

Today: Neural Networks

Problem: How to compute gradients?

$$s = f(x; W_1, W_2) = W_2 \max(0, W_1 x) \quad \text{Nonlinear score function}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM Loss on predictions}$$

$$R(W) = \sum_k W_k^2 \quad \text{Regularization}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda R(W_1) + \lambda R(W_2) \quad \text{Total loss: data loss + regularization}$$

If we can compute $\frac{\partial L}{\partial W_1}, \frac{\partial L}{\partial W_2}$ then we can learn W_1 and W_2

(Bad) Idea: Derive $\nabla_W L$ on paper

$$s = f(x; W) = Wx$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \sum_{j \neq y_i} \max(0, W_{j,:} \cdot x + W_{y_i,:} \cdot x + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda \sum_k W_k^2$$

$$= \frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, W_{j,:} \cdot x + W_{y_i,:} \cdot x + 1) + \lambda \sum_k W_k^2$$

$$\nabla_W L = \nabla_W \left(\frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, W_{j,:} \cdot x + W_{y_i,:} \cdot x + 1) + \lambda \sum_k W_k^2 \right)$$

Problem: Very tedious: Lots of matrix calculus, need lots of paper

Problem: What if we want to change loss? E.g. use softmax instead of SVM? Need to re-derive from scratch =(

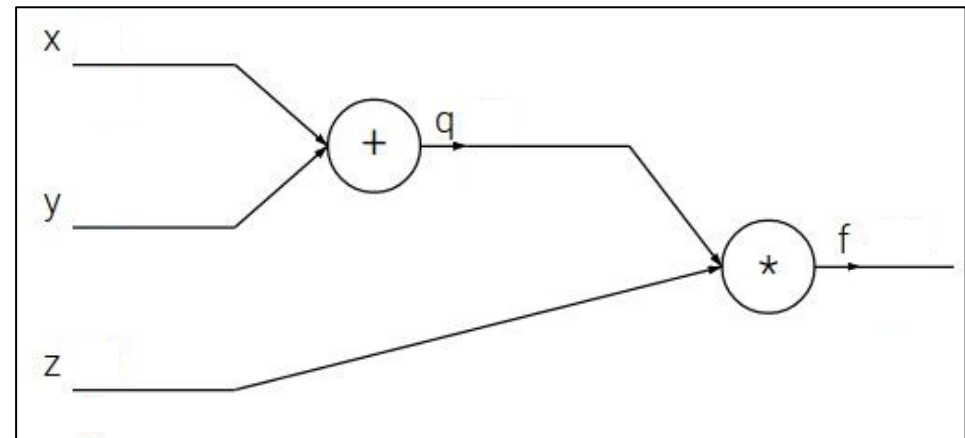
Problem: Not feasible for very complex models!

Backpropagation: a simple example

$$f(x, y, z) = (x + y)z$$

Backpropagation: a simple example

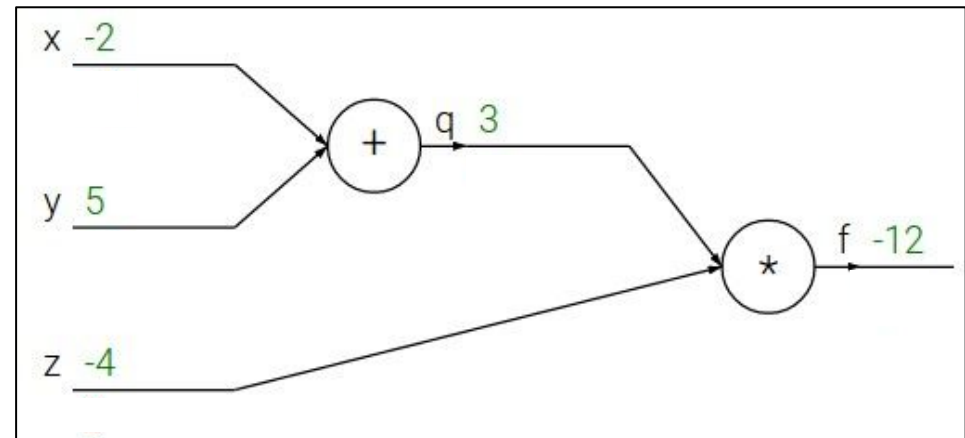
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e.g. $x = -2$, $y = 5$, $z = -4$



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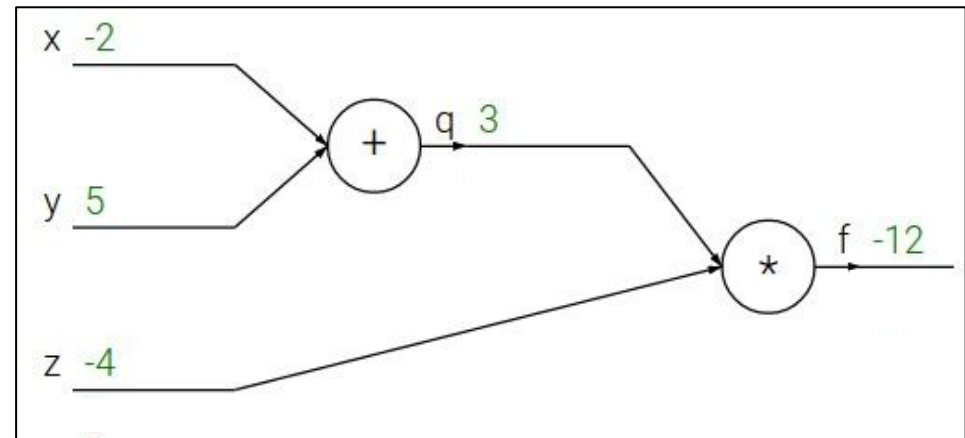
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Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



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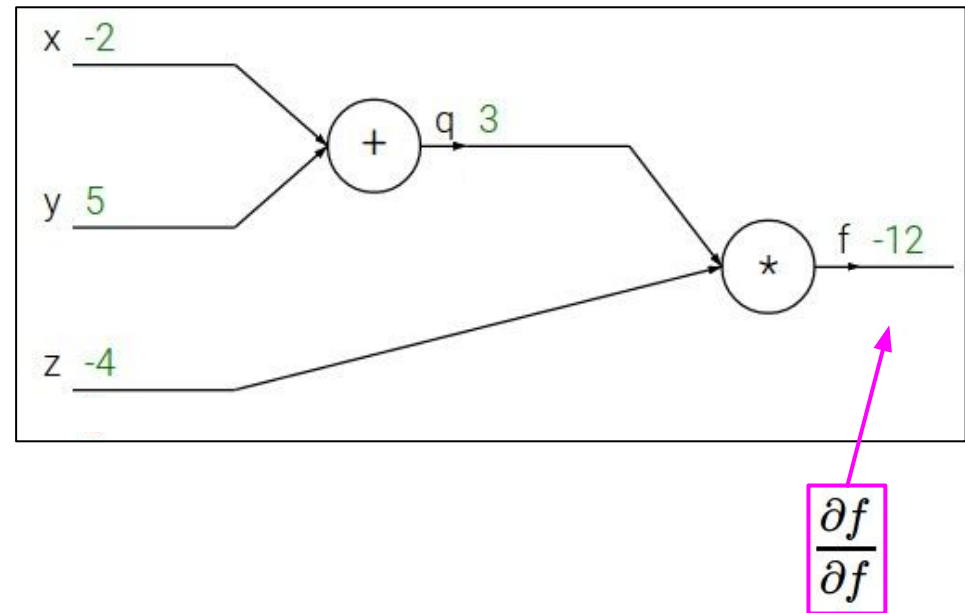
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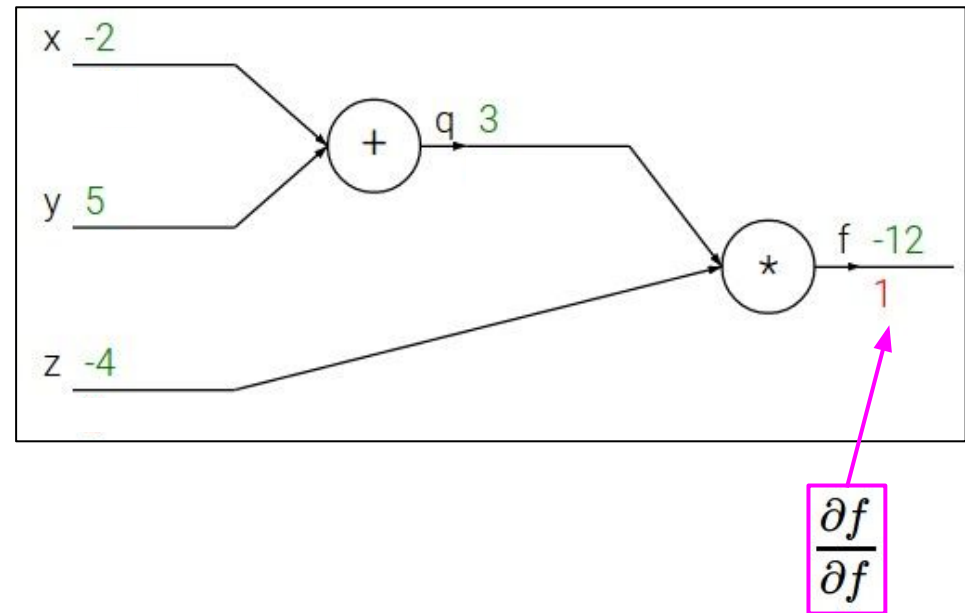
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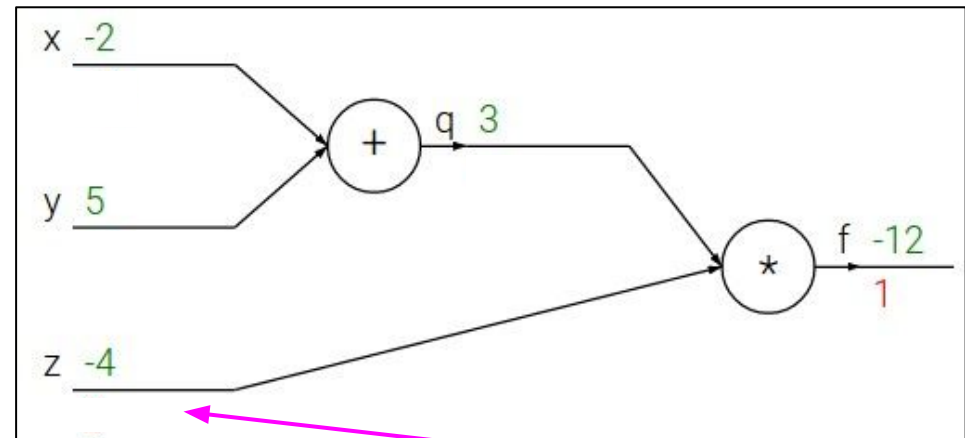
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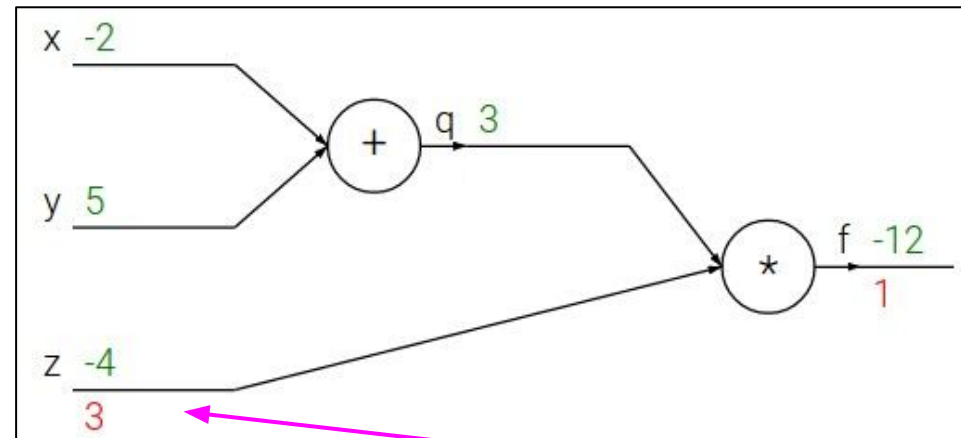
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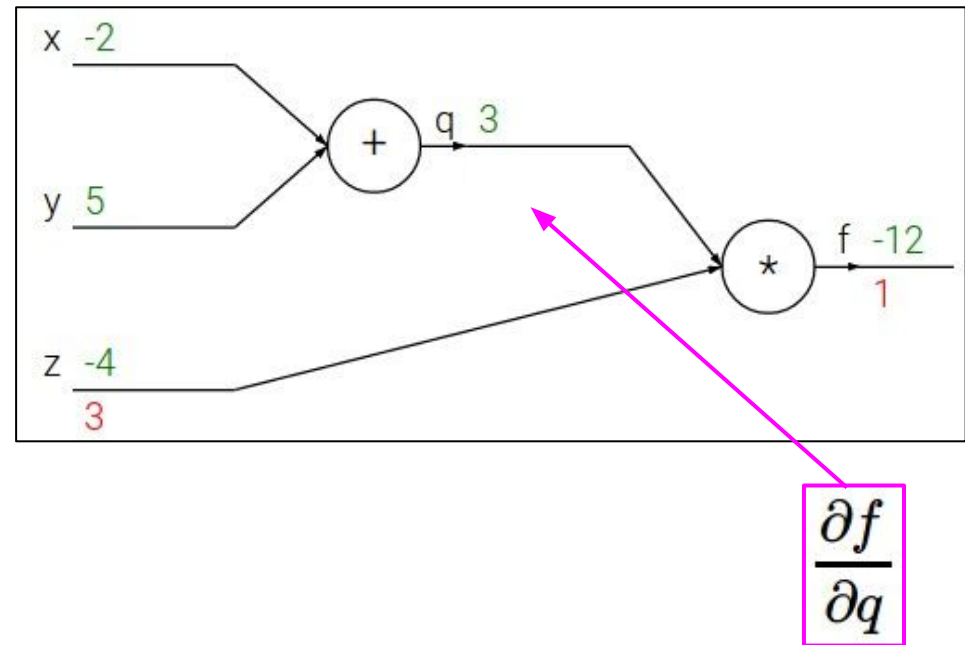
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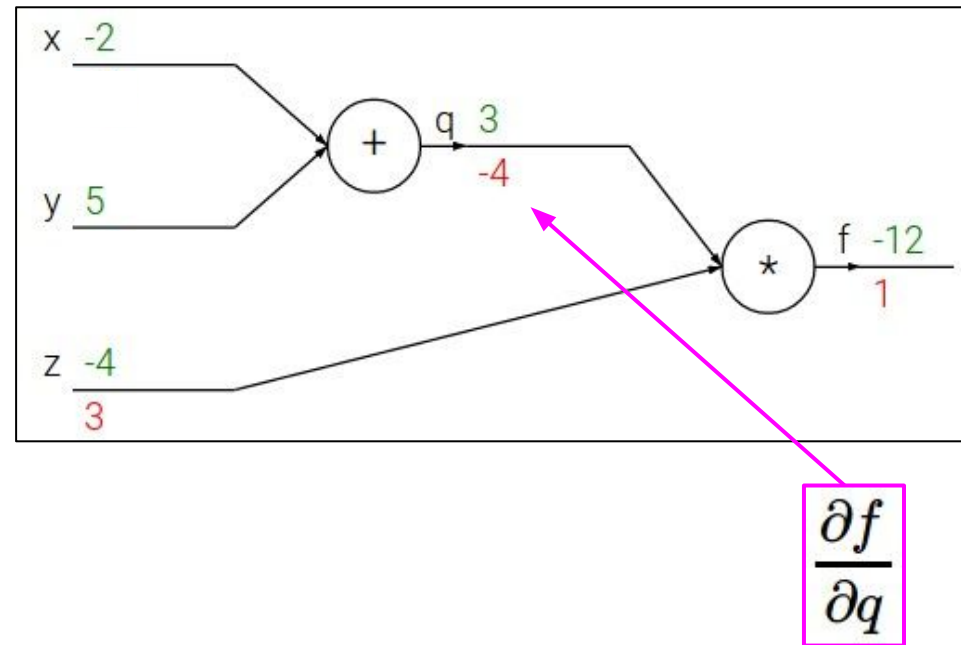
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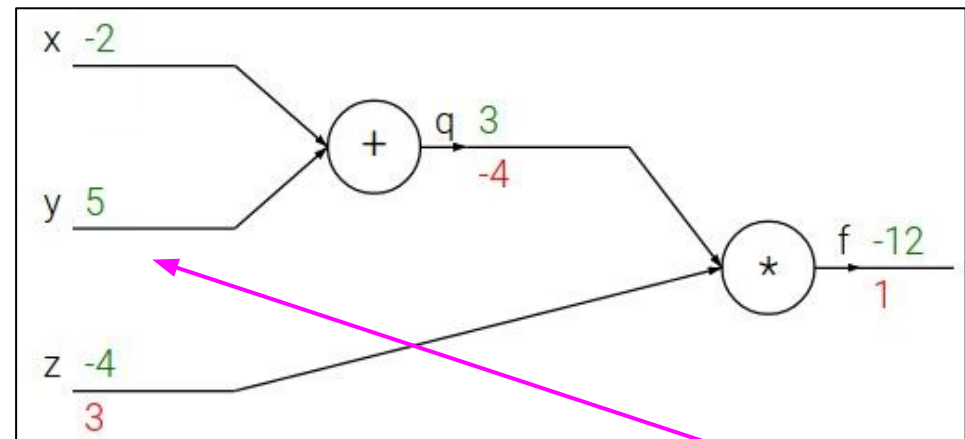
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Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial y}$$

Upstream
gradient

Local
gradient

$$\frac{\partial f}{\partial y}$$

Backpropagation: a simple example

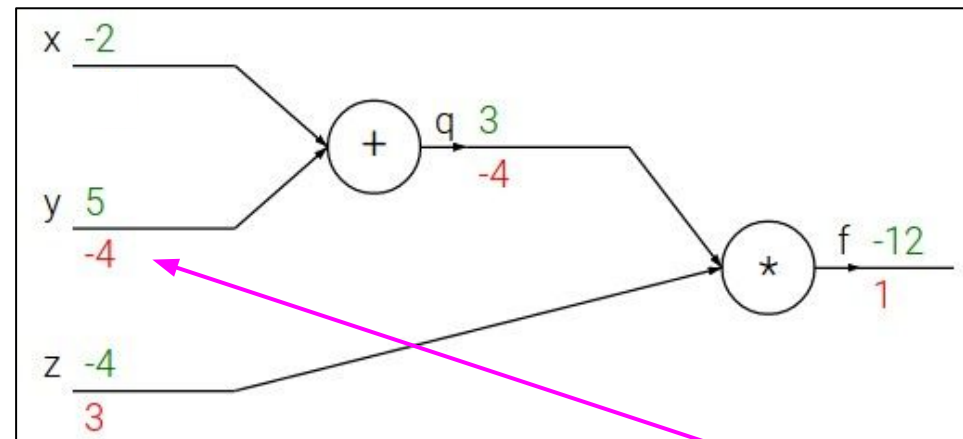
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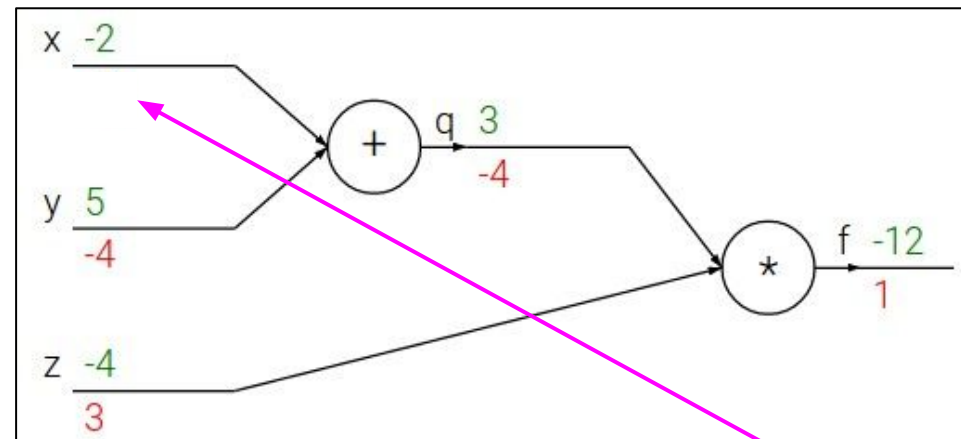
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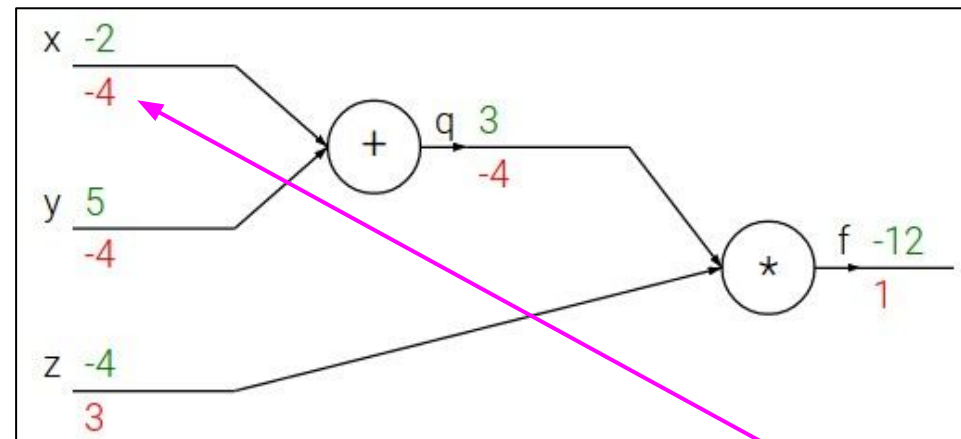
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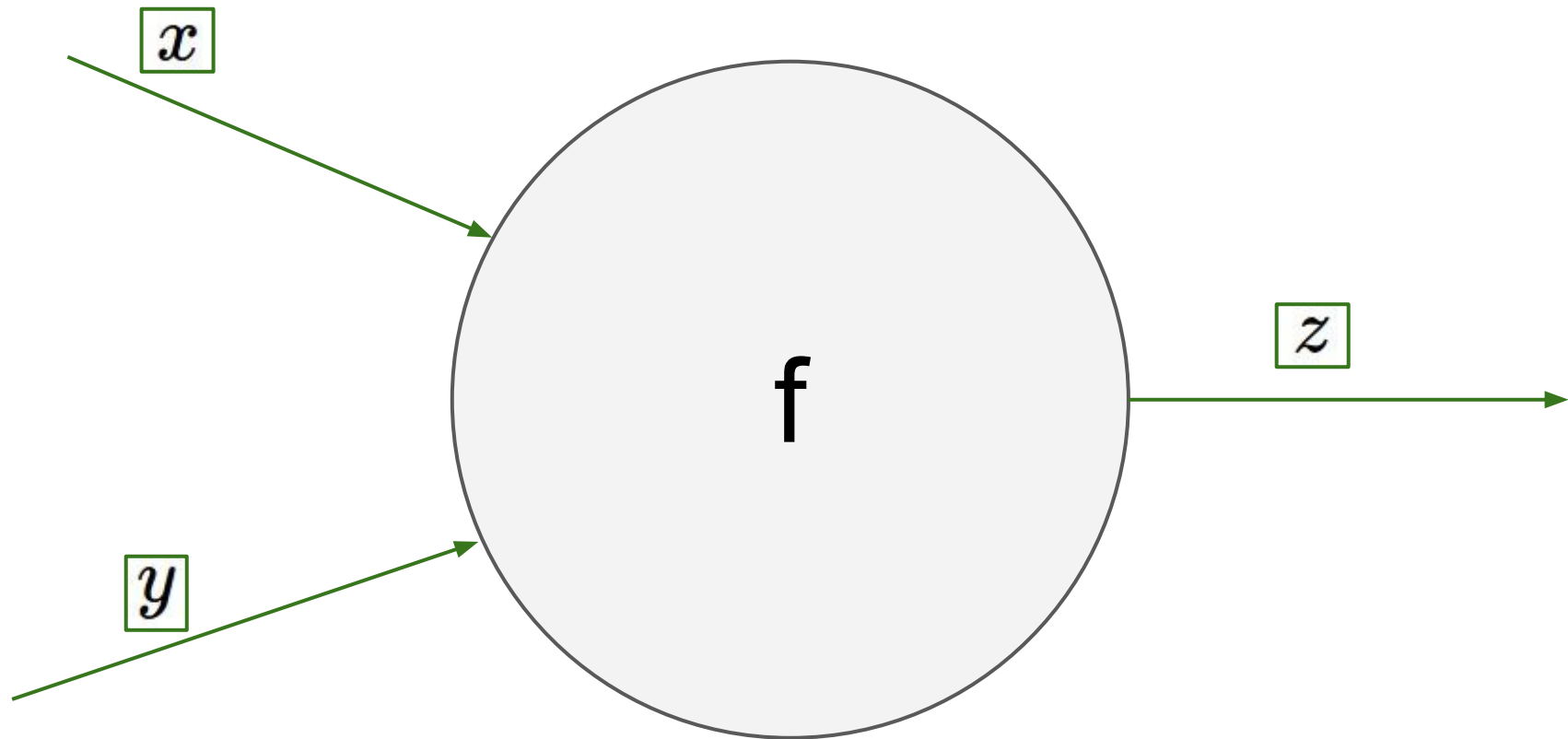
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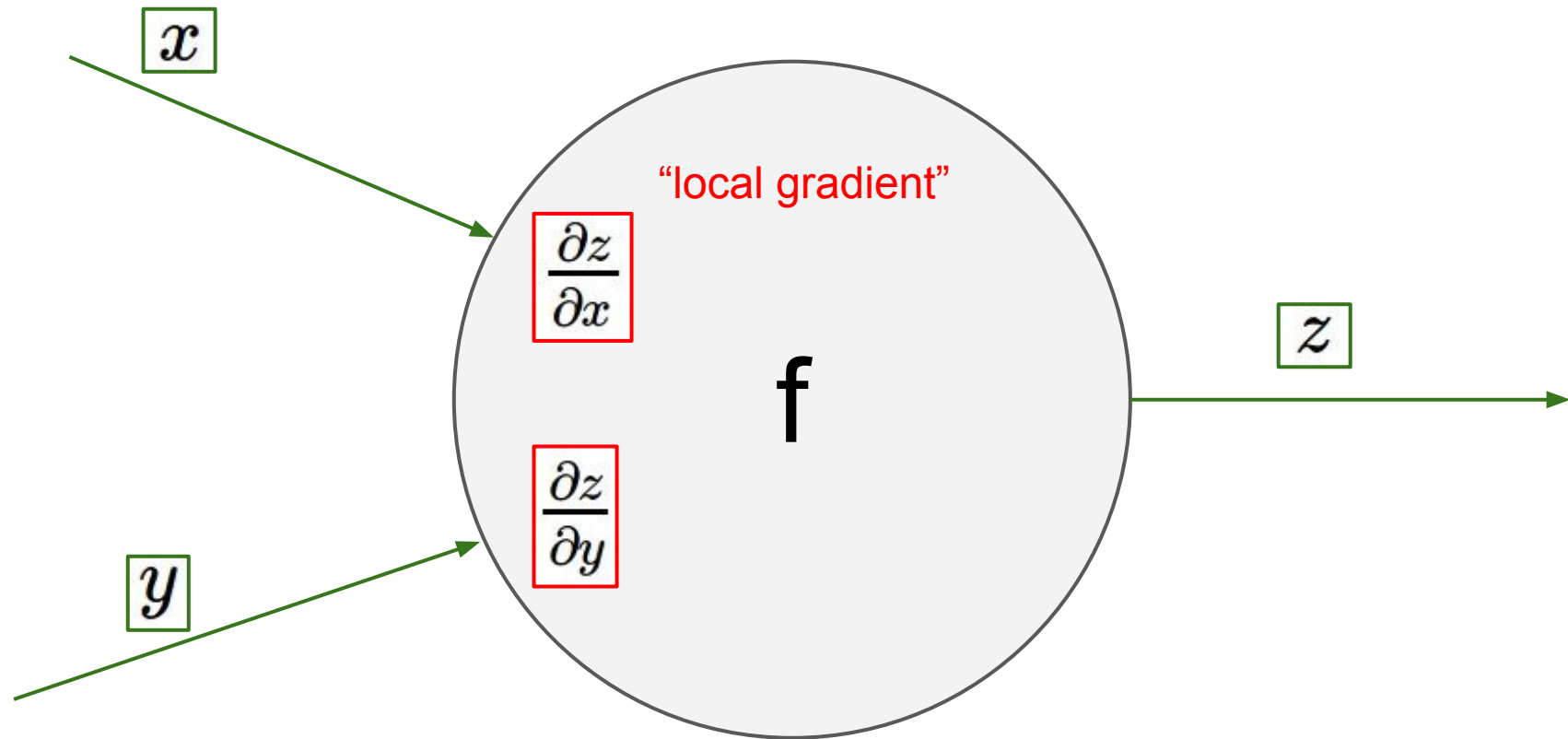
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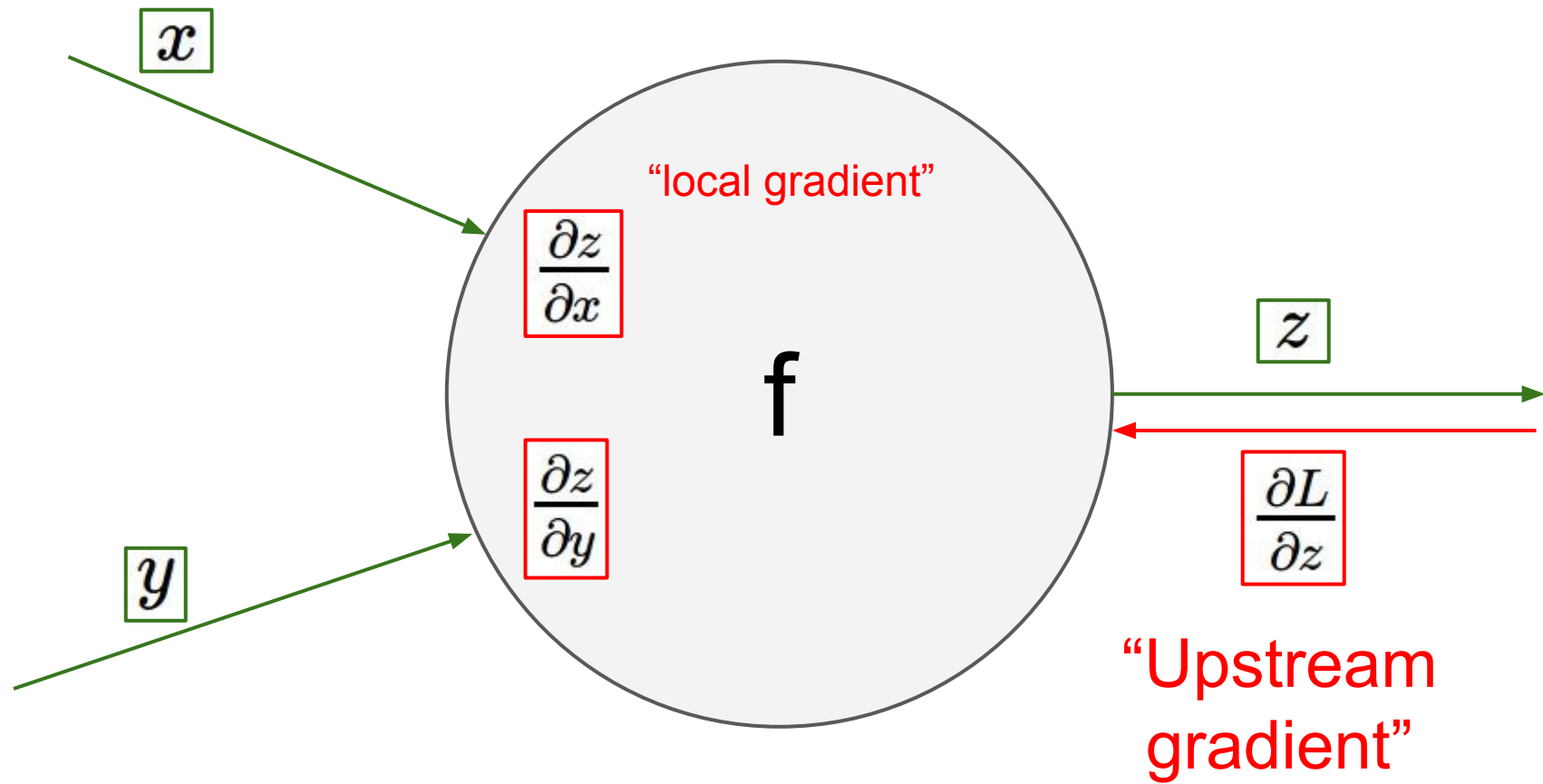
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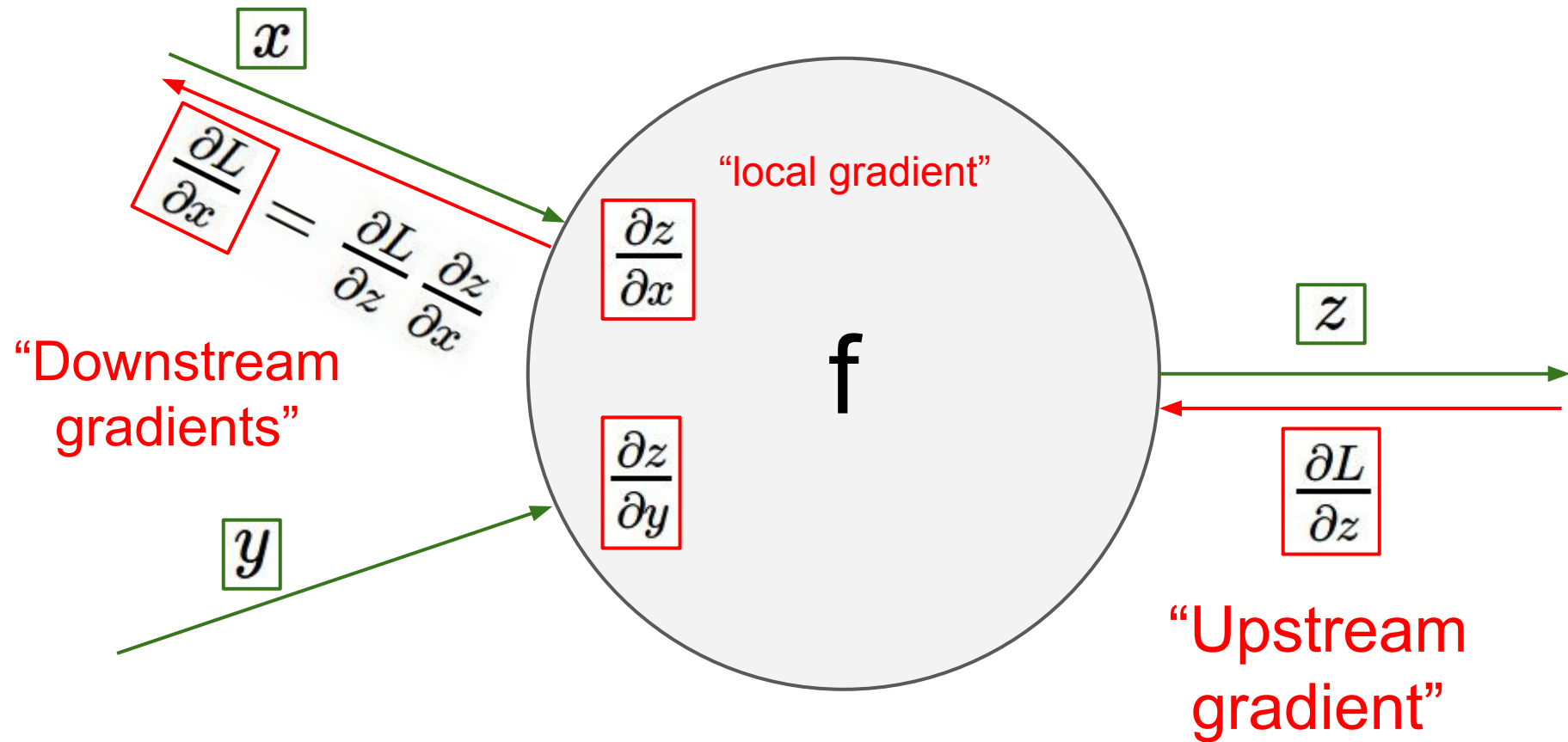
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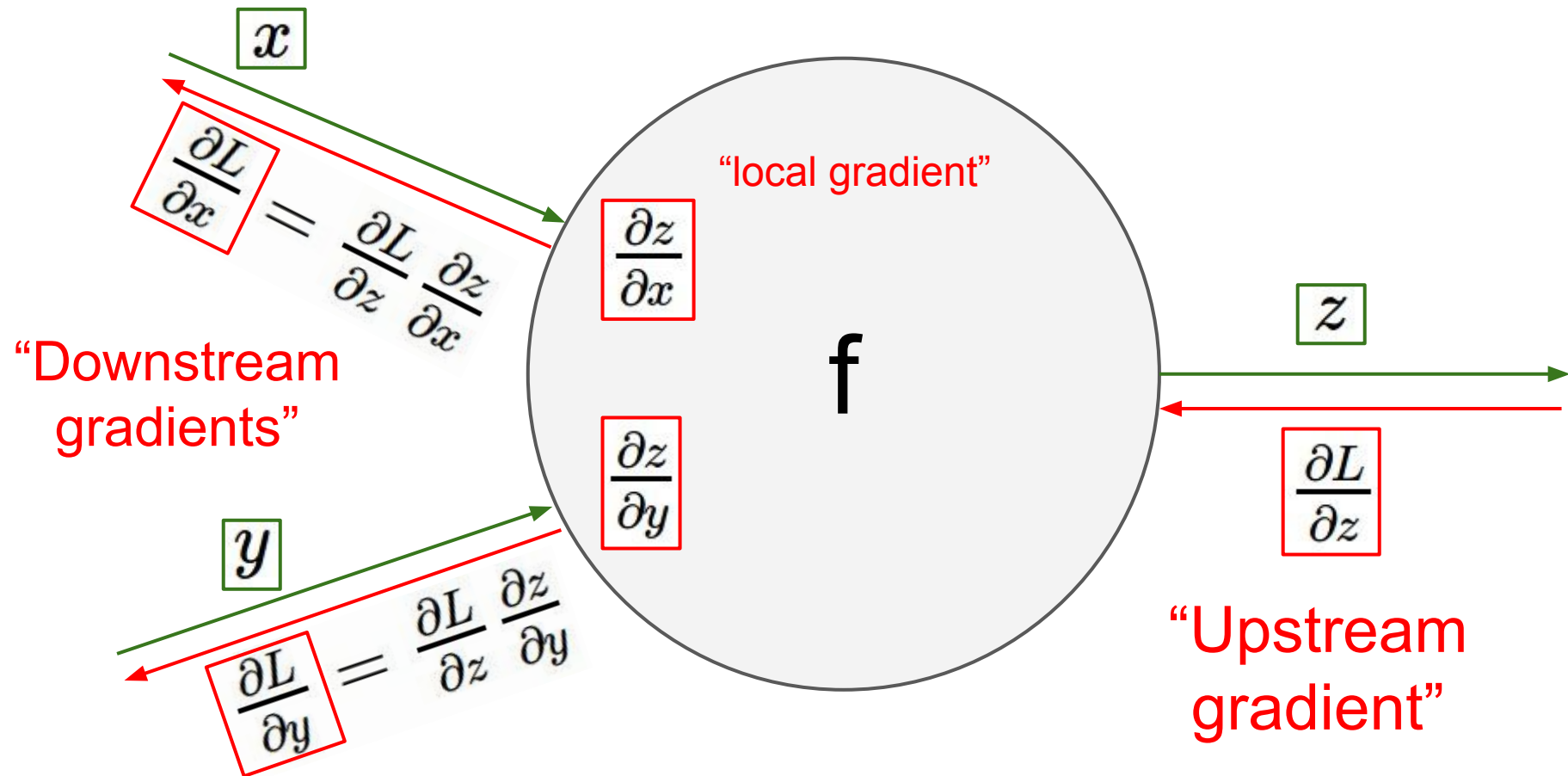
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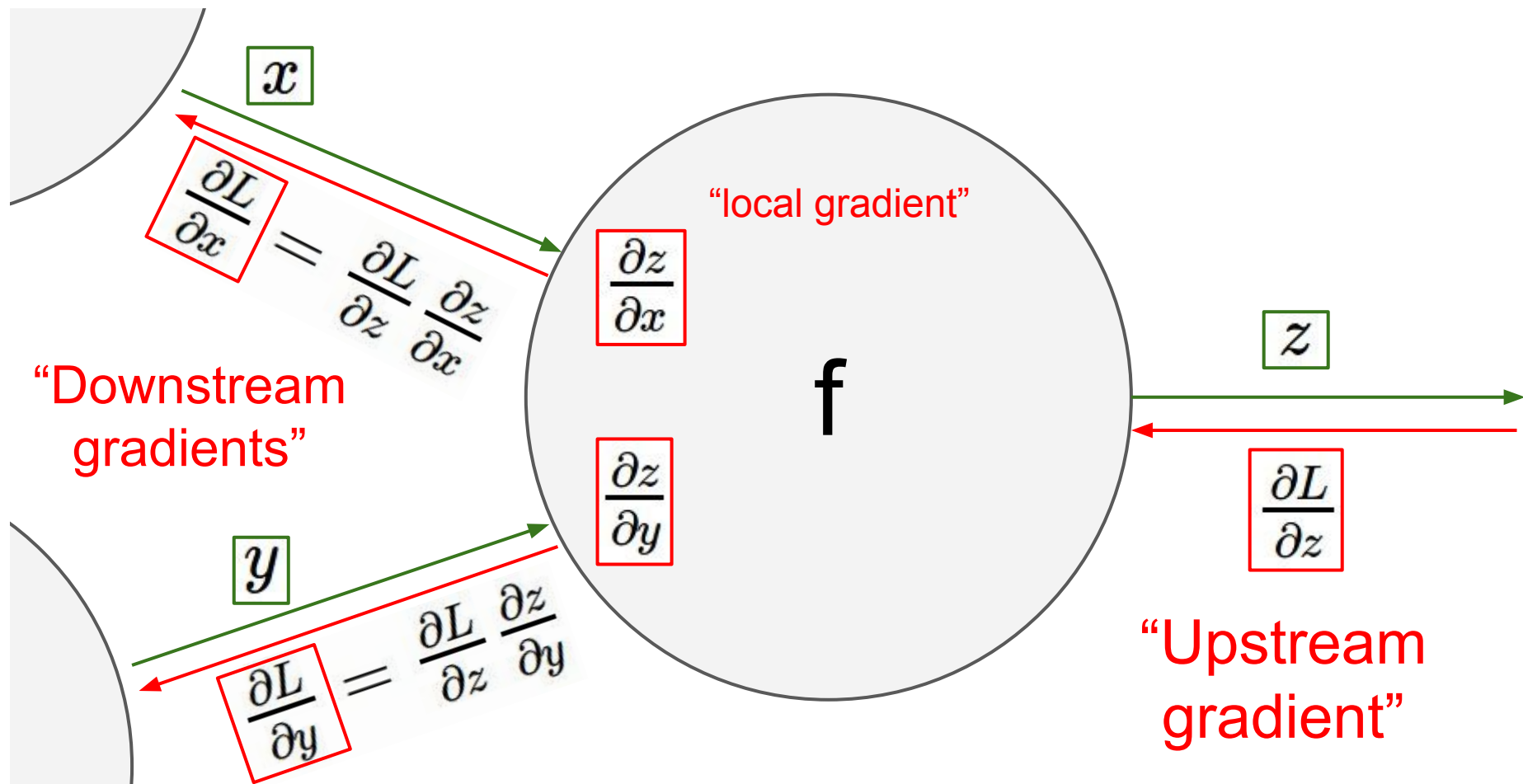




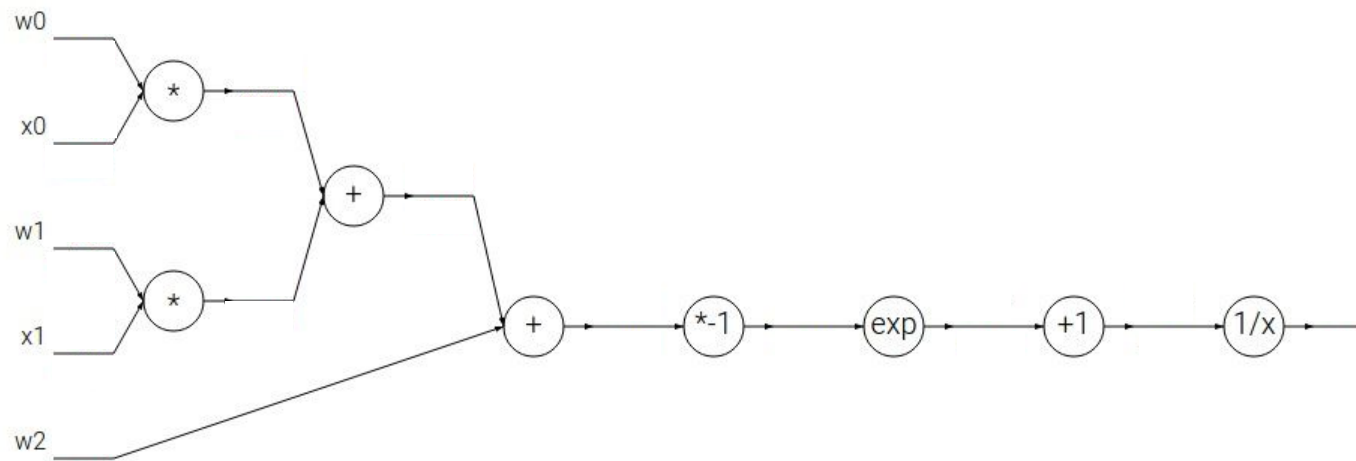




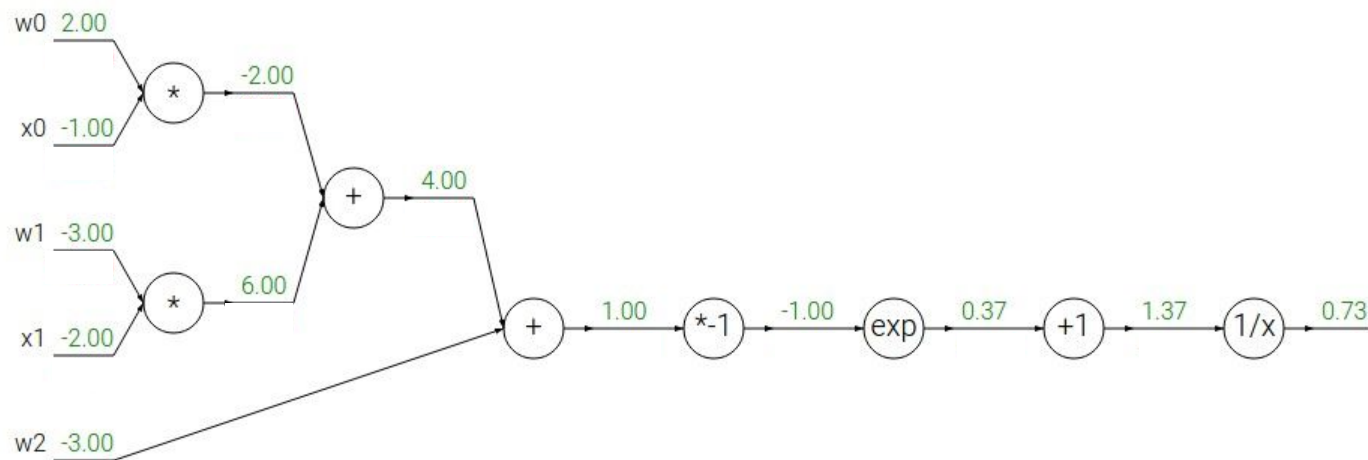




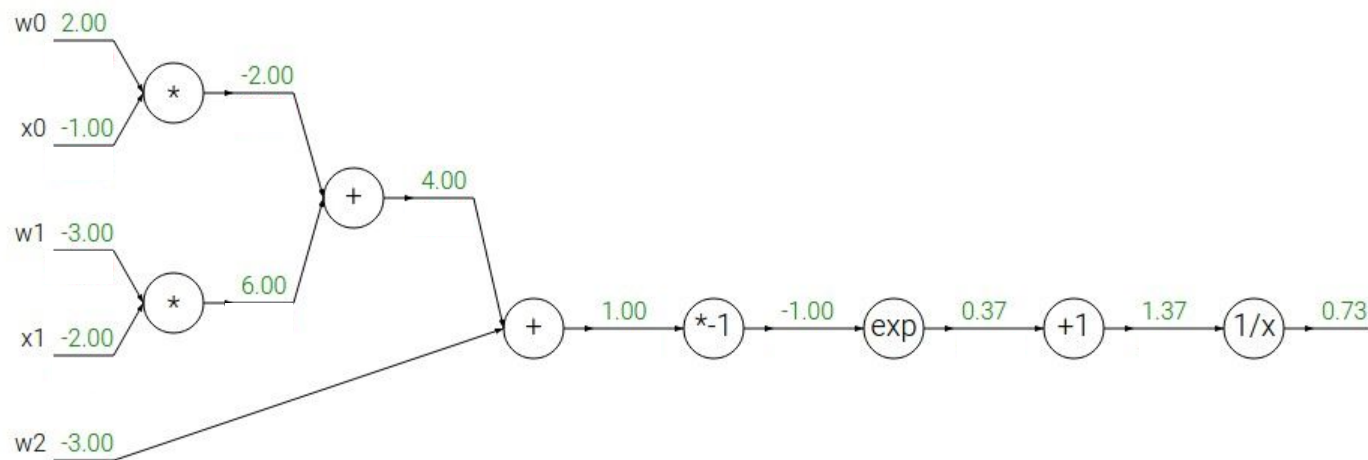
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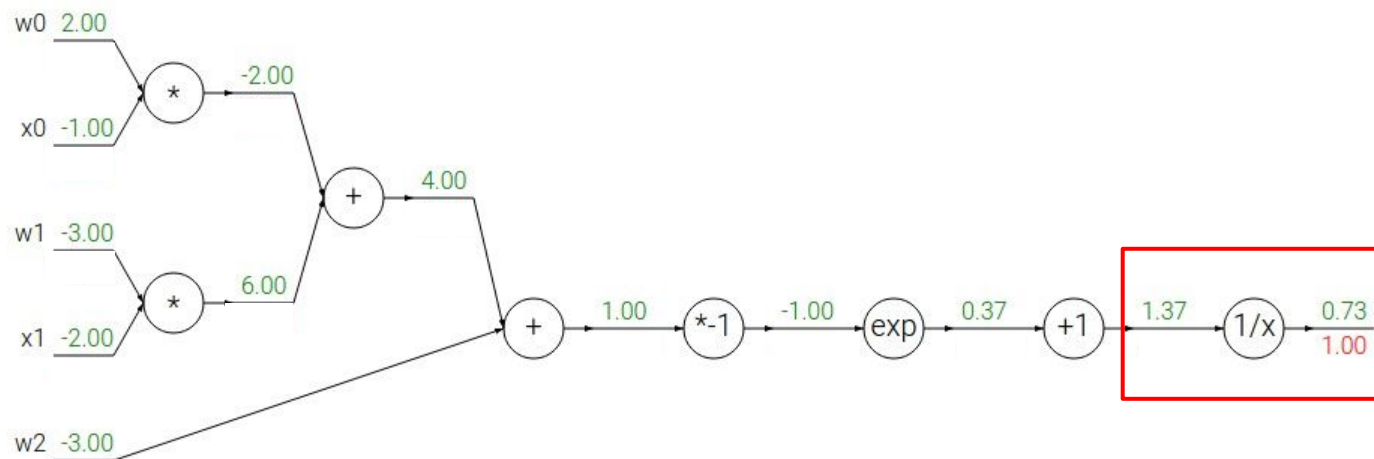


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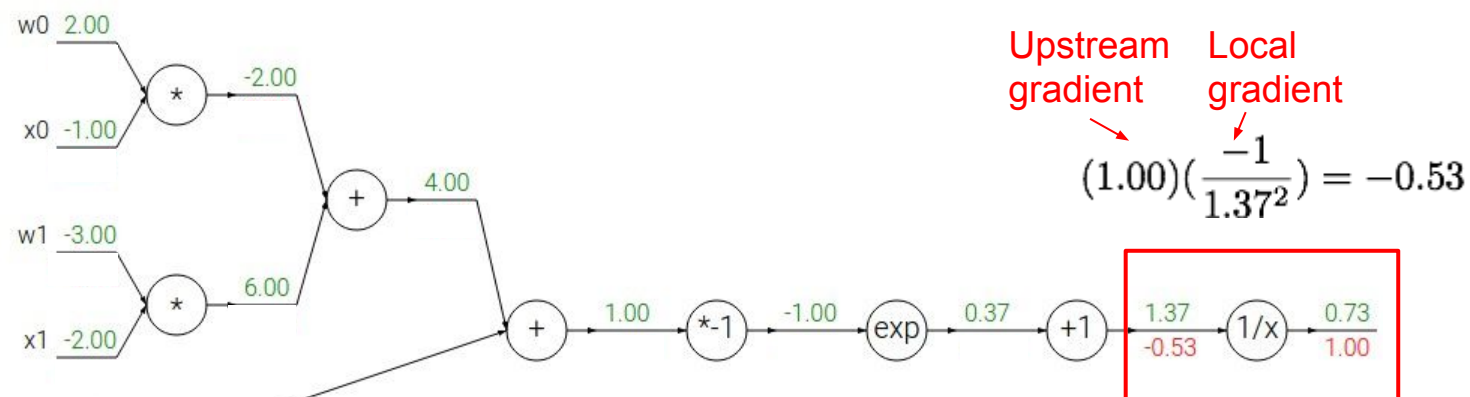
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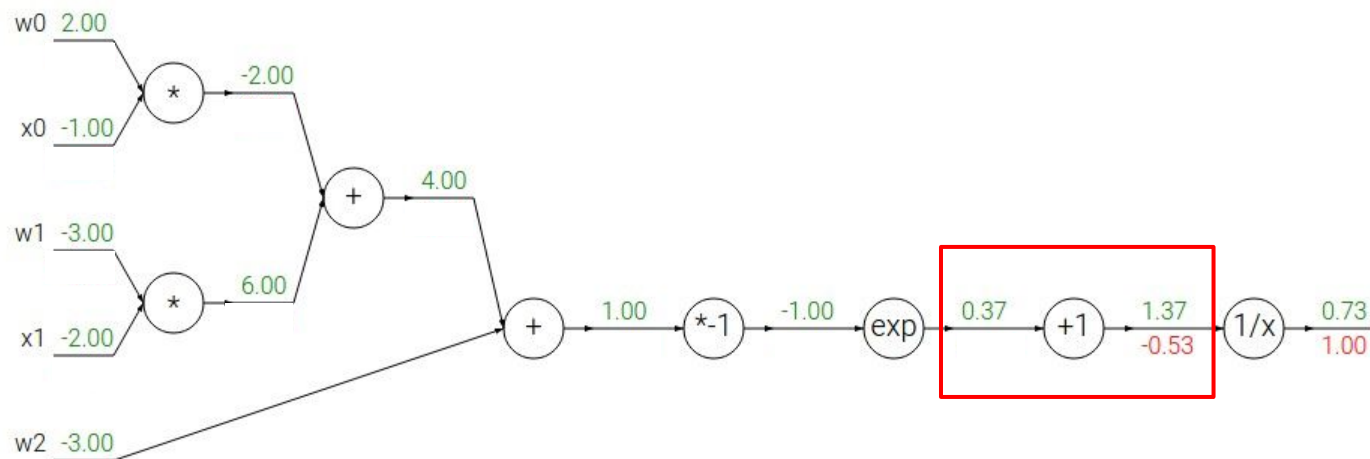
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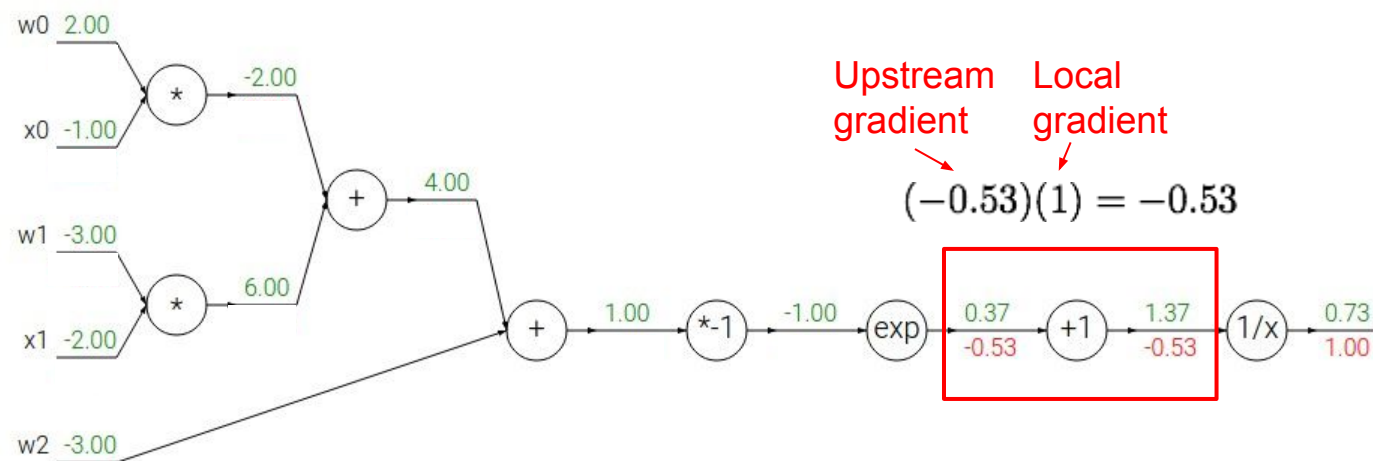
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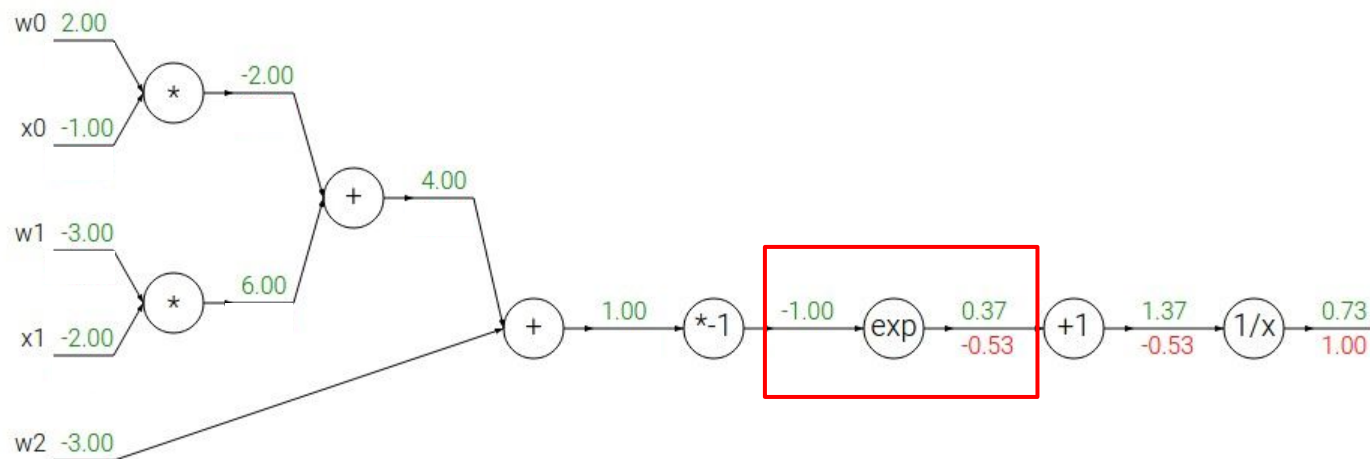
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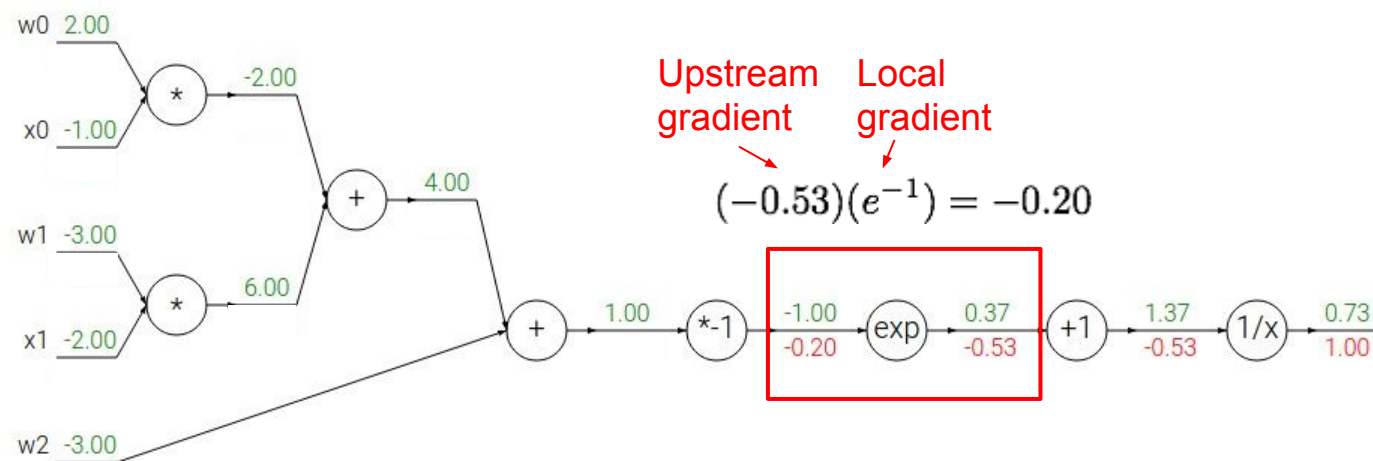
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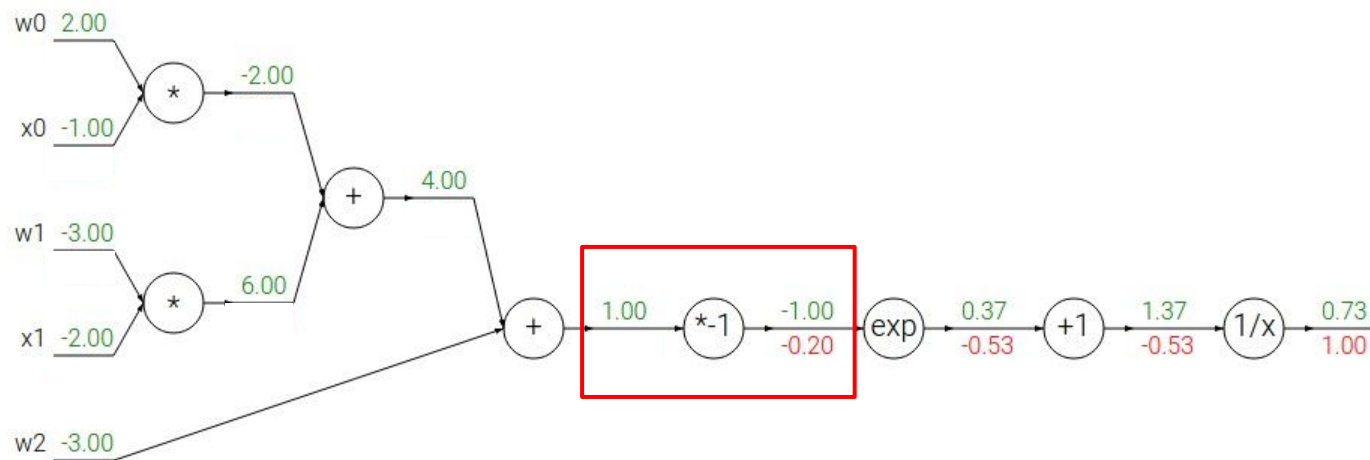
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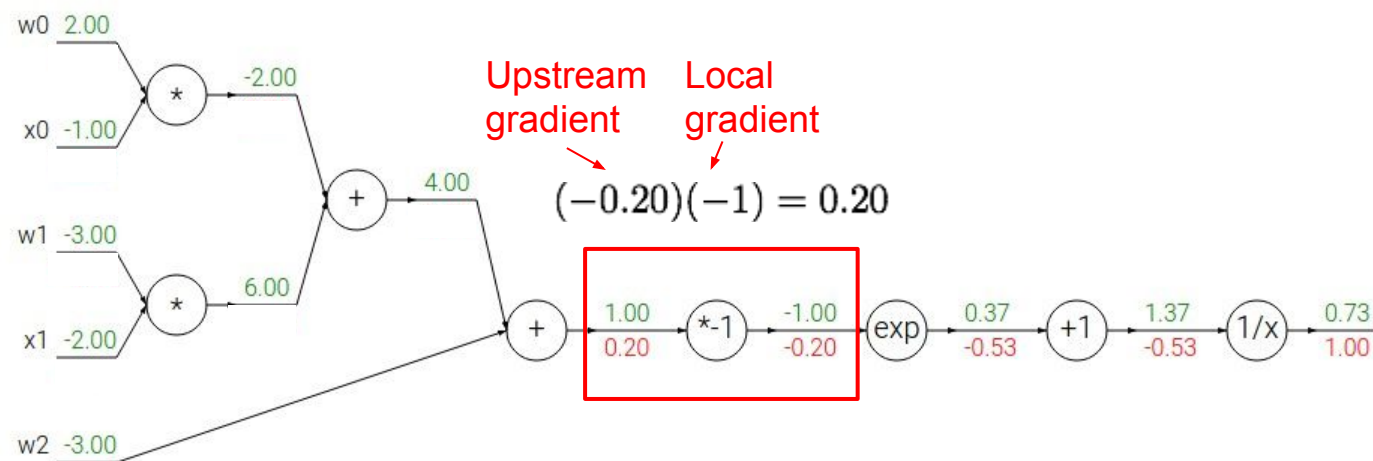
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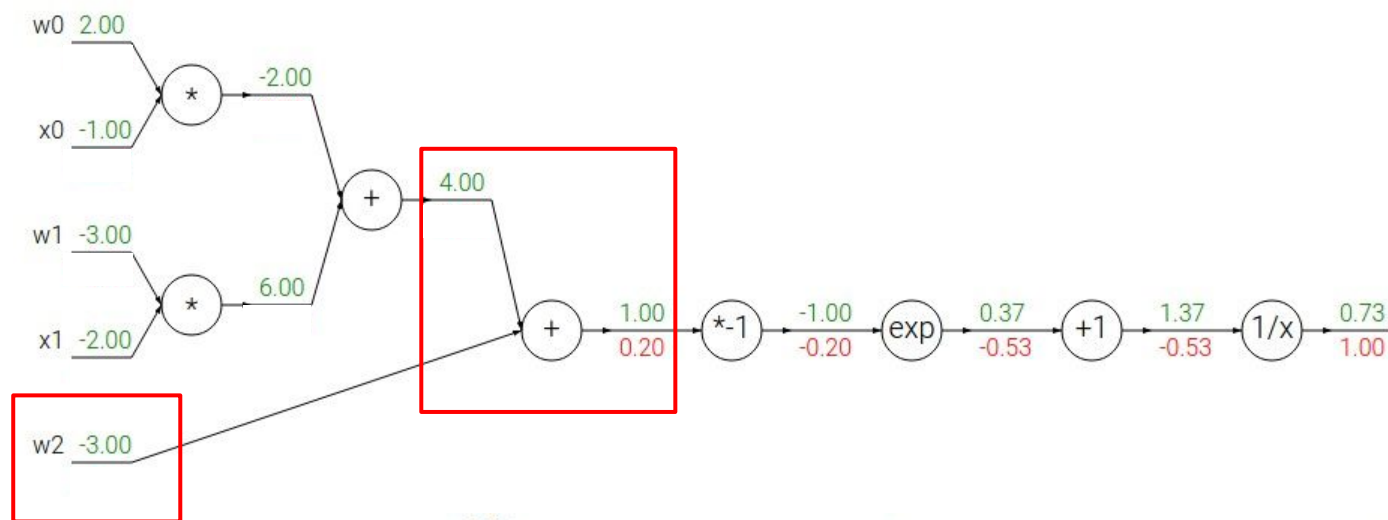
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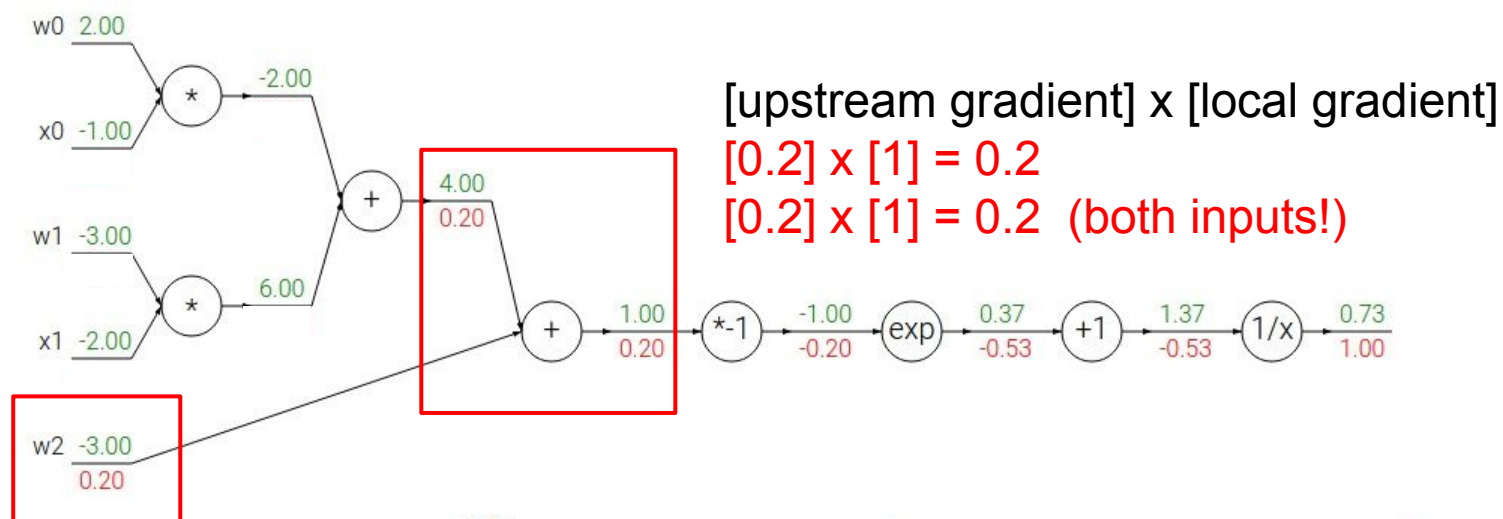
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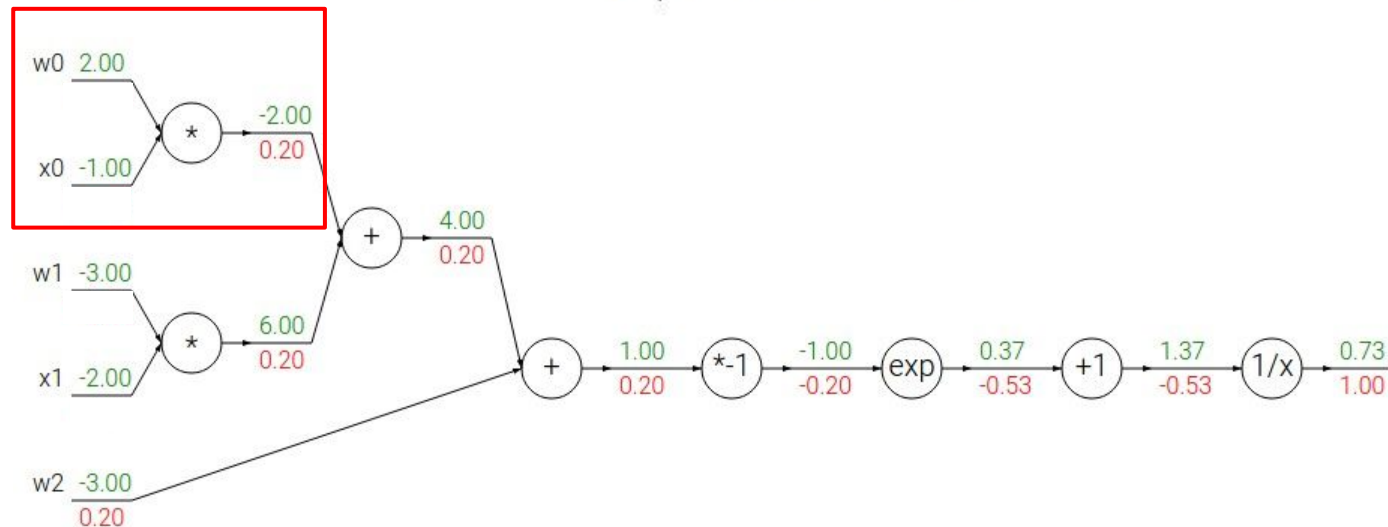
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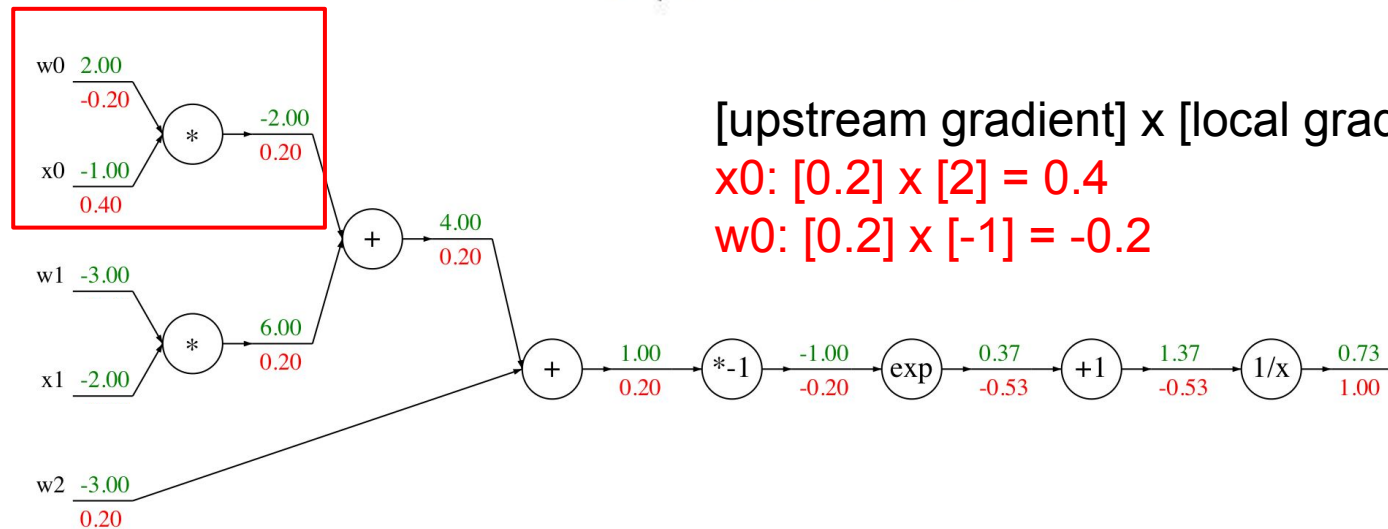
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Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2 x_2)}}$$



[upstream gradient] x [local gradient]

$$x_0: [0.2] \times [2] = 0.4$$

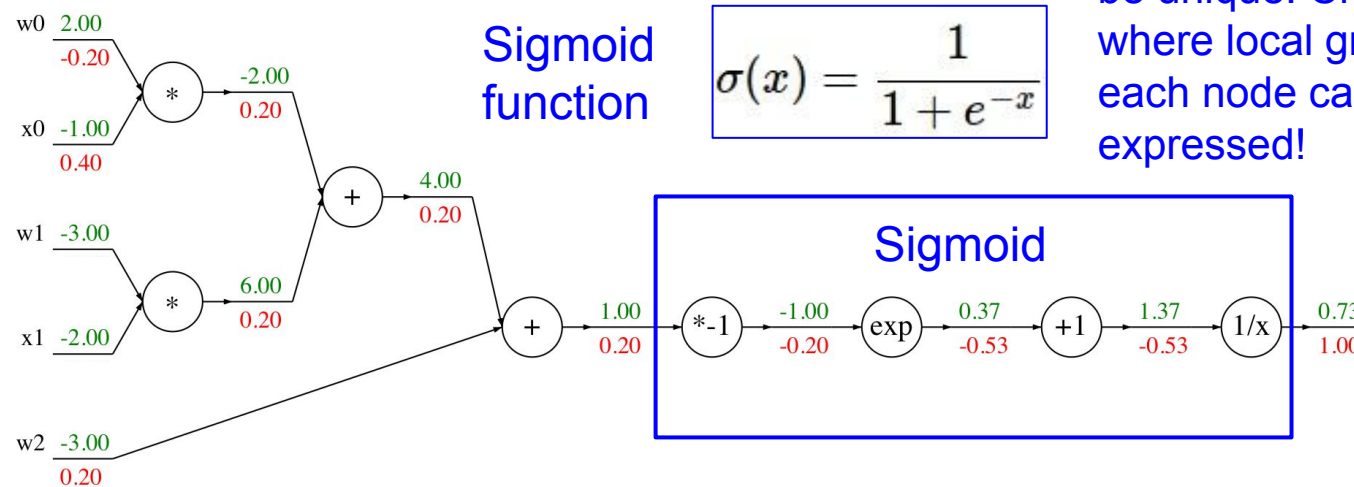
$$w_0: [0.2] \times [-1] = -0.2$$

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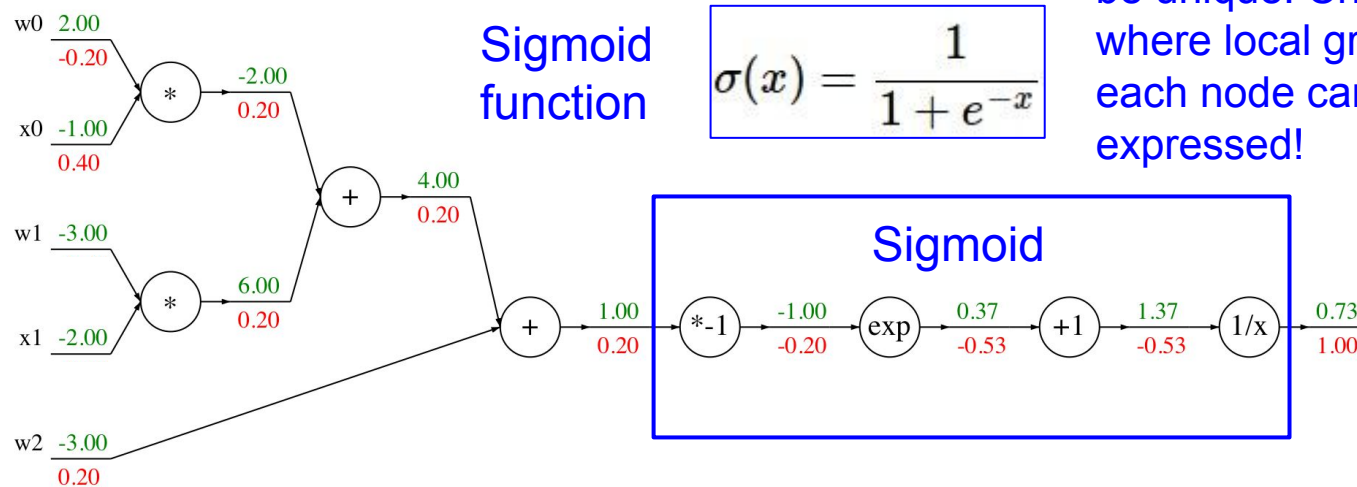
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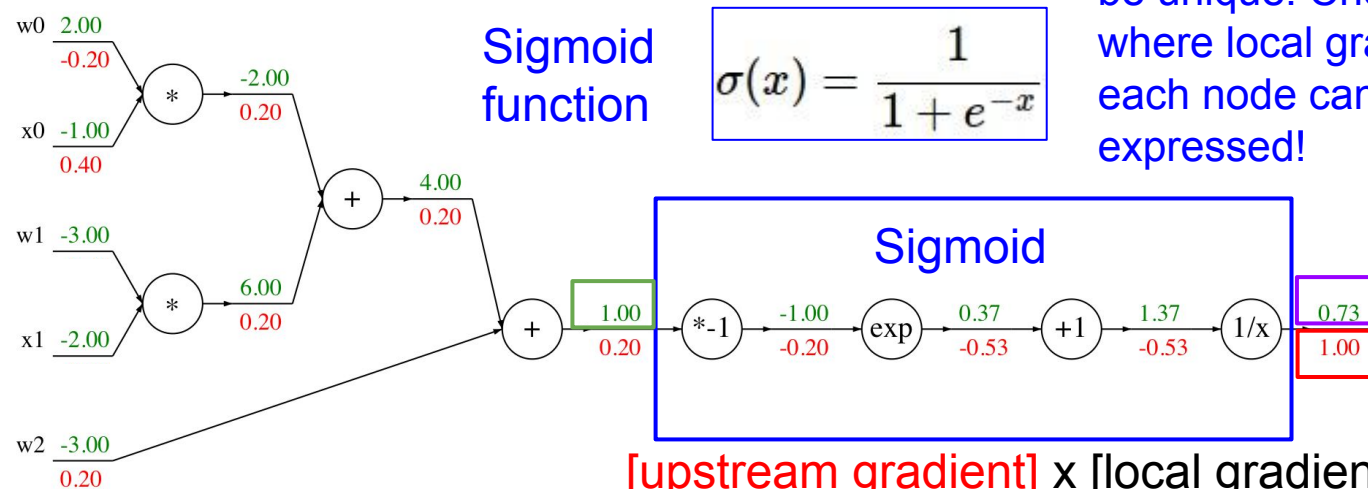
Sigmoid local gradient:

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1 + e^{-x})^2} = \left(\frac{1 + e^{-x} - 1}{1 + e^{-x}} \right) \left(\frac{1}{1 + e^{-x}} \right) = (1 - \sigma(x)) \sigma(x)$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$

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Sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

Sigmoid

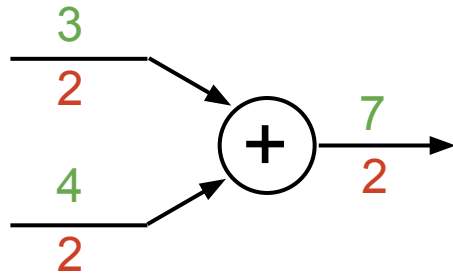
[upstream gradient] x [local gradient]
 $[1.00] \times [(1 - 0.73) (0.73)] = 0.2$

Sigmoid local gradient:

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1 + e^{-x})^2} = \left(\frac{1 + e^{-x} - 1}{1 + e^{-x}} \right) \left(\frac{1}{1 + e^{-x}} \right) = (1 - \sigma(x)) \sigma(x)$$

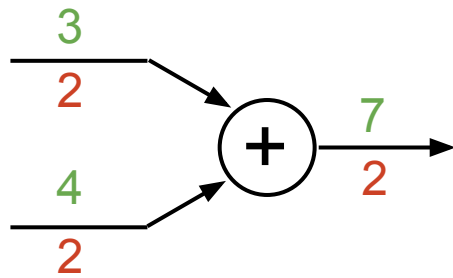
Patterns in gradient flow

add gate: gradient distributor

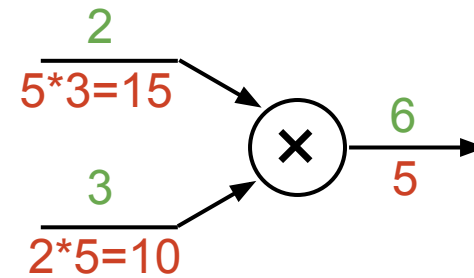


Patterns in gradient flow

add gate: gradient distributor

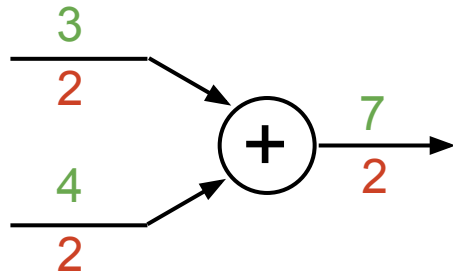


mul gate: “swap multiplier”

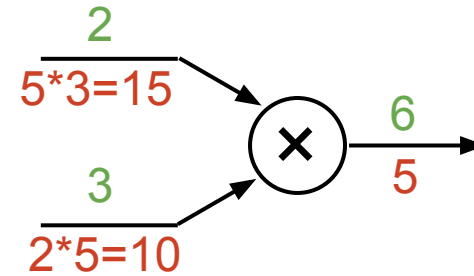


Patterns in gradient flow

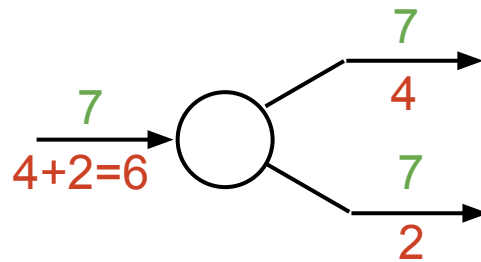
add gate: gradient distributor



mul gate: “swap multiplier”

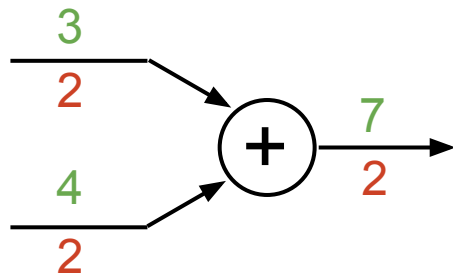


copy gate: gradient adder

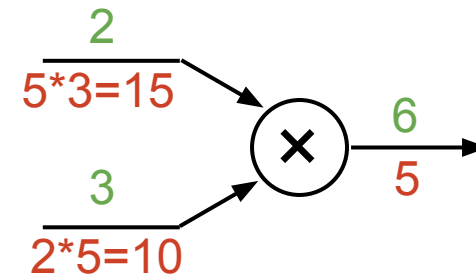


Patterns in gradient flow

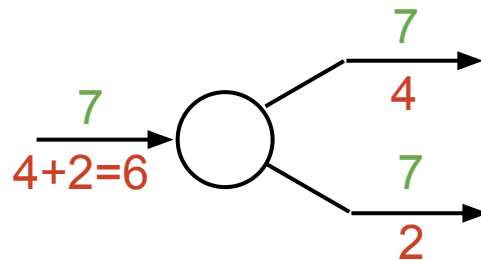
add gate: gradient distributor



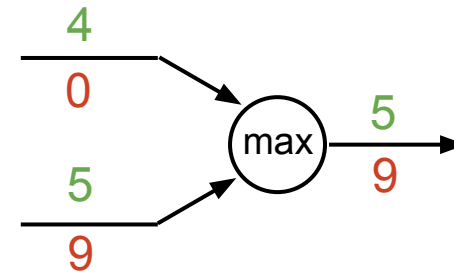
mul gate: “swap multiplier”



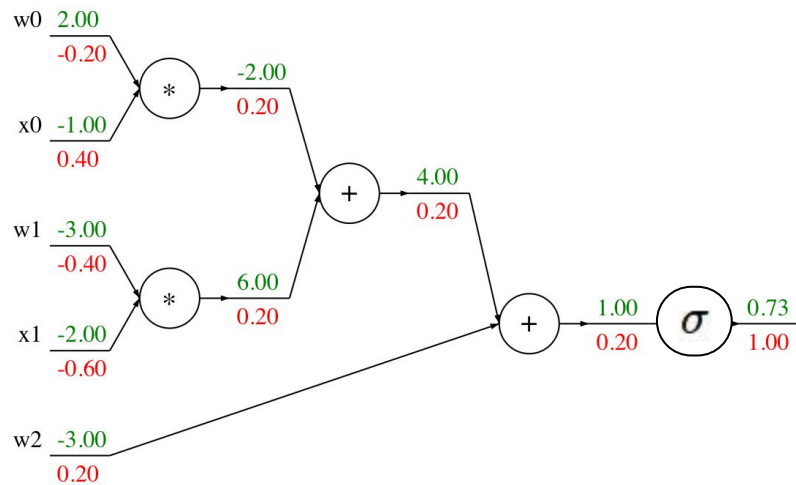
copy gate: gradient adder



max gate: gradient router



Backprop Implementation: “Flat” code



Forward pass:
Compute output

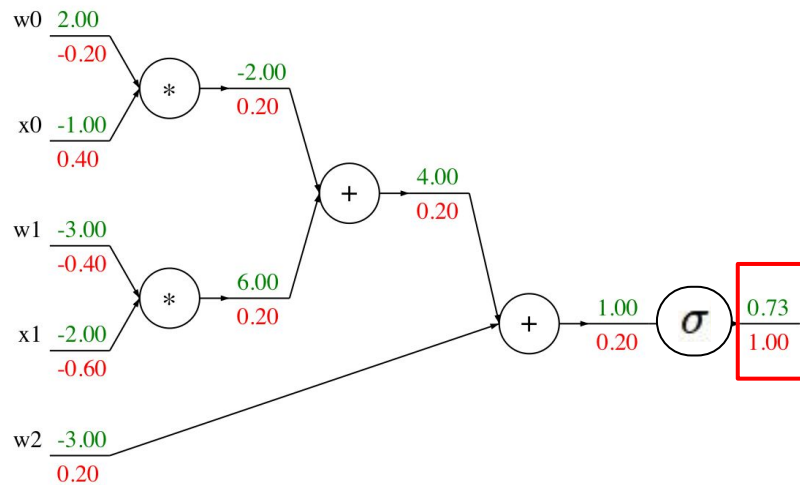
```
def f(w0, x0, w1, x1, w2):
```

```
    s0 = w0 * x0
    s1 = w1 * x1
    s2 = s0 + s1
    s3 = s2 + w2
    L = sigmoid(s3)
```

Backward pass:
Compute grads

```
    grad_L = 1.0
    grad_s3 = grad_L * (1 - L) * L
    grad_w2 = grad_s3
    grad_s2 = grad_s3
    grad_s0 = grad_s2
    grad_s1 = grad_s2
    grad_w1 = grad_s1 * x1
    grad_x1 = grad_s1 * w1
    grad_w0 = grad_s0 * x0
    grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

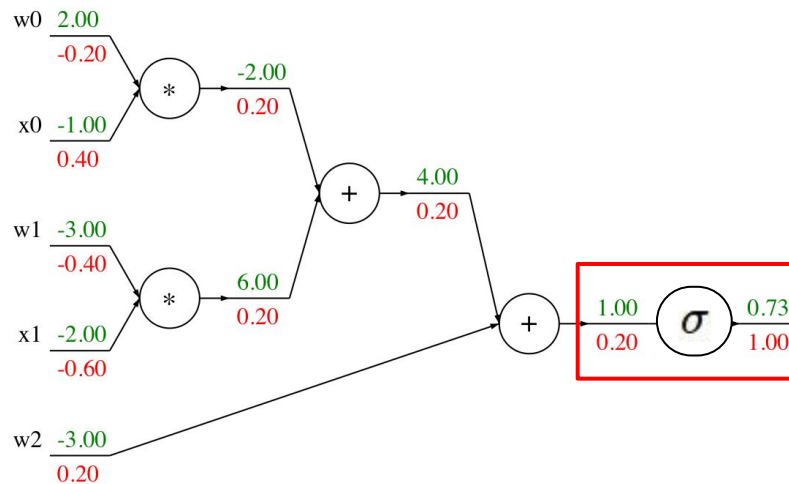
```
    s0 = w0 * x0
    s1 = w1 * x1
    s2 = s0 + s1
    s3 = s2 + w2
    L = sigmoid(s3)
```

Base case

```
    grad_L = 1.0
```

```
    grad_s3 = grad_L * (1 - L) * L
    grad_w2 = grad_s3
    grad_s2 = grad_s3
    grad_s0 = grad_s2
    grad_s1 = grad_s2
    grad_w1 = grad_s1 * x1
    grad_x1 = grad_s1 * w1
    grad_w0 = grad_s0 * x0
    grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

```
    s0 = w0 * x0
```

```
    s1 = w1 * x1
```

```
    s2 = s0 + s1
```

```
    s3 = s2 + w2
```

```
    L = sigmoid(s3)
```

Sigmoid

```
    grad_L = 1.0
```

```
    grad_s3 = grad_L * (1 - L) * L
```

```
    grad_w2 = grad_s3
```

```
    grad_s2 = grad_s3
```

```
    grad_s0 = grad_s2
```

```
    grad_s1 = grad_s2
```

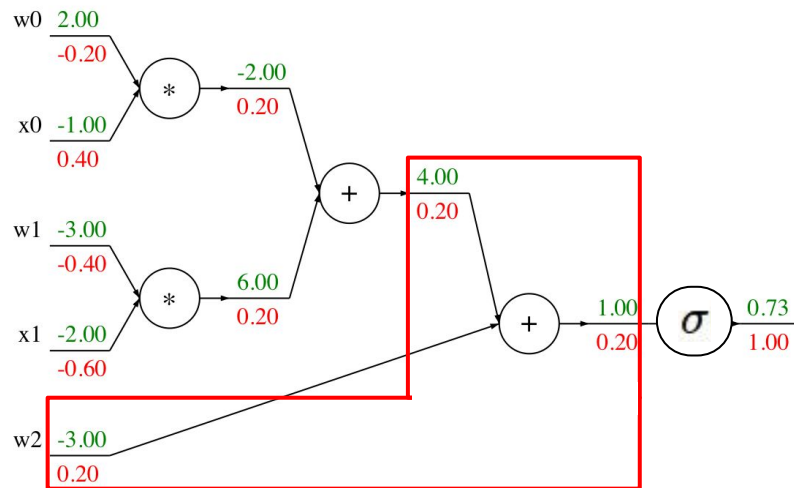
```
    grad_w1 = grad_s1 * x1
```

```
    grad_x1 = grad_s1 * w1
```

```
    grad_w0 = grad_s0 * x0
```

```
    grad_x0 = grad_s0 * w0
```


Backprop Implementation: “Flat” code



Forward pass:
Compute output

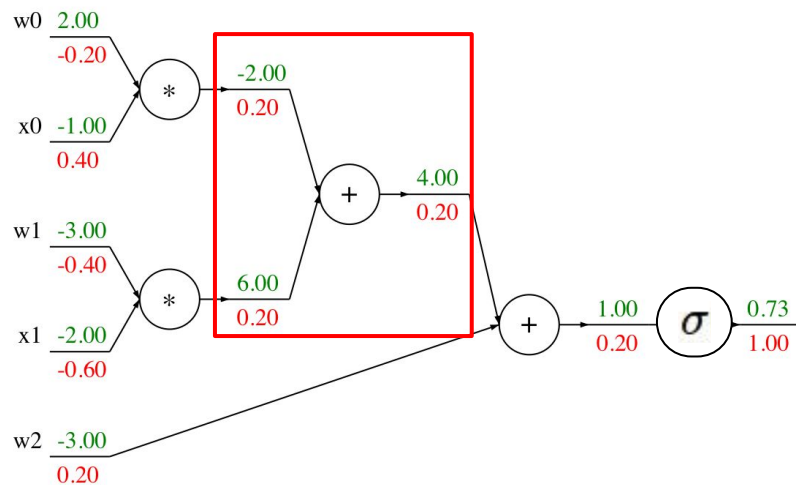
```
def f(w0, x0, w1, x1, w2):
```

```
    s0 = w0 * x0
    s1 = w1 * x1
    s2 = s0 + s1
    s3 = s2 + w2
    L = sigmoid(s3)
```

Add gate

```
    grad_L = 1.0
    grad_s3 = grad_L * (1 - L) * L
    grad_w2 = grad_s3
    grad_s2 = grad_s3
    grad_s0 = grad_s2
    grad_s1 = grad_s2
    grad_w1 = grad_s1 * x1
    grad_x1 = grad_s1 * w1
    grad_w0 = grad_s0 * x0
    grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



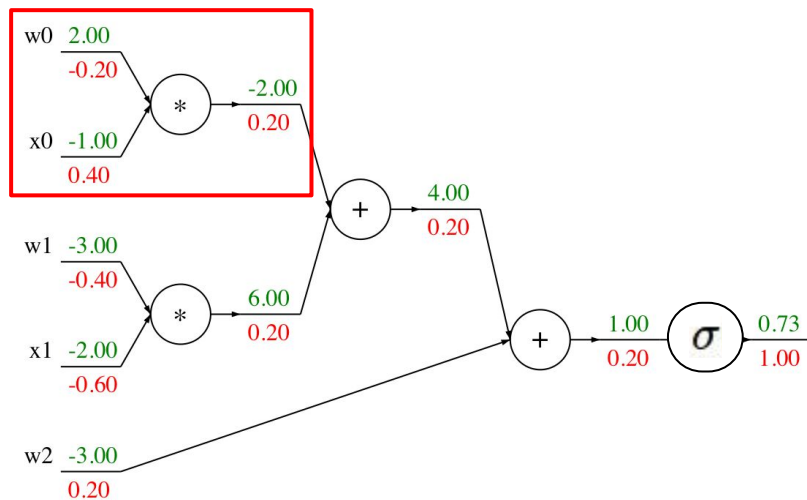
Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
    s0 = w0 * x0
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```

Add gate

```
grad_L = 1.0
grad_s3 = grad_L * (1 - L) * L
grad_w2 = grad_s3
grad_s2 = grad_s3
grad_s0 = grad_s2
grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
grad_w0 = grad_s0 * x0
grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

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    s0 = w0 * x0
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    s2 = s0 + s1
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```

```
    grad_L = 1.0
```

```
    grad_s3 = grad_L * (1 - L) * L
```

```
    grad_w2 = grad_s3
```

```
    grad_s2 = grad_s3
```

```
    grad_s0 = grad_s2
```

```
    grad_s1 = grad_s2
```

```
    grad_w1 = grad_s1 * x1
```

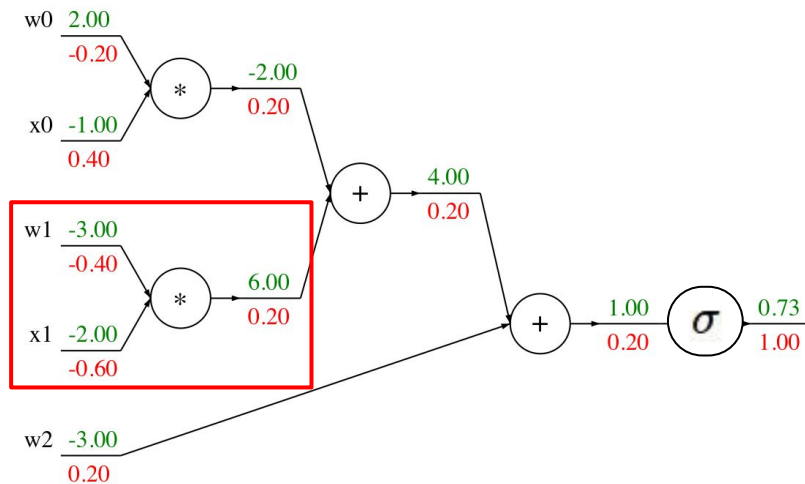
```
    grad_x1 = grad_s1 * w1
```

```
    grad_w0 = grad_s0 * x0
```

```
    grad_x0 = grad_s0 * w0
```

Multiply gate

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

```
    s0 = w0 * x0
```

```
    s1 = w1 * x1
```

```
    s2 = s0 + s1
```

```
    s3 = s2 + w2
```

```
    L = sigmoid(s3)
```

```
    grad_L = 1.0
```

```
    grad_s3 = grad_L * (1 - L) * L
```

```
    grad_w2 = grad_s3
```

```
    grad_s2 = grad_s3
```

```
    grad_s0 = grad_s2
```

```
    grad_s1 = grad_s2
```

```
    grad_w1 = grad_s1 * x1
```

```
    grad_x1 = grad_s1 * w1
```

```
    grad_w0 = grad_s0 * x0
```

```
    grad_x0 = grad_s0 * w0
```

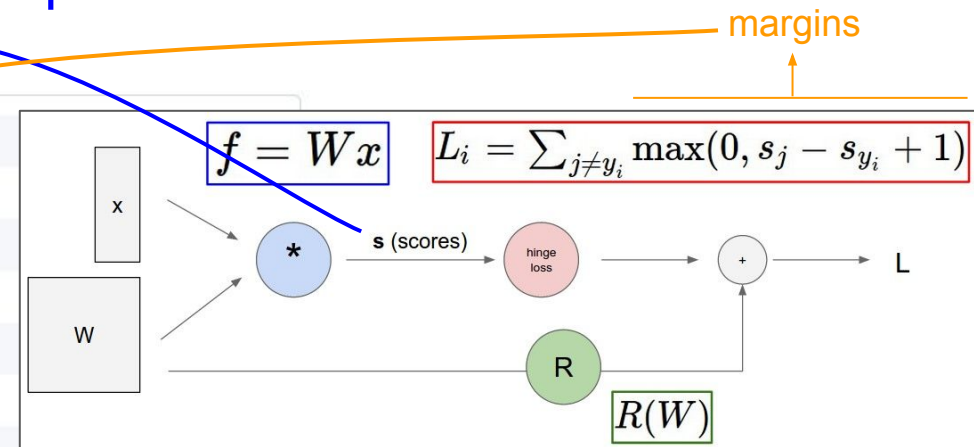
Multiply gate

“Flat” Backprop: Do this for assignment 1!

Stage your forward/backward computation!

E.g. for the SVM:

```
# receive W (weights), X (data)
# forward pass (we have 8 lines)
scores = #...
margins = #...
data_loss = #...
reg_loss = #...
loss = data_loss + reg_loss
# backward pass (we have 5 lines)
dmargins = # ... (optionally, we go direct to dscores)
dscores = #...
dW = #...
```



“Flat” Backprop: Do this for assignment 1!

E.g. for two-layer neural net:

```
# receive W1,W2,b1,b2 (weights/biases), X (data)
# forward pass:
h1 = #... function of X,W1,b1
scores = #... function of h1,W2,b2
loss = #... (several lines of code to evaluate Softmax loss)
# backward pass:
dscores = #...
dh1,dW2,db2 = #...
dW1,db1 = #...
```

Recap: Vector derivatives

Scalar to Scalar

$$x \in \mathbb{R}, y \in \mathbb{R}$$

Regular derivative:

$$\frac{\partial y}{\partial x} \in \mathbb{R}$$

If x changes by a small amount, how much will y change?

Vector to Scalar

$$x \in \mathbb{R}^N, y \in \mathbb{R}$$

Derivative is **Gradient**:

$$\frac{\partial y}{\partial x} \in \mathbb{R}^N \quad \left(\frac{\partial y}{\partial x} \right)_n = \frac{\partial y}{\partial x_n}$$

For each element of x , if it changes by a small amount then how much will y change?

Recap: Vector derivatives

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Derivative is **Gradient**:

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For each element of x , if it changes by a small amount then how much will y change?

Vector to Vector

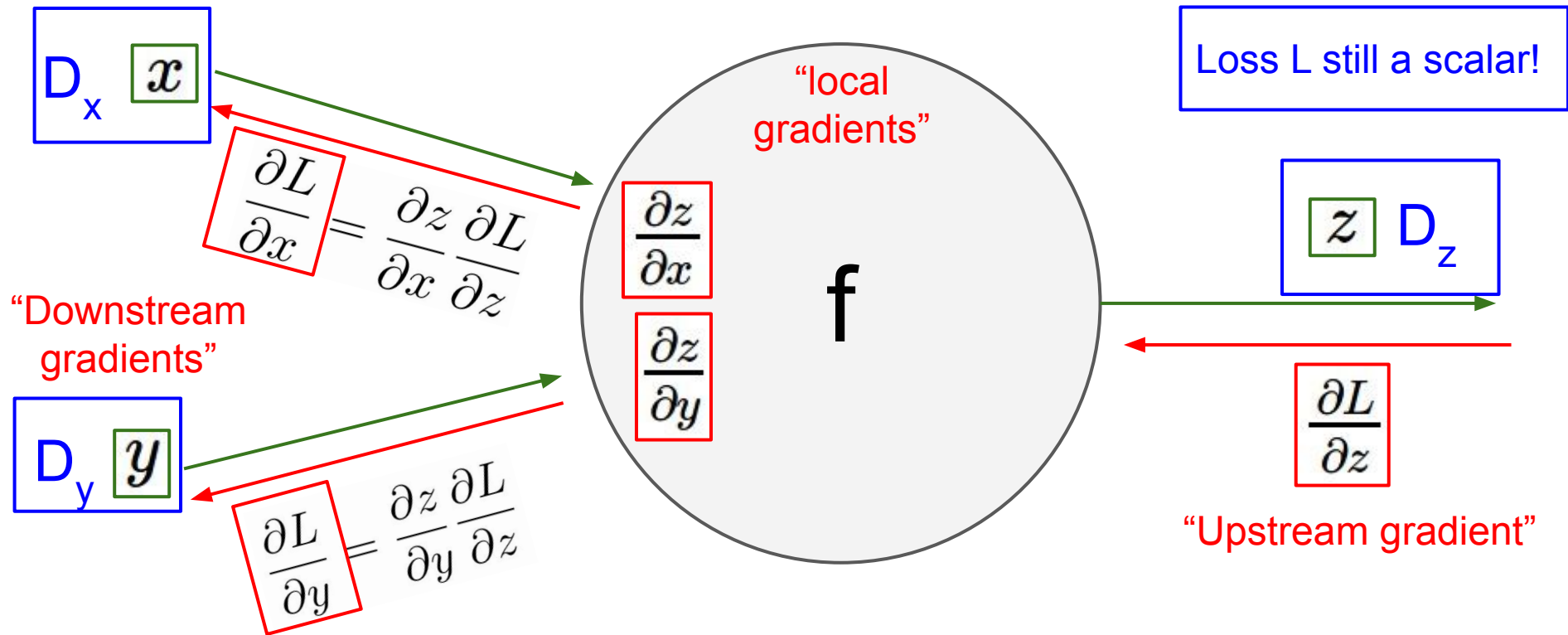
$$x \in \mathbb{R}^N, y \in \mathbb{R}^M$$

Derivative is **Jacobian**:

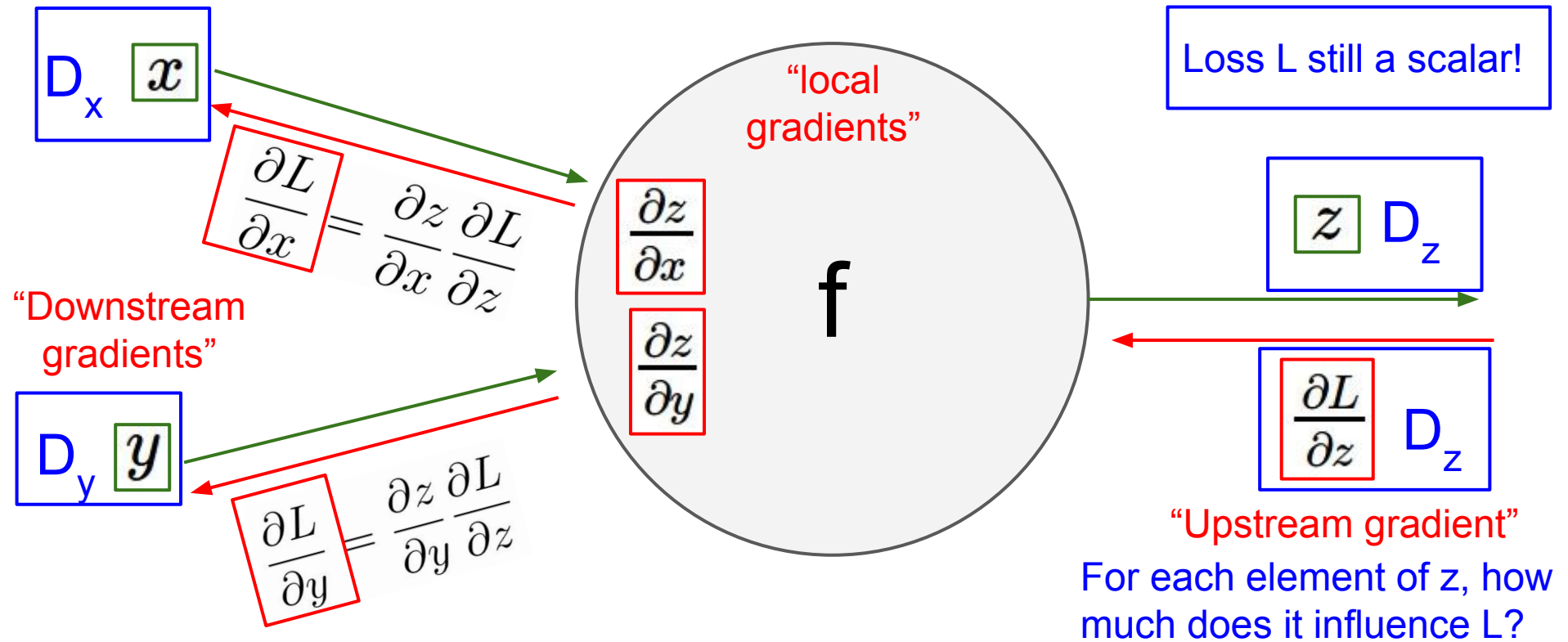
$$\frac{\partial y}{\partial x} \in \mathbb{R}^{N \times M} \quad \left(\frac{\partial y}{\partial x} \right)_{n,m} = \frac{\partial y_m}{\partial x_n}$$

For each element of x , if it changes by a small amount then how much will each element of y change?

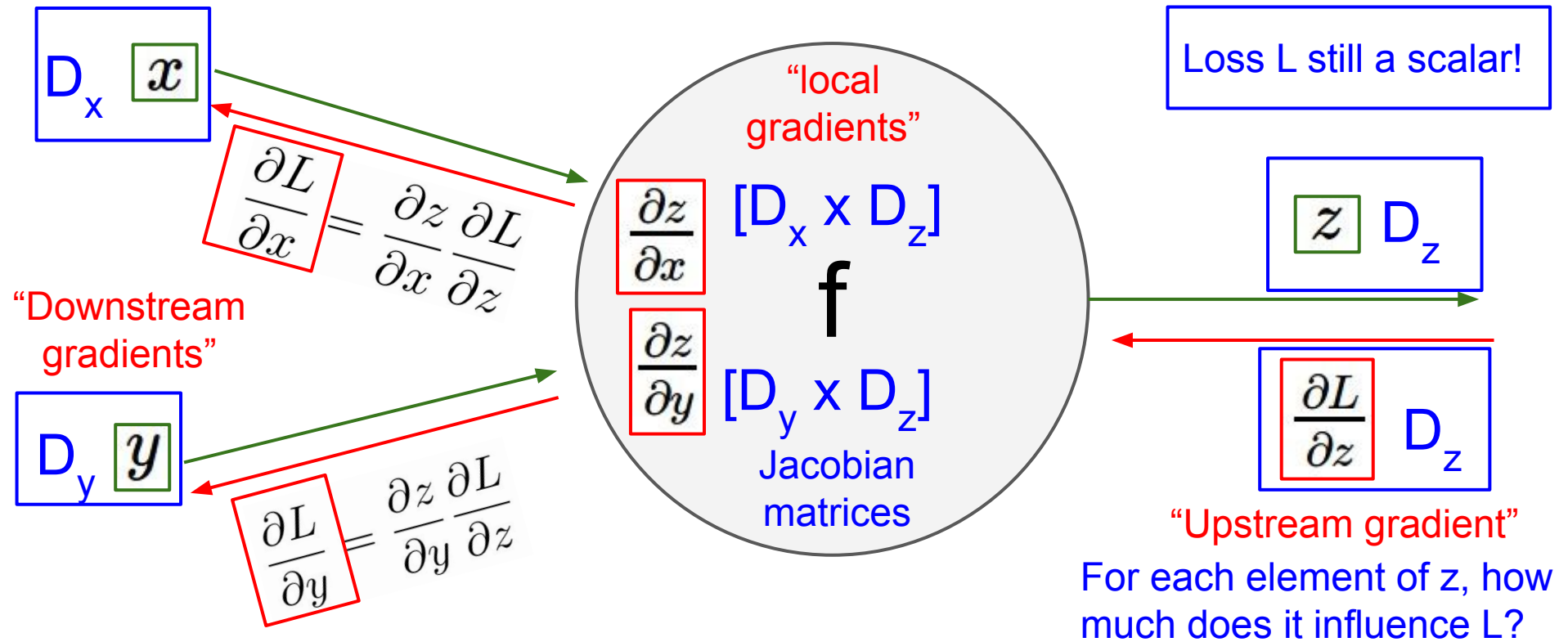
Backprop with Vectors



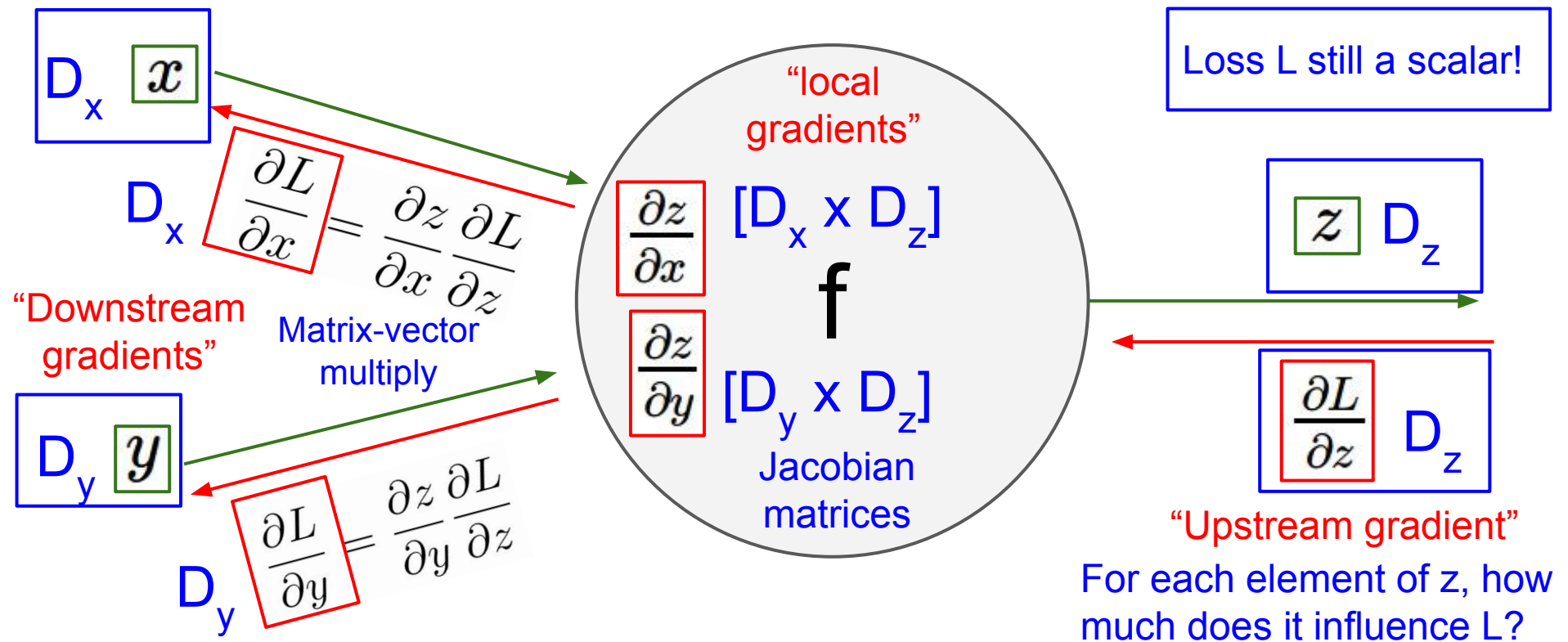
Backprop with Vectors



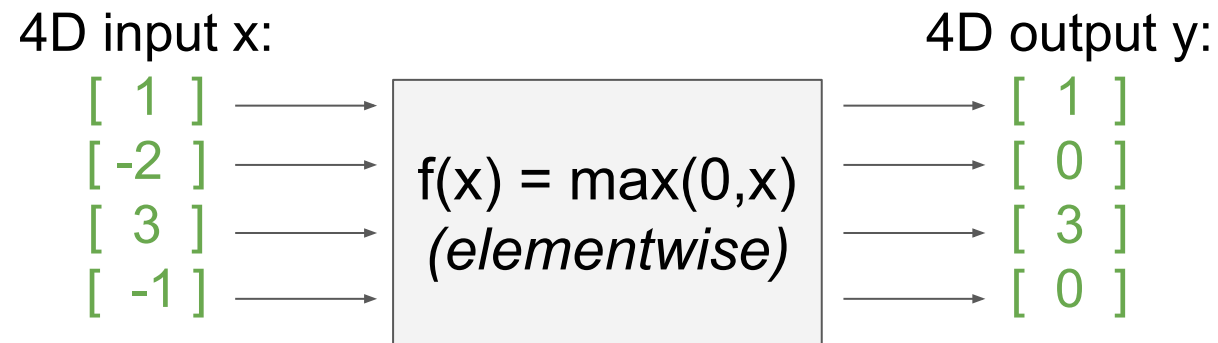
Backprop with Vectors



Backprop with Vectors



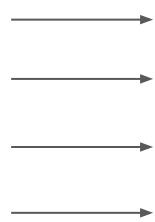
Backprop with Vectors



Backprop with Vectors

4D input x:

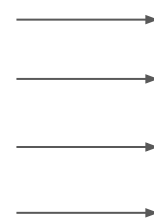
$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$



$f(x) = \max(0, x)$
(*elementwise*)

4D output y:

$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$



4D dL/dy:

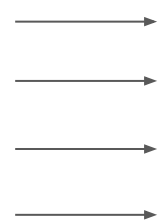
$\begin{bmatrix} 4 \\ -1 \\ 5 \\ 9 \end{bmatrix}$

Upstream
gradient

Backprop with Vectors

4D input x:

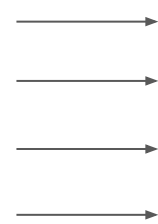
$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$



$f(x) = \max(0, x)$
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4D output y:

$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$



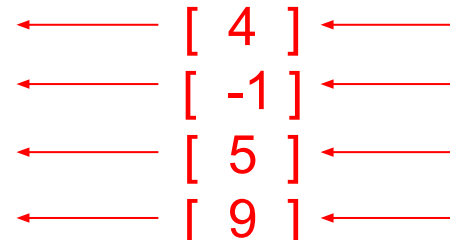
Jacobian dy/dx

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4D dL/dy :

$\begin{bmatrix} 4 \\ -1 \\ 5 \\ 9 \end{bmatrix}$

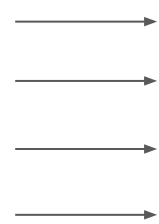
Upstream
gradient



Backprop with Vectors

4D input x:

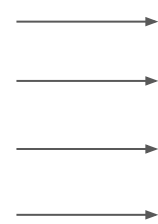
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$f(x) = \max(0, x)$
(*elementwise*)

4D output y:

$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

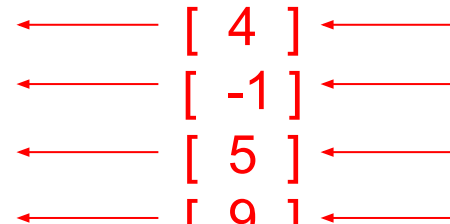


$\begin{bmatrix} dy/dx \end{bmatrix} \begin{bmatrix} dL/dy \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 5 \\ 9 \end{bmatrix}$

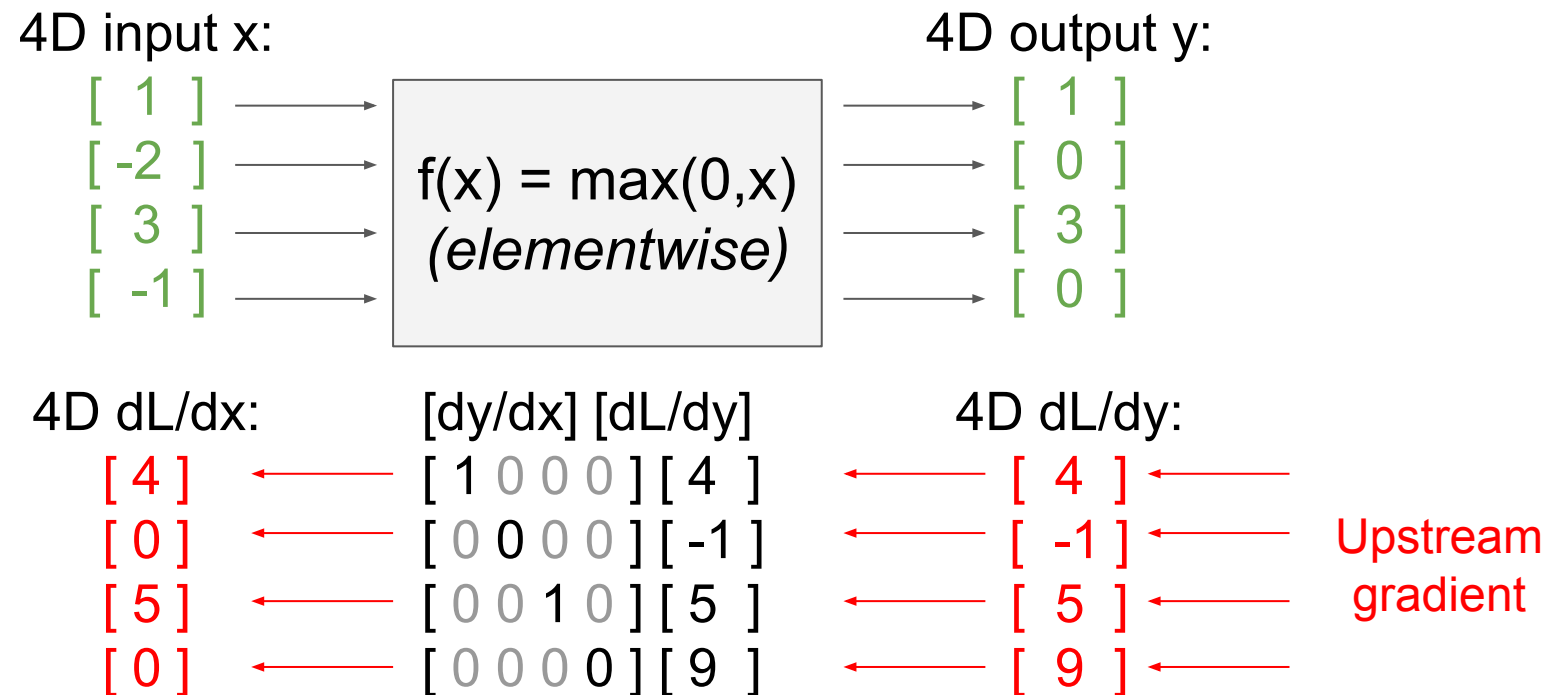
4D dL/dy :

$\begin{bmatrix} 4 \\ -1 \\ 5 \\ 9 \end{bmatrix}$

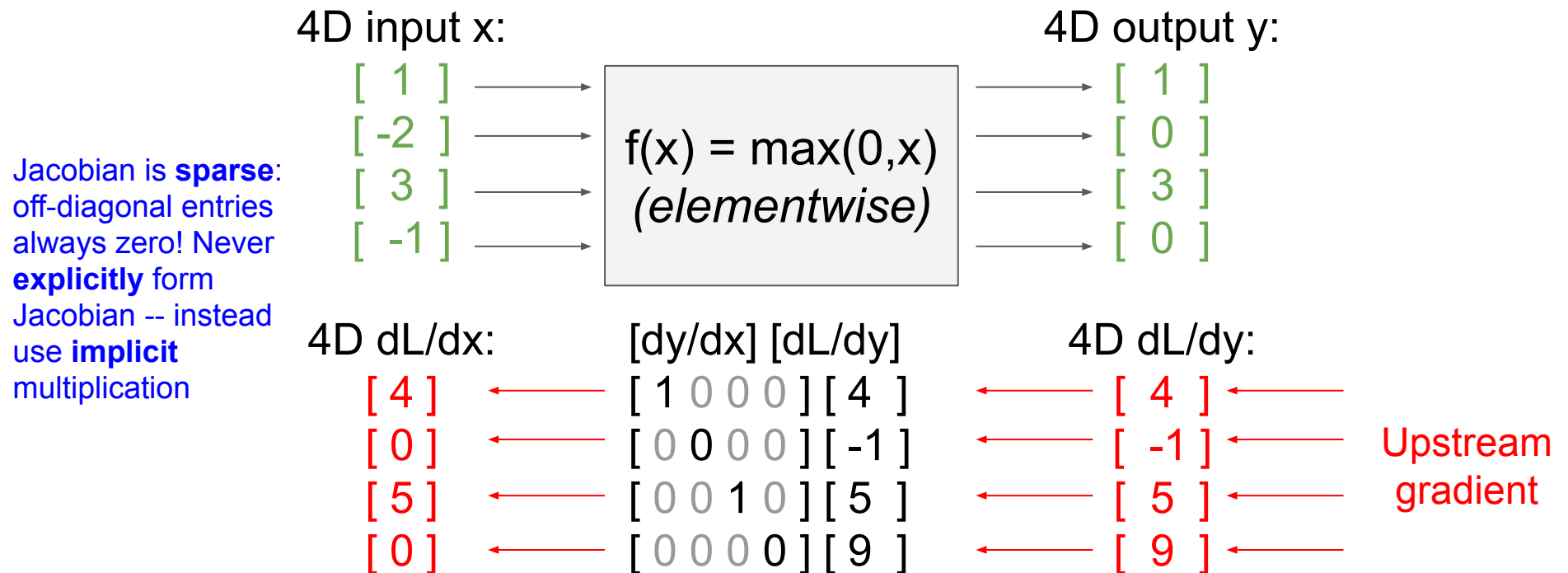


Upstream
gradient

Backprop with Vectors



Backprop with Vectors



Backprop with Vectors

Jacobian is **sparse**:
off-diagonal entries
always zero! Never
explicitly form
Jacobian -- instead
use **implicit**
multiplication

4D input x:

$\begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$

$f(x) = \max(0, x)$
(*elementwise*)

4D output y:

$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$

4D dL/dx:

$\begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \end{bmatrix}$

$[dy/dx] [dL/dy]$

$$\left(\frac{\partial L}{\partial x}\right)_i = \begin{cases} \left(\frac{\partial L}{\partial y}\right)_i & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

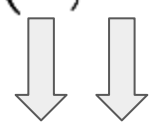
4D dL/dy:

$\begin{bmatrix} 4 \\ -1 \\ 5 \\ 9 \end{bmatrix}$

Upstream
gradient

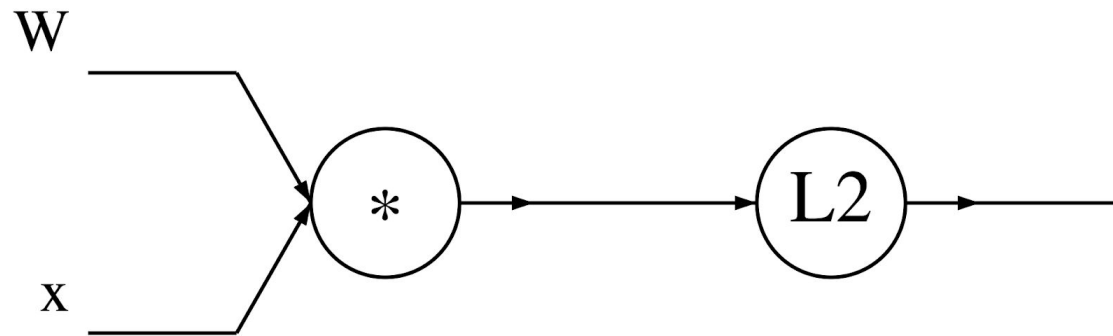
A vectorized example: $f(x, W) = ||W \cdot x||^2 = \sum_{i=1}^n (W \cdot x)_i^2$

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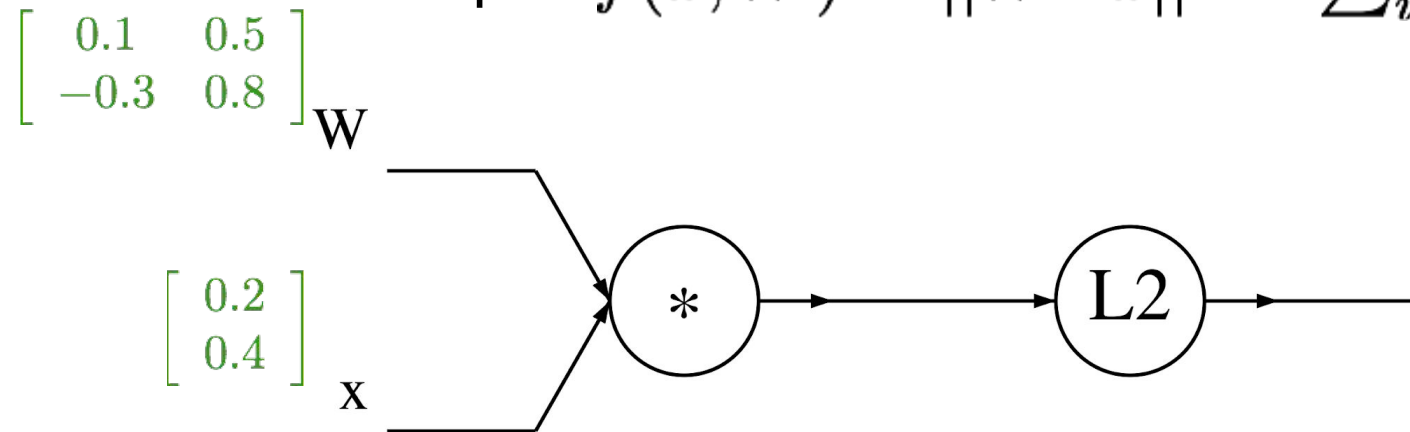


$\in \mathbb{R}^n \quad \in \mathbb{R}^{n \times n}$

A vectorized example: $f(x, W) = ||W \cdot x||^2 = \sum_{i=1}^n (W \cdot x)_i^2$



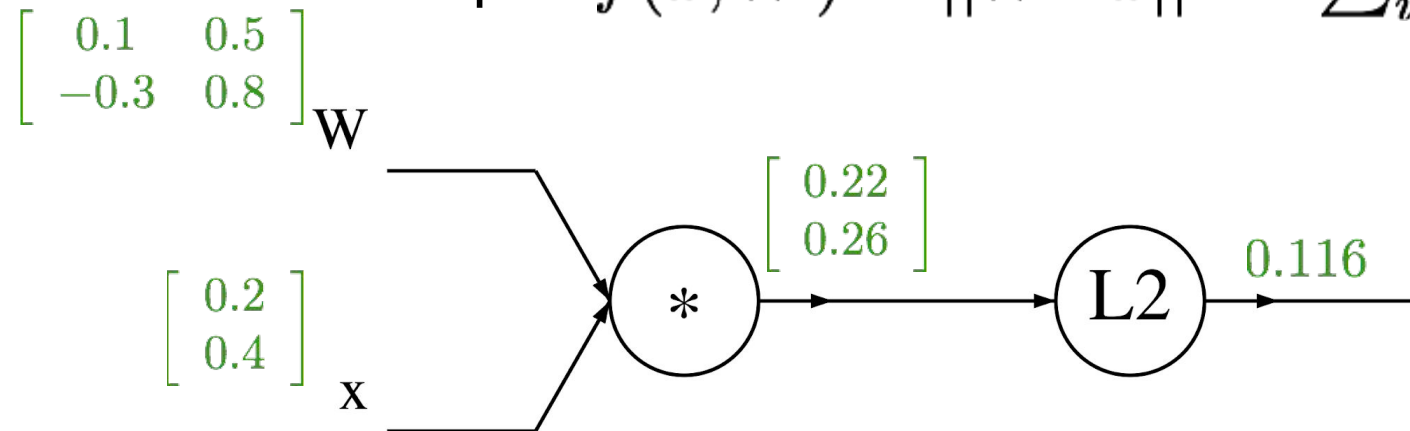
A vectorized example: $f(x, W) = ||W \cdot x||^2 = \sum_{i=1}^n (W \cdot x)_i^2$



$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = ||q||^2 = q_1^2 + \cdots + q_n^2$$

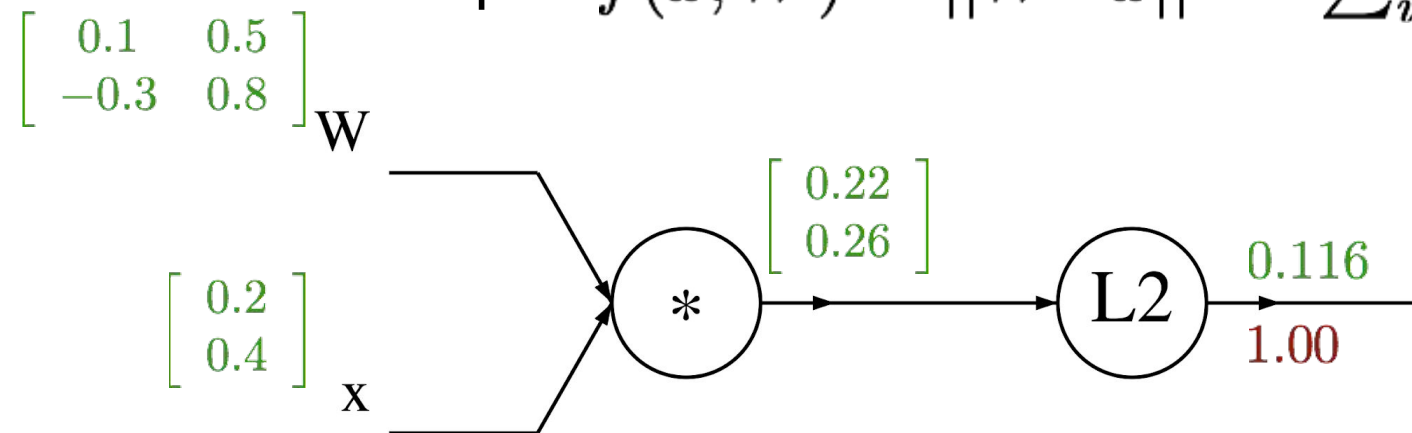
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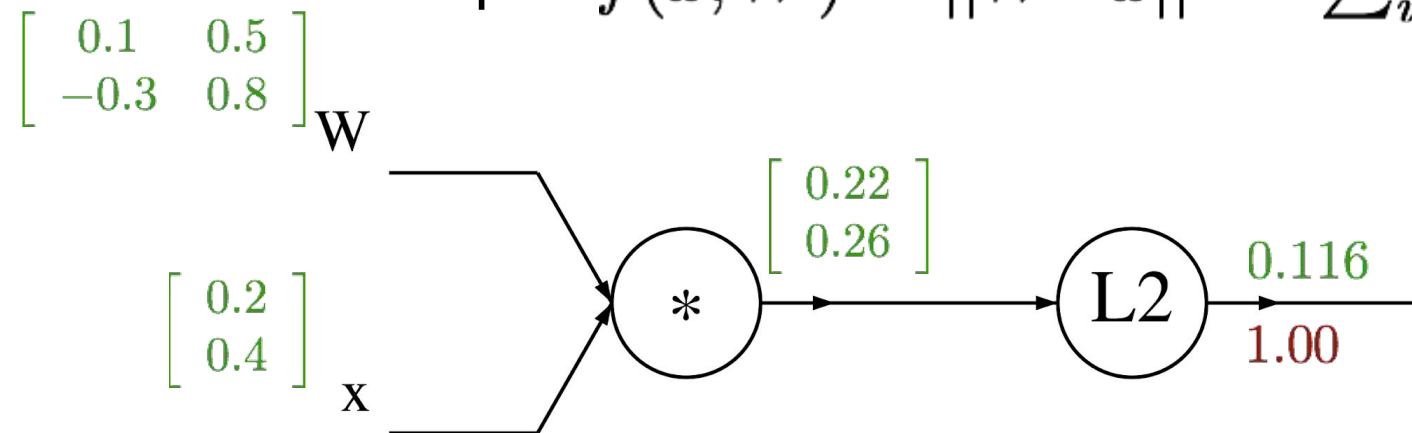
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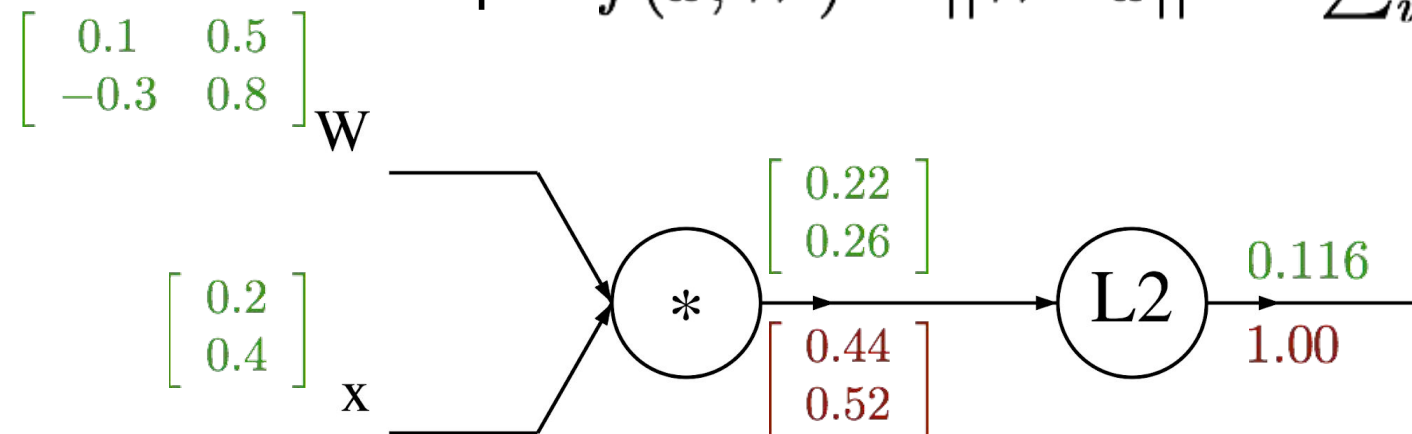
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$$f(q) = ||q||^2 = q_1^2 + \cdots + q_n^2$$

$$\frac{\partial f}{\partial q_i} = 2q_i$$

$$\nabla_q f = 2q$$

A vectorized example: $f(x, W) = ||W \cdot x||^2 = \sum_{i=1}^n (W \cdot x)_i^2$



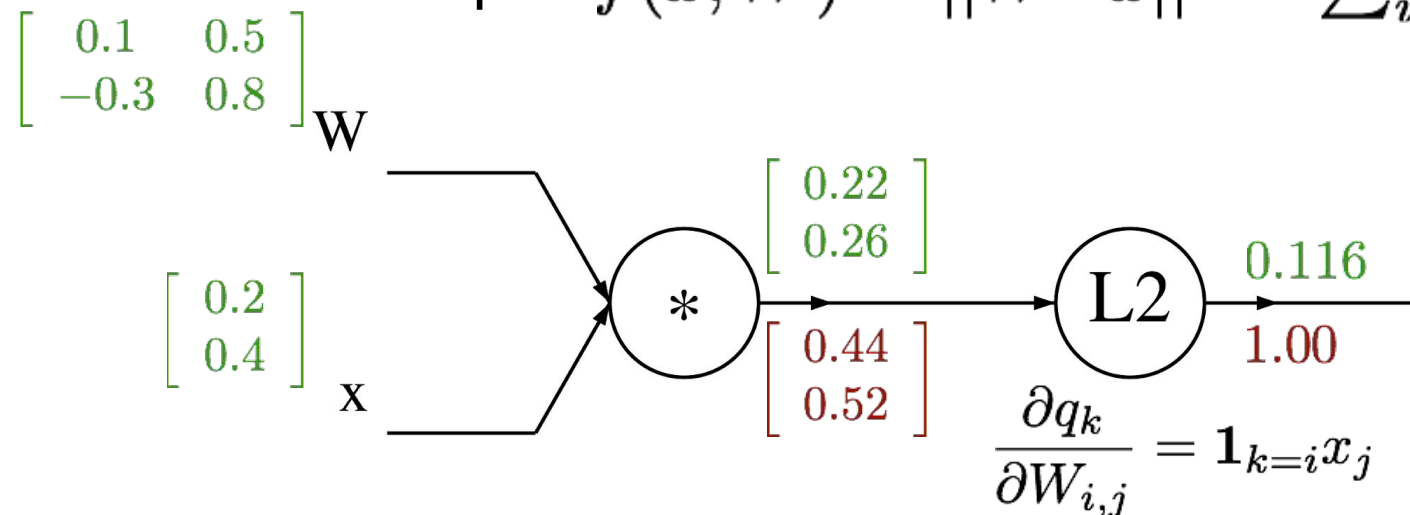
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$$\frac{\partial f}{\partial q_i} = 2q_i$$

$$\nabla_q f = 2q$$

A vectorized example: $f(x, W) = ||W \cdot x||^2 = \sum_{i=1}^n (W \cdot x)_i^2$

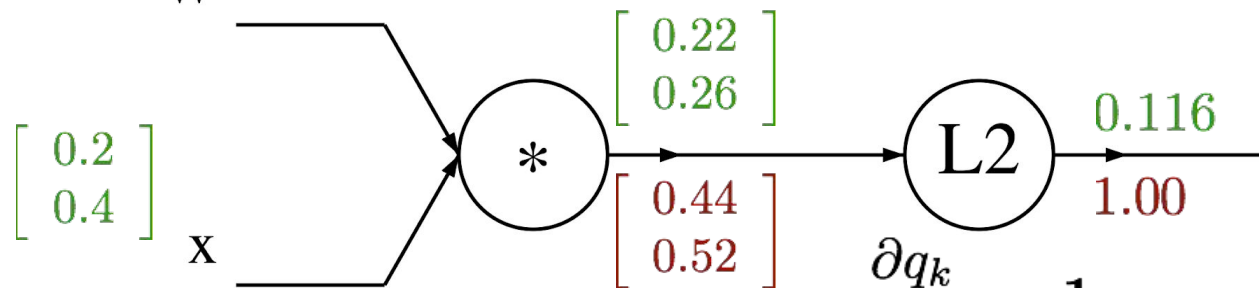


$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = ||q||^2 = q_1^2 + \cdots + q_n^2$$

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$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$



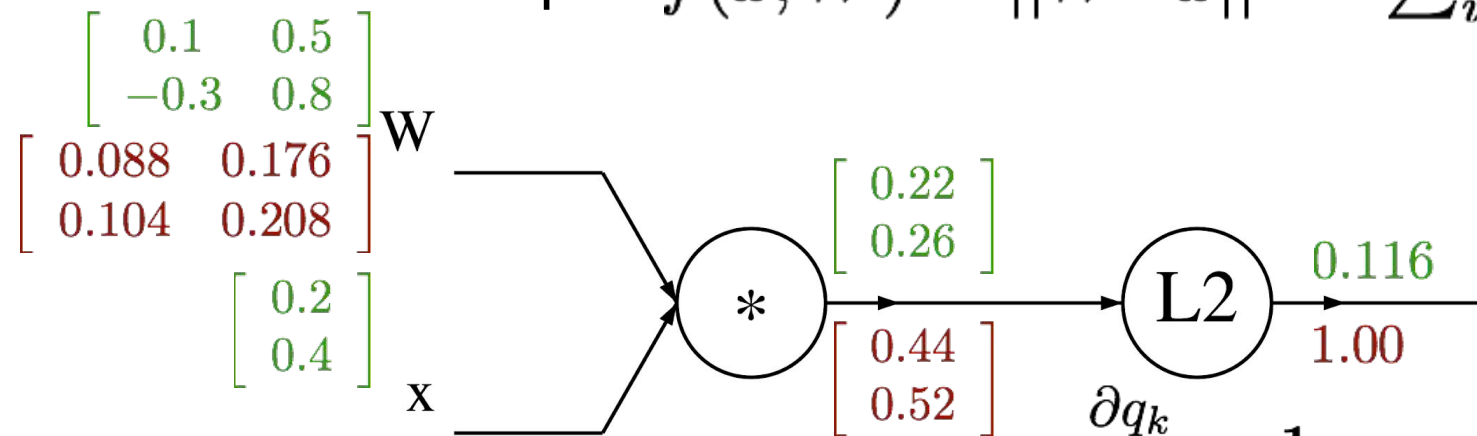
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$$f(q) = ||q||^2 = q_1^2 + \cdots + q_n^2$$

$$\frac{\partial q_k}{\partial W_{i,j}} = \mathbf{1}_{k=i}x_j$$

$$\begin{aligned} \frac{\partial f}{\partial W_{i,j}} &= \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial W_{i,j}} \\ &= \sum_k (2q_k)(\mathbf{1}_{k=i}x_j) \\ &= 2q_i x_j \end{aligned}$$

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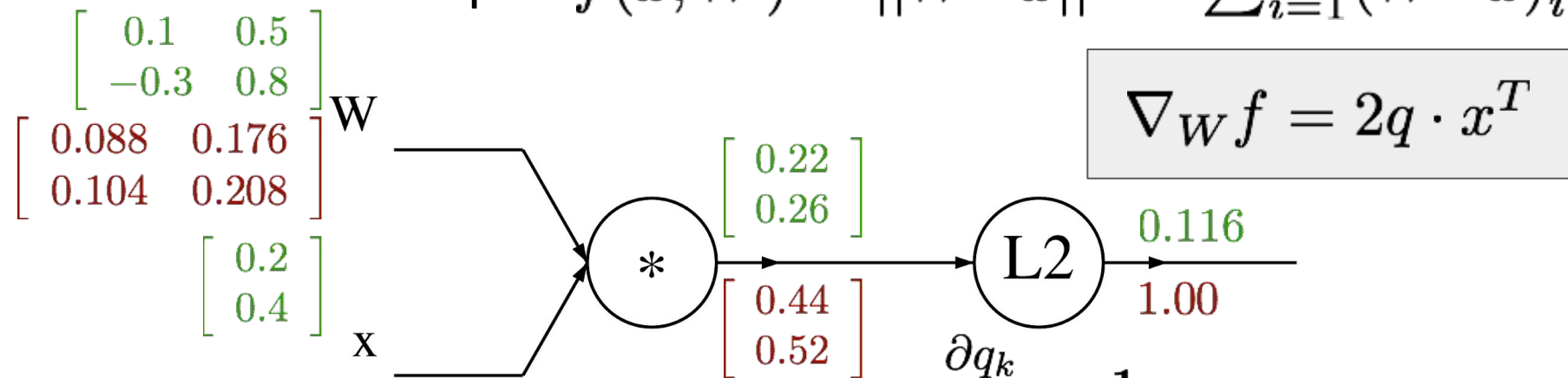


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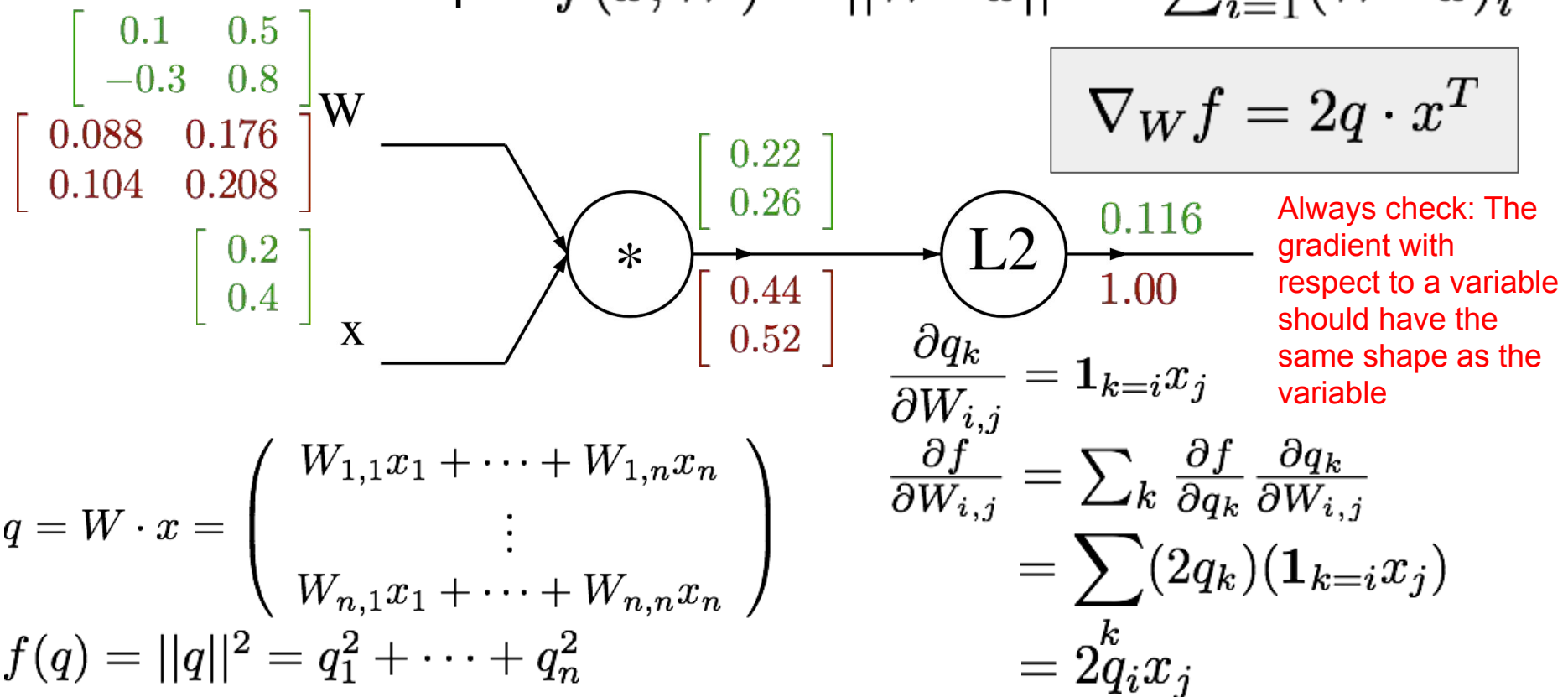


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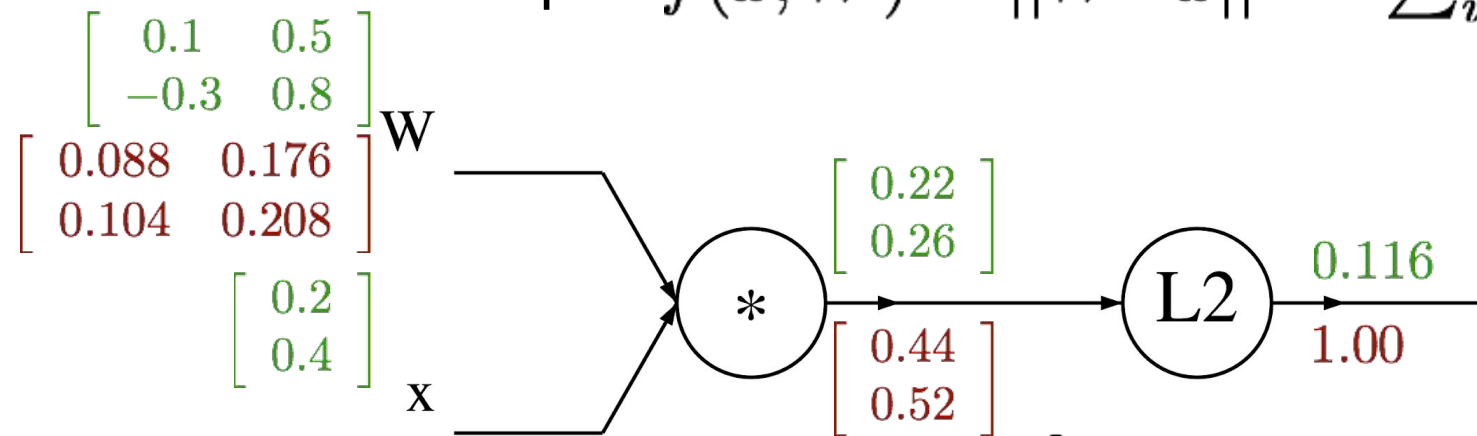
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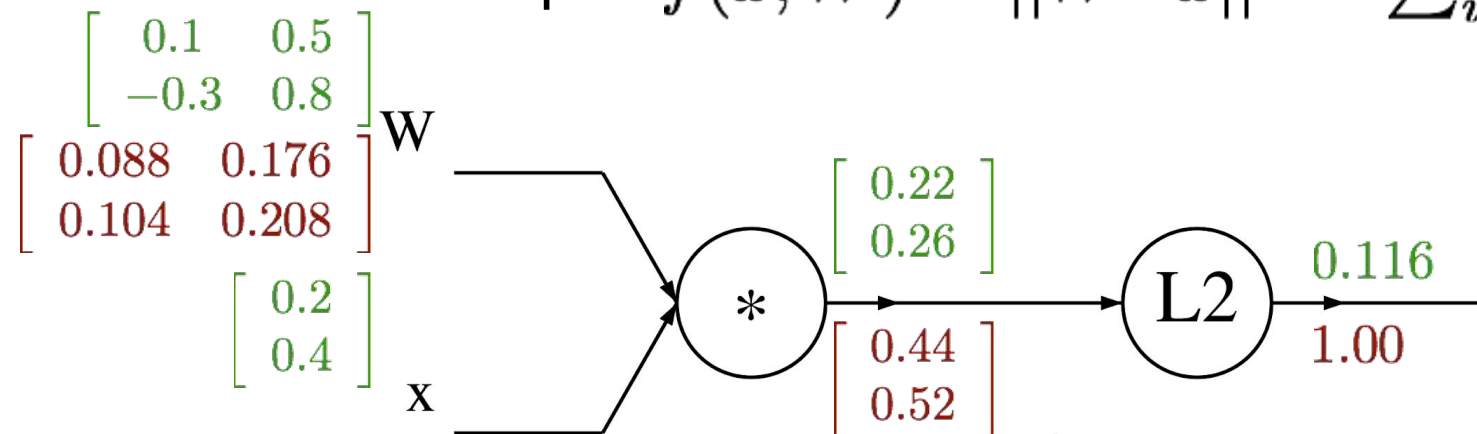


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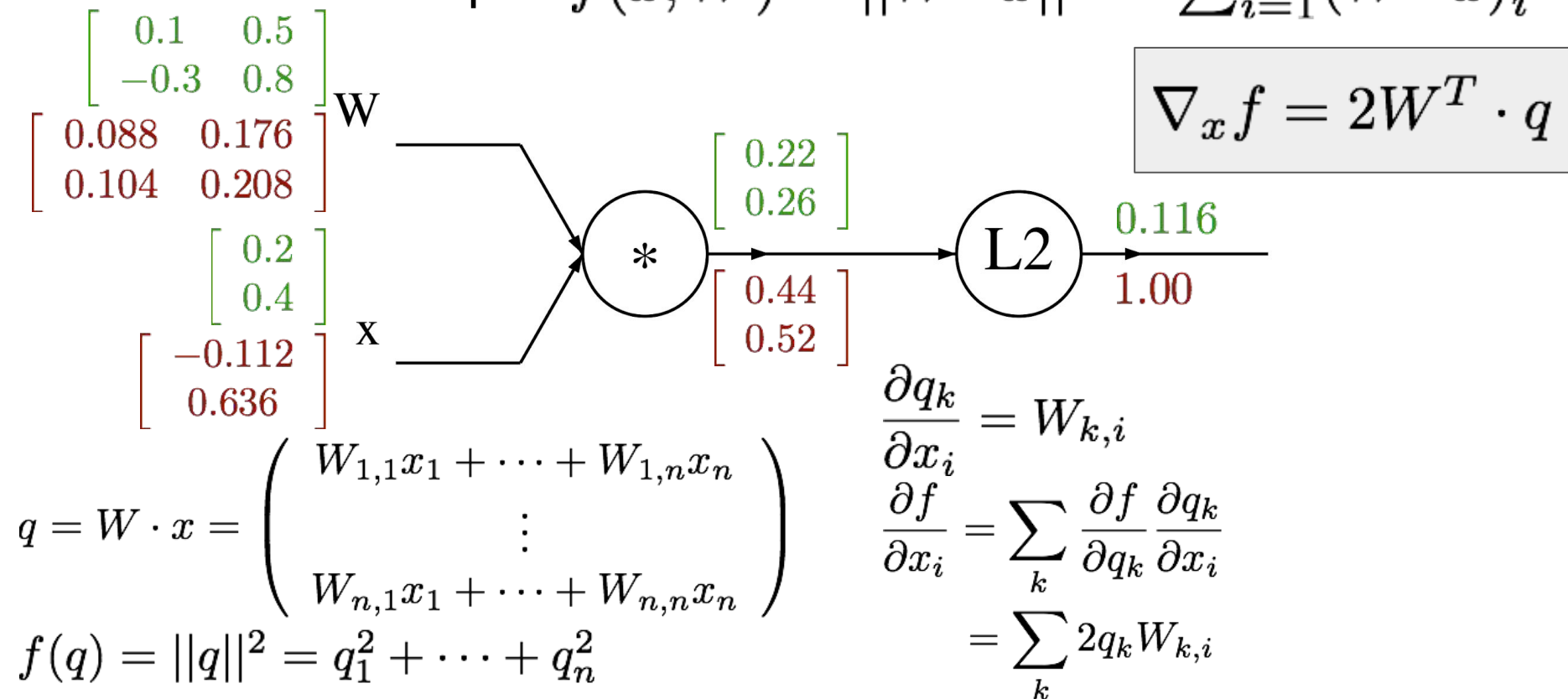


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In discussion section: A matrix example...

$$z_1 = XW_1$$

$$h_1 = \text{ReLU}(z_1)$$

$$\hat{y} = h_1 W_2$$

$$L = ||\hat{y}||_2^2$$

$$\frac{\partial L}{\partial W_2} = ?$$

$$\frac{\partial L}{\partial W_1} = ?$$

