in agreement with [1, 31, 3].

4.8 Multi partite isotropic states

In this last subsection we obtain robustness of entanglement for a n-partite d-levels system. Let us consider the following mixture of completely random state $\rho_0 = I/d^n$ and maximally entangled state $|\psi^+\rangle$

$$\rho = (1-r)\frac{I}{d^n} + r\left|\psi^+\right\rangle\left\langle\psi^+\right|, \qquad 0 \le r \le 1, \tag{4-107}$$

where I denotes identity operator in d^n-dimensional Hilbert space and $|\psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\cdots i\rangle$. The separability properties of the state (4-107) is considered in Ref. [39]. It is shown that the above state is separable iff $0 \le r \le r_0 = (1 + d^{n-1})^{-1}$.

Again to obtain the optimal robustness of entanglement with respect to accessible separable region, that is, density matrices of multi partite isotropic type with $0 \le r' \le r_0$, all we need is to choose an arbitrary point ρ'_s in separable region $0 \le r' \le r_0$. Then the SDP method of

optimization of $-Tr(\rho\Lambda)$ with respect to $\rho'_s - \Lambda\rho > 0$ yields

$$\hat{\Lambda} = \frac{1 + r'(d^n - 1)}{1 + r(d^n - 1)}, \tag{4-108}$$

where the second equality follows from the fact that parameters r' and r are restricted to the

regions $r' \in (0, r_0)$ and $r \in (r_0, 1)$, respectively.

Therefore, for the corresponding parameter s we get

$$s = \frac{(r - r')(d^n - 1)}{1 + r'(d^n - 1)}, \tag{4-109}$$

which is a monotonic decreasing function of r'. Hence, the optimal robustness of entanglement

is $s = \frac{(r - r_0)(d^n - 1)}{1 + r_0(d^n - 1)}$ which corresponds to the choice of multi partite isotropic type separability

matrix with $r' = r_0$. On the other hand we have

$$\rho'' = \rho - (s + 1)\rho'_s = \frac{1}{d^n - 1}(I - \left|\psi^+\right\rangle\left\langle\psi^+\right|), \tag{4-110}$$

which corresponds to separable states with $r = \frac{1}{1 - d^n}$ in Eq.(4-107).

Again both separable state ρ'_s and ρ'' are at the boundary of separable region in agreement

with [1, 31, 3].

5 Appendix

In the definition of the robustness we have to minimize over all separable states. But almost in

all of examples, we have considered a specific set of separable states, i.e, the diagonal separable

states (in the basis that the entangled state itself is diagonal). Here in this appendix we try to

show that the minimum robustness of an entangled diagonal density matrix in a given basis,

with respect to the set of separable diagonal states in the same basis, is also minimum over

the separable sets of off-diagonal extension of these diagonal separable states. In [1], it has

been shown that the robustness given in equation (4-17) is minimum over all Bell decomposable

states. In the following, we generalize it and show that for any orthogonal basis the off-diagonal

elements of ρ'_s and ρ'' (in basis that ρ is diagonal) play no role in robustness. First we consider the diagonal entangled density matrices in some orthonormal basis and we consider the generic separable states ρ'_s and ρ'' defined by

$$\rho'_s = \sum_i p'_i |\chi_i\rangle\langle\chi_i| + \sum_{i,j} a_{ij} |\chi_i\rangle\langle\chi_j| \tag{5-111}$$

$$\rho'' = \sum_i p''_i |\chi_i\rangle\langle\chi_i| + \sum_{i,j} b_{ij} |\chi_i\rangle\langle\chi_j|, \tag{5-112}$$

where $a_{ii} = b_{ii} = 0$ and $|\chi_i\rangle, i = 1, 2, ..$ are orthogonal states. Then the pseudomixture equation (4-23) implies that the following equations must hold

$$p_i = (1 + s)p'_i - sp''_i, \tag{5-113}$$

$$(1 + s)a_{ij} - sb_{ij} = 0. \tag{5-114}$$

Now one can easily obtain robustness of ρ relative to ρ'_s as

$$s = \frac{\| \rho - \rho'_s \|}{\| \rho'_s - \rho'' \|}$$

$$= \sqrt{\frac{\sum_i (p_i - p'_i)^2 + Tr(A^2)}{\sum_i (p'_i - p''_i)^2 + Tr((A-B)^2)}}$$

$$= \sqrt{\frac{\sum_i (p_i - p'_i)^2 + Tr(A^2)}{\sum_i (p'_i - p''_i)^2 + \frac{1}{s^2} Tr(A^2)}} \tag{5-115}$$

where A and B are Hermitian matrices with non vanishing off-diagonal matrix elements a_{ij} and b_{ij}, respectively, and in the last line we have used equation (5-114). By solving equation (5-115) for robustness s we get

$$s = \sqrt{\frac{\sum_i (p_i - p'_i)^2}{\sum_i (p'_i - p''_i)^2}}. \tag{5-116}$$

Equation (5-115) shows that off-diagonal elements of ρ'_s and ρ'' (in basis that ρ is diagonal) play no role in robustness. Now we consider the general case of diagonal entangled density matrices in some non-orthogonal basis. Obviously in this case the entangled density matrix ρ can be written as

$$\rho = \sum_i p_i |\eta_i\rangle\langle\eta_i| = \sum_{i,j} \tilde{p}_{ij} |\tilde{\eta}_i\rangle\langle\tilde{\eta}_i|, \tag{5-117}$$

where $|\eta_i\rangle, i = 1, 2, ...$ are non-orthogonal states and $|\tilde{\eta}_i\rangle, i = 1, 2, ..$ are dual of $|\eta_i\rangle, i = 1, 2, ...$, i.e., we have $\langle\tilde{\eta}_i||\eta_j\rangle = \delta_{ij}, i, j = 1, 2, .. $. Also the separable states ρ'_s and ρ'' can be written as

$$\rho'_s = \sum_i p'_i|\eta_i\rangle\langle\eta_i| + \sum_{i,j} a_{ij}|\eta_i\rangle\langle\eta_j| = \sum_i \tilde{p}'_i|\tilde{\eta}_i\rangle\langle\tilde{\eta}_i| + \sum_{i,j} \tilde{a}_{ij}|\tilde{\eta}_i\rangle\langle\tilde{\eta}_j| \tag{5-118}$$

$$\rho'' = \sum_i p''_i|\eta_i\rangle\langle\eta_i| + \sum_{i,j} b_{ij}|\eta_i\rangle\langle\eta_j| = \sum_i \tilde{p}''_i|\tilde{\eta}_i\rangle\langle\tilde{\eta}_i| + \sum_{i,j} \tilde{b}_{ij}|\tilde{\eta}_i\rangle\langle\tilde{\eta}_j|, \tag{5-119}$$

where $a_{ii} = b_{ii} = \tilde{a}_{ii} = \tilde{b}_{ii} = 0$. again the pseudomixture equation (4-23) implies that the following equations must hold

$$p_i = (1 + s)p'_i - sp''_i, \tag{5-120}$$

$$0 = (1 + s)a_{ij} - sb_{ij}, \tag{5-121}$$

$$\tilde{p}_{ii} = (1 + s)\tilde{p}'_i - s\tilde{p}''_i, \tag{5-122}$$

$$\tilde{p}_{ij} = (1 + s)\tilde{a}_{ij} - s\tilde{b}_{ij}. \tag{5-123}$$

Now the robustness of ρ relative to ρ'_s can be easily obtained by using above Equations as

$$s = \frac{\|\rho - \rho'_s\|}{\|\rho'_s - \rho''\|}$$

$$= \sqrt{\frac{\sum_i (p_i - p'_i)(\tilde{p}_{ii} - \tilde{p}'_i)}{\sum_i (p'_i - p''_i)(\tilde{p}'_i - \tilde{p}''_i)}}. \tag{5-124}$$

Again this equation shows that off-diagonal elements of ρ'_s and ρ'' in non-orthogonal basis play no role in robustness.

6   conclusion

Using the elegant method of convex semidefinite optimization method, we have been able to obtain the robustness of some set of mixed density matrices with respect to some accessible separable set. In this method we have been able to calculate the robustness without using any kind of the space of density matrices, where the results that obtained are in agreement with those of norm-method of ref[2].

Also using SDP method we have shown that the separable density matrices contributing to the robustness lie at the boundary of accessible separable region.

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Figure Captions

Figure 1: All BD states are defined as points interior to tetrahedral. Vertices P_1, P_2, P_3 and P_4 denote projectors corresponding to Bell states given in Eqs. (4-8) to (4-11), respectively. Octahedral corresponds to separable states.

Figure 2: Entangled tetrahedral corresponding to singlet state. Point t denotes a generic state ρ and points t' and t'' are on the separable boundary planes.

Figure 3: The space of a generic two qubit density matrix is represented by a tetrahedral. Vertices P_i for $i = 1, 2, 3, 4$ correspond to pure states defined by $\rho = \lambda_i \left| x_i' \right\rangle \left\langle x_i' \right|$. Irregular octahedral corresponds to separable states. Separable planes $\mathcal{S}_1$ and $\mathcal{S}_1'$ are shown explicitly.