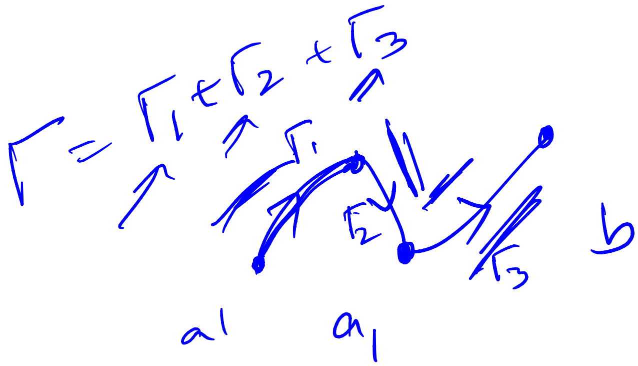
Lecture 28

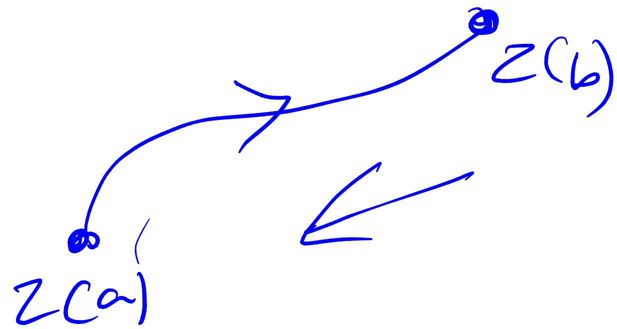
Real and Complex Analysis

MTL-122

• $z = z(t)$, $a \le t \le b$ represent 'C' (contour), from

$z_1 = z(a)$ to $z_2 = z(b)$





• $f(z) \rightarrow$ piecewise continuous on C

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt}$$

$C \longrightarrow -C$

$(z_1 \rightarrow z_2)$ $\qquad$ $(z_2 \rightarrow z_1)$

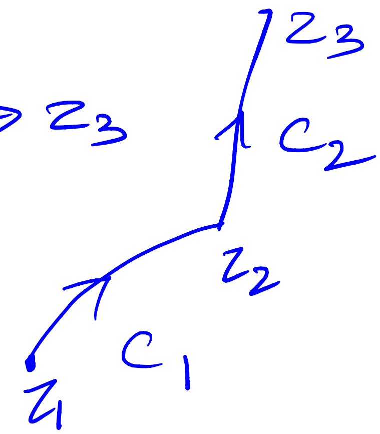
$\left\{ C : z(t),\ a \le t \le b \right\}$ $\qquad$ $\left\{ -C : z(-t),\ -b \le t \le -a \right\}$.

Properties

- $$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$$

- C : $\quad C_1 \quad\quad C_2$

  $\quad\quad z_1 \to z_2 \quad z_2 \to z_3$



$C : C_1 + C_2$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

C : $C_1 + C_2$ (sum of legs of C_1 and C_2)

$$\Rightarrow \int_{C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$$

$$\int_C [f(z)+g(z)]\,dz$$

$$= \int_C f(z)\,dz + \int_C g(z)\,dz$$

linearity of contour integration.

$$\left| \int_C f(z)\,dz \right|$$

$$= \left| \int_a^b f(z(t))\, z'(t)\,dt \right|$$

$$\leq \int_a^b |f(z(t))\, z'(t)\,dt|$$

Let $M > 0$ s.t

$$|f(z)| \le M \quad \forall z \in C.$$

(contour)

$$\left| \int_C f(z)\,dz \right| \le M \int_a^b |z'(t)|\,dt.$$

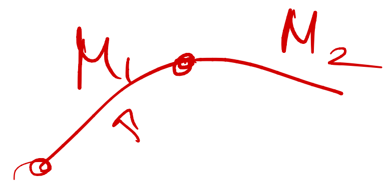
[L: Length of the contour.]

$$= ML.$$

Remark: Suppose f is continuous fn. of real variable 't', $a \le t \le b$.

$\Rightarrow$ f is bdd & $\exists\, t_0 \in [a,b]$ s.t $f(t_0) = \sup_{a \le t \le b} |f(t)|$

$\Rightarrow$ If f is cont on C then $f(z(t))$ is cont. on $[a,b]$. (?)

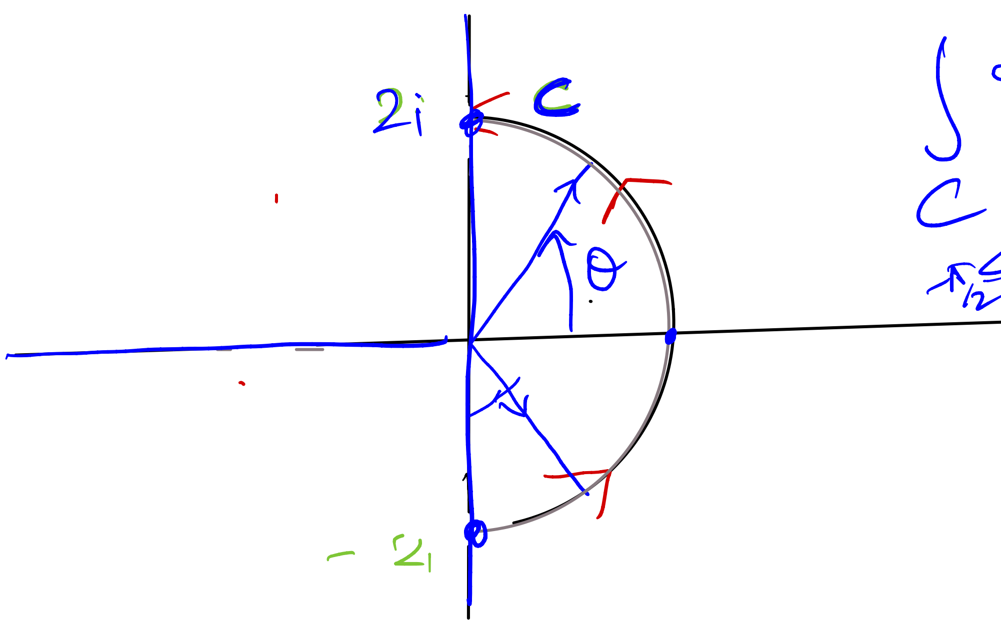


Example

$$I = \int_C \bar{z}\, dz$$

C: Right hand half

$z = 2e^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$



$$\int_C \frac{dz}{z} = \pi i$$

$-\pi/2 \le \theta \le \pi/2$

$f(z) = \bar{z}$

$C: z(\theta) = 2e^{i\theta}, \quad -\pi/2 \le \theta \le \pi/2$

Defn.

$$\int_C f(z)\,dz = \int_{-\pi/2}^{\pi/2} f(z(\theta))\, z'(\theta)\, d\theta$$

$$\left[\; z'(\theta) = \frac{d}{d\theta}\left(2e^{i\theta}\right) = 2ie^{i\theta} \;\checkmark \;\right]$$

$$= \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}}\; 2ie^{i\theta}\, d\theta$$

$$= 4i \int_{-\pi/2}^{\pi/2} d\theta = 4\pi i \;\checkmark$$

- $z: \quad |z| = 2$

$$z\bar{z} = |z|^2 = 4$$

$$\bar{z} = \frac{4}{z}$$

Calculate:

$f(z) = \frac{1}{z}$

$$I = \int_C \frac{dz}{z} = \frac{1}{4} \int_C \frac{4}{z}\, dz$$

$$= \frac{1}{4} \int_C \bar{z}\, dz$$

$$= \frac{1}{4} \cdot 4\pi i = \pi i$$

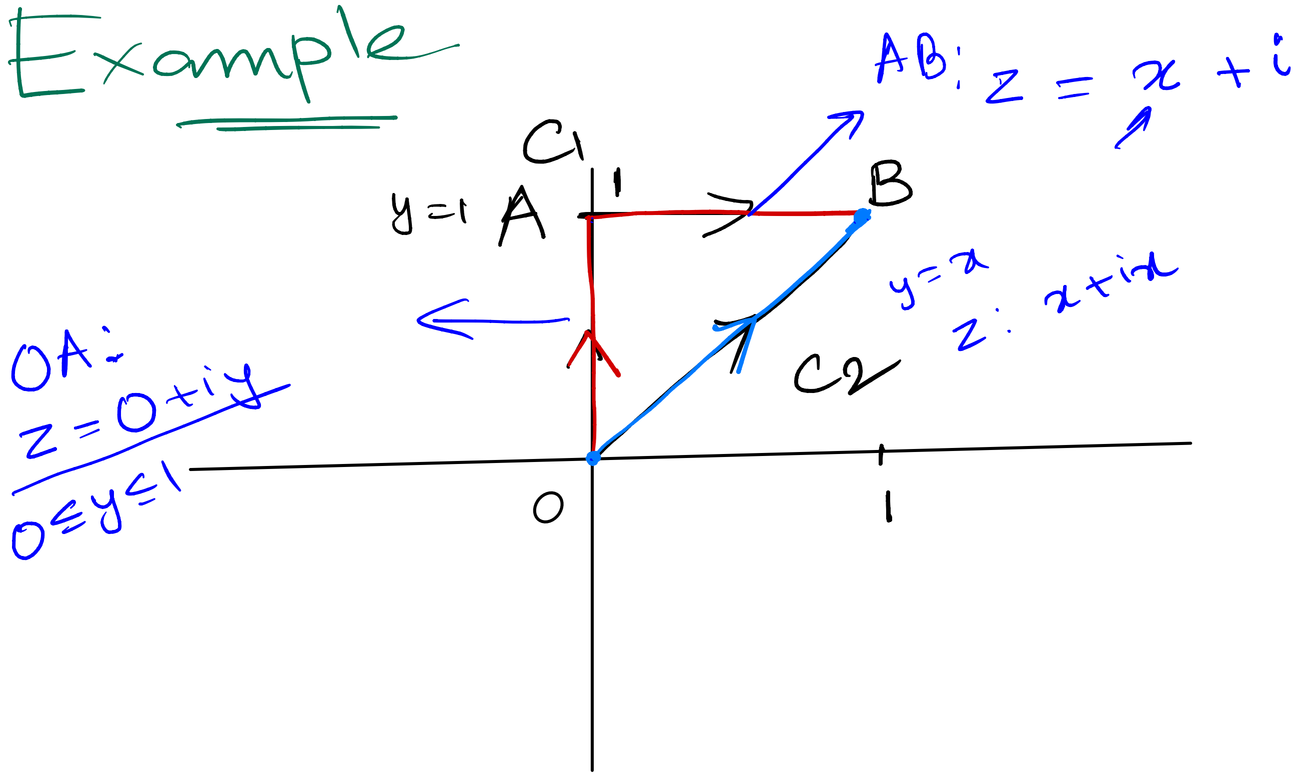
Exercise:

Find $\int_{C_1} \frac{dz}{z}$,

$C_1: \quad z = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$

$2e^{i\theta} \qquad 3e^{i\theta}$.

Example



C_1: OAB $\longrightarrow$ OA + OB

C_2: OB $(y = x)$

$$f(z) = \underline{y - x} - i3x^2, \quad z = x + iy$$

$$\int_{C_1} f(z) \quad \& \quad \int_{C_2} f(z)$$

$$\int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{OB} f(z)\,dz$$

$$= \int_0^1 iy\,dy + \int_0^1 (1 - x - i3x^2)\,dx$$

$$\left[\because \int_a^b f(z(t))\,z'(t)\,dt \right]$$

$$= \frac{i}{2} + \left(1 - \frac{1}{2} - i3\frac{1}{3}\right)$$

$$= \frac{i}{2} + \left(\frac{1}{2} - i\right)$$

$$= \frac{1}{2} - \frac{i}{2} = \frac{1-i}{2}$$

$$\int_{C_1} f(z)\,dz = \frac{1-i}{2}$$

$$\int_{C_2} f(z)\,dz, \qquad C_2: \; z(x) = x + ix$$

$$= \int_{OB} f(z)\,dz = -\int_0^1 i3x^2 (1+i)\,dx.$$

$$= (1-i)$$

Remark:

$$\int_{C_1} f \neq \int_{C_2} f \, dz$$

So,

$$C = C_2 - C_1$$

$$\int_C f(z)\,dz = \int_{C_2 - C_1} f(z)\,dz$$

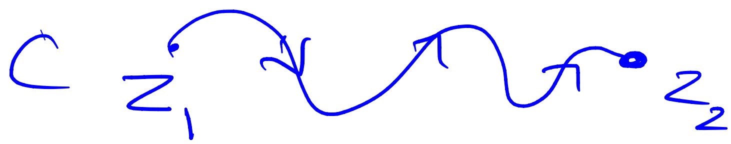
$$= \int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz$$

$$= \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz$$

$$\neq 0$$

Example

$C: z = z(t), \quad a \le t \le b.$



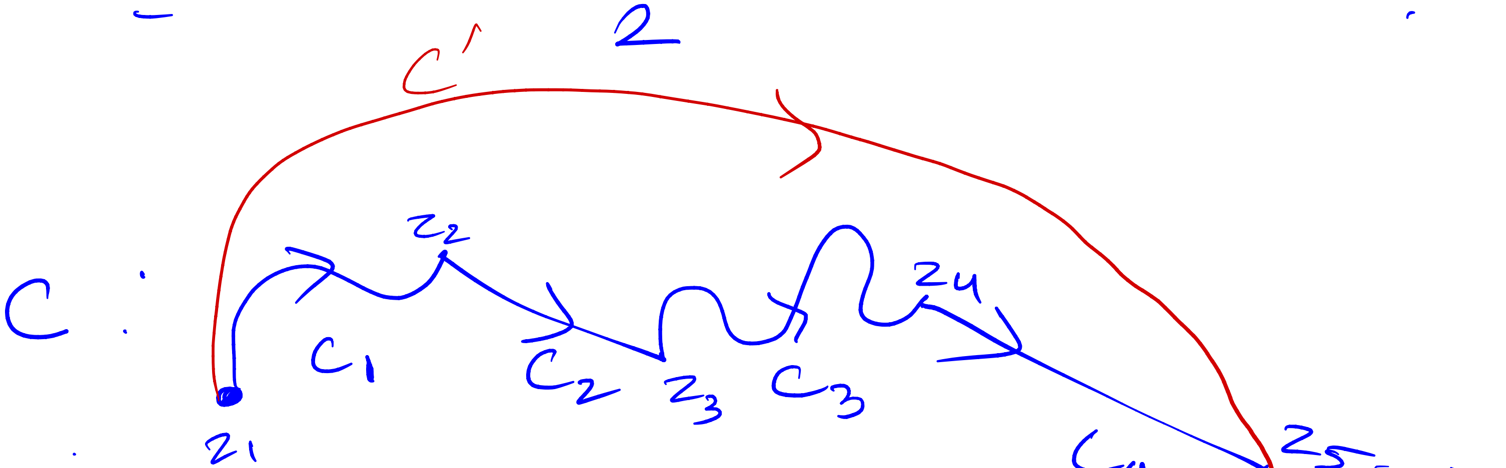
$f(z) = z$

$$\int_C f(z)\,dz = \int_C z\,dz.$$

$$= \int_a^b \underline{(z(t))\, z'(t)}\, dt$$

$$= \left[\frac{[z(t)]^2}{2} \right]_a^b$$

$$= \frac{z(b)^2 - z(a)^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$



$C : C_1 + C_2 + C_3 + C_4$

$$\int_C f(z)\,dz$$

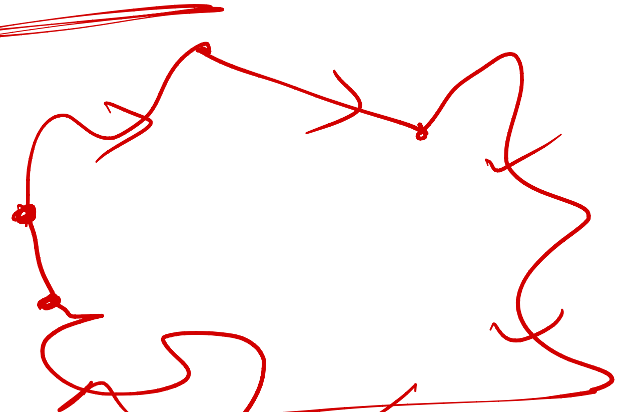
$$= \int_{C_1} z\,dz + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \frac{z_2^2 - z_1^2}{2} + \frac{z_3^2 - z_2^2}{2} + \frac{z_4^2 - z_3^2}{2} + \frac{z_5^2 - z_4^2}{2}$$

$$= \frac{z_5^2 - z_1^2}{2}$$

$$z_1 = z_5$$

$$\int_C z\,dz = 0$$



if the initial and final pts are same.