Lecture - 30

MTL-122 Real & Complex Analysis

Theorem:

Let $w = f(z)$ be an analytic function on simply connected domain D. Let Γ be a simple closed contour (+tive). Then for every pt z_0 in Γ

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} \, dz$$

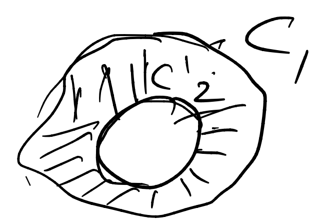
Proof: Begin with $\frac{f(z)}{z - z_0}$

So $\frac{f(z)}{z - z_0} \rightarrow$ analytic everywhere in D except z_0.

C_r : the circle inside Γ with center z_0 & radius r.

Using the last corollary in the last lecture we get $\frac{f(z)}{z-z_0}$ is analytic.

$$\int_\Gamma \frac{f(z)}{z-z_0}\,dz = \int_{C_r}\frac{f(z)}{z-z_0}\,dz.$$



$$= \int_{C_r}\frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\,dz$$

$$= \int_{C_r}\frac{f(z_0)}{z-z_0}\,dz + \int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\int_{C_r}\frac{dz}{z-z_0} + \int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\,2\pi i + \int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\left[\because \int_{B(0,r)}\frac{dz}{z} = 2\pi i\right]$$

$$\left|\int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right|$$

$$M_r = \max_{z\in C_r}\left|f(z)-f(z_0)\right|$$

& , $|z-z_0| = r$

C_r

$$\hookrightarrow \left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right|$$

$$\leq \int_{C_r} \frac{|f(z)-f(z_0)|}{|z-z_0|}\, dz$$

$$\leq \frac{M_r}{r} \int_{C_r} dz$$

$$= \frac{M_r}{r} \cdot 2\pi r = 2\pi M_r.$$

$f \to$ continuous at z_0 (?)

[because f is analytic in D & $z_0 \in D$]

$\Rightarrow$ $r \to 0^+$ then

$M_r \to 0$

$$\int_{\Gamma} \frac{f(z)}{z-z_0}\,dz = f(z_0)\,2\pi i + \lim_{r\to 0^+} \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

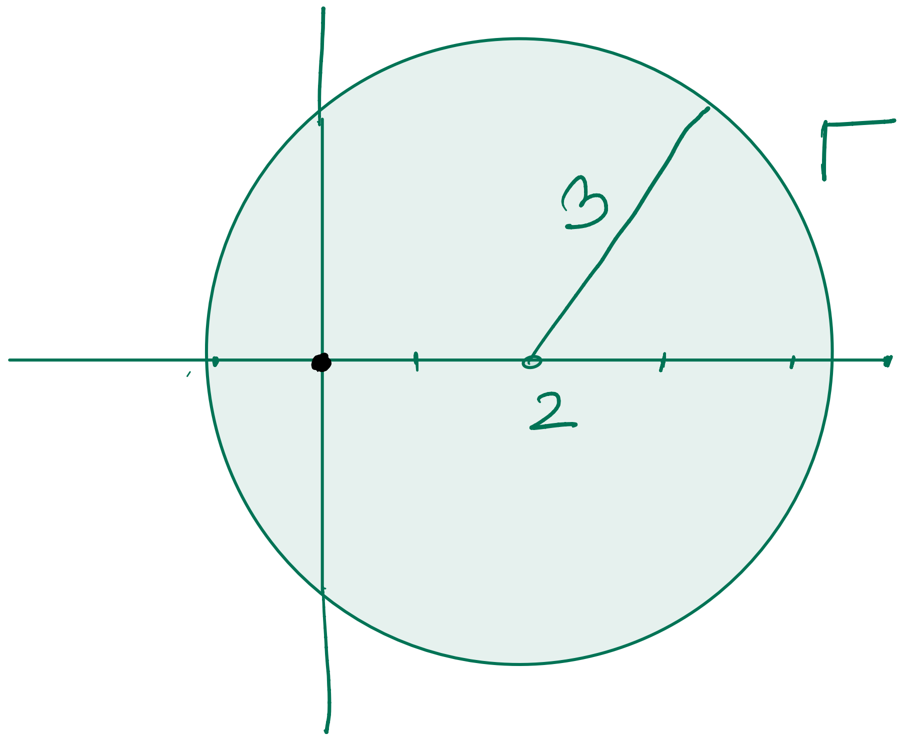
$$= f(z_0)\cdot 2\pi i$$

$$\Rightarrow \boxed{f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z-z_0}\,dz}$$

Example

Compute $\int_{\tilde{\Gamma}} \frac{e^z + \sin z}{z}\,dz$,

$\Gamma: \{z \in \mathbb{C} \,;\, |z-2| = 3\}$



$f(z) = e^z + \sin z \Rightarrow$ analytic

$$\int_\Gamma \frac{f(z)}{z-0} = 2\pi i\, f(0) = \underline{\underline{2\pi i}}$$

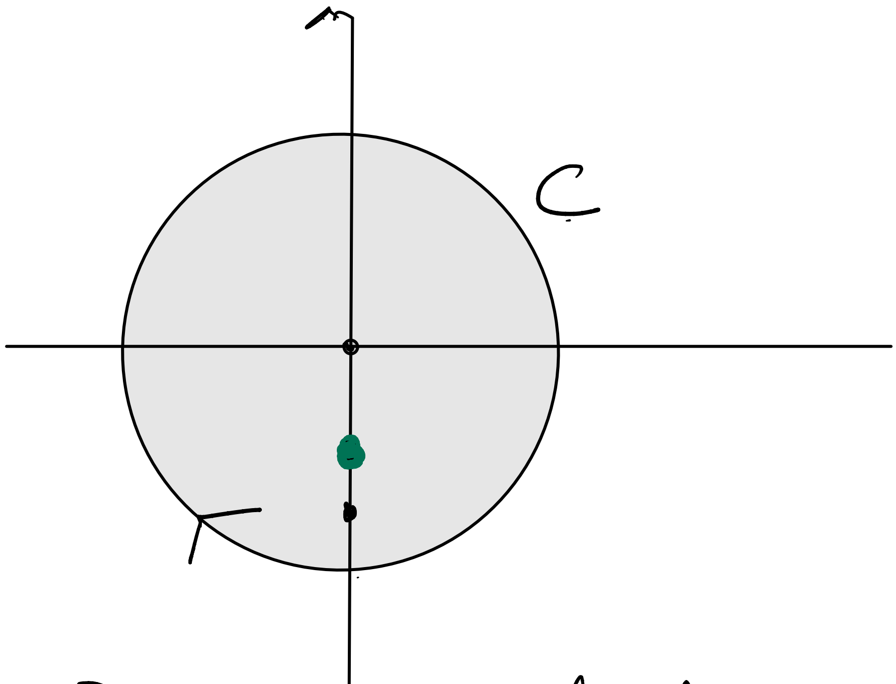
$$\left[f(0) = e^0 + \sin 0 = 1 \right]$$

Example. Compute

$$\int_C \frac{z^2 e^z}{2z+i}\, dz, \quad C: \text{unit circle}$$

centered at origin in clockwise direction.

Soln.



$f(z) = z^2 e^z \rightarrow$ analytic everywhere on & inside C.

$$\int_C \frac{f(z)}{2z+i} = \int_C \frac{f(z)}{2(z+i/2)} = -\frac{1}{2}\int_{-C} \frac{f(z)}{z-(-i/2)}$$

$$= -\frac{1}{2} \cdot 2\pi i \, f\left(-\frac{i}{2}\right) = \frac{\pi i}{4} e^{-i/2}$$

Theo. Let $w=f(z)$ be holomorphic on a simply connected domain D. Let Γ be a simple closed contour in D oriented once in the counter-clockwise direction. Then for all z inside Γ,

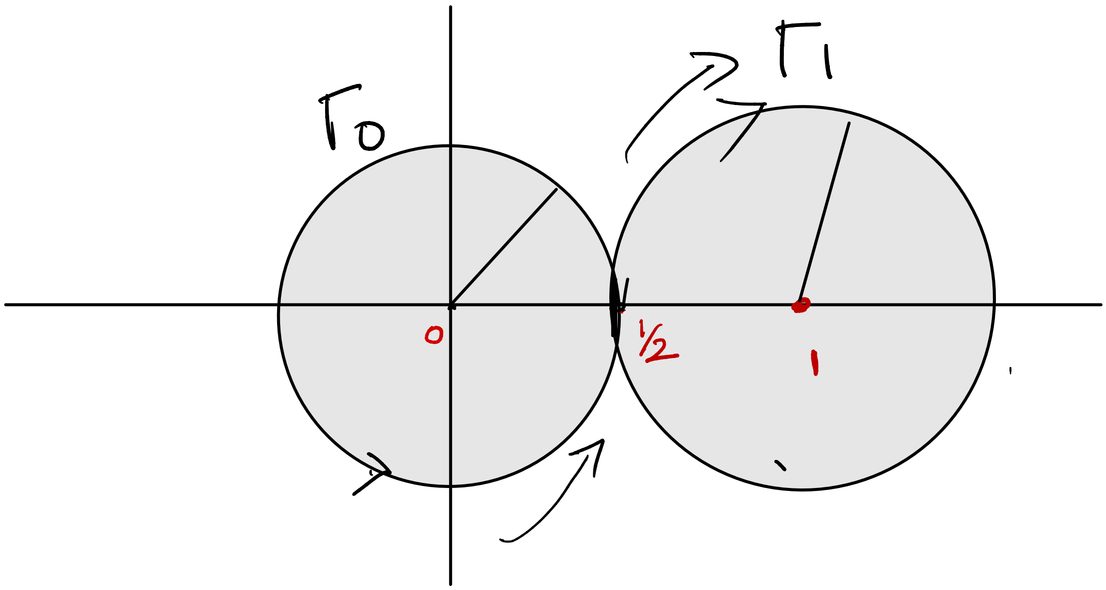
$$\boxed{f^{n}(z) = \frac{n!}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta.}$$

Example

$$\Gamma = \Gamma_0 \cup \Gamma_1$$

$$\Gamma_0 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1/2 \right\} \rightarrow \text{counter clockwise}$$

$$\Gamma_1 = \left\{ z \in \mathbb{C} \;\middle|\; |z-1| = 1/2 \right\} \rightarrow \text{clockwise.}$$



$\rightarrow$ Γ : ∞ contour.

Compute : $\displaystyle\int_{\Gamma} \frac{2z+1}{z(z-1)^2}\,dz$

$$= \int_{\Gamma_0} \frac{2z+1}{z(z-1)^2}\,dz + \int_{\Gamma_1} \frac{2z+1}{z(z-1)^2}\,dz$$

$f(z) = \dfrac{2z+1}{(z-1)^2}$ analytic inside & on Γ_0.

$g(z) = \dfrac{2z+1}{z}$ " " " Γ_1

$$= \int_{\Gamma_0} \frac{f(z)}{z-0} + \int_{\Gamma_1} \frac{g(z)}{(z-1)^2}\,dz$$

$$= 2\pi i\, f(0) - \int_{-\Gamma_1} \frac{g(z)}{(z-1)^2}\,dz$$

$$= 2\pi i \frac{2.0+1}{(0-1)^2} - 2\pi i \frac{d}{dz}\left(\frac{2z+1}{z}\right)\Big|_{z=1}$$

$$= 2\pi i + 2\pi i \left(\frac{1}{z^2}\right)\Big|_{z=1}$$

$$= 4\pi i$$

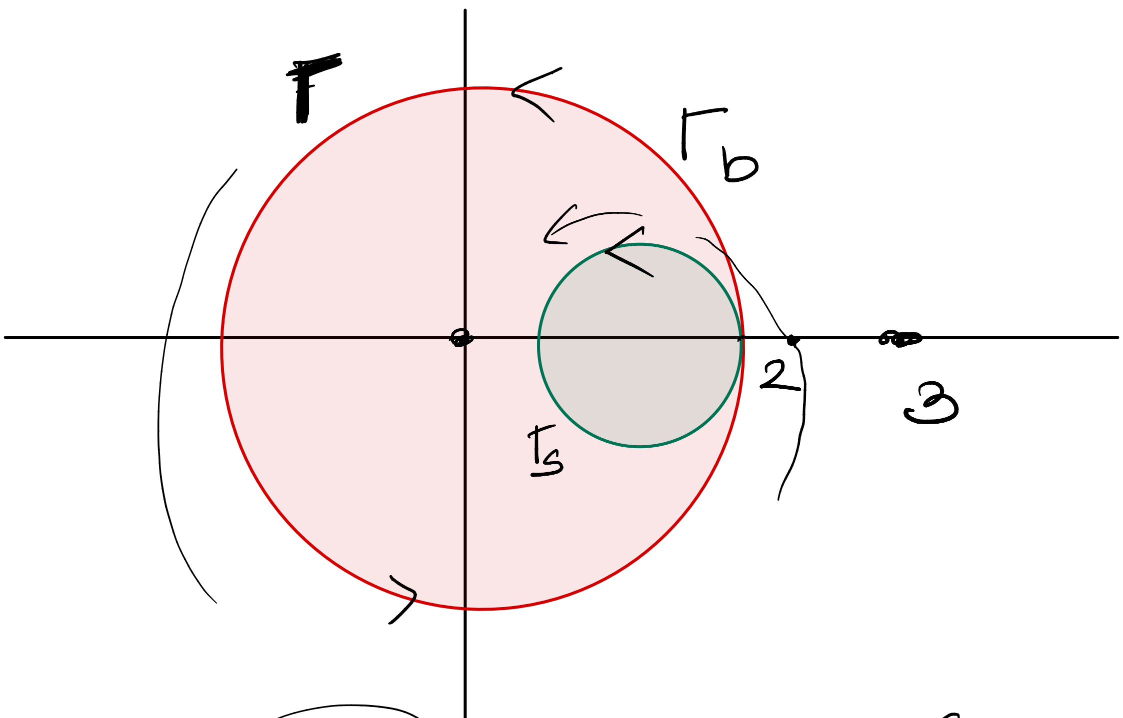
Thus: $\displaystyle\int_{\Gamma} \frac{2z+1}{z(z-1)^2}\,dz = 4\pi i$

Cor. Let $w = f(z)$ be a holomorphic fn on D. Then $f', f'', f''', \ldots, f^{(n)}, \ldots$ exist are holomorphic on D.

Example

Γ : self intersecting at 2.
(counter clockwise)

Compute $\displaystyle\int_\Gamma \frac{\cos z}{z^2(z-3)}\,dz$



$$\int_{\Gamma_s} \frac{\cos z}{z^2(z-3)}\,dz = 0$$ (Cauchy integral theo)

$$\int_{\Gamma_b} \frac{\cos z}{z^2(z-3)}\, dz$$

$$= \int_{\Gamma_b} \frac{\frac{\cos z}{z-3}}{(z-0)^2}\, dz$$

$$= 2\pi i\, f'(0), \quad f(z) = \frac{\cos z}{z-3}$$

$$f'(z) = \frac{-(z-3)\sin z - \cos z}{(z-3)^2}$$

& hence,

$$\int_{\Gamma} \frac{\cos z}{z^2(z-3)}\, dz = \int_{\Gamma_b} \frac{\cos z}{z^2(z-3)}\, dz$$

$$= -\frac{2}{9}\pi i$$