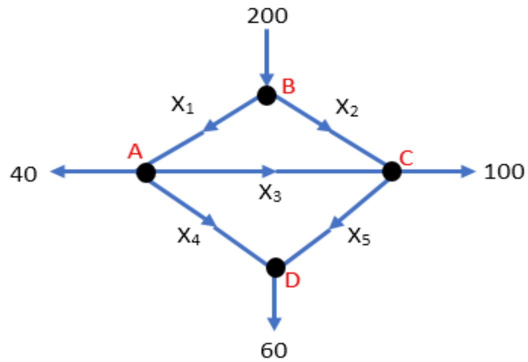
Problem #1 Traffic flow system



(a). The assumption (Input flow =Output flow) is valid (same as Kirchhoff law for current flow in electrical circuits). the assumption could be wrong when there was a traffic congestion or traffic jam in some intersection points.

Other simplifications that we have adopted are: the system is linear, all vehicles have the same normal average speed, no time delay, no traffic jam, one-way roads, and the input flow to the traffic network is equal to the output flow from the traffic network (=200).

(b).

Intersection Point	Input flow (Vehicles/m)	Output flow (Vehicles/m)	Formula (Input=Output)
A	X_1	X_3+X_4+40	$X_1- X_3-X_4=40$
B	200	X_1+X_2	$X_1+X_2=200$
C	X_2+X_3	X_5+100	$X_2+X_3- X_5=100$
D	X_4+X_5	60	$X_4+X_5=60$

(c). The traffic flow for the road system is (where X_3 and X_5 are free values) :

$X_1 = X_3-X_5 +100$

$X_2 =- X_3+ X_5+100$

$X_4 =- X_5+60$

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A = {{1, 0, -1, -1, 0}, {1, 1, 0, 0, 0}, {0, 1, 1, 0, -1}, {0, 0, 0, 1, 1}};
X = {x1, x2, x3, x4, x5};
B = {40, 200, 100, 60};
MatrixForm[A].MatrixForm[X] == MatrixForm[B]
MatrixForm[augmented = Transpose[Join[Transpose[A], {B}]]]
MatrixForm[rowreduce = RowReduce[augmented]]
MatrixForm[rowreduce [[All, {1, 2, 3, 4, 5}]].X == rowreduce [[All, 6]]]
```

Out[256]=
$$\begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{pmatrix} == \begin{pmatrix} 40 \\ 200 \\ 100 \\ 60 \end{pmatrix}$$

Out[257]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{pmatrix}$$

Out[258]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[259]//MatrixForm=
$$\{x1 - x3 + x5,\ x2 + x3 - x5,\ x4 + x5,\ 0\} == \{100,\ 100,\ 60,\ 0\}$$

(d). If X_4 is zero, then the traffic flow will be:

$X_1 = X_3+40$

$X_2 =- X_3+160$

$X_5=60$

(e). If X_4 is zero, then the minimum value of X_1 will be:

We have $X_1 = X_3+40$, or $X_3 = X_1-40$

But $X_3 \geq 0$, Thus minimum value of X_1 is **40** vehicles/min

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Plot[{x - 40, -x + 160}, {x, 0, 200}, PlotRange → {{0, 200}, {0, 200}},
 AxesLabel → {"X1,X2", "X3"}, PlotLegends → {"X1=X3+40", "X2=-X3+160"}, Filling → Axis]
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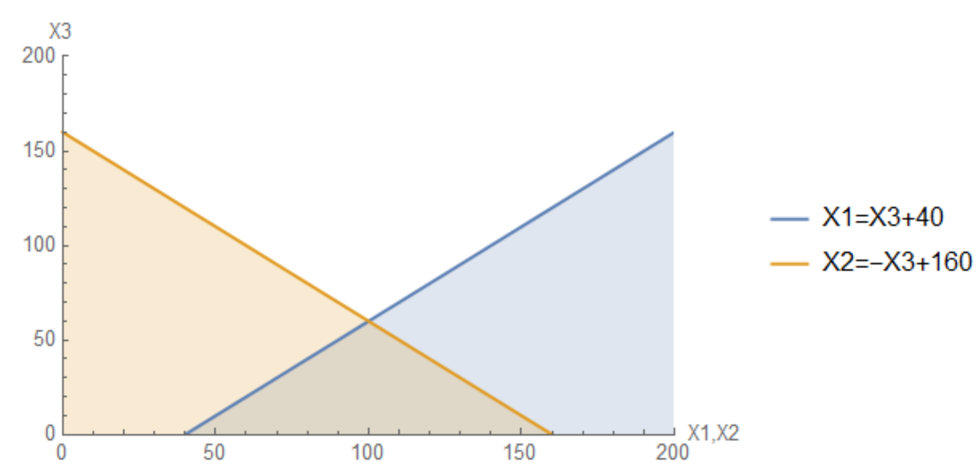

Problem #2 Fibonacci Sequence

a. Assuming the time is for 1 year, then the sequence (rabbits growth) will be:

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RSolve[{p[n + 2] == p[n + 1] + p[n], p[1] == p[2] == 1}, p[n], n]
Table[Fibonacci[n], {n, 12}]
ListLinePlot[t = Table[{n, Fibonacci[n]}, {n, 12}], AxesLabel → {n, p[n]},
 Epilog → {PointSize[Large], Red, Point[t]}]
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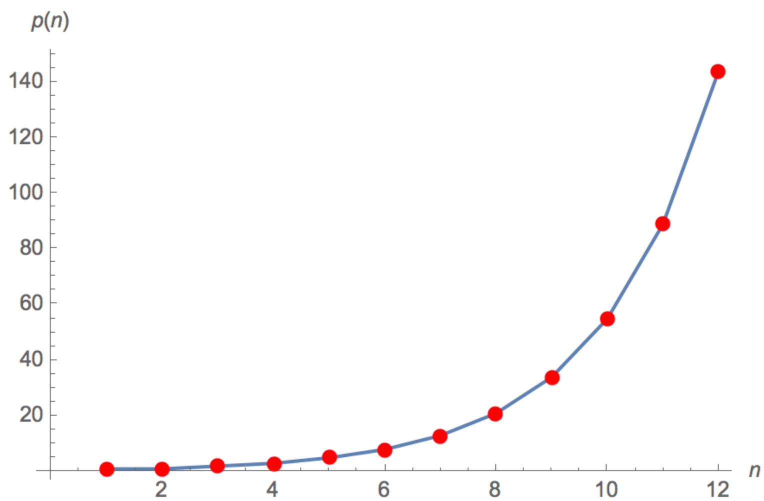
{{p[n] → Fibonacci[n]}}

{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144}
```



b. From the graph, it seems that Fibonacci Sequence p(n) is growth exponentially when n increases.

c. The restrictions for the model are the initial values are p[1] = p[2] = 1, which means that if we want to know any term, then we must also know the previous two terms.

d. We can improve this model by deriving an exponential function for p[n] which does not need any initial values **[Ref 2]**:

$p[n+2] = p[n+1] + p[n]$ or $p[n+2] - p[n+1] - p[n] = 0$

This function is a difference equation and we can rewrite it as

$x^{n+2} - x^{n+1} - x^{n} = 0$

Dividing by x^{n} results

$$x^2 - x - 1 = 0$$

Which is a second-degree equation with the roots:

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Solve[x^2 - x - 1 == 0, x]
```

$$x_1 = \frac{1+\sqrt{5}}{2} \;\;,\;\; x_2 = \frac{1-\sqrt{5}}{2}$$

$$\left\{\left\{x \to \frac{1}{2}\left(1-\sqrt{5}\right)\right\}, \left\{x \to \frac{1}{2}\left(1+\sqrt{5}\right)\right\}\right\}$$

So, the general solution of this difference equation is: $f[n] = a x_1{}^n + b x_2{}^n$

Where a and b are constants and can be found by set p[0]=0, and p[1] = 1

$$0 = a + b$$

$$1 = a\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2}$$

Solving these two equations results

$$a = \frac{1}{\sqrt{5}} \;\;,\;\; b = -\frac{1}{\sqrt{5}}$$

```
Solve[a + b == 0 && a * (1 + Sqrt[5]) / 2 + b * (1 - Sqrt[5]) / 2 == 1, {a, b}]
```

$$\left\{\left\{a \to \frac{1}{\sqrt{5}}, b \to -\frac{1}{\sqrt{5}}\right\}\right\}$$

Thus, $f[n] = a x_1{}^n + b x_2{}^n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$

```
f[n_] := ((1 + Sqrt[5]) / 2) ^n / Sqrt[5] - ((1 - Sqrt[5]) / 2) ^n / Sqrt[5]
ListLinePlot[Table[{n, f[n]}, {n, 12}], AxesLabel → {n, "f[n]"}]
Table[N[f[n]], {n, 12}]
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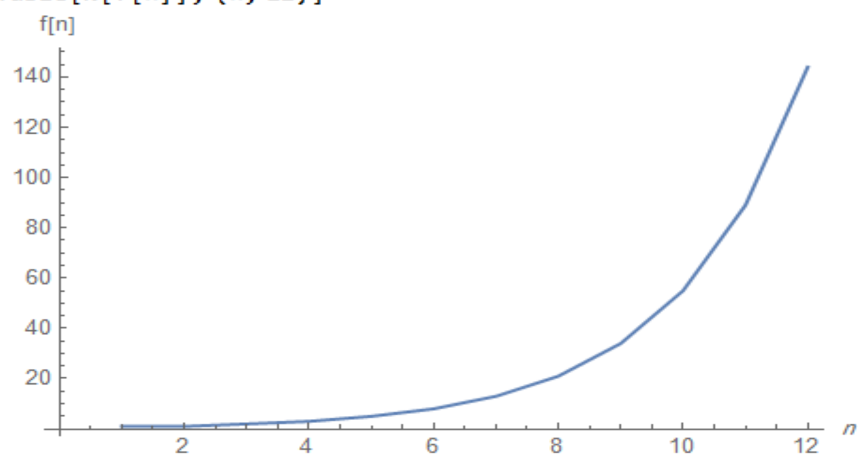

{1., 1., 2., 3., 5., 8., 13., 21., 34., 55., 89., 144.}

When n goes to infinity, this function f(n) converges to a constant value:

$$\lim_{n\to\infty}\frac{f(n+1)}{f(n)} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{5}}\left[x_1\right]^{n+1} - \frac{1}{\sqrt{5}}\left[x_2\right]^{n+1}}{\frac{1}{\sqrt{5}}\left[x_1\right]^{n} - \frac{1}{\sqrt{5}}\left[x_2\right]^{n}} = \lim_{n\to\infty}\frac{\left[x_1\right]^{n+1} - \left[x_2\right]^{n+1}}{\left[x_1\right]^{n} - \left[x_2\right]^{n}} = \lim_{n\to\infty}\frac{x_1(1 - \left[\frac{x_2}{x_1}\right]^{n+1})}{1 - \left[\frac{x_2}{x_1}\right]^{n}}$$

But $x_2 < 1$, so

$$\lim_{n\to\infty}\frac{f(n+1)}{f(n)} = \frac{x_1(1-0)}{1-0} = x_1 = \frac{1+\sqrt{5}}{2} = 1.618033$$

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Limit[f[n + 1] / f[n], n → Infinity]
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$\frac{1}{2}\left(1+\sqrt{5}\right)$

1.61803

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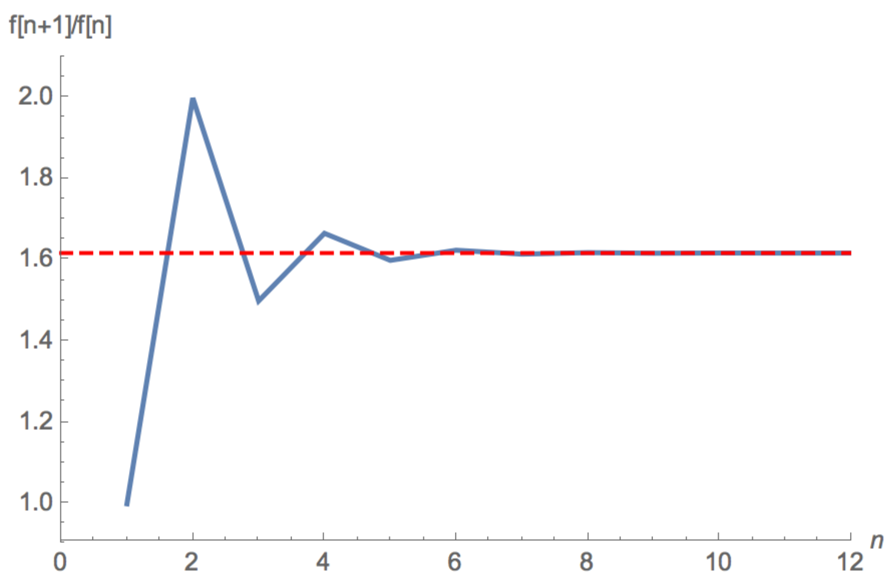
limitfun = ListLinePlot[Table[{n, f[n + 1] / f[n]}, {n, 12}],
   AxesLabel → {n, "f[n+1]/f[n]"}, PlotRange → {{0, 12}, {0.9, 2.1}}, PlotStyle → {Thick}];
limitval = ListLinePlot[Table[{n, 1/2 (1 + √5)}, {n, 0, 12}], PlotStyle → {Red, Dashed}];
Show[limitfun, limitval]
```



References:

[1] Mathematica 11 Documentation

[2] Difference Equations, https://www.math.ksu.edu/math240/math240.s11/homodiffeqns06.pdf