Lecture 21.

MTL 122

An analytic fn is infinitely differentiable.

$$\boxed{f^{n}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} dz}$$

$n = 0$, $0! = 1$.

$$\oint_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{n}(z_0).$$

Converse of Cauchy's Int formula

= Morera's theorem.

Suppose f is continuous in a domain D & has antiderivative F.

$$\Rightarrow \quad f(z) = F'(z)$$

$$\Downarrow$$

F is analytic.

$\Downarrow$

F' is analytic

$\Downarrow$

$F'' \Longrightarrow F'' \ldots$

$\Downarrow$

- f is analytic.

- Path independence lemma

f has antiderivative $\Longleftrightarrow$ its loop integrals in D $\oint_\Gamma f = 0$

$\Rightarrow$ f is continuous in D & $\forall$ loop integrals of f vanishes in D

$\Rightarrow$ f is analytic.

$f \to$ analytic in D.

'C_R' circle of radius R

Let $|f(z)| \leq M$

By C.I.F.

$$|f^n(z_0)| \leq \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{n!}{2\pi i} \frac{M}{R^{n+1}} \oint_C dz$$

$$= \frac{n!}{2\pi i} \frac{M}{R^{n+1}} 2\pi R$$

$$\boxed{|f^{(n)}(z_0)| \leq \frac{n!\, M}{R^n}} \to \text{Cauchy estimate.}$$

Suppose f is entire &

let f be bdd.

From Cauchy estimate

for any $R > 0$

$$|f'(z_0)| \leq M/R$$

Given any $\epsilon > 0$, R large.

$M/R < \epsilon$ so that

$$|f'(z_0)| < \epsilon$$

$\therefore |f'(z_0)| = 0$

$\Rightarrow f'(z_0) = 0$

$\Rightarrow f$ is constant.

Liouville's Theo.

A bdd entire fn is constant.

'N'

$$P(z) = z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0.$$

$\exists R > 0$ s.t $|z| \geq R$

$|P(z)| \geq c|z|^N$, $0 < c < 1$

depends on R such a way

that as $R \to \infty$

this $c \to 1$.

[Hint: $A \geq 1$.

$A = \max\{|a_0| \ldots |a_{N-1}|, 1\}$

$$P(z) = z^N \left(1 + \frac{a_{N-1}}{z} + \cdots + \frac{a_0}{z^N}\right]$$

- $$\left|\frac{a_{N-1}}{z} + \cdots \frac{a_0}{z^N}\right| \leq \frac{NA}{|z|}$$

$$\left|1 + \underbrace{\frac{a_{N-1}}{z} + \cdots \frac{a_0}{z^N}}\right| \geq 1 - |\_\_|$$

$$\geq 1 - \frac{NA}{|z|}.$$

- $|z| \geq \underline{R \geq NA > 1}$.

$$\left|1+\frac{a_{N-1}}{z}+\cdots\frac{a_0}{z^N}\right|$$

$$\geq 1-\frac{NA}{R}.$$

- $$|P(z)| \geq \left(\frac{R-NA}{R}\right)|z|^N.$$

$$\underbrace{\left(\frac{R-NA}{R}\right)}_{\substack{\shortparallel \\ 0<c<1}}$$

Fundamental theo of algebra

$$\boxed{\text{Every non const polynomial has at least one zero.}}$$

- $P(z) \longrightarrow$ has no zero.

$\Rightarrow \dfrac{1}{P(z)}$ is an entire fn.

Let R,

$|z| \geq R$.

Previous proof

$\Rightarrow |P(z)| \geq c|z|^N$
$0 < c < 1$.

$\Rightarrow \left|\frac{1}{P(z)}\right| = \frac{1}{|P(z)|}$

$\leq \frac{1}{c|z|^N} \leq \frac{1}{cR^N}$. — 1

$|z| \leq R$.

$\frac{1}{P(z)}$ is contionuous.

$\Rightarrow \frac{1}{P(z)}$ is bdd.

$M = \max_{|z| \leq R} \frac{1}{|P(z)|}$ — 2

By 1 & 2

$\Rightarrow \dfrac{1}{P(z)}$ is bdd.

Seq & Series

- $z_1, z_2, \ldots, z_n, \ldots$

$$\longrightarrow z \quad \text{as } n \to \infty$$

$\epsilon > 0 \quad \exists\, n_0$

$$|z_n - z| < \epsilon \quad \forall\, n > n_0$$

$$z_n = x_n + i y_n \to z = x + iy$$

$$\Leftrightarrow x_n \to x, \quad y_n \to y.$$

- $$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

$\longrightarrow S$ ~~co~~ iff

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N$$

$\longrightarrow S$ as $N \to \infty$.

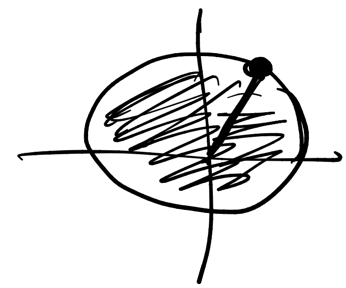
$$\sum_{n=1}^{\infty} z_n = S.$$

- Comparison test
- Ratio test
- Root test

Have a look

Power Series

$$\sum_{n=1}^{\infty} a_n (z-z_0)^n$$



Theo. If a p.s

$\sum a_n z^n$ converges at $z=z_1$ $(z_1 \neq 0)$

then it is absolutely convergent at each pt z in a open disk $|z|<|z_1|$

Theo.

Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

$\exists R \in [0,\infty]$ s.t

1) $\sum a_n (z-z_0)^n$ converges

absolutely on
$\{z \in \mathbb{C} \mid |z-z_0|<R\}$

2) the series diverges
on $\{z \in \mathbb{C} \mid |z-z_0|>R\}$

3) the series converges
uniformly on every
subdisk of the
disk of convergence.
and $\boxed{R = \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{a_n}}}$

Taylor series

$w = f(z) \to$ analytic

fn. at z_0.

$$\sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!}(z-z_0)^n.$$

$z_0 = 0 \rightarrow$ Maclaurin Series

<u>Theo.</u> Let $w = f(z)$ be a holomorphic fn on the open disk $\{z \in \mathbb{C} \mid |z-z_0| < R\}$. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!}(z-z_0) \longrightarrow f$$

uniformly

on every closed subdisk. $\{z \in \mathbb{C} \mid |z-z_0| \leq \rho\}$

Ex.

$$\underline{\mathrm{Log}(z)}, \quad z_0 = 1.$$

$z = 1.$

$$f'(z) = \frac{1}{z} \Rightarrow f'(1) = 1$$

$$f''(z) = -\frac{1}{z^2} \Rightarrow f''(1) = -1$$

$$f'''(z) = \frac{2}{z^3} \Rightarrow f'''(1) = 2$$

$\vdots$

$$\mathrm{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (z-1)^n, \quad |z-1| < 1.$$

$\Rightarrow$

- f is holomorphic at 1 (why). (Ex.)
- $\{z \in \mathbb{C} \mid |z-1| < 1\}$ (why)

Theo

$w = f(z) \rightarrow$ holomorphic at z_0.

Then the taylor series.

f' at $z_0 \longrightarrow$ T.S f at z_0

by differentiating term by term. and it converges at the same disk.

Ex. $\sin z$, $\cos z$

$f(z) = \cancel{\sin}\ \sin z$. at z_0

$f'(0) = 1$, $f''(0) = 0$ — — —

$\Rightarrow \sin z = z - \frac{z^3}{3!} + \cdots, z \in \mathbb{C}$.

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

Ex. $f(z) = \tan z$ (Notes)

Theo. $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$
$$|z-z_0| < R.$$

then f is holomorphic on $\{z \in \mathbb{C} \mid |z-z_0| < R\}$.

- A power series with positive radius of convergence is a Taylor series.