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Separability criteria *via* wavelet transform on homogenous spaces and projective representations

Research Article

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Abstract: The intimate connection between the Banach space wavelet reconstruction method for each unitary representation of a given group and homogenous space, and the quantum entanglement description using group theory were both studied in our previous articles. Here, we present a universal description of quantum entanglement using group theory and non-commutative characteristic functions for homogenous space and projective representation of compact groups on Banach spaces for some well known examples, such as: Moyal representation for a spin; Dihedral and Permutation groups.

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1. Introduction

Entanglement is one of the most fascinating features of quantum mechanics. As Einstein, Podolsky and Rosen [1] pointed out, the quantum states of two physically separated systems that interacted in the past can defy our intuitions about the outcome of the local measurements. Moreover, it has recently been recognized that entanglement is a very important resource in quantum information processing [2]. A bipartite mixed state is said to be separable [3] (not entangled) if it considered to be a convex combination of pure product states.

A separability criterion is based on a simple property which can be shown to hold for every separable state. If a state does not satisfy this property, it is deemed to be entangled. However, the reverse does not necessarily imply that the state is separable. One of the first and most widely used related criteria is the Positive Partial Transpose (PPT) criterion, introduced by Peres [4]. Furthermore, the necessary and sufficient condition for separability in $H_2 \otimes H_2$ and $H_2 \otimes H_3$ was shown by Horodeckis [5], which was based on the previous work by Woronowicz [6]. However, in higher dimensions, there are PPT states which are not at all entangled, as was first shown in [7], based on [6]. These states are called bound-entangled because they have a peculiar property from which no entan-

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glement can be distilled by local operations [8, 9]. Another approach to distinguish separable states from entangled ones involves the so called entanglement witness (EW) [9–11]. Some entanglement measures and best- separable state conditions using semi-definite programming method are given in [11–15]. However,there has been no investigation of the separability problem in them. On the other hand, previously the method of characteristic functions has been successfully applied for studying non-classical states of quantum harmonic oscillator but not for studying entanglement genuine quantum features of quantum states [16]. Korbicz and Lewenstein defined the characteristic functions applied for testing the entanglement of states by choosing a compact group G and a set of its irreducible unitary representations as the main ingredients of the mathematical representation of the state space. Although they do not present any new entanglement test, their results offer a new point of view on the separability problem. Moreover, they were able to translate the positivity of the partial transpose (PPT) criterion [4] into the group theoretical language [17]. We have also obtained a group theoretical approach to quantum entanglement and tomography with wavelet transform in [18].

A general framework has already been presented for the unification of the Hilbert space wavelets transformation and quasi-distributions and tomographic transformation associated with a given pure quantum states [19]. Here, in this manuscript, we are trying to establish the intimate connection between the quantum entanglement using group theory and non-commutative characteristic functions for homogenous space and projective representation of group on Banach spaces for some well known examples, such as: Moyal representation for a spin, Dihedral and permutation groups, all of which can be represented by density matrices. For density matrices, one may define the norm as $tr()$ which implies the absence of a scalar product in the density matrix space (so it is not a Hilbert space but a Banach space) [20, 21].

The paper is organized as follows: Following this introduction as Sec. 1, in Sec. 2 we define wavelet transform based on homogeneous space and projective representation spaces on Banach space. In Sec. 3, a brief recapitulation of group theoretical approach to entanglement for irreducible representation of any compact group is studied. In Sec. 4, we study the group theoretical approach to entanglement associated with the unitary irreducible representation on homogenous space for Moyal representation of a spin and the projective representation of Permutation and Dihedral groups by using the Banach space wavelet transform method.

2. Wavelet transform in Banach spaces on homogeneous space and based on projective representation of group

2.1. Wavelet transform on homogeneous space

The following is a brief recapitulation of some aspects of the theory of wavelets on homogeneous space. We only mention the concepts that will be needed in the sequel, a more detailed treatment may be found in[1] [22–24], for example. Let G be a locally compact group with the left Haar measure $d\mu$ and H be a closed subgroup of G. Let $\hat{\pi}$ be a continuous representation of a group and $X = G/H$ a homogeneous space.

We could define a representation for homogeneous space $X \times X$ in the space $\mathcal{L}(\mathcal{B})$ of bounded linear operators $\mathcal{B} \to \mathcal{B}$:

$$\hat{\pi} : X \times X \to \mathcal{L}(\mathcal{L}(\mathcal{B})) : \hat{O} \to U(x_1)\hat{O}U\left(x_2^{-1}\right), \qquad (1)$$

where if x_1 is equal to x_2, the representation is called adjoint representation, and, if x_2 is equal to identity operator, it is called left representation of homogeneous space.

Let $\mathcal{L}(\mathcal{B})$ be the space of bounded linear operator $\mathcal{B} \to \mathcal{B}$ in Banach space. We will say that $b_0 \in \mathcal{B}$ is a vacuum vector if for all $h \in H_1 \times H_2$, we have $\hat{\pi}(h)b_0 = \chi(h)b_0$ and also the set of vectors $b_{x_1,x_2} = \hat{\pi}(x_1,x_2)b_0$ forms a family of coherent states, if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional) and a vector $b_0 \in \mathcal{B}$ (called vacuum vector) such that:

$$C(b_0, b_0') = \int_X \left\langle \hat{\pi}(x_1^{-1}, x_2^{-1})b_0, l_0 \right\rangle \left\langle \hat{\pi}(x_1, x_2)b_0', l_0' \right\rangle d\mu(x_1, x_2), \qquad (2)$$

is non-zero and finite, which is known as the admissibility relation.

If the subgroup H is non-trivial, one does not need to know the wavelet transform on the whole group G, but it should be defined on only the homogeneous space G/H, then the reduced wavelet transform $\mathcal{W}$ to a homogeneous space of

[1] *V. V. Kisil, Wavelets in Applied and Pure Mathematics, lecture note (2003)*

function L_2 is defined by a representation $\hat{\pi}$ of $G \times G$ on $\mathcal{B}$ and a test functional $l_0 \in \mathcal{B}^*$ such that[1]:

$$\begin{aligned} \mathcal{W} : \mathcal{B} \to L_2(X \times X) : \hat{O} \to \hat{O}(x_1, x_2) &= \left[\mathcal{W}\hat{O}\right](x_1, x_2) \\ &= \left\langle \hat{\pi}(x_1^{-1}, x_2^{-1})\hat{O}, l_0 \right\rangle = \left\langle \hat{O}, \hat{\pi}^*(x_1, x_2) l_0 \right\rangle, \\ &\forall x_1, x_2 \in X, \end{aligned} \tag{3}$$

where $\hat{\pi}^*$ is dual of $\hat{\pi}$.

2.2. Wavelet transform based on projective representation

Let G be a locally compact group with left Haar measure $d\mu$, $Z(H)$ be a center of group H, U be a continuous representation of the group G and $X = G/Z(H)$ be a central extension. In the last subsection we saw that $\hat{\pi}$ has been lifted to an ordinary representation of G.
Let $\mathcal{L}(\mathcal{B})$ be the space of bounded linear operator $\mathcal{B} \to \mathcal{B}$ in Banach space. We will say that $b_0 \in \mathcal{B}$ is a vacuum vector if for all $h \in Z(H)$ we have $\hat{\pi}(h)b_0 = \chi(h)b_0$ and also the set of vectors $b_x = \hat{\pi}(x)b_0$ forms a family of coherent states, if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional) and a vector $b_0 \in \mathcal{B}$ (called vacuum vector) such that:

$$C(b_0, b_0') = \int_X \langle \hat{\pi}\left(x^{-1}\right) b_0, l_0 \rangle \left\langle \hat{\pi}(x) b_0', l_0' \right\rangle d\mu(x), \tag{4}$$

is non-zero and finite, which is known as the admissibility relation.
If the center $Z(H)$ is non-trivial, one does not need to know the wavelet transform on the whole group G, but it should be defined on only the central extension $G/Z(H)$, then the reduced wavelet transform $\mathcal{W}$ to a central extension space of function L_2 is defined by a projective representation $\hat{\pi}$ of $G \times G$ on $\mathcal{B}$ and a test functional $l_0 \in \mathcal{B}^*$ such that[1]:

$$\begin{aligned} \mathcal{W} : \mathcal{B} \to L_2(X) : \hat{O} \to \hat{O}(x) &= \left[\mathcal{W}\hat{O}\right](x) \\ &= \left\langle \hat{\pi}\left(x^{-1}\right) \hat{O}, l_0 \right\rangle = \left\langle \hat{O}, \hat{\pi}^*(x) l_0 \right\rangle, \quad \forall x \in X, \end{aligned} \tag{5}$$

where $\hat{\pi}^*$ is dual of $\hat{\pi}$.

3. Quantum entanglement *via* group theory with wavelet transform on Banach space

In order to reformulate the separability problem in terms of the group theoretical language [17], let G be a compact group, with an irreducible unitary representation U acting on separable Hilbert space $\mathcal{H}$, and let us assume that ρ is separable, *i.e.*, there exists a decomposition of type $\rho = \sum_i p_i |u_i\rangle\langle u_i| \otimes |v_i\rangle\langle v_i|$. According to [25, 26] the definition of the characteristic or the sampling functions $\Phi_\rho(g_1, g_2)$ is as follows:

$$\Phi_\rho(g) \equiv tr\left(\rho U(g)\right)$$

On the other hand, the wavelet transformation in Banach space for an arbitrary operator $\hat{O}$ has the following form:

$$\begin{aligned} \mathcal{W} : \mathcal{B} \mapsto F(g) : \hat{O} \mapsto \hat{\phi}(g) &= \left\langle \hat{O}, l_g \right\rangle \\ &= \left\langle \hat{O}, U(g) l_0 \right\rangle = \left\langle \hat{O} U(g)^\dagger, l_0 \right\rangle = tr\left(\hat{O} U(g)^\dagger\right). \end{aligned} \tag{6}$$

One can see that the characteristic function and the wavelet transformation are the same.
Now the characteristic function in Banach space with above density matrix and for irreducible representation $U(g) \equiv U_1(g_1) \otimes U_2(g_2)$ is obtained as:

$$\Phi_\rho(g_1, g_2) = tr\left(\rho U(g)\right) = \sum_i p_i K_i(g_1)\eta_i(g_2), \tag{7}$$

where $K_i(g_1) = \langle u_i \mid U(g_1) u_i \rangle$, $\eta_i(g_2) = \langle v_i \mid U(g_2) v_i \rangle$.
We state the following results which are standard and are presented in [17] which are needed in the sequel.

Theorem 3.1.

Let G be a compact Kinematical group and π, τ are irreducible representation. A state ρ is separable iff its characteristic function can be written in the form of:

$$\Phi_\rho(g_1, g_2) = \sum_i p_i K_i(g_1)\eta_i(g_2),$$

where $K_i, \eta_i \in \mathcal{P}_1(G)$ ($\mathcal{P}_1(G)$ is the space of all normalized positive definite functions on G) and the equality holds almost everywhere w.r.t. the Haar measure dg on $G \times G$.

Theorem 3.2.

Let G be a compact Kinematical group and π, τ are irreducible representations at G and ρ is an arbitrary state in $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$: The condition ($\rho$ is separable) $\Rightarrow$ $\tilde{\phi}_\rho \in \mathcal{P}(G \otimes G)$ leads either to PPT criterion for ρ when $\pi \sim \bar{\pi}$ or is empty otherwise. Where $\bar{\pi}(g) \equiv \pi(g^{-1})$ and $\tilde{\phi}(g_1, g_2) \equiv \phi(g_1^{-1}, g_2)$.

In the next section, we apply our method to some well known examples.

3.1. Moyal-type representations for a spin

For a spin s, in [29], 'Stratonovich-Weyl' correspondence, as a rule which maps each operator $\widehat{O}$ on the $(2s+1)$-dimensional Hilbert space $\mathcal{H}_s$ to a function on the phase space of the classical spin, $\mathcal{S}^2$, is defined. A *discrete* Moyal formalism is defined as: [30]:

$$\hat{\Delta}_{\mathbf{n}} = \widehat{U}_{\mathbf{n}} \hat{\Delta}_{\mathbf{n}_z} \hat{U}_{\mathbf{n}}^{\dagger}, \tag{8}$$

where $\hat{U}_{\mathbf{n}}$ represents a rotation which maps the vector $\mathbf{n}_z$ to $\mathbf{n}$ and the associated kernel are:

$$\hat{\Delta}_{\mathbf{n}} \;=\; |s,\mathbf{n}\rangle\langle s,\mathbf{n}| \equiv |\mathbf{n}\rangle\langle\mathbf{n}|, \tag{9}$$

$$\hat{\Delta}^{\mathbf{n}} = \sum_{m=-s}^{s} \Delta^m |m,\mathbf{n}\rangle\langle m,\mathbf{n}|. \tag{10}$$

In the wavelet notation, the Banach space is $(2s+1)^2$-dimensional and group is $SU(2)$, the subgroup is $U(1)$ and the measure is $d\mu(n) = \frac{2s+1}{4\pi}\, d(\mathbf{n})$ and the unitary irreducible representation of the group is U_n which is the result of the adjoint representation on any operators in Banach space:

$$\hat{\pi}(n)\hat{\rho} = \hat{U}_n \hat{\rho} \hat{U}_n^{\dagger}. \tag{11}$$

Then the wavelet transform in this Banach space with respect to the test functional,

$$l_0(\hat{\rho}) = \mathrm{Tr}\left(\hat{\rho}\sum_m \hat{\Delta}^m |m,n_z\rangle\langle m,n_z|\right), \tag{12}$$

is given by:

$$\begin{aligned}\mathcal{W}\hat{\rho} = \phi(n) &= \left\langle \hat{\pi}^{\dagger}(n)\hat{O}, l_0\right\rangle \\ &= \mathrm{Tr}\left(\hat{U}_n^{\dagger}\hat{\rho}\hat{U}_n \sum_m \hat{\Delta}^m \mid m,n_z\rangle\langle m,n_z \mid\right),\end{aligned} \tag{13}$$

then the characteristic function has the following form:

$$\phi(n) = \mathrm{Tr}\left(\hat{\rho}\hat{U}_n \sum_m \hat{\Delta}^m \mid m,n_z\rangle\langle m,n_z \mid \hat{U}_n^{\dagger}\right) = \mathrm{Tr}\left(\hat{O}\hat{\Delta}^n\right).$$

In the wavelet notation for the two partite spin systems, the irreducible representation $SU(2)\times SU(2)$ is $\hat{\Pi}(n,n') = \hat{\Pi}(n)\otimes\hat{\Pi}(n')$ and the test functional is defined as:

$$l_0(\hat{\rho}) = \mathrm{Tr}\left(\hat{\rho}\sum_m \hat{\Delta}^m |m,n_z\rangle\langle m,n_z| \otimes \sum_{m'} \hat{\Delta}^{m'} |m',n'_z\rangle\langle m',n'_z|\right), \tag{14}$$

then the characteristic function is shown as:

$$\phi(n,n') = \left\langle \hat{\pi}(n,n')^{\dagger}\hat{O}, l_0\right\rangle = \mathrm{Tr}\left(\hat{U}_n^{\dagger}\otimes\hat{U}_{n'}^{\dagger}\hat{\rho}\hat{U}_n\otimes\hat{U}_{n'} \sum_m \hat{\Delta}^m \mid m,n_z\rangle\langle m,n_z \mid \otimes \sum_{m'} \hat{\Delta}^{m'} \mid m',n'_z\rangle\langle m',n'_z \mid\right), \tag{15}$$

and we have:

$$\phi(n,n') = \mathrm{Tr}\left(\hat{\rho}\hat{U}_n\otimes\hat{U}_{n'} \sum_m \hat{\Delta}^m \mid m,n_z\rangle\langle m,n_z \mid \otimes \sum_{m'} \hat{\Delta}^{m'} \mid m',n'_z\rangle\langle m',n'_z \mid \hat{U}_n^{\dagger}\otimes\hat{U}_{n'}^{\dagger}\right) = \mathrm{Tr}\left(\hat{\rho}\hat{\Delta}^n\otimes\hat{\Delta}^{n'}\right). \tag{16}$$

In order to study the separability of the state ρ, let us consider the three-dimensional representation of $SO(3)$ group as a rotation U_n:

$$U_1(g_1) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}, \qquad U_2(g_2) = \begin{pmatrix} \lambda'_{11} & \lambda'_{12} & \lambda'_{13} \\ \lambda'_{21} & \lambda'_{22} & \lambda'_{23} \\ \lambda'_{31} & \lambda'_{32} & \lambda'_{33} \end{pmatrix}, \tag{17}$$

where λ_{ij} and $\lambda'_{ij}, i,j = 1,2,3$ are defined by using three Euler angles. Some $3\otimes 3$ un-normalized separable states [9, 11] are defined as:

$$\rho_m = \sum_k |\psi_{km}\rangle\langle\psi_{km}| = \sum_l |l\rangle\langle l| \otimes |l+m\rangle\langle l+m|,$$

$$\rho'_m = \sum_k |\psi_{mk}\rangle\langle\psi_{mk}| = \sum_{l,l',k} \omega^{m(l-l')}|l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|,$$

$$\rho''_n = \sum_k |\psi_{nk,k}\rangle\langle\psi_{nk,k}| = \sum_{l,l',k} \omega^{nk(l-l')}|l\rangle\langle l'| \otimes |l+k\rangle\langle l'+k|, \tag{18}$$

where $n = 0, 1, 2,\ m = 0, 1, 2.$ One can show that any convex sum of these states is separable and lies at the boundary of the separable region [9]. We can obtain the characteristic function $\Phi(g_1, g_2)$ for any convex sum of the above separable states but we consider a particular simple case ρ_0. Then the characteristic function $\Phi(g_1, g_2)$ for ρ_0 is obtained as:

$$\phi(g_1; g_2) = \mathrm{Tr}\left(\hat{\rho}_0 \hat{U}_\lambda \otimes \hat{U}_{\lambda'} \sum_{m=0}^{2} \hat{\Delta}^m \mid m\rangle\langle m \mid \otimes \sum_{m'=0}^{2} \hat{\Delta}^{m'} \mid m'\rangle\langle m' \mid \hat{U}_\lambda^\dagger \otimes \hat{U}_{\lambda'}^\dagger\right). \tag{19}$$

With some calculations we have:

$$\Phi(g_1, g_2) = \sum_{m=0}^{2} \Delta^m \left(\sum_{i=1}^{3} \lambda_{im}\lambda_{mi}\right) \sum_{m'=0}^{2} \Delta^{m'} \left(\sum_{j=1}^{3} \lambda_{jm'}\lambda_{m'j}\right) = \sum_{m=0}^{2} \Delta^m \sum_{m'=0}^{2} \Delta^{m'} = 1. \tag{20}$$

Using the definition of

$$K_i(g_1) = \langle i|U_1(g_1)|i\rangle = \sum_{j=1}^{3} \lambda_{ji}\lambda_{ij} = 1 \ \text{ and } \ \eta_i(g_2) = \langle i|U_2(g_2)|i\rangle = \sum_{j=1}^{3} \lambda_{ji}\lambda_{ij} = 1,$$

the characteristic function is rewritten as:

$$\Phi(g_1, g_2) = \sum_{i=1}^{3} p_i K_i(g_1)\eta_i(g_2) = 1 \qquad \text{for} \qquad p_i = \frac{1}{3}, \tag{21}$$

which is in agreement with Theorem 1 and we show that this state is separable. Just the same way we say that ρ_1 and ρ_2 are also separable states.
At the end of this section we introduce some possible bound entangled states. For this we first introduce the PPT operators. We consider the following $3 \otimes 3$ density matrix

$$\rho = p\left(|\psi_{00}\rangle\langle\psi_{00}|\right) + \frac{1-p}{3}(\mu_1\rho_1 + \mu_2\rho_2), \tag{22}$$

where it is positive for $p \geq 0, \mu_1, \mu_2 \leq 1$ and $\sum_{i=0}^{2} \mu_i = 1$. Now in order to obtain the PPT density matrix, we need to use the relations $\rho_i^{T_A} = \rho$ and substitute the partial transpose of Bell state projection operator $|\psi_{00}\rangle\langle\psi_{00}|$:

$$\left(|\psi_{00}\rangle\langle\psi_{00}|\right)^{T_A} = \frac{1}{3} \sum_{m,l=0}^{2} \omega^{ml} |\psi_{m,l}\rangle\langle\psi_{m,3-l}|, \omega = e^{\frac{2\pi\iota}{3}}, \tag{23}$$

in the partial transpose of Eq. (22). Now, the positivity of the partial transpose of density matrix (22) implies that:

$$p \leq \max\left\{\frac{\sqrt{\mu_1\mu_2}}{1 + \sqrt{\mu_1\mu_2}}\right\}. \tag{24}$$

For $\mu_1 = \mu_2 = \frac{1}{2}$ the parameter p is optimal and equal to $p = \frac{1}{3}$. As it is shown in [18], we obtain the characteristic function $\Phi(g_1, g_2)$ for this bound entangled state as:

$$\Phi(g_1,g_2) = \mathrm{Tr}[\rho U(g_1,g_2)] = p\left(\lambda_{11}\lambda'_{11}\left(\hat{\Delta}^0\right)^2 + \lambda_{22}\lambda'_{22}\left(\hat{\Delta}^1\right)^2 + \lambda_{33}\lambda'_{33}\left(\hat{\Delta}^2\right)^2\right)$$
$$+\frac{(1-p)}{3}\left(\mu_1\lambda_{11}\lambda'_{22}\hat{\Delta}^0\hat{\Delta}^1 + \mu_2\lambda_{11}\lambda'_{33}\hat{\Delta}^0\hat{\Delta}^2 + \mu_2\lambda_{22}\lambda'_{11}\hat{\Delta}^0\hat{\Delta}^1 + \mu_1\lambda_{22}\lambda'_{33}\hat{\Delta}^1\hat{\Delta}^2 + \mu_1\lambda_{33}\lambda'_{11}\hat{\Delta}^0\hat{\Delta}^2 + \mu_2\lambda_{33}\lambda'_{22}\hat{\Delta}^1\hat{\Delta}^2\right). \quad (25)$$

Comparing the characteristic function (25) with separable characteristic function (7), one can see that the function (25) cannot be represented as a convex mixture of products of positive definite functions $K_i(g_1)$ and $\eta_i(g_2)$, depending on parameters (λ_{ii}) and (λ'_{ii}) respectively.

3.2. Quantum entanglement based on projective representation of permutation group

Now we consider the projective representations of the symmetric (permutation) groups that have long been known to mathematicians, but received little attention from physicists. Such representations were overlooked in physics much like projective representations of the rotation groups in the early days of quantum mechanics. One especially useful presentation of the symmetric group S_n on n elements is given by:

$$S_n = \left\{t_1,\ldots,t_{n-1} : t_i^2 = 1, \left(t_j t_{j+1}\right)^3 = 1, t_m t_l = t_l t_m\right\}, \quad (26)$$

where $1 \le i \le n-1, 1 \le j \le n-2, m \le l-2$. Here t_i are transpositions,

$$t_1 = (12), t_2 = (23), \ldots, t_{n-1} = (n-1 \;\; n). \quad (27)$$

Closely related to S_n is group $\tilde{S}_n$,

$$\tilde{S}_n = \{z, t'_{1;2}, \ldots, t'_{n-1;n} \mid z^2 = 1, z t'_{i;i+1} = t'_{i;i+1} z,$$
$${t'_{1;2}}^2 = z, \left(t'_{j;j+1} t'_{j+1;j+2}\right)^3 = z,$$
$$t'_{m;m+1} t'_{l;l+1} = z t'_{l;l+1} t'_{m;m+1}\}, \quad (28)$$

where $1 \le i \le n-1, 1 \le j \le n-2, m \le l-2$.
A celebrated theorem of Schur (Schur, 1911 [31]) states the following:

(i) Group $\tilde{S}_n$ has the order $2(n!)$.

(ii) The subgroup $\{1, z\}$ is central, and is in the commutator subgroup of $\tilde{S}_n$, provided that $n = 4$.

(iii) $\tilde{S}_n/\{1, z\} \simeq S_n$.

(iv) If $n < 4$, then every projective representation of S_n is projectively equivalent to a linear representation.

(v) If $n \le 4$, then every projective representation of S_n is projectively equivalent to a representation $\hat{\pi}$,

$$\rho(S_n) = \{\hat{\pi}(t_1), \ldots, \hat{\pi}(t_{n-1}) : \hat{\pi}(t_i)^2 = z,$$
$$\left(\hat{\pi}(t_j)\hat{\pi}(t_{j+1})\right)^3 = z, \hat{\pi}(t_m)\hat{\pi}(t_l) = z\hat{\pi}(t_l)\hat{\pi}(t_m)\}, \quad (29)$$

where $1 \le i \le n-1, 1 \le j \le n-2, m \le l-2$ and $z = \pm 1$.

In the case $z = +1$, $\hat{\pi}$ is a linear representation of S_n. The group $\tilde{S}_n$ is called the representation group for S_n. The most elegant way to construct a projective representation $\hat{\pi}(S_n)$ of S_n is by using the complex Clifford algebra $\mathrm{Cliff}_C(V, g) \equiv \mathcal{C}_n$ associated with the real vector space $V = n\mathcal{R}$,

$$\{\gamma_i, \gamma_j\} = 2g(\gamma_i, \gamma_j) \quad (30)$$

Here, $\{i\}_{i=1}^n$ is an orthonormal basis of V with respect to the symmetric bilinear form:

$$g(\gamma_i, \gamma_j) = +\delta_{ij}. \quad (31)$$

Clearly, any subspace $\bar{V}$ of $V = nR$ generates a subalgebra $\mathrm{Cliff}_C(\bar{V}, \bar{g})$, where $\bar{g}$ is the restriction of g to $\bar{V} \times \bar{V}$. A particularly interesting case is realized when $\bar{V}$ is:

$$\bar{V} = \left\{\sum_{m=1}^n \alpha^m \gamma_m : \sum_{m=1}^n \alpha^m = 0\right\} \quad (32)$$

of codimension one, with the corresponding subalgebra denoted by $\bar{C}_{n-1}$ [32]. If we consider a special basis $\{t'_m\}_{m=1}^{n-1} \subset \bar{V}$ (which is not orthonormal) defined by:

$$t'_{m;m+1} = \frac{1}{\sqrt{2}}(\gamma_m + \gamma_{m+1}) \quad m = 1, \ldots n-1, \quad (33)$$

then the group generated by this basis is isomorphic to $\tilde{S}_n$. This can be seen by mapping t_i to t'_i and z to -1, and by noticing that:

1) For $m = 1, ..., n-1$:

$$t'^2_{m;m+1} = -1. \tag{34}$$

2) For $n-2 \geq j$:

$$\left(t'_{j;j+1} t'_{j+1;j+2}\right)^3 = -1. \tag{35}$$

3) For $n-1 \geq q > m+1$:

$$t'_{m;m+1} t'_{q;q+1} = -t'_{q;q+1} t'_{m;m+1}, \tag{36}$$

as can be checked through direct calculation. One choice for the matrices is provided by the following construction (Brauer and Weyl, 1935 [33]):

$$\begin{cases} \gamma_{2m-1} = \sigma_3 \otimes ...\sigma_3 \otimes (\sigma_1) \otimes 1... \otimes 1, \\ \gamma_{2m} = \sigma_3 \otimes ...\sigma_3 \otimes (\sigma_2) \otimes 1... \otimes 1, \\ m = 1, 2, 3, ...k, \end{cases} \tag{37}$$

for $n = 2k$. Here σ_1, σ_2 occur in the m-th position, the product involves k factors, and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. If $n = 2k+1$, we first add one more matrix,

$$\gamma_{2k+1} = \sigma_3 \otimes ... \otimes \sigma_3 \quad (k \ \text{factors}). \tag{38}$$

An irreducible module of $\bar{C}_{n-1}$ restricts that representation to the irreducible representation of $\tilde{S}_n$, since $\{t'_{i;i+1}\}_{i=1}^{n-1}$ generates $\bar{C}_{n-1}$ as an algebra [32]. The simplest (irreducible) non-trivial projective representations of S_n are already surprisingly intricate and have dimensions which grow exponentially with n. They are intimately related to spinor representations of $SO(n)$[34].

Now, we try to obtain the characteristic function *via* wavelet transforms in Banach space based on the projective representation of permutation group (spinor representation of permutation group). In order to do so, we need to choose the set:

$$\{\hat{\pi}_{i_1,...,i_m} = \gamma_1^{i_1}\gamma_2^{i_2}\gamma_3^{i_3}...\gamma_m^{i_m} \quad , \quad i_1, i_2, ..., i_m = \{0, 1\}\},$$

as a projective representation and the identity operator as a vacuum vector. So, the corresponding wavelet transformation takes the following form:

$$\begin{aligned} &\mathcal{W} : \mathcal{B} \mapsto L_2(G) : \hat{\rho} \mapsto \hat{\rho}(i_1, ..., i_m) \\ &= \langle \hat{\rho}, l_{(i_1,...,i_m)} \rangle = \langle \hat{\rho}, \hat{\pi}_{i_1,...,i_m} l_0 \rangle = \mathrm{Tr}\left(\hat{\rho}\hat{\pi}^{\dagger}_{i_1,...,i_m}\right). \end{aligned} \tag{39}$$

In the wavelet notation for the two-partite permutation group, for simplicity, we reduce our considerations on the irreducible representation of a non-abelian group $S_3 \times S_3$:

$$\begin{aligned} &\hat{\pi}(g_1, g_2) = \hat{\pi}_{i_1,i_2} \otimes \hat{\pi}_{j_1,j_2} \\ &\qquad \text{with} \quad \hat{\pi}_{i_1,i_2} = \gamma_1^{i_1}\gamma_2^{i_2}, i_1, i_2 = \{0, 1, 2\}, \end{aligned} \tag{40}$$

then the characteristic function is defined as:

$$\phi(i_1, i_2; j_1, j_2) = \left\langle \hat{\rho}, l_{(i_1,i_2;j_1,j_2)} \right\rangle = \left\langle \hat{\rho}, \hat{\pi}_{i_1,i_2;j_1,j_2} l_0 \right\rangle, \tag{41}$$

using test functional l_0 *via* trace function the characteristic function is reduced to:

$$\phi(i_1, i_2; j_1, j_2) = \left\langle \hat{\rho}\hat{\pi}^{\dagger}_{i_1,i_2;j_1,j_2}, l_0 \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{\pi}^{\dagger}_{i_1,i_2;j_1,j_2}\right). \tag{42}$$

In order to study the separability criteria for any given state ρ, let us consider the $3 \otimes 3$ Werner states

$$\begin{aligned} \rho_f = \frac{1}{24}\left((3-f)I + (3f-1)\sum_{i,j} |ij\rangle\langle ji|\right), \\ -1 \leq f \leq 1, \end{aligned} \tag{43}$$

where I is the identity operator. The characteristic function (42) for the above-mentioned $3 \otimes 3$ Werner states has a matrix form $\Phi_{\alpha\alpha',\beta\beta'} = \phi(g_{\alpha}^{-1} g_{\beta}, g_{\alpha'}^{-1} g_{\beta'})$, where this matrix elements are determined using the group multiplication table [17].

Due to theorem 2, a function $\phi \in \mathcal{P}(S_3 \times S_3)$ is separable if $\Phi^T \geq 0$ ($\Phi_{\alpha\alpha',\beta\beta'} \geq 0 \Rightarrow \Phi_{\beta\alpha',\alpha\beta'} \geq 0$). Then by some simple but vast calculations one can show that Φ^T will be positive if $0 \leq f \leq 1$ *i.e.*, the Werner states is separable if $0 \leq f \leq 1$.

3.3. Quantum entanglement of dihedral group based on projective representation

The dihedral group D_n of order $2n$ is defined by [32, 35]:

$$D_n = \left\langle a, b | a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \right\rangle. \tag{44}$$

Let ε be a primitive nth root of 1 and let:

$$\chi : D_n \times D_n \to c^*, \tag{45}$$

be defined by [36]:

$$\chi\left(a^i, a^j b^k\right) = 1 \quad and \quad \chi\left(a^i b, a^j b^k\right) = \varepsilon^j, \tag{46}$$

for all $i, j \in \{0, 1, 2, ..., n-1\}$ and $k \in \{0, 1\}$.
For n=2m is even, for each $r \in \{1, ..., m-1\}$ put:

$$A_r = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{-r} \end{pmatrix}, \quad A_m = \begin{pmatrix} \varepsilon^m & 0 \\ 0 & \varepsilon^m \end{pmatrix}$$

$$B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{47}$$

and for n=2m+1 is odd [37], for each $r \in \{1, ..., m\}$ put:

$$A_r = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{-r} \end{pmatrix}, \quad B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{48}$$

and let $\hat{\pi}_r : D_n \to GL(2, c)$ be defined by:

$$\hat{\pi}(i, j) = \hat{\pi}_r \left(a^i b^j\right) = A_r^i B_r^j, \quad i \in \{0, 1, ..., n-1\}, \qquad j = \{0, 1\}. \tag{49}$$

Now we try to obtain the characteristic function *via* wavelet transform in Banach space based on the projective representation. In order to do so, we need to choose the set $\hat{\pi}(i, j)$ as a projective representation and the identity operator as a vacuum vector. So the corresponding wavelet transformation takes the following form:

$$\mathcal{W} : \mathcal{B} \mapsto L_2(G) : \hat{\rho} \mapsto \hat{\rho}(i, j) = \left\langle \hat{\rho}, l_{(i,j)} \right\rangle = \langle \hat{\rho}, \hat{\pi}(i, j) l_0 \rangle = \mathrm{Tr}\left(\hat{\rho}\hat{\pi}^\dagger(i, j)\right). \tag{50}$$

In the wavelet notation for the two partite permutation group, for simplicity we reduce our considerations on the irreducible representation $D_3 \times D_3$ is $\hat{\pi}(g_1, g_2) = \hat{\pi}_{i_1,i_2} \otimes \hat{\pi}_{j_1,j_2}$, with $\pi_{i_1,i_2} = A_r^{i_1} B_r^{i_2}, i_1, i_2 = \{0, 1, 2\}$, and the characteristic function is defined as

$$\phi\left(i_1, i_2; j_1, j_2\right) = \left\langle \hat{\rho}, l_{(i_1,i_2;j_1,j_2)} \right\rangle = \left\langle \hat{\rho}, \hat{\pi}_{i_1,i_2;j_1,j_2} l_0 \right\rangle, \tag{51}$$

using test functional l_0 *via* trace function the characteristic function is reduced to

$$\phi\left(i_1, i_2; j_1, j_2\right) = \left\langle \hat{\rho}\hat{\pi}^\dagger_{i_1,i_2;j_1,j_2}, l_0 \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{\pi}^\dagger_{i_1,i_2;j_1,j_2}\right). \tag{52}$$

Considering the $3 \otimes 3$ Werner states similar to the permutation group (43) and obtaining the matrix of characteristic function, one can see that the Werner states will be separable if $0 \le f \le 1$.

4. Conclusions

The universal description of quantum entanglement has been considered using group theory and non-commutative characteristic functions for homogenous space and the projective representation of the group on Banach spaces for some of the well known examples, such as: Moyal representation for a spin, Dihedral and permutation groups. Entanglement consideration for other homogenous spaces and the projective representation of the groups is under investigation.

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