Theory of Computing

CSE - 203

Introduction to Grammars

Grammar

A grammar **G** can be formally written as a 4-tuple (N, T, S, P) where −

- **N** or $\mathbf{V_N}$ is a set of variables or non-terminal symbols.
- **T** or **∑** is a set of Terminal symbols.
- **S** is a special variable called the Start symbol, $S \in N$
- **P** is Production rules for Terminals and Non-terminals. A production rule has the form $\alpha \rightarrow \beta$, where α and β are strings on $V_N \cup \sum$ and least one symbol of α belongs to V_N.

Example

Grammar G1 − ({S, A, B}, {a, b}, S, {S → AB, A → a, B → b})

Here,

- **S, A,** and **B** are Non-terminal symbols;
- **a** and **b** are Terminal symbols
- **S** is the Start symbol, $S \in N$
- Productions, **P : S → AB, A → a, B → b**

Example

If there is a grammar

G: N = {S, A, B} T = {a, b} P = {S → AB, A → a, B → b}

Here **S** produces **AB**, and we can replace **A** by **a**, and **B** by **b**. Here, the only accepted string is **ab**, i.e.,

$$L(G) = \{ab\}$$

Construction of a Grammar Generating a Language

We'll consider some languages and convert it into a grammar G which produces those languages.

Example

Problem − Suppose, $L(G) = \{a^m b^n \mid m \geq 0 \text{ and } n > 0\}$. We have to find out the grammar **G** which produces **L(G)**.

Solution

Since L(G) = {$a^m b^n$ | m ≥ 0 and n > 0}

the set of strings accepted can be rewritten as −

L(G) = {b, ab,bb, aab, abb, …….}

Here, the start symbol has to take at least one 'b' preceded by any number of 'a' including null.

To accept the string set {b, ab, bb, aab, abb, …….}, we have taken the productions −

S → aS, S → B, B → b and B → bB
S → B → b (Accepted)
S → B → bB → bb (Accepted)
S → aS → aB → ab (Accepted)
S → aS → aaS → aaB → aab (Accepted)
S → aS → aB → abB → abb (Accepted)

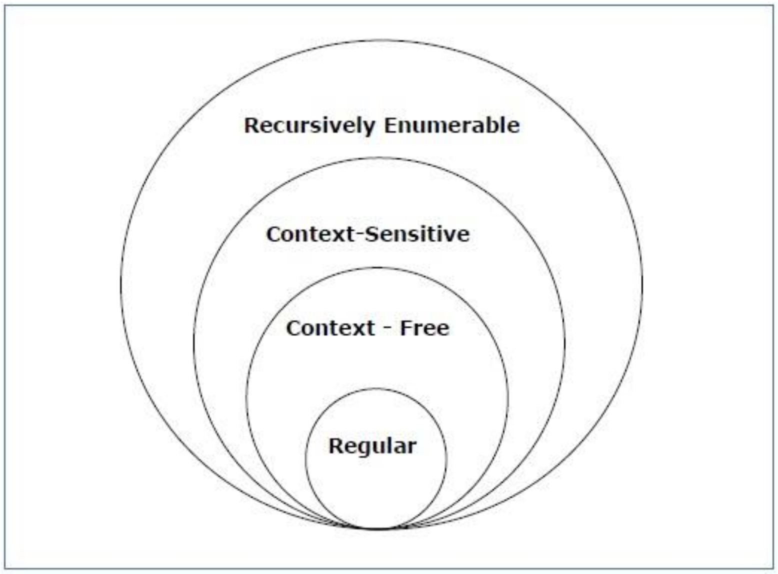
Thus, we can prove every single string in L(G) is accepted by the language generated by the production set.

Hence the grammar −
G: ({S, A, B}, {a, b}, S, {S → aS | B, B → b | bB})

According to Noam Chomosky, there are four types of grammars − Type 0, Type 1, Type 2, and Type 3. The following table shows how they differ from each other −

Grammar Type	**Grammar Accepted**	**Language Accepted**	**Automaton**
Type 0	Unrestricted grammar	Recursively enumerable language	Turing Machine
Type 1	Context-sensitive grammar	Context-sensitive language	Linear-bounded automaton
Type 2	Context-free grammar	Context-free language	Pushdown automaton
Type 3	Regular grammar	Regular language	Finite state automaton

The following illustration shows the scope of each type of grammar −



Type - 3 Grammar

Type-3 grammars generate regular languages. Type-3 grammars must have a single non-terminal on the left-hand side and a right-hand side consisting of a single terminal or single terminal followed by a single non-terminal.

The productions must be in the form **X → a or X → aY** where **X, Y ∈ N** (Non terminal) and **a ∈ T** (Terminal)

The rule **S → ε** is allowed if **S** does not appear on the right side of any rule.

Example

X → ε
X → a | aY
Y → b

Type - 2 Grammar

Type-2 grammars generate context-free languages.

The productions must be in the form **A → γ** where **A ∈ N** (Non terminal) and **γ ∈ (T ∪ N)*** (String of terminals and non-terminals).

These languages generated by these grammars are be recognized by a non-deterministic pushdown automaton.

Example

S → X a

X → a

X → aX

X → abc

X → ε

Type - 1 Grammar

Type-1 grammars generate context-sensitive languages. The productions must be in the form

α A β → α γ β

where **A ∈ N** (Non-terminal) and **α, β, γ ∈ (T ∪ N)*** (Strings of terminals and non-terminals)

The strings **α** and **β** may be empty, but **γ** must be non-empty.

The rule **S → ε** is allowed if S does not appear on the right side of any rule. The languages generated by these grammars are recognized by a linear bounded automaton.

Example

AB → AbBc

A → bcA

B → b

Type - 0 Grammar

Type-0 grammars generate recursively enumerable languages. The productions have no restrictions. They are any phase structure grammar including all formal grammars.

They generate the languages that are recognized by a Turing machine.

The productions can be in the form of **α → β** where **α** is a string of terminals and nonterminals with at least one non-terminal and **α** cannot be null. **β** is a string of terminals and non-terminals.

Example

S → ACaB

Bc → acB

CB → DB

aD → Db

Pumping Lemma for Regular Grammars

Theorem

Let L be a regular language. Then there exists a constant **'c'** such that for every string **w** in **L** –

$$|w| \geq c$$

We can break **w** into three strings, **w = xyz**, such that −

- $|y| > 0$
- $|xy| \leq c$
- For all $k \geq 0$, the string xy^kz is also in L.

Applications of Pumping Lemma

Pumping Lemma is to be applied to show that certain languages are not regular. It should never be used to show a language is regular.

- If **L** is regular, it satisfies Pumping Lemma.
- If **L** does not satisfy Pumping Lemma, it is non-regular.

Method to prove that a language L is not regular

- At first, we have to assume that **L** is regular.
- So, the pumping lemma should hold for **L**.
- Use the pumping lemma to obtain a contradiction −
    - Select **w** such that $|w| \geq c$
    - Select **y** such that $|y| \geq 1$
    - Select **x** such that $|xy| \leq c$
    - Assign the remaining string to **z.**
    - Select **k** such that the resulting string is not in **L.**

Hence L is not regular.

Problem

Prove that $L = \{a^ib^i \mid i \geq 0\}$ is not regular.

Solution −

- At first, we assume that **L** is regular and n is the number of states.
- Let w = a^nb^n. Thus $|w| = 2n \geq n$.
- By pumping lemma, let w = xyz, where $|xy| \leq n$.
- Let $x = a^p$, $y = a^q$, and $z = a^rb^n$, where $p + q + r = n$, $p \neq 0$, $q \neq 0$, $r \neq 0$. Thus $|y| \neq 0$.
- Let k = 2. Then $xy^2z = a^pa^{2q}a^rb^n$.
- Number of as = $(p + 2q + r) = (p + q + r) + q = n + q$
- Hence, $xy^2z = a^{n+q}\ b^n$. Since $q \neq 0$, xy^2z is not of the form a^nb^n.
- Thus, xy^2z is not in L. Hence L is not regular.

Homework

1. Find a CFG that generates the following languages:
    a. $L(G) = \{ a^n b^m c^m d^{2n} \mid n \geq 0, m > 0\}$.
    b. $L(G) = \{ a^n b^m \mid 0 \leq n \leq m \leq 2n\}$.

2. Which language generates the grammar G given by the following production rules?

    a. $S \rightarrow aSa \mid aBa$
       $B \rightarrow bB \mid b$

    b. $S \rightarrow abScB \mid \varepsilon$
       $B \rightarrow bB \mid b$