Theory of Computing

Problem set

Context Free Grammar

Problem 1.
Which language generates the grammar G given by the productions?
$S \to aSa \mid aBa$
$B \to bB \mid b$

<u>Solution</u>
$L(G) = \{ a^n b^m a^n \mid n > 0, m > 0\}$.

The rule $S \to aSa$ recursively builds an equal number of *a's* on each end of the string. The recursion is terminated by the application of the rule $S \to aBa$, ensuring at least one leading and one trailing *a*. The recursive *B* rule then generates any number of *b's*. To remove the variable B from the string and obtain a sentence of the language, the rule $B \to b$ must be applied, forcing the presence of at least one *b*.

Problem 2.
Find a CFG that generates the language:
$L(G) = \{ a^n b^m c^m d^{2n} \mid n \geq 0, m > 0\}$.

<u>Solution</u>
The relationship between the number of leading *a's* and trailing *d's* in the language indicates that the recursive rule is needed to generate them. The same is true for *b's* and *c's*. Derivations in the grammar generate strings in an outside-to-inside manner.
$S \to aSdd \mid A$
$A \to bAc \mid bc$

Problem 3.
Find a CFG that generates the language
$L(G) = \{ a^n b^m \mid 0 \leq n \leq m \leq 2n\}$.

<u>Solution</u>
$S \to aSb \mid aSbb \mid \varepsilon$
The first recursive rule of G generates a trailing *b* for every *a*, while the second generates two *b*'s for each *a*. Thus there is at least one *b* for every *a* and at most two, as specified in the language.

Problem 4.
Consider the grammar
$S \to abScB \mid \varepsilon$
$B \to bB \mid b$
What language does it generate?

Solution

The language of the grammar consists of the set $L(G) = \{ (ab)^n (cb^m)^n \mid n \geq 0, m > 0\}$.

Problem 5

Construct context free grammars to accept the following languages. $\Sigma = \{0, 1\}$

a. {w | w starts and ends with the same symbol}
b. {w | w is odd}
c. {w | w is odd and its middle symbol is 0}
d. $\{0^n1^n \mid n > 0\} \cup \{0^n1^{2n} \mid n>0\}$
e. $\{0^i1^j2^k \mid i \neq j \text{ or } j \neq k\}$
f. Binary strings with twice as many 1s as 0s.

Solution

a.
S → 0A0 | 1A1
A → 0A | 1A | ε

b.
S → 0A | 1
A → 0S | 1S | ε

c.
S → 0 | 0S0 | 0S1 | 1S0 | 1S1

d.
S → 0A1 | 0B11
A → 0A1 | ε
B → 0B11 | ε

e.
S → AC | BC | DE | DF
A → 0 | 0A | 0A1
B → 1 | B1 | 0B1
C → 2 | 2C
D → 0 | 0D
E → 1 | 1E | 1E2
F → 2 | F2 | 1F2

f.
S → ε | 0S1S1S | 1S0S1S | 1S1S0S

Ambiguity

Problem 6

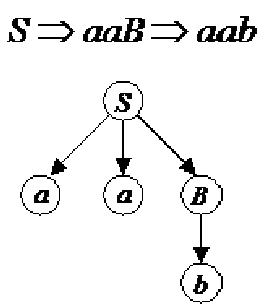
Explain why the grammar below is ambiguous.
S → 0A | 1B
A → 0AA | 1S | 1
B → 1BB | 0S | 0

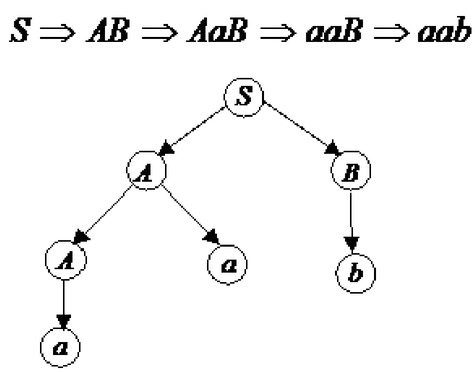
Solution
The grammar is ambiguous because we can find strings which have multiple derivations:
$S \Rightarrow 0A \Rightarrow 00AA \Rightarrow 001S1 \Rightarrow 0011B1 \Rightarrow 001101$
$S \Rightarrow 0A \Rightarrow 00AA \Rightarrow 0011S \Rightarrow 00110A \Rightarrow 001101$

Problem 7
Given the following ambiguous context free grammar
$S \rightarrow Ab \mid aaB$
$A \rightarrow a \mid Aa$
$B \rightarrow b$

Solution
Find the string s generated by the grammar that has two leftmost derivations. Show the derivations. The string s = aab has the following two leftmost derivations

$S \Rightarrow aaB \Rightarrow aab$
$S \Rightarrow AB \Rightarrow AaB \Rightarrow aaB \Rightarrow aab$





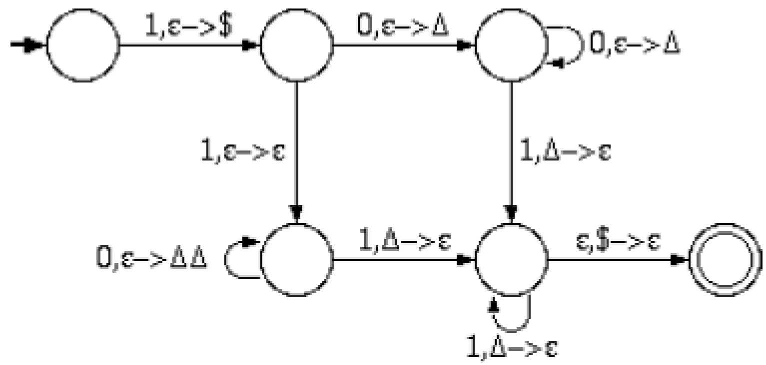
DPDA

Problem 8
Construct deterministic pushdown automata to accept the following languages.

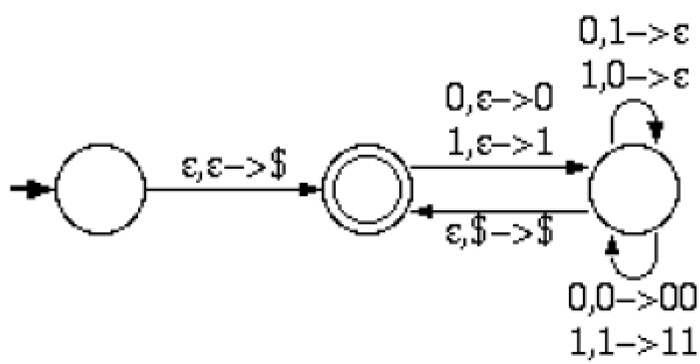
a. $\{10^n1^n \mid n>0\} \cup \{110^n1^{2n} \mid n>0\}$
b. Binary strings that contain an equal number of 1s and 0s
c. Binary strings with twice as many 1s as 0s
d. Binary strings that start and end with the same symbol and have the same number of 0s as 1s.
e. $L = \{a^nb^m : m \geq n + 2\}$

Solution

a.

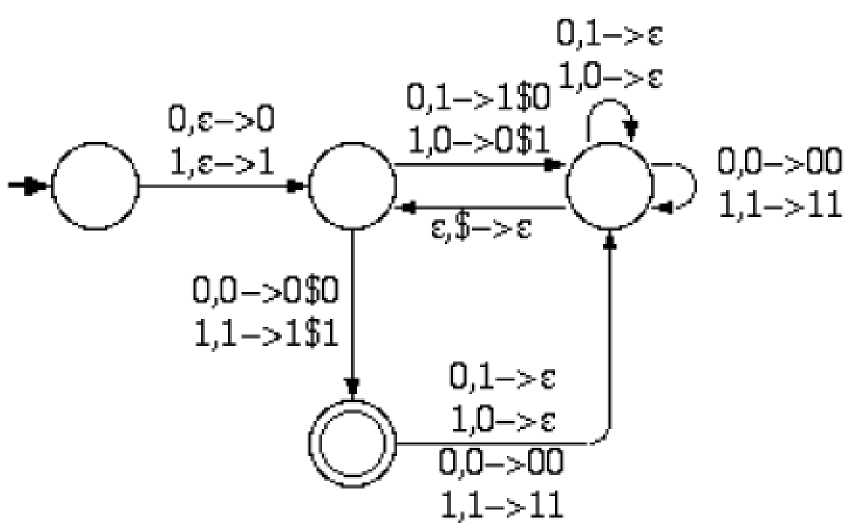


b.



c. Do it yourself

d.



e. Do it yourself

Pumping Lemma

Problem 9. Show that the following language on $\Sigma = \{a, b, c\}$ is not context-free.
$L = \{a^n b^j c^k : k = jn\}$.

Solution

Assume that $L = \{a^n b^j c^k : k = jn\}$ is a context free language.

Let $w = a^m b^m c^{m^2}$, $w \in L$

By the Pumping Lemma w can be decomposed as w = uvxyz with $|vxz| \le m$

and $|vy| \ge 1$ such that $uv^i xy^i z \in L, i \ge 0$

case 1

$$\underbrace{aaa\ldots a}_{uvxy}\underbrace{ab\cdots bc\cdots c}_{z}$$

If i=0, $uv^0 xy^0 z = a^{m-|vy|} b^m c^{m^2} \notin L$

case 2

$$\underbrace{aaa\ldots a}_{uv}\underbrace{a}_{x}\underbrace{b\ldots}_{y}\underbrace{bc\ldots}_{z}$$

If i=0, $uv^0 xy^0 z = a^{m-|v|} b^{m-|y|} c^{m^2} \notin L$

case 3

$$\underbrace{aaa\ldots ab}_{u}\underbrace{b\ldots b}_{vxy}\underbrace{bc\ldots c}_{z}$$

If i=0, $uv^0 xy^0 z = a^m b^{m-|vy|} c^{m^2} \notin L$

case 4

$$\underbrace{aaa\ldots ab}_{u}\underbrace{b\ldots b}_{v}\underbrace{bc}_{x}\underbrace{c\ldots c}_{yz}$$

If i=0, $uv^0xy^0z = a^m b^{m-|v|} c^{m^2-|y|} \notin L$

case 5

$$\underbrace{aaa\ldots ab\ldots bc}_{u}\underbrace{c\ldots c}_{vxyz}$$

If i=0, $uv^0xy^0z = a^m b^m c^{m^2-|vy|} \notin L$

case 6 v or y containing ab or bc

If I > 0, uv^ixy^iz would be $a\ldots ab\ldots ba\ldots ab\ldots b\ldots c\ldots c$

Or $a\ldots ab\ldots bc\ldots cb\ldots c$

Contrary to the assumption.

Language is not context free.

Problem 10. Show that the following language on Σ = {a, b} is not context-free:
L = { $ww^Ra^{|w|}$: w ∈{a, b}*}

Solution

Assume that L = { $ww^Ra^{|w|}$: w ∈ {a, b}*}is a context free language.

Let $w = a^m b^m$, then $ww^R a^{|w|} = a^m b^m b^m a^m a^{2m} \in L$

By the Pumping Lemma w can be decomposed as w = uvxyz with |vxz| ≤m

and |vy| ≥ 1 such that $uv^ixy^iz \in L, i \geq 0$

case 1

$$\underbrace{a\ldots ab\ldots b}_{uvxy}\underbrace{b\ldots ba\ldots aa\ldots a}_{z}$$

If i=0, |w| = 2m - |vy| is less than $|w^R|$. So $uv^0xy^0z \notin L$.

case 2

$$\underbrace{a\ldots ab\ldots}_{uvxy}\underbrace{b\ldots ba\ldots aa\ldots a}_{z}$$

If i=0, $|w^R| = 2m - |vy|$ is less than $|w|$. So $uv^0xy^0z \notin L$.

case 3

$$\underbrace{a\ldots ab\ldots bb\ldots ba\ldots aa}_{u}\underbrace{b\ldots\ldots a}_{vxyz}$$

If i=0, $\left|a^{|w|}\right| = 2m - |vx| < |w|$. So $uv^0xy^0z \notin L$.

case 4

$$\underbrace{a\ldots ab}_{u}\underbrace{b\ldots\ldots b}_{vxy}\underbrace{ba\ldots aa\ldots a}_{z}$$

If i=0, $ww^R = 4m - |vy| < 2\left|a^{|w|}\right| = 4m$. So $uv^0xy^0z \notin L$.

case 5

$$\underbrace{a\ldots ab\ldots bb\ldots ba}_{u}\ldots\underbrace{a\ldots\ldots aa}_{vxy}\underbrace{a\ldots a}_{z}$$

If i=0, $w^R a^{|w|} = 4m - |vy| < 2|w| = 4m$. So $uv^0xy^0z \notin L$.

This is contrary to the assumption.

Language L is therefore not context free.

Parse Tree

Problem 11:

Consider the following grammar with start symbol *E*:

$E \rightarrow E+T \mid E-T \mid T$
$T \rightarrow T*F \mid T/F \mid F$
$F \rightarrow (E) \mid id$

Construct a parse tree for the following string:

id(id-id)/id*

Solution

$E \Rightarrow T \Rightarrow T*F \Rightarrow F*F \Rightarrow id*F \Rightarrow id*T/F \Rightarrow id*F/F \Rightarrow id*(E)/F \Rightarrow id*(E\text{-}T)/F \Rightarrow id*(T\text{-}T)/F \Rightarrow$
$id*(F\text{-}T)/F \Rightarrow id*(id\text{-}T)/F \Rightarrow id*(id\text{-}F)/F \Rightarrow id*(id\text{-}id)/F \Rightarrow id*(id\text{-}id)/id$

Chomsky Normal Form

Problem 12:

Proof that any context-free language is generated by a context-free grammar in Chomsky normal form.

Solution Book (2.9)

Problem 13:

Convert the following grammar in Chomsky normal form

$S \rightarrow ASA \mid aB$
$A \rightarrow B \mid S$
$B \rightarrow b \mid \varepsilon$

Solution Book (2.10)

Turing Machine

Problem 14:

Construct a Turing machines that accepts the following languages on the alphabet {a, b}.

a. $L = \{01^*0\}$
b. $L = \{a^n b^n c^n \mid n \geq 1\}$

Solution sheet-2

Problem 15

Describe/give formal definition to

Context Free Language (CFL)	Non Context Free Language (NCFL)
Context Free Grammar (CFG)	Pumping Lemma for NCFL
Ambiguity	PushDown Automata (PDA)
Chomsky Normal Form (CNF)	Turing Machine (TM)

Syllabus

1. Book (2.1, 2.2, 2.3, 3.1)
2. Sheet-1, sheet-2
3. Problem set