# Backtracking and Branch and Bound

# Introduction

- Solutions to many combinatorial optimization problems include exhaustive search
  - Optimal solution desired at cost of speed
  - Exhaustive-search technique suggests generating all candidate solutions and then identifying the one (or the ones) with a desired property
- Backtracking can be used
  - To reduce the cost of search
  - To list all possible solutions for a combinatorial problem

# Backtracking: Overview

- Systematic/intelligent way to iterate through all the possible configurations of a search space
  - Configurations may represent
    - all possible arrangements of objects (permutations)
    - all possible ways of building a collection of them (subsets)
  - Configurations must be generated only once, and potential configurations must not be missed
- Model combinatorial search solution as a vector $a = (a_1, a_2, ..., a_k)$
  - Vector might represent an arrangement where $a_i$ contains the ith element of the permutation
  - Or represent a given subset *S*, where $a_i$ is true if and only if the *i*th element of the universe is in *S*.

# Backtracking: Overview

- Strategy
  - At each step during backtracking
    - try to extend a given partial solution $a = (a_1, a_2, ..., a_k)$ by adding another element at the end
    - Test if the extending lead to a solution or not
    - If solution not found explore if proceeding further will lead to a solution or if we have to go back to a previous partial solution
- Constructs a tree of partial solutions
  - Each node represents a partial solution
  - Edge indicates an advancement of a solution

# Search Space Tree

- A rooted tree where each level represents a choice in the solution space that depends on
  - the level above and
  - any possible solution is represented by some path starting out at the root and ending at a leaf
- Root represents state where no partial solution has been made
- A leaf represents the state where all choices making up a solution have been made

# Backtracking Overview

- constructs a tree of partial solutions, where each vertex represents a partial solution
  - This tree also called a “state-space tree”
  - A node in a state-space tree is ***promising*** if it corresponds to a partial solution that may still lead to a complete solution;
    - its child is generated by adding the first remaining legitimate option for the next component of a solution, and the processing moves to this child
  - Otherwise, it is called ***nonpromising***
    - Leaves represent nonpromising solutions or dead-ends
    - algorithm backtracks to the node's parent to consider the next possible option for its last component

# Backtracking Overview

- Corresponds to doing a DFS of the state-space tree
- Backtrack-DFS(A, k)

```
if A = (a1, a2, ..., ak) is a solution, report it.
else
    k = k + 1
    compute Sk
    while Sk = ∞ do
        ak = an element in Sk
        Sk = Sk − ak // Sk is a finite set where ak belongs to
        Backtrack-DFS(A, k)
```

# Backtracking - Procedure

- backtrack(int a[], int k, data input) {

```
if (is_a_solution(a, k, input) process_solution(a, k, input)
else {
    k=k+1;
    construct_candidates(a,k,input,c,ncandidates);
    for (i=0; i<ncandidates; i++) {
        a[k] = c[i];
        make_move(a,k,input);
        backtrack(a,k,input);
        unmake_move(a,k,input);
        if (finished) return; /* terminate early *
    }
}
}
```

# Backtracking: Procedure Details

- is a solution(a,k,input):
  - tests whether the first k elements of vector a from a complete solution for the given problem
- construct candidates(a,k,input,c,ncandidates):
  - fills an array c with the complete set of possible candidates for kth position of a, given contents of first k − 1 positions
- process solution(a,k,input):
- make move(a,k,input) and unmake move(a,k,input)
  - Modify data structure in response to latest move

# Problem 1: Constructing Subsets

- How many subsets are there of an n-element set, say the integers {1, . . . , n}?
  - there are $2^n$ subsets of n elements
- Solution
  - set up an array/vector of n cells that represents a subset
  - The value of ai is true or false and signifies whether the $i^{th}$ item is in the given subset.
  - The termination happens when k=n

# Solution

- void subsets(int k, boolean[] a) {
  - if (k == N) {

    Do something ... maybe print it.

    Return; }

  // A[k] is not in the subset.

  a[k] = false;

  subsets(k + 1, a);

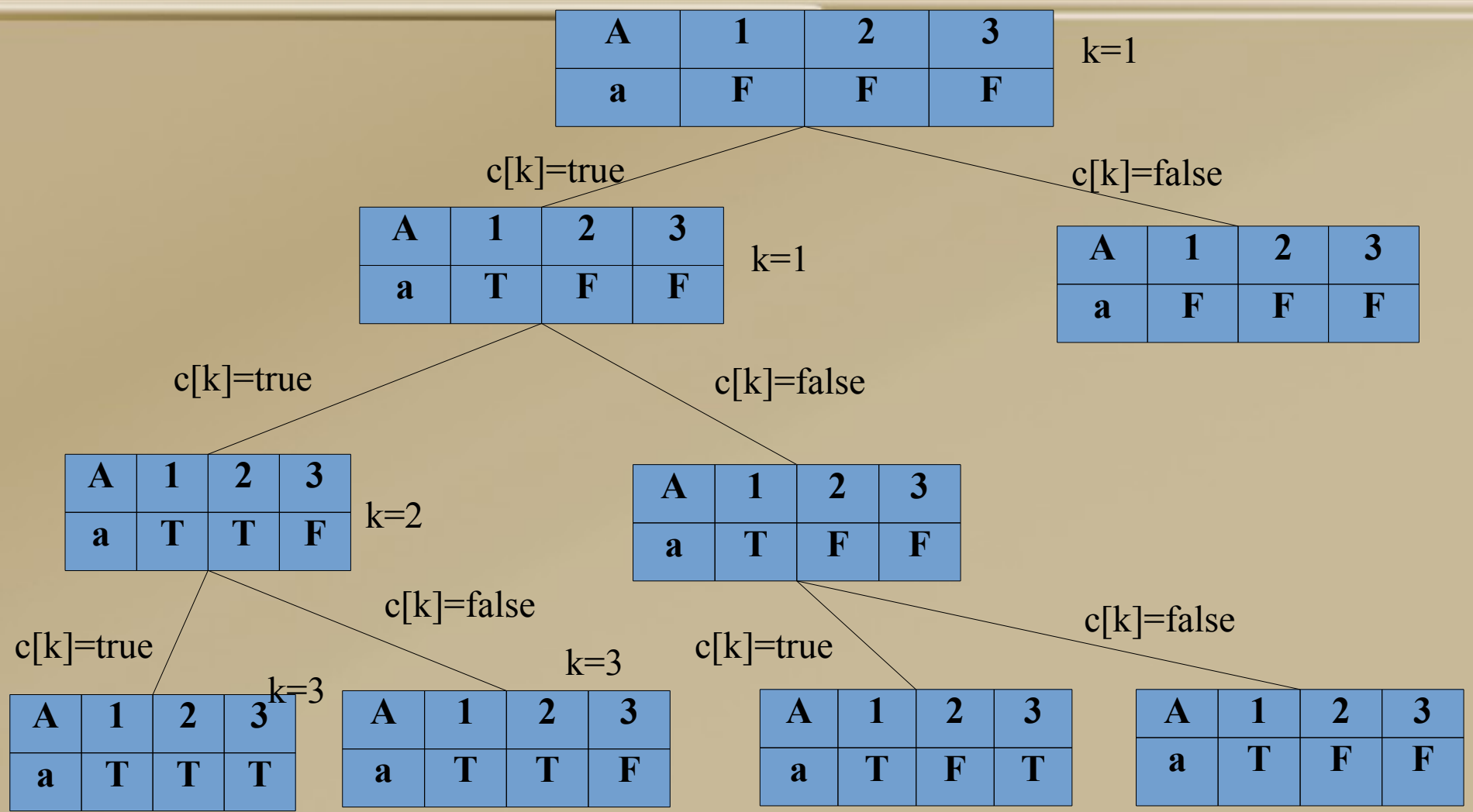
  // A[k] is in the subset.

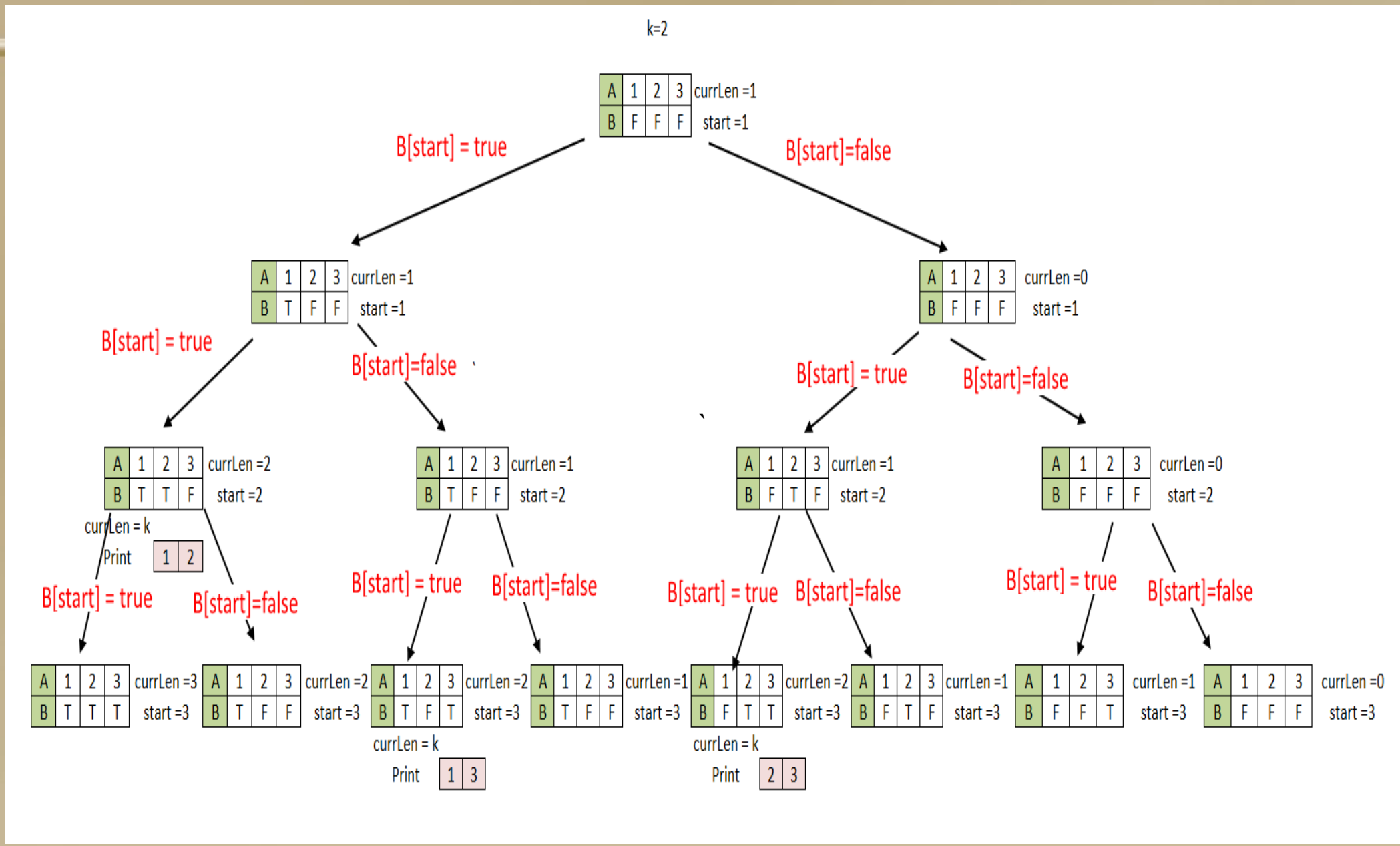
  a[k] = true;

  subsets(k + 1, a);

}

# Example


A 1 2 3
a F F F
k=1
c[k]=true
c[k]=false
A 1 2 3
a T F F
k=1
A 1 2 3
a F F F
c[k]=true
c[k]=false
A 1 2 3
a T T F
k=2
A 1 2 3
a T F F
c[k]=true
c[k]=false
c[k]=true
c[k]=false
A 1 2 3
a T T T
k=3
A 1 2 3
a T T F
k=3
A 1 2 3
a T F T
A 1 2 3
a T F F


k=2
A 1 2 3 currLen =1
B F F F start =1
B[start] = true
B[start]=false
A 1 2 3 currLen =1
B T F F start =1
A 1 2 3 currLen =0
B F F F start =1
B[start] = true
B[start]=false
B[start] = true
B[start]=false
A 1 2 3 currLen =2
B T T F start =2
A 1 2 3 currLen =1
B T F F start =2
A 1 2 3 currLen =1
B F T F start =2
A 1 2 3 currLen =0
B F F F start =2
currLen = k
Print 1 2
B[start] = true
B[start]=false
B[start] = true
B[start]=false
B[start] = true
B[start]=false
B[start] = true
B[start]=false
A 1 2 3 currLen =3
B T T T start =3
A 1 2 3 currLen =2
B T F F start =3
A 1 2 3 currLen =2
B T F T start =3
A 1 2 3 currLen =1
B T F F start =3
A 1 2 3 currLen =1
B F T T start =3
A 1 2 3 currLen =2
B F T F start =3
A 1 2 3 currLen =1
B F F T start =3
A 1 2 3 currLen =0
B F F F start =3
currLen = k
Print 1 3
currLen = k
Print 2 3

# Solution

- is_a_solution(int a[], int k, int n) {

  return (k == n); /* is k == n? */

  }

- construct_candidates(int a[], int k, int n, int c[], int ncandidates){

  c[0] = TRUE;

  c[1] = FALSE;

  ncandidates = 2;

  }

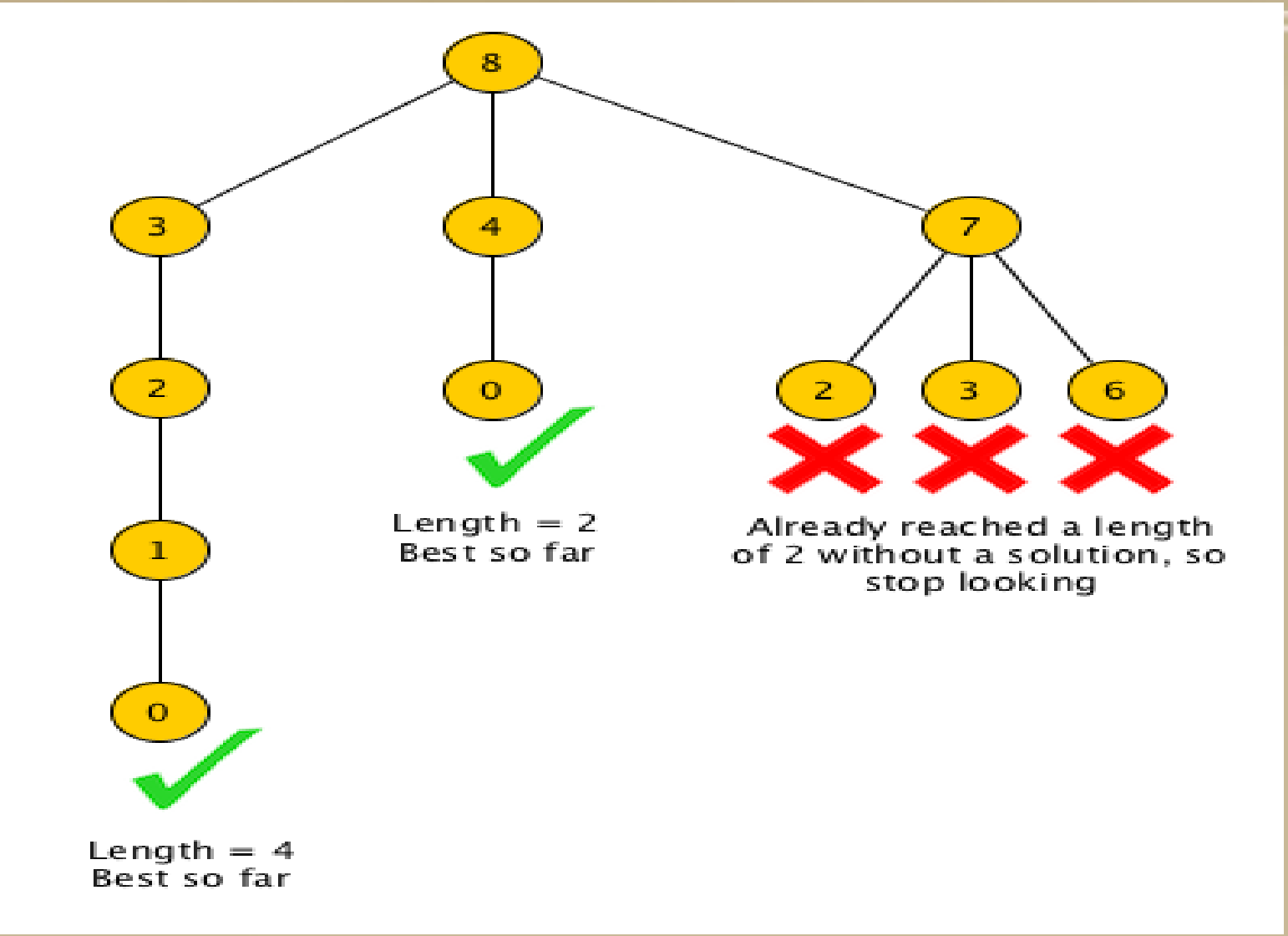
# Solution

```
process_solution(int a[], int k) {
    int i; /* counter */
    printf("{");
    for (i=1; i<=k; i++)
        if (a[i] == TRUE) printf(" %d",i);
    printf(" }\n");
}
```

- generate_subsets(int n)

```
{
    int a[NMAX]; /* solution vector */
    backtrack(a,0,n);
}
```

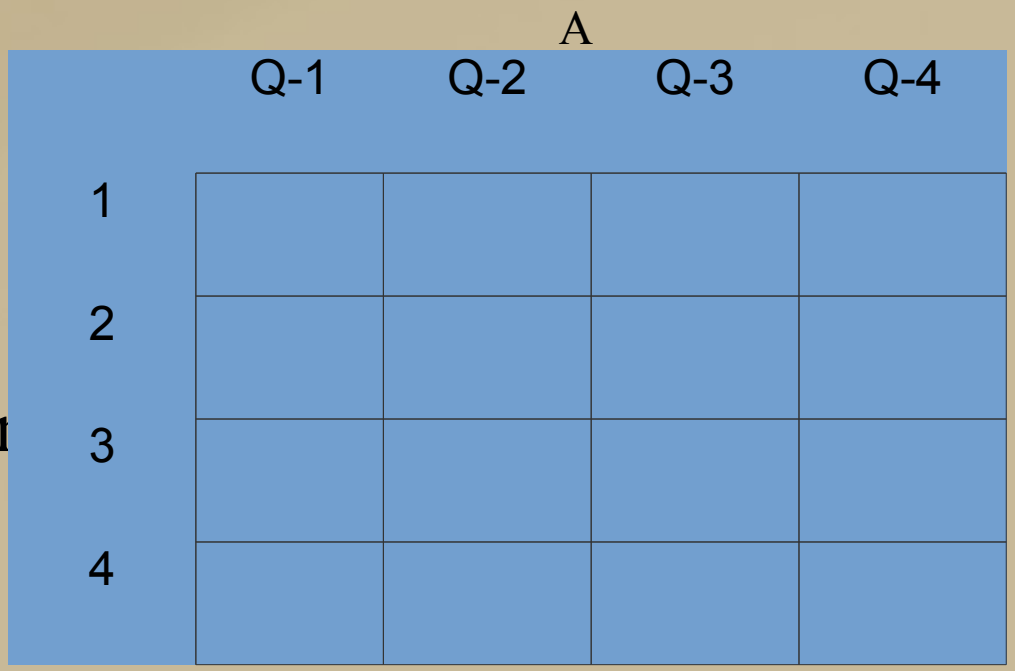
# Another Example




http://cs.lmu.edu/~ray/notes/backtracking/

# N-Queens Problem

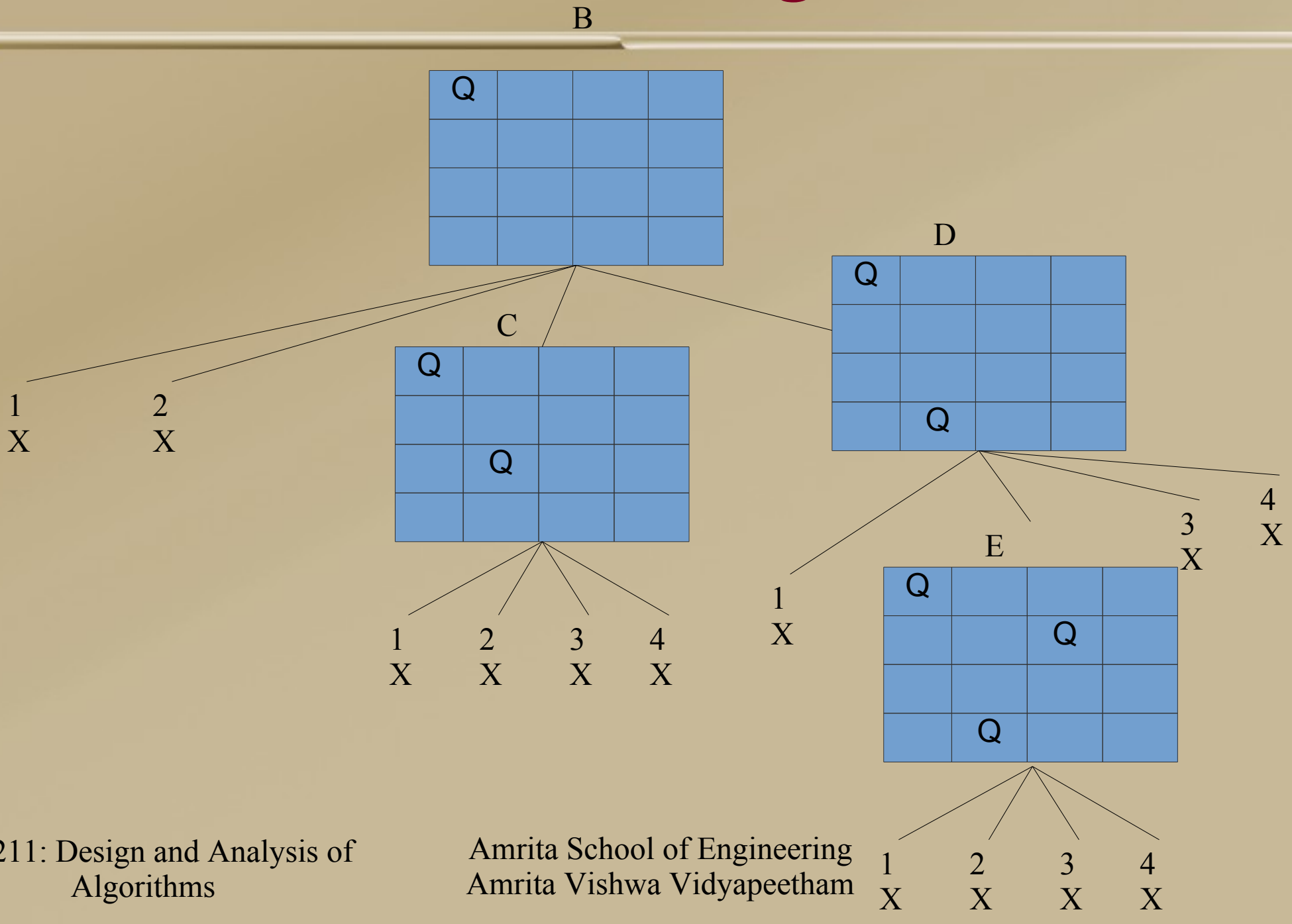
- n-Queens Problem Place n queens on an n×n chessboard so that no two queens attack each other
  - Two queens cannot be in the same column, row, or diagonal
- Solution trivial for n=1
  - No solution for n=2 or 3
- 4 Queens Problem
  - Each queen to be placed in its own column



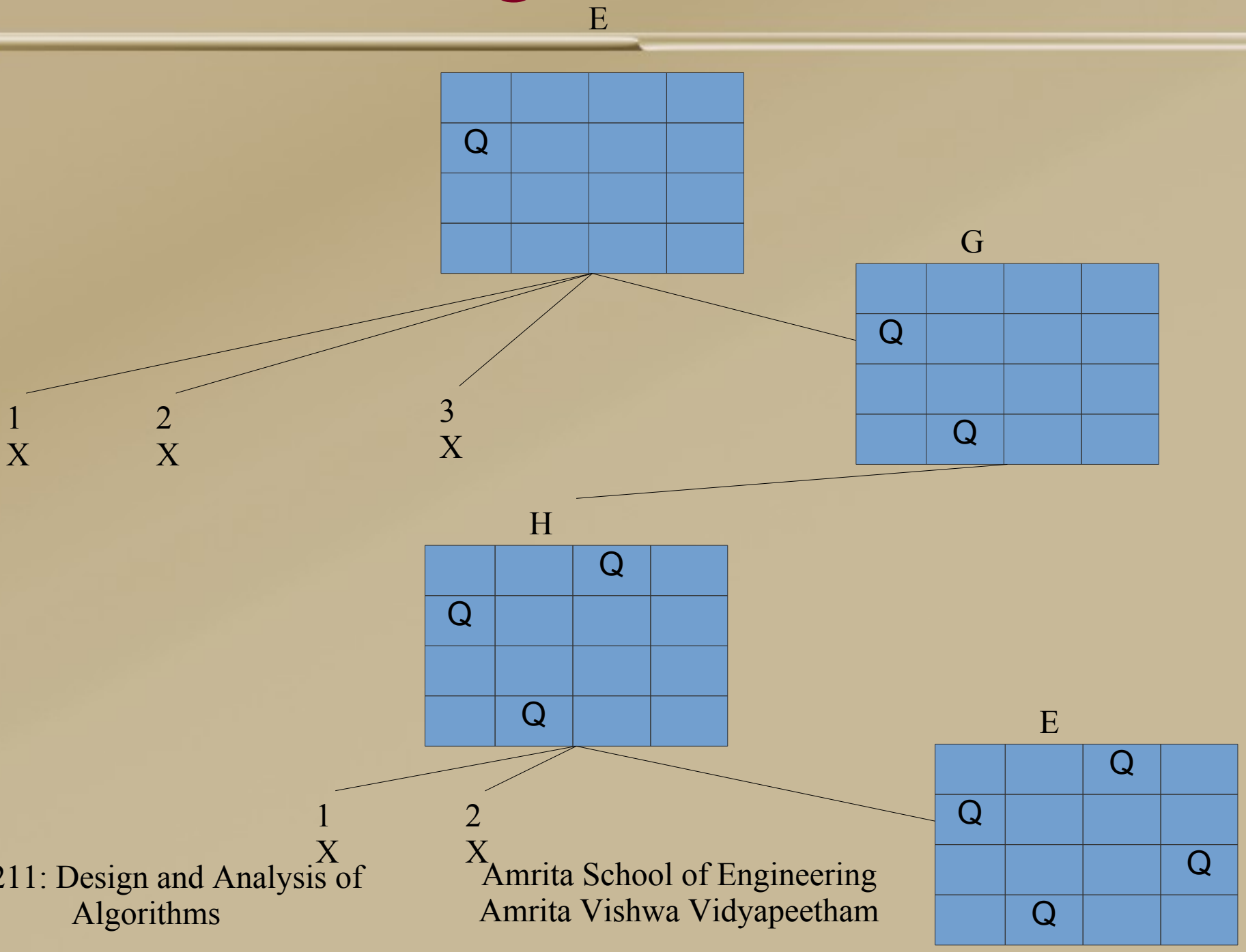
# 4-Queens- Backtracking Solution

- Start with queen1 and place in first possible position – [1,1], place queen2, in rows 1 and 2 of the second column
  - Not acceptable
  - Acceptable solution is row 3 and column 2
- State space tree
  - Each node is a configuration for the column and possible row
  - X denotes an unacceptable configuration

# 4-Queens: Backtracking: Dead-end



# Backtracking: Possible Solution



# Pseudocode

```
tryConfig(i):
    for j = 1 to n:
        if safe then:
            select jth candidate;
            set queen
            if i < n then:
                tryConfig(i+1);
            else
                record solution
            remove queen
```

Src: http://www.brian-borowski.com/software/nqueens/

# Performance

- Exhaustive Search
  - Number of placements = 16! / 4!(16 − 4)!
  - = (16 · 15 · 14 · 13)/(4 · 3 · 2) = 1820.
- Backtracking
  - Consider the number of combinations of n objects taken at k at a time, consider only queens placed at different columns – solution candidates = $4^4$ = 256
  - Queens must be at different rows
    - Solution candidates = 4! = 24
    - For 8-queens problem solution candidates = 40,320

# Branch and Bound

- In an optimization problem
  - *Feasible solution* is a point in the problem's search space that satisfies all the problem's constraints
  - *Optimal solution* is a feasible solution with best value to objective function
- Backtracking stops when solution is infeasible
  - This idea can be strengthened

# Branch and Bound

- Two aspects required in this approach
  - a way to provide, for every node of a state-space tree, a bound on the best value of the objective function on any solution that can be obtained
  - The value of best solution seen so far
- Principle Idea
  - If node's bound value is not better than the best seen so far node is non promising, hence pruned.
  - no solution obtained from the node can yield a better solution than the one already available.

# Search Space Pruning

- A search along a path is terminated if
  - The value of the node's bound is not better than the value of the best solution seen so far
  - Constraints of solution already violated, hence node represents no feasible solution
  - The subset of feasible solutions represented by the node consists of a single point (and hence no further choices can be made )

# 0-1 Knapsack Problem

- Construct a search space tree
  - if there are N possible items to choose from, then the kth level represents state where it has been decided which of the first k items have or have not been included in the knapsack.
    - The path shows the choices made for the first k items ie the selection from first k items
  - branch going to the left indicates the inclusion of the next item while a branch to the right indicates its exclusion

# O-1 Knapsack

- At each node record
  - total weight w of the selection
  - the total value v of this selection
  - Upper bound b
    - $b = v + ( W - w) (v_{i+1} / w_{i+1})$
    - $v$ – total value of items already selected
    - $W$-$w$ – remaining capacity of knapsack
    - $v_{i+1} / w_{i+1}$ - best per unit payoff among the remaining items

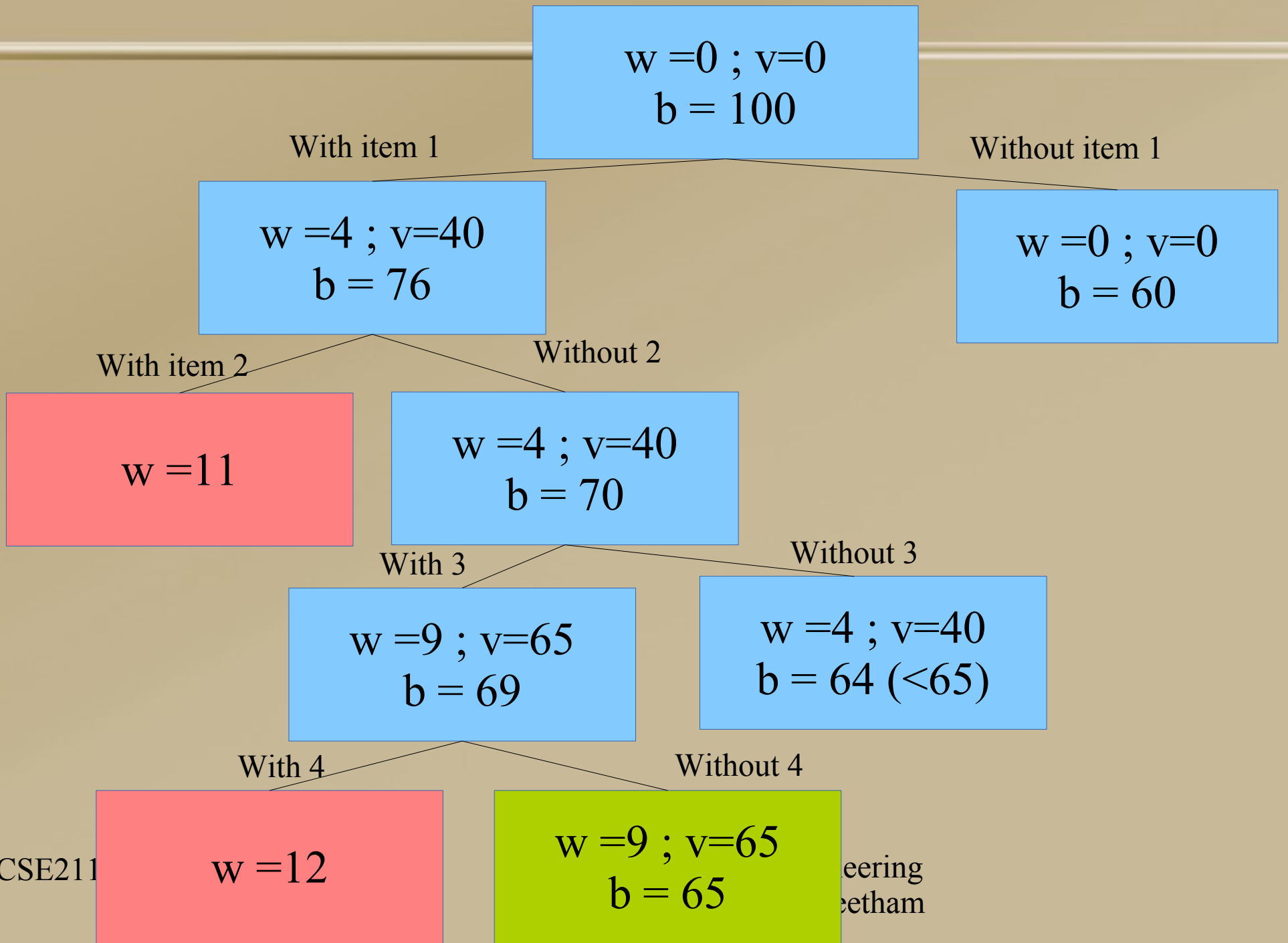
# Example

| Item | Weight | Value |
|---|---|---|
| 1 | 4 | 40 |
| 2 | 7 | 42 |
| 3 | 5 | 25 |
| 4 | 3 | 12 |

- Capacity of Knapsack – 10

w =0 ; v=0
b = 100

# Solution


w =0 ; v=0
b = 100
With item 1
Without item 1
w =4 ; v=40
b = 76
w =0 ; v=0
b = 60
With item 2
Without 2
w =11
w =4 ; v=40
b = 70
With 3
Without 3
w =9 ; v=65
b = 69
w =4 ; v=40
b = 64 (<65)
With 4
Without 4
w =12
w =9 ; v=65
b = 65

# References


- Steven Skiena, "The Algorithm Design Manual", Springer, 2008
- Anany Levitin, "Design and Analysis of Algorithms", 2nd Edition, 2006, Addison Wesley
- http://www.seas.gwu.edu/~ayoussef/cs212/branchandbou
- http://ocw.mit.edu/courses/sloan-school-of-management/