Units, Vectors and Motion

A measurement necessarily involves a reference frame and therefore units. In 1960, during the eleventh Conférence Générale des Poids et Mesures (CGPM), the International System of Units, the SI, was developed. It now includes two classes of units The seven **base unites** and the **derived units**.

	SI base unit	
Base quantity	**Name**	**Symbol**
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Electricity and magnetism		Mass and related quantities	
potential difference, U :	volt (V = W/A)	density : ρ	$kg.m^{-3}$
electrical capacitance, C :	farad (F = C/V)	volume : V	m^3
electrical resistance, R :	ohm (Ω = V/A)	force : F	newton (N)
inductance, L :	henri (H = Wb/A)	torque : M	N.m
quantity of electricity, Q :	coulomb (C = A.s)	pressure : p	pascal (Pa)
power, P :	watt (W = J/s)	dynamic volume : v	m^3
energy, W :	joule (J = N.m)	mass flow-rate : qm	$kg.s^{-1}$
magnetic induction, B :	tesla (T = Wb/m2)	volume flow-rate : qv	$m^3.s^{-1}$
electric field, E :	volt per metre (V/m)	air flow-rate : V	$m.s^{-1}$
magnetic field strength, H :	ampère per metre (A/m)		
electric conductance, G :	siemens (S = A/V)		

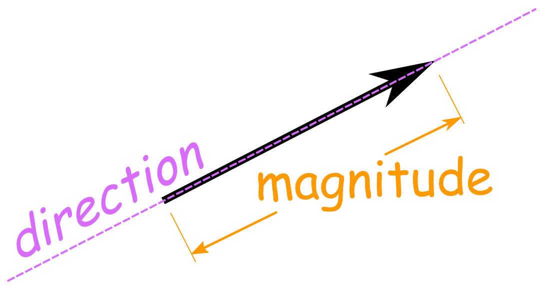
The 20 SI prefixes used to form decimal multiples and submultiples of SI units are given bellow

Factor	Name	Symbol	Factor	Name	Symbol
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	µ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f
10^{1}	deka	da	10^{-18}	atto	a

Vectors

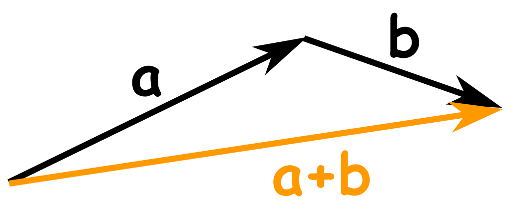
This is a vector:

A vector has **magnitude** (size) and **direction**:

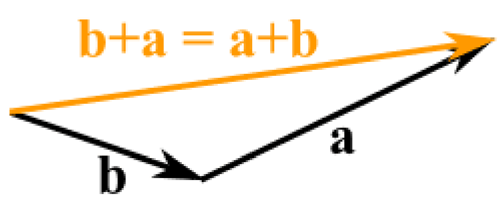


The length of the line shows its magnitude and the arrowhead points in the direction.

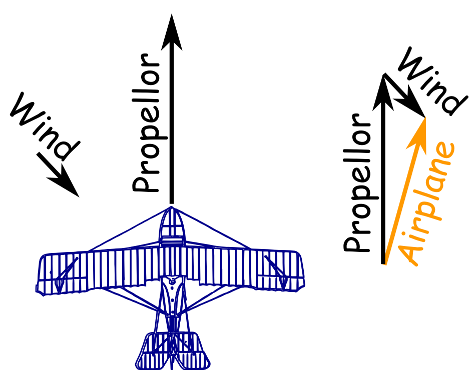
We can add two vectors by joining them head-to-tail:



And it doesn't matter which order we add them, we get the same result:

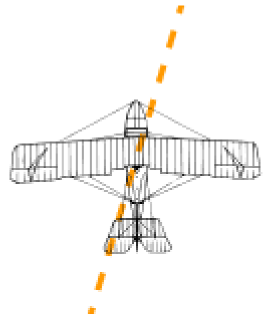


Example: A plane is flying along, pointing North, but there is a wind coming from the North-West.



The two vectors (the velocity caused by the propeller, and the velocity of the wind) result in a slightly slower ground speed heading a little East of North.

If you watched the plane from the ground it would seem to be slipping sideways a little.



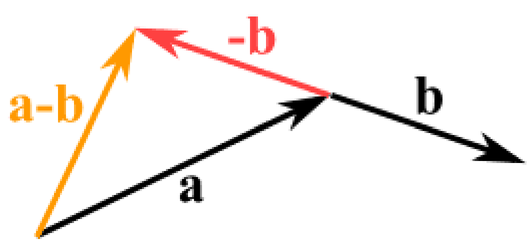
Have you ever seen that happen? Maybe you have seen birds struggling against a strong wind that seem to fly sideways. Vectors help explain that.

Velocity, acceleration, force and many other things are vectors.

Subtracting

We can also subtract one vector from another:

- first we reverse the direction of the vector we want to subtract,
- then add them as usual:

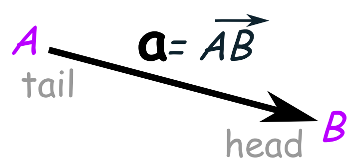


a − b

Notation

A vector is often written in **bold**, like **a** or **b**.

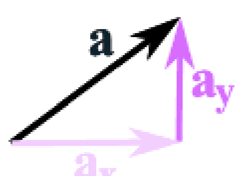
A vector can also be written as the letters of its head and tail with an arrow above it, like this:



Calculations

Now ... how do we do the calculations?

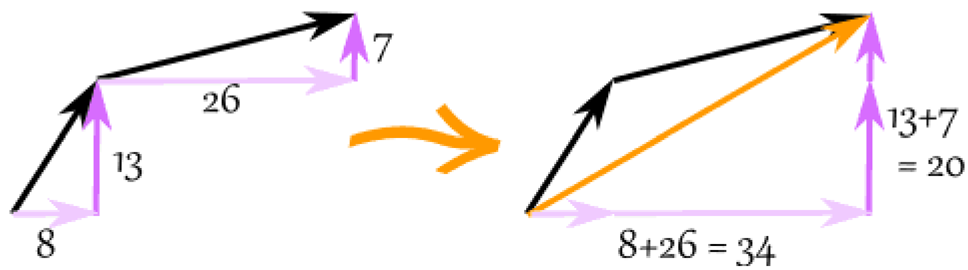
The most common way is to first break up vectors into x and y parts, like this:



The vector **a** is broken up into
the two vectors $\mathbf{a}_x$ and $\mathbf{a}_y$

Adding Vectors

We can then add vectors by **adding the x parts** and **adding the y parts**:



The vector (8,13) and the vector (26,7) add up to the vector (34,20)

Example: add the vectors a = (8, 13) and b = (26, 7)

c = **a** + **b**

c = (8,13) + (26,7) = (8+26,13+7) = (34,20)

Subtracting Vectors

To subtract, first reverse the vector we want to subtract, then add.

Example: subtract k = (4,5) from v = (12,2)

a = **v** + −**k**

a = (12,2) + −(4,5) = (12,2) + (−4,−5) = (12−4,2−5) = (8,−3)

Magnitude of a Vector

The magnitude of a vector is shown by two vertical bars on either side of the vector: |**a**|

OR it can be written with double vertical bars (so as not to confuse it with absolute value): ||**a**||

We use Pythagoras' theorem to calculate it: $|\mathbf{a}| = \sqrt{(x^2 + y^2)}$

Example: what is the magnitude of the vector b = (6,8) ?

$|\mathbf{b}| = \sqrt{(6^2 + 8^2)} = \sqrt{(36+64)} = \sqrt{100} = 10$

A vector with magnitude 1 is called a Unit Vector.

Vector vs Scalar

A scalar has magnitude (size) only.

Scalar: just a number (like 7 or −0.32) ... definitely not a vector.

A **vector** has **magnitude and direction**, and is often written in **bold**, so we know it is not a scalar:

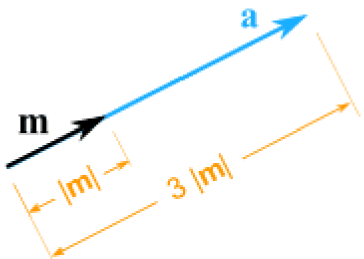
- so **c** is a vector, it has magnitude and direction
- but c is just a value, like 3 or 12.4

Example: kb is actually the scalar k times the vector b.

Multiplying a Vector by a Scalar

When we multiply a vector by a scalar it is called "scaling" a vector, because we change how big or small the vector is.

Example: multiply the vector m = (7,3) by the scalar 3

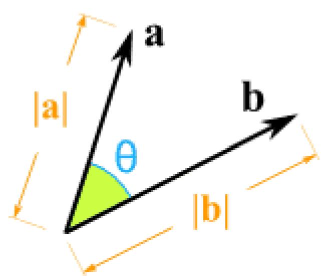


a = 3**m** = (3×7,3×3) = (21,9)

It still points in the same direction, but is 3 times longer

(And now you know why numbers are called "scalars", because they "scale" the vector up or down.)

Multiplying a Vector by a Vector (Dot Product and Cross Product)

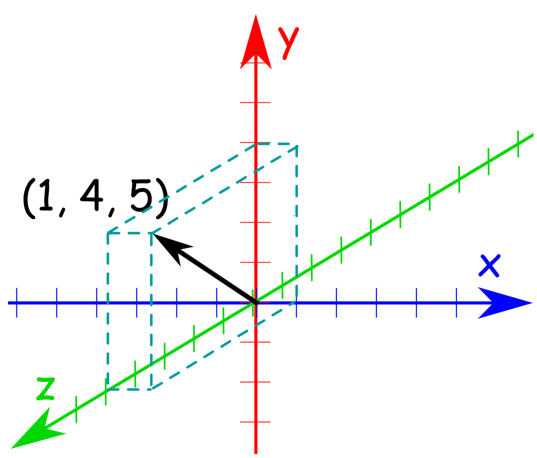


How do we **multiply two vectors** together? There is more than one way!

- The scalar or Dot Product (the result is a scalar).
- The vector or Cross Product (the result is a vector).

More Than 2 Dimensions

Vectors also work perfectly well in 3 or more dimensions:



The vector (1,4,5)

Example: add the vectors a = (3,7,4) and b = (2,9,11)

c = **a** + **b**

c = (3,7,4) + (2,9,11) = (3+2,7+9,4+11) = (5,16,15)

Example: what is the magnitude of the vector w = (1,−2,3) ?

$|\mathbf{w}| = \sqrt{(1^2 + (-2)^2 + 3^2)} = \sqrt{(1+4+9)} = \sqrt{14}$

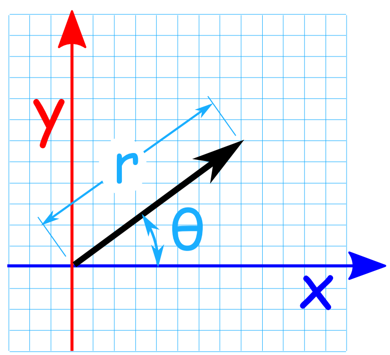
Here is an example with 4 dimensions (but it is hard to draw!):

Example: subtract (1,2,3,4) from (3,3,3,3)

(3,3,3,3) + −(1,2,3,4)
= (3,3,3,3) + (−1,−2,−3,−4)
= (3−1,3−2,3−3,3−4)
= (2,1,0,−1)

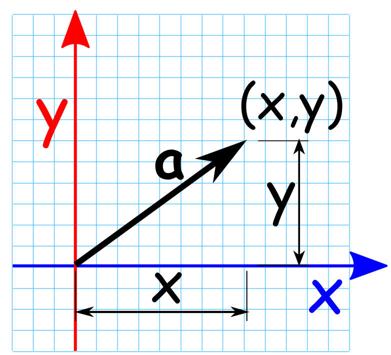
Magnitude and Direction

We may know a vector's magnitude and direction, but want its x and y lengths (or vice versa):



Vector **a** in Polar Coordinates

<=>



Vector **a** in Cartesian Coordinates

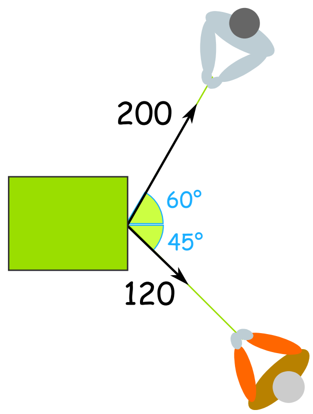
You can read how to convert them at Polar and Cartesian Coordinates, but here is a quick summary:

From Polar Coordinates (r,θ) to Cartesian Coordinates (x,y)

- $x = r \times \cos(\theta)$
- $y = r \times \sin(\theta)$

From Cartesian Coordinates (x,y) to Polar Coordinates (r,θ)

- $r = \sqrt{(x^2 + y^2)}$
- $\theta = \tan^{-1}(y / x)$



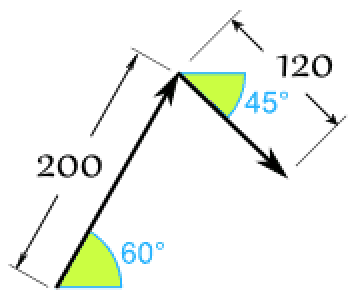
An Example

Sam and Alex are pulling a box.

- Sam pulls with 200 Newtons of force at 60°
- Alex pulls with 120 Newtons of force at 45° as shown

What is the combined force, and its direction?

Let us add the two vectors head to tail:



First convert from polar to Cartesian (to 2 decimals):

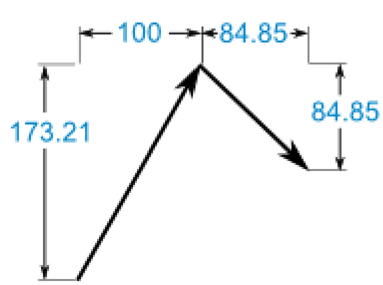
Sam's Vector:

- $\mathbf{x = r \times \cos(\,\theta\,) = 200 \times \cos(60°) = 200 \times 0.5 = 100}$
- $\mathbf{y = r \times \sin(\,\theta\,) = 200 \times \sin(60°) = 200 \times 0.8660 = 173.21}$

Alex's Vector:

- $\mathbf{x = r \times \cos(\,\theta\,) = 120 \times \cos(-45°) = 120 \times 0.7071 = 84.85}$
- $\mathbf{y = r \times \sin(\,\theta\,) = 120 \times \sin(-45°) = 120 \times -0.7071 = -84.85}$

Now we have:



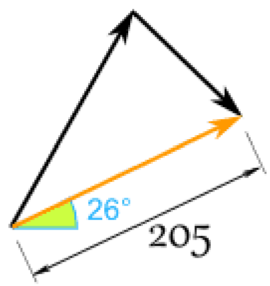
Add them:

(100, 173.21) + (84.85, −84.85) = (184.85, 88.36)

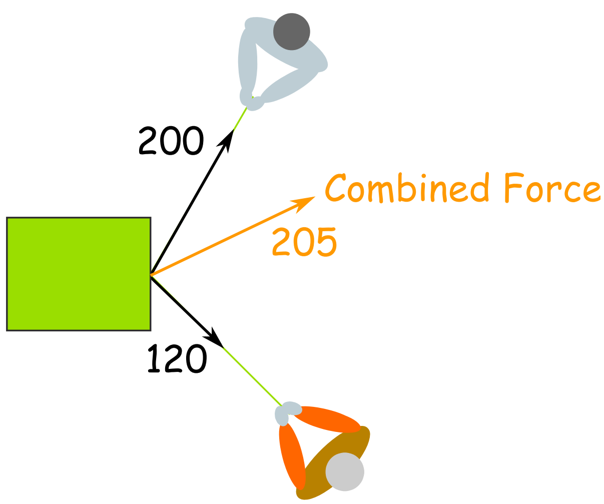
That answer is valid, but let's convert back to polar as the question was in polar:

- $r = \sqrt{(x^2 + y^2)} = \sqrt{(184.85^2 + 88.36^2)} = 204.88$
- $\theta = \tan^{-1} (y / x) = \tan^{-1} (88.36 / 184.85) = 25.5°$

  And we have this (rounded) result:



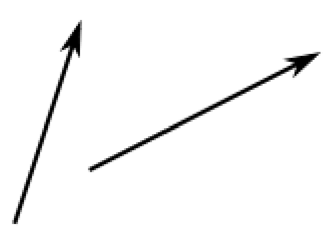
  And it looks like this for Sam and Alex:



They might get a better result if they were shoulder-to-shoulder!

Dot Product

Here are two vectors:



They can be multiplied using the "Dot Product"

Calculating

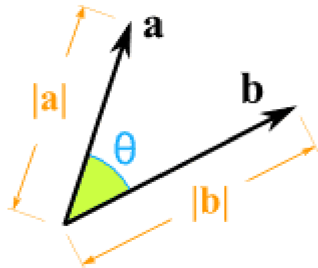
The Dot Product gives a **number** as an answer (a "scalar", not a vector).

The Dot Product is written using a central dot:

$$\mathbf{a} \cdot \mathbf{b}$$

This means the Dot Product of **a** and **b**

We can calculate the Dot Product of two vectors this way:

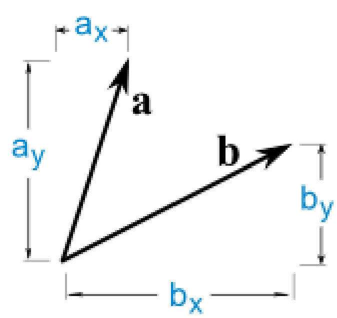


$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

Where:
$|\mathbf{a}|$ is the magnitude (length) of vector **a**
$|\mathbf{b}|$ is the magnitude (length) of vector **b**
θ is the angle between **a** and **b**

So we multiply the length of **a** times the length of **b**, then multiply by the cosine of the angle between **a** and **b**

OR we can calculate it this way:

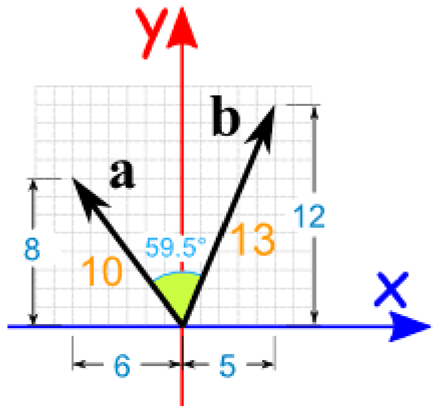


$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$

So we multiply the x's, multiply the y's, then add.

Both methods work!

Example: Calculate the dot product of vectors a and b:



$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

$\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times \cos(59.5°)$
$\mathbf{a} \cdot \mathbf{b} = 10 \times 13 \times 0.5075...$
$\mathbf{a} \cdot \mathbf{b} = 65.98... = 66$ (rounded)

or we can calculate it this way:

$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$

$\mathbf{a} \cdot \mathbf{b} = -6 \times 5 + 8 \times 12$
$\mathbf{a} \cdot \mathbf{b} = -30 + 96$
$\mathbf{a} \cdot \mathbf{b} = 66$

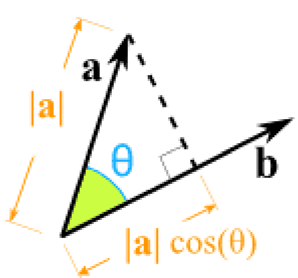
Both methods came up with the same result (after rounding)

Also note that we used **minus 6** for a_x (it is heading in the negative x-direction)

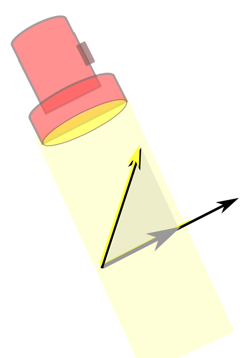
Why cos(θ) ?

OK, to multiply two vectors it makes sense to multiply their lengths together ***but only when they point in the same direction.***

So we make one "point in the same direction" as the other by multiplying by cos(θ):

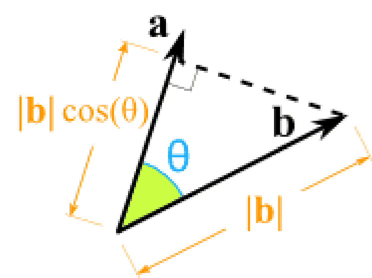


We take the component of **a** that lies alongside **b**



Like shining a light to see where the shadow lies

THEN we multiply !



It works exactly the same if we "projected" **b** alongside **a** then multiplied:

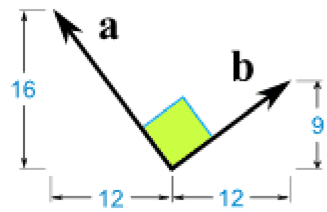
Because it doesn't matter which order we do the multiplication:

$|\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta) = |\mathbf{a}| \times \cos(\theta) \times |\mathbf{b}|$

Right Angles

When two vectors are at right angles to each other the dot product is **zero**.

Example: calculate the Dot Product for:



$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(90°)$
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times 0$
$\mathbf{a} \cdot \mathbf{b} = 0$

or we can calculate it this way:

$\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y$

$\mathbf{a} \cdot \mathbf{b} = -12 \times 12 + 16 \times 9$
$\mathbf{a} \cdot \mathbf{b} = -144 + 144$
$\mathbf{a} \cdot \mathbf{b} = 0$

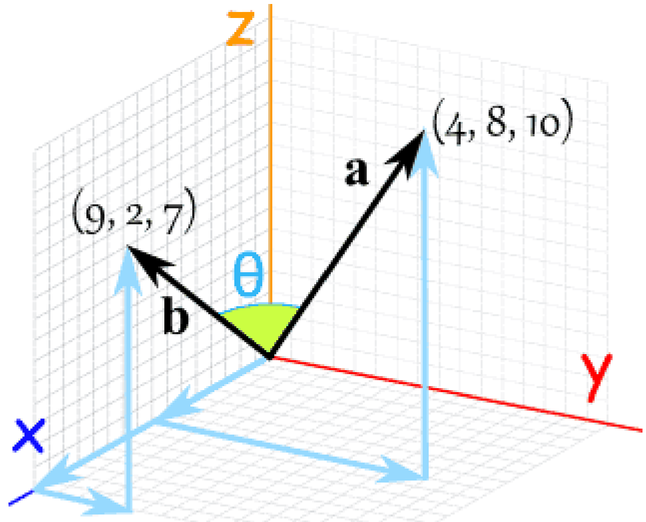
This can be a handy way to find out if two vectors are at right angles.

Three or More Dimensions

This all works fine in 3 (or more) dimensions, too.

And can actually be very useful!

Example: Sam has measured the end-points of two poles, and wants to know the angle between them:



We have 3 dimensions, so don't forget the z-components:

a · **b** = $a_x \times b_x + a_y \times b_y + a_z \times b_z$

a · **b** = 9 × 4 + 2 × 8 + 7 × 10
a · **b** = 36 + 16 + 70
a · **b** = 122

Now for the other formula:

a · **b** = |**a**| × |**b**| × cos(θ)

But what is |**a**| ? It is the magnitude, or length, of the vector **a**. We can use Pythagoras:

- |**a**| = $\sqrt{(4^2 + 8^2 + 10^2)}$
- |**a**| = √(16 + 64 + 100)
- |**a**| = √180

Likewise for |**b**|:

- |**b**| = $\sqrt{(9^2 + 2^2 + 7^2)}$
- |**b**| = √(81 + 4 + 49)
- |**b**| = √134

And we know from the calculation above that **a** · **b** = 122, so:

a · **b** = |**a**| × |**b**| × cos(θ)

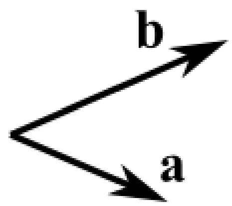
122 = √180 × √134 × cos(θ)
cos(θ) = 122 / (√180 × √134)
cos(θ) = 0.7855...
$\theta = \cos^{-1}(0.7855...) = 38.2...°$

Done!

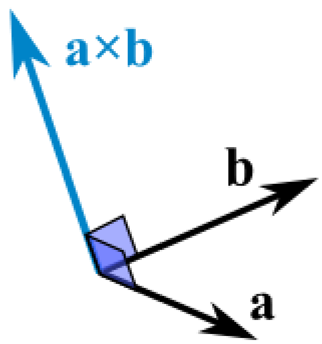
I tried a calculation like that once, but worked all in angles and distances ... it was very hard, involved lots of trigonometry, and my brain hurt. The method above is much easier.

Cross Product

Two vectors can be multiplied using the "Cross Product"



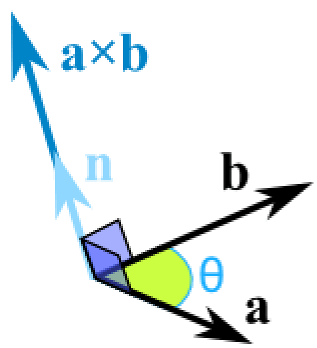
The Cross Product **a** × **b** of two vectors is **another vector** that is at right angles to both:



And it all happens in 3 dimensions!

Calculating

We can calculate the Cross Product this way:



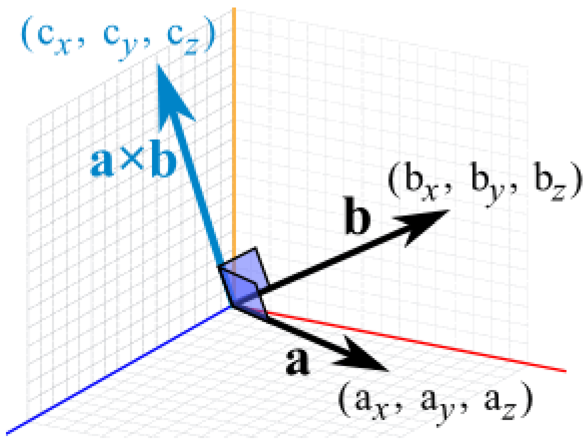
a × **b** = |**a**| |**b**| sin(θ) **n**

- |**a**| is the magnitude (length) of vector **a**
- |**b**| is the magnitude (length) of vector **b**
- θ is the angle between **a** and **b**
- **n** is the unit vector at right angles to both **a** and **b**

So the **length** is: the length of **a** times the length of **b** times the sine of the angle between **a** and **b**,

Then we multiply by the vector **n** to make sure it heads in the right **direction** (at right angles to both **a** and **b**).

OR we can calculate it this way:



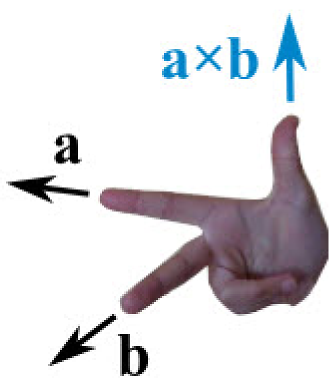
When **a** and **b** start at the origin point (0,0,0), the Cross Product will end at:

- $\mathbf{c_x = a_yb_z - a_zb_y}$
- $\mathbf{c_y = a_zb_x - a_xb_z}$
- $\mathbf{c_z = a_xb_y - a_yb_x}$

Example: The cross product of a = (2,3,4) and b = (5,6,7)

- $c_x = a_yb_z - a_zb_y = 3\times7 - 4\times6 = -3$
- $c_y = a_zb_x - a_xb_z = 4\times5 - 2\times7 = 6$
- $c_z = a_xb_y - a_yb_x = 2\times6 - 3\times5 = -3$

Answer: **a** × **b** = (−3,6,−3)



Which Direction?

The cross product could point in the completely opposite direction and still be at right angles to the two other vectors, so we have the: **"Right Hand Rule"**

With your right-hand, point your index finger along vector **a**, and point your middle finger along vector **b**: the cross product goes in the direction of your thumb.