Lecture 20

MTL 122

Real and Complex Analysis.

Boundedness $\qquad A \subseteq X$

Theo. $f_n : A \to \mathbb{R}$ is bounded on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A

$\Rightarrow f : A \to \mathbb{R}$ is bdd.

Proof: $\epsilon = 1$

Defn of U.C $\exists N_\epsilon \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < 1 \quad \forall x \in A$$

if $n > N_\epsilon$

Choose $n_0 > N_\epsilon$

Since f_n is bdd $\forall n$.

$\exists\ M_{n_0}$ s.t

$$|f_{n_0}(x)| \le M_{n_0} \quad \forall x \in A.$$

$$\Rightarrow \quad |f(x)| = |f(x) - f_{n_0}(x) + f_{n_0}(x)|$$

$$\le \underbrace{|f(x) - f_{n_0}(x)|} + |f_{n_0}(x)|$$

$$< 1 + M_{n_0}$$

$$\forall x \in A.$$

$\Rightarrow$ f is bdd.

Ex: $f_n : (0,1) \to \mathbb{R}$.

$$f_n(x) = \frac{n}{nx+1}$$

'f_n' $\not\to$ uniformly

$f(x) = 1/x \to$ limiting fn.

$f_n \to f$ pointwise as $n \to \infty$

f_n ~~are~~ bdd.

If $f_n \xrightarrow[\text{unif conv.}]{} f$ as $n \to \infty$

f must be bdd.

$[a, 1)$, $0 < a < 1$

$\hookrightarrow$ $f_n \to f$ uniformly

How?

Let $\underline{a \le x < 1}$

$$|f_n(x) - f(x)|$$

$$= \left| \frac{n}{nx+1} - \frac{1}{x} \right|$$

$$= \frac{1}{x(nx+1)} < \frac{1}{nx^2} \leq \frac{1}{na^2}$$

$$< \epsilon \quad \text{if}$$

with $\left[N > \frac{1}{a^2 \epsilon} \right]$

$$n > N$$

$$\Rightarrow f_n \longrightarrow f \text{ on } [a, 1)$$

uniformly.

bdd

bdd

Note $|f(x)| = \left|\frac{1}{x}\right| \leq \frac{1}{a}$

$\forall x \in [a, 1]$

$\Rightarrow$ f is bdd.

Continuity

Theo. (f_n), $f_n : A \to \mathbb{R}$

$f_n \xrightarrow[\text{uniform}]{} f$, f_n are continuous

$\Rightarrow$ $f : A \to \mathbb{R}$ is continuous.

Pf: $c \in A$, $\epsilon > 0$ (given)

Every $n \in \mathbb{N}$:

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \quad (1)$$

Choose $n_1 \in \mathbb{N}$

$$|f_{n_1}(x) - f(x)| < \epsilon/3, \quad \forall x \in A$$

holds.

$$|f(x) - f(c)| < \frac{2\epsilon}{3} + |f_{n_1}(x) - f_{n_1}(c)|$$

$\forall x \in A$

'f_n' are continuous $\forall x \in A$
$\forall n$

$\exists \delta > 0$ s.t

Defn of conti:

$$|f_{n_1}(x) - f_{n_1}(c)| < \epsilon/3$$

if $|x - c| < \delta$
& $x \in A$.

$$\Rightarrow |f(x) - f(c)| < \epsilon$$

if $|x - c| < \delta$.
$x \in A$

$\Rightarrow f$ is cont on A.

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{x \to c} f(x)$$

$$= f(c)$$

$$\lim_{n \to \infty} \lim_{x \to c} f_n(x) = \lim_{n \to \infty} f_n(c)$$

$$= f(c)$$

Exchange in the order of limits

$$\boxed{\lim_{n \to \infty} \lim_{x \to c} f_n(x) = \lim_{x \to c} \lim_{n \to \infty} f_n(x)}$$

Differentiability

" U. conv. ~~of~~ of differentiable fn. (~~does not~~) $\not\Rightarrow$ anything about the convergence of the derivatives or the differentiability of their limit.

Ex. $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \frac{x}{1+nx^2}$$

$f_n \longrightarrow 0$ uniformly. (?)

if $$|f_n(x)| = \frac{1}{\sqrt{n}}\left(\frac{\sqrt{n}\,|x|}{1+nx^2}\right)$$

$$= \frac{1}{\sqrt{n}}\left(\frac{t}{1+t^2}\right)$$

where $t = \sqrt{n}\,|x|$.

$$\frac{t}{1+t^2} < \frac{1}{2} ?$$

$$\left((1-t)^2 \geq 0 \Rightarrow 2t < 1+t^2\right)$$

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}} \quad \forall x \in \mathbb{R}$$

$\epsilon > 0$ choose

$$N = 1/4\epsilon^2$$

$$|f_n(x)| < \epsilon \quad \forall x \in \mathbb{R}$$
$$n > N_\epsilon$$

$$\Rightarrow \quad f_n \xrightarrow[\text{unif}]{} 0$$

Each 'f_n' is diff.

$$f_n'(x) = \frac{1 - nx^2}{(1+nx^2)^2}$$

$$f_n' \longrightarrow g \quad \text{as } n \to \infty$$

pointwise

$$g(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

'g' is discontinous.

$$\left.\begin{array}{l} f_n \xrightarrow{\text{unif.}} 0 \\ f_n'(0) \longrightarrow 1 \end{array}\right\} \text{interchange is not possible.}$$

Theo. f_n, $f_n : (a,b) \to \mathbb{R}$,

differentiable

- $f_n \to f$ pointwise
- $f_n' \to g$ uniformly

$\Big\{ f, g : (a,b) \to \mathbb{R}$

$\Rightarrow$ f is differentiable on (a, b) & $f' = g$.

Integration

Theo. $(f_n) \to$ seq of cont. fn. $[a, b]$

* $f_n \xrightarrow[\text{uniformly}]{} f$

$\Rightarrow$ f is conti. &

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Pf. f is conti.

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx$$

$\epsilon > 0$.

$f_n \xrightarrow[\text{uniform}]{} f$

$\exists \; N_\epsilon \in \mathbb{N}$

$$|f_n(x) - f(x)| < \frac{\epsilon}{(b-a)}$$

$\forall n \geq N$.

$\forall x \in [a,b]$

Hence $\forall \; n \geq N$.

$$\left| \int_a^b f_n(x) - \int_a^b f(x) \right|$$

$$< \epsilon \qquad \forall n \geq N$$

$\forall x \in [a,b]$