L1

Note Title 02-02-2021

Optimization – a process of maximization / minimization

The fn. to optimize is objective function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

like $f(x) = x^3 - x^2 + 1$

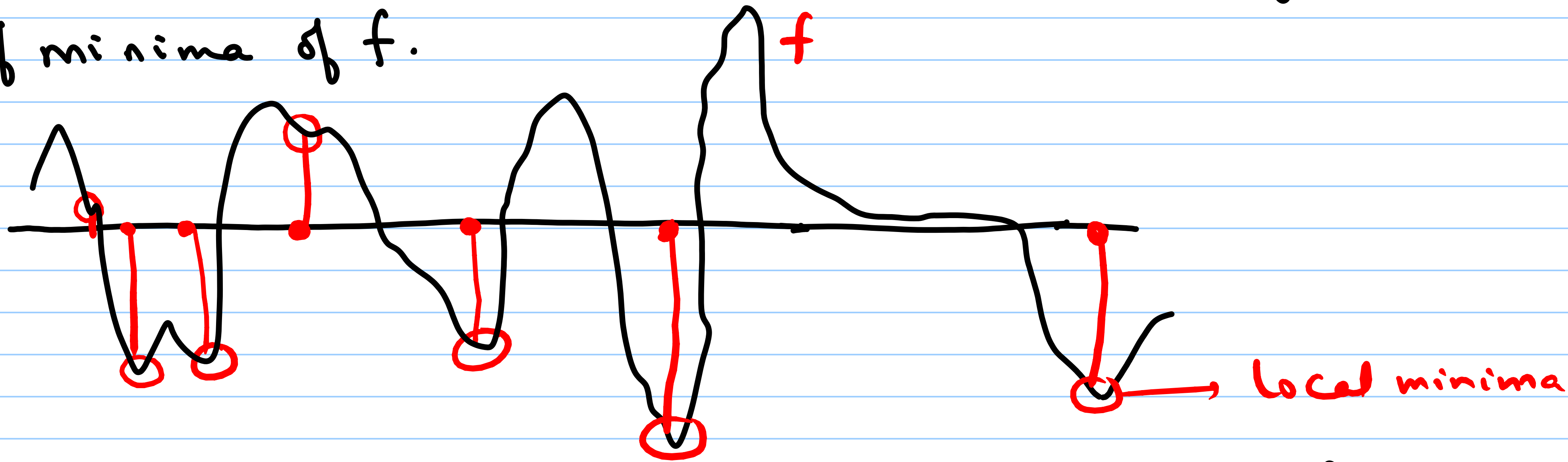
or $f(x) = x e^x + x^3$ & so

The technique that we studied in our earlier days based on derivative is that we first find the critical points by solving $f'(x) = 0$, and then we apply either the change of sign in f in nbd of the critical pt or if f'' exists then check the sign of f'' at the critical point to classify it as pt of maxima or pt of minima or neither of the two.

Note the derivative based method always yield local optimal solution.

Point of local minima - $\bar{x} \in \mathbb{R}$ is point of local min. of f if $\exists$ a nbd of $\bar{x}$ (in $\mathbb{R}$ it is an open interval $(\bar{x}-\delta, \bar{x}+\delta)$) say $N_\delta(\bar{x})$ such that $f(\bar{x}) \leq f(x)$, $\forall x \in N_\delta(\bar{x})$.

If $f(\bar{x}) \leq f(x)$ $\forall x \in \mathbb{R}$, then $\bar{x}$ is absolute or global point of minima of f.



We have also learned how to optimize $f: R^n \to R$ using first order partial derivatives and then see the nbd behaviour or second order partial derivatives to classify critical points.

Next, think of $f: S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

If we wish to talk about point of local minima now then that $\underline{\bar{x} \in S}$ and $\exists$ a nbd $N_\delta(\bar{x})$ in $\mathbb{R}^n$ such that

$$f(\bar{x}) \leq f(x), \ \forall \ x \in N_\delta(\bar{x}) \cap S$$

So, the point of local minima is to be searched not in $\mathbb{R}^n$ but in S.

This S puts constraint on the search space.

Recall what we use to do in $\mathbb{R}$

$$f : [a, b] \longrightarrow \mathbb{R}$$

In (a, b), we use to go for $f'(x) = 0$ and apply the classical approach to see $\bar{x} \in (a, b)$ & then classify.

And thereafter we use to check at the end points a and b separately as the concept of two side derivatives at the end points is not defined.

The same approach we use to adopt for function

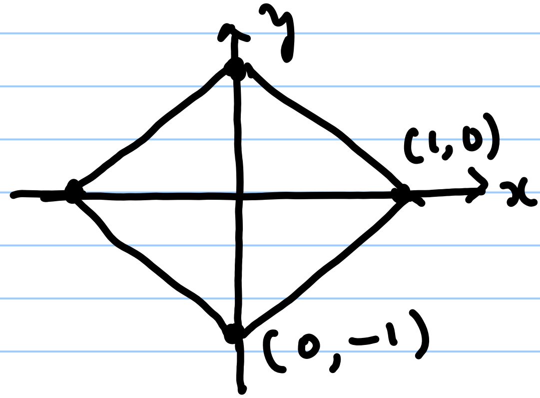
$f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, we talk about $\nabla f(x) = \begin{pmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{pmatrix}$

we find critical points, check which are in the interior S and then go for their classification.

And then we also check how f behaves on the boundary of S to see for other critical points and their nature

$$f(x, y) = x^2 - xy + y^2$$

$$(x, y) : |x| + |y| \leq 1$$



$$\nabla f(x, y) = \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = 0$$

$\Rightarrow (x, y) = (0, 0) \rightarrow$ critical point

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \succ 0$$

$\Rightarrow (0, 0)$ is the point of local minima of f & $f(0,0) = 0$

Now, at the boundary, we need to check seperately.

Also, at the four corners, $(\pm 1, 0)$ or $(0, \pm 1)$, $f \equiv 1$

And $x + y = 1$, $f(x,y) = x^2 - x(1-x) + (1-x)^2 = 3x^2 - 3x + 1$

$\nabla f = \frac{\partial f}{\partial x} = 6x - 3 = 0 \Rightarrow x = 1/2 \Rightarrow y = 1/2$ & $\frac{\partial^2 f}{\partial x^2} > 0$

and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}$ local minima

Linear programming problem (LPP)

(P) $\max / \min \ f(x)$ objective function

subject to $g_i(x) \leq 0, \ i = 1, 2, ---, m$ constraints or constraint functions

$S = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ i = 1, 2, ---, m \}$ decision variables

$f : \mathbb{R}^n \to \mathbb{R}, \quad g_i : \mathbb{R}^n \to \mathbb{R}$

The problem (P) is constrained optimization problem

In case if $S = \mathbb{R}^n$, then (P) is unconstrained optimization problem.

If all the objective fn. and constraint fns. are linear functions of x, then (P) is called a LPP

$$\min/\max \ f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \leftarrow$$ c_i's are cost coeff.

subject to

m constraint → (resources)

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{array} \right.$$

non-negativity constraint $\{ \ x_1, x_2, \ldots, x_n \ge 0$

In matrix form, we can write

$$\min / \max \quad z \equiv f(x) = c^T x$$

subject to $\quad Ax \leq b$

$\qquad\qquad\qquad x \geq 0$

$c : n \times 1$, $A : m \times n$, $x : n \times 1$, $b : m \times 1$

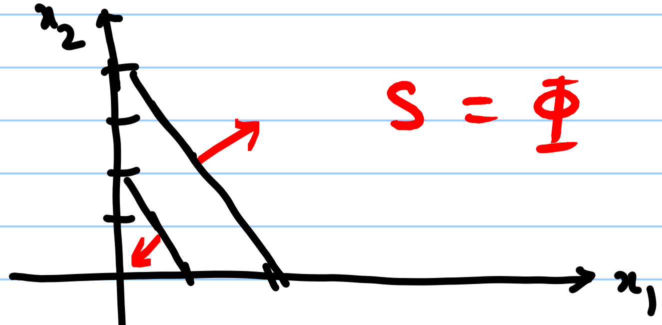
The set $S = \{ x \in \mathbb{R}^n : Ax \leq b, \; x \geq 0 \} \rightarrow$ feasible set

The non-negativity condition $x \geq 0$ can be dropped.

The set S may be an empty set. In that case, we say the LPP is <u>infeasible</u>.

Note that S is a closed set though it need not be bounded.

$$S = \{ (x_1, x_2) : \; 2x_1 + x_2 \geq 4, \;\; x_1 + \tfrac{1}{2}x_2 \leq 1, \;\; x_1, x_2 \geq 0 \}$$



$S = \Phi$

$$S = \{ (x_1, x_2) : \; x_1 + x_2 \geq 1$$
$$-2x_1 + x_2 \leq 2, \;\; x_1, x_2 \geq 0 \}$$

S is unbounded

S is a convex set

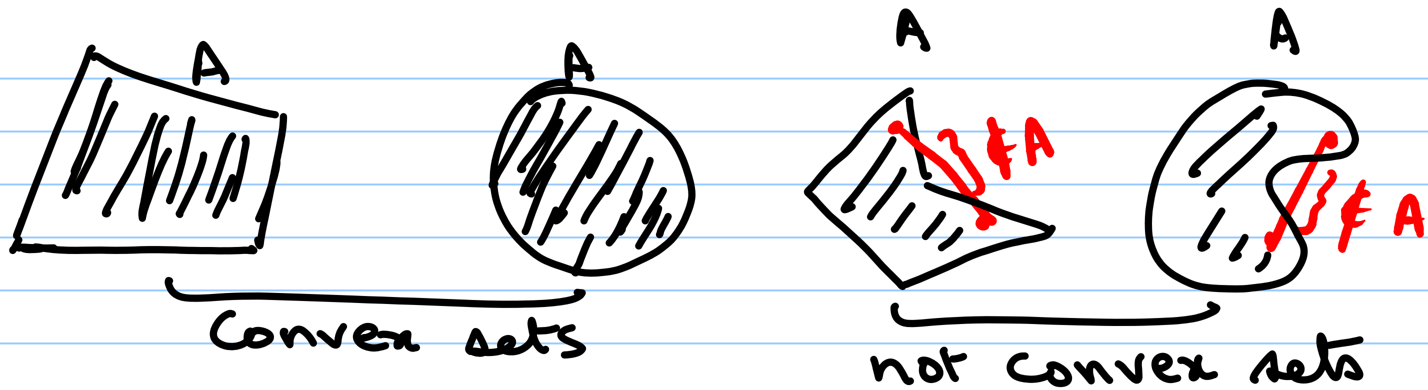
What is a convex set ?

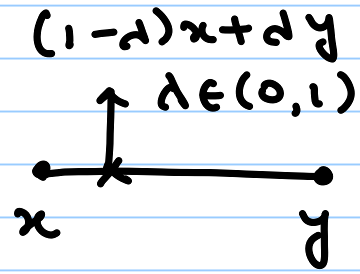
Let $A \subseteq \mathbb{R}^n$. A is called a convex set if for every $x_1, x_2 \in A$ and any $\lambda \in [0,1]$,

$$(1-\lambda)x_1 + \lambda x_2 \in A$$

or the line segment connecting any two pts in A lies completely within the set A.

Note: empty set and sin ton set are convex sets.





S is a convex set

Proof:- If S is Φ or singleton set then S is convex.

Let $x, y \in S$ and $\lambda \in [0,1]$.

$\Rightarrow$ $x, y \geq 0$ and $Ax \leq b$, $Ay \leq b$

$\Rightarrow$ $(1-\lambda)x + \lambda y \geq 0$ and $(1-\lambda)Ax \leq (1-\lambda)b$, $\lambda Ay \leq \lambda b$

$\Rightarrow \quad A((1-\lambda)x) \leq (1-\lambda)b \quad$ and $\quad A(\lambda y) \leq \lambda b$

$\Rightarrow \quad A((1-\lambda)x+\lambda y) \leq b \quad \& \quad (1-\lambda)x+\lambda y \geq 0$

$\Rightarrow \quad (1-\lambda)x+\lambda y \in S$

In fact S is a polyhedron.

A polyhedron is an intersection of a finite number of half spaces.

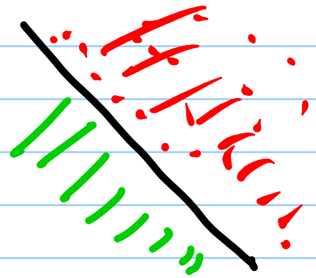
What is a half space?

A hyperplane is a plane in n-dimension. Its equation is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$$

If we draw this hyperplane in n-dimensional space $\mathbb{R}^n$, then it will divide the $\mathbb{R}^n$ into two half spaces

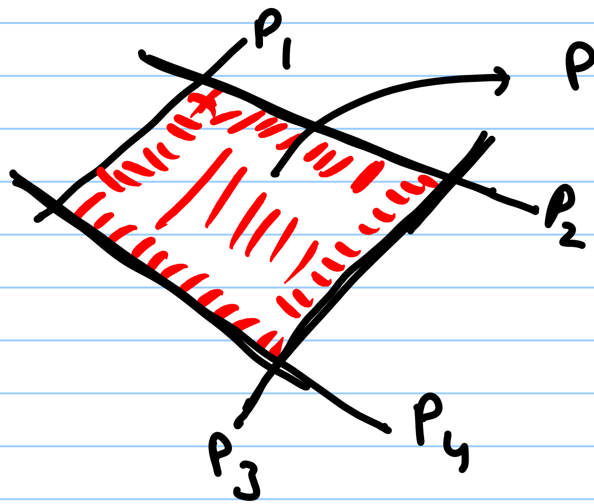
- points that are above and on the hyperplane
- points that are below and on the hyperplane

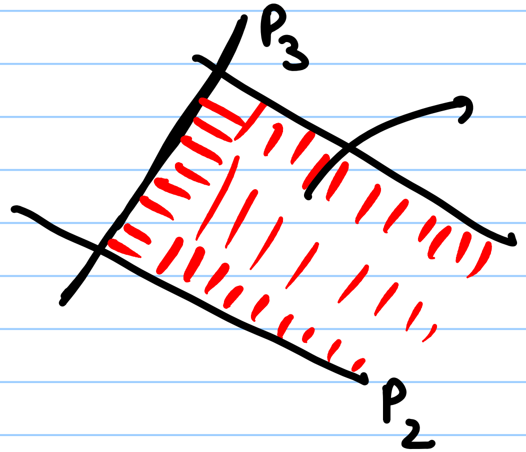


$$\{x \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \cdots + a_nx_n \le d\}$$

$$\{x \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \cdots + a_nx_n \ge d\}$$

Note each of the two half spaces so generated by one hyperplane is a convex set.





A bounded polyhedron is called a polytope.

Coming to $S = \{ x \in \mathbb{R}^n : Ax \le b, \ -x \le 0 \}$

we can take

$$S = S_1 \cap S_2 \cap \cdots \cap S_m \cap S_{m+1} \cdots \cap S_{m+n}$$

intersection of finite number of half spaces where

$S_1 = \{ x \in \mathbb{R}^n : a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \}$

$S_2 = \{ x \in \mathbb{R}^n : a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \}$

$\vdots$

$S_m = \{ x \in \mathbb{R}^n : a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \}$

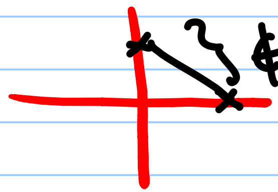
$S_{m+1} \{ x \in \mathbb{R}^n : -x_1 \le 0 \}, \cdots, S_{m+n} = \{ x \in \mathbb{R}^n : -x_n \le 0 \}$

So, S is a polyhedron.

Moreover, S is an intersection of finite number of half spaces, each of the half space is convex, so S is a convex set.

- Intersection of convex sets is convex
- Union of convex sets may not be convex

eg: $\{(x,y) : y=0\} \cup \{(x,y) : x=0\} = A$



$\} \notin A$

- Intersection of convex sets is convex

Let $\{A_\alpha : \alpha \in \Lambda\}$ be convex sets in $\mathbb{R}^n$

Let $A = \bigcap_{\alpha \in \Lambda} A_\alpha$

If $A = \Phi$ or singleton, then A is a convex set.

Let $x, y \in A \Rightarrow x, y \in A_\alpha \;\forall\, \alpha \in \Lambda$

$\Rightarrow (1-\lambda)x + \lambda y \in A_\alpha \quad \forall\, \alpha \in \Lambda, \;\forall\, \lambda \in [0,1]$

$\Rightarrow (1-\lambda)x + \lambda y \in A, \;\forall\, \lambda \in [0,1]$

$\Rightarrow A$ is a convex set.

Recall the LPP

(P) $\max \; z = c^T x$

subject to $x \in S = \{ x \in \mathbb{R}^n : Ax \leq b, \; x \geq 0 \}$.

Four cases arises:

- (P) is infeasible $\Leftrightarrow S = \Phi$
- (P) is unbounded
- (P) has a unique optimal soln.
- (P) has alternate optimal solns.

In last case, (P) has infinitely many optimal solns.

(P) is called unbounded if we can find a sequence of feasible $\langle x_n \rangle$, $x_n \in S$, $\forall n$, such that $Z(x_n) \to +\infty$ as $n \to \infty$.

A point $\bar{x} \in S$ is called an optimal solution of (P) if
$$Z(\bar{x}) = c^T \bar{x} \geq c^T x = z(x), \quad \forall x \in S.$$

In case if (P) has alternate optimal solution say $\bar{x} \in S$ and $\hat{x} \in S$ both are optimal solutions of (P), then any point on the line segment of $\bar{x}$ and $\hat{x}$ is also an optimal soln of (P).

$\because \quad \bar{x}, \hat{x} \in S \Rightarrow (1-\lambda)\bar{x} + \lambda\hat{x} \in S, \quad \forall \lambda \in [0,1]$

and
$$c^T((1-\lambda)\bar{x} + \lambda\hat{x}) = (1-\lambda)\, c^T\bar{x} + \lambda\, c^T\hat{x}$$
$$= (1-\lambda)\, c^T\bar{x} + \lambda\, c^T\bar{x}$$
$$(\because\ c^T\bar{x} = c^T\hat{x})$$
$$= c^T\bar{x} = \max_{x \in S} z(x)$$

$\Rightarrow (1-\lambda)\bar{x} + \lambda\hat{x}$ also yields the maximum value of the objective fn. $z(x)$, $\forall \lambda \in [0,1]$

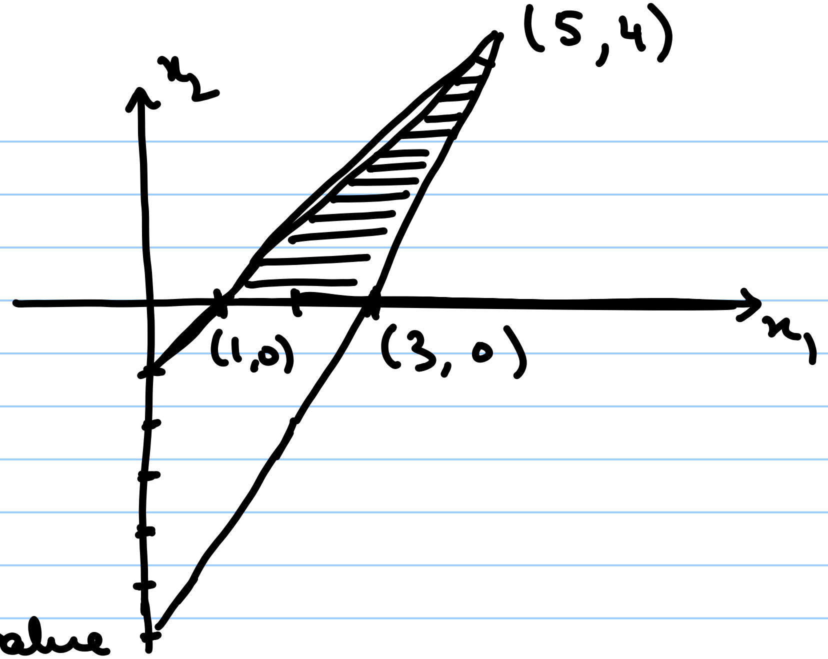
$\therefore$ (P) has infinitely many optimal solns and the set of all optimal solns of (P) is a convex set.

$$\max \ z = 6x_1 - 2x_2$$

subject to $\quad x_1 - x_2 \le 1$

$$2x_1 - x_2 \le 6$$

$$x_1, x_2 \geq 0$$

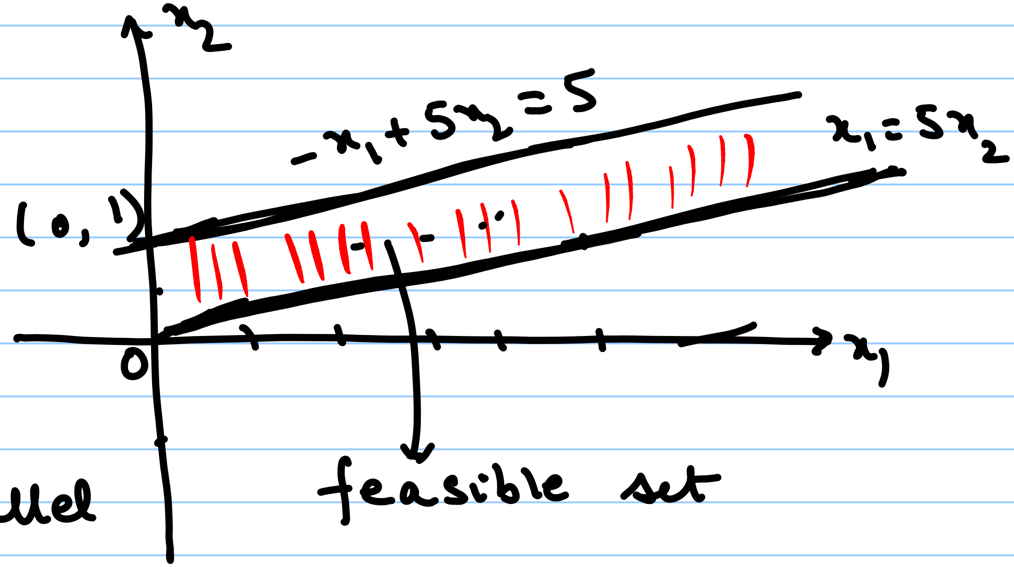


unique optimal solution $(x_1^* = 5, \ x_2^* = 4)$ and optimal value $z^* = 22$.

$\max\ z = x_1 - x_2$

subject to

$$x_1 - 5x_2 \le 0$$
$$-x_1 + 5x_2 \le 5$$
$$x_1,\ x_2 \ge 0$$



lines $\left.\begin{array}{l} x_1 = 5x_2 \\ x_1 = 5x_2 - 5 \end{array}\right\}$ are parallel

S is an unbounded set.

and (P) is unbounded problem as within S, $x_2 = x_1/5$
and $x_1 - x_2 = 4/5\, x_1 \to \infty$ along the line $x_1 = 5x_2$

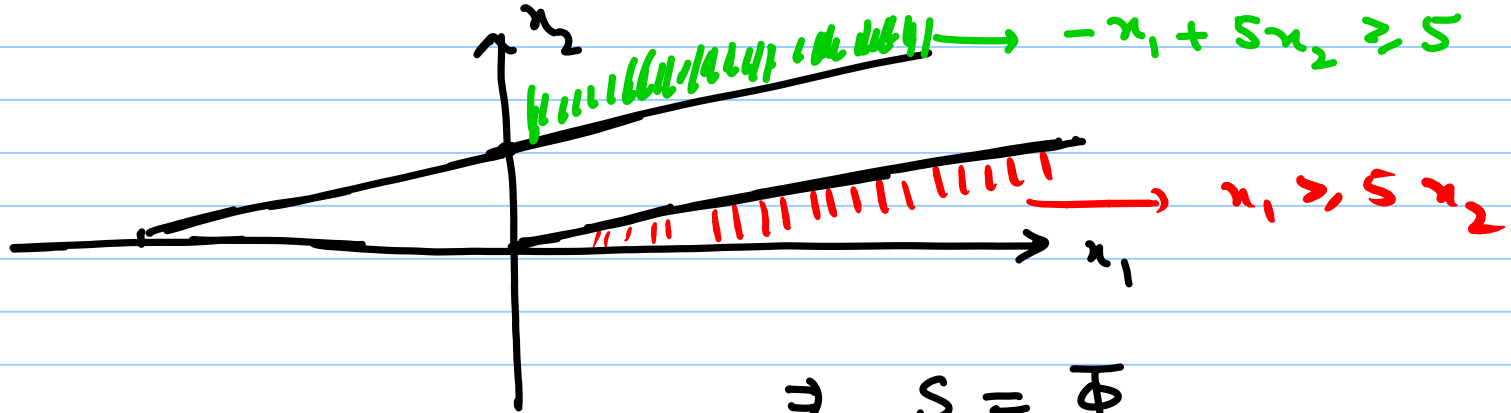
If in the same problem we change the objective fn. to

$$\left.\begin{array}{l} \max \; x_1 - 10x_2 \\ \frac{x_1}{5} \le x_2 \le 1 + \frac{x_1}{5} \\ x_1 \ge 0 \end{array}\right\} \Rightarrow \begin{array}{c} -10 - x_1 \le x_1 - 10x_2 \le -x_1 \\ \text{and } x_1 \ge 0 \end{array}$$

Then $(0,0)$ is the unique optimal soln. with optimal value $z^* = 0$.

In the same problem if we change the inequalities in the constraints and consider the set

$$S = \{(x_1, x_2) : x_1 - 5x_2 \geq 0, \ -x_1 + 5x_2 \geq 5, \ x_1, x_2 \geq 0\}$$



$$\Rightarrow \quad S = \Phi$$

and (P) is infeasible.

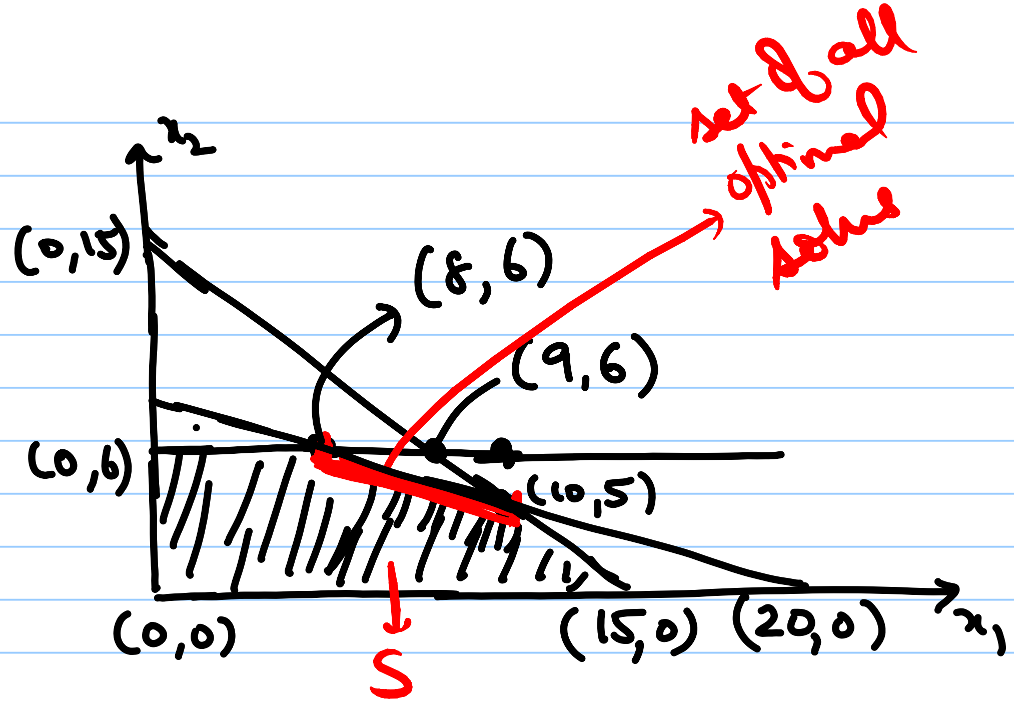
max $2x_1 + 4x_2$

subject to $x_1 + x_2 \le 15$

$x_1 + 2x_2 \le 20$

$x_2 \le 6$

$x_1, x_2 \ge 0$



At $(8,6)$

$Z = 16 + 24 = 40$

At $(10,5)$, $Z = 20 + 20 = 40$

The problem has alternate optimal solns and optimal value $z^* = 40$.

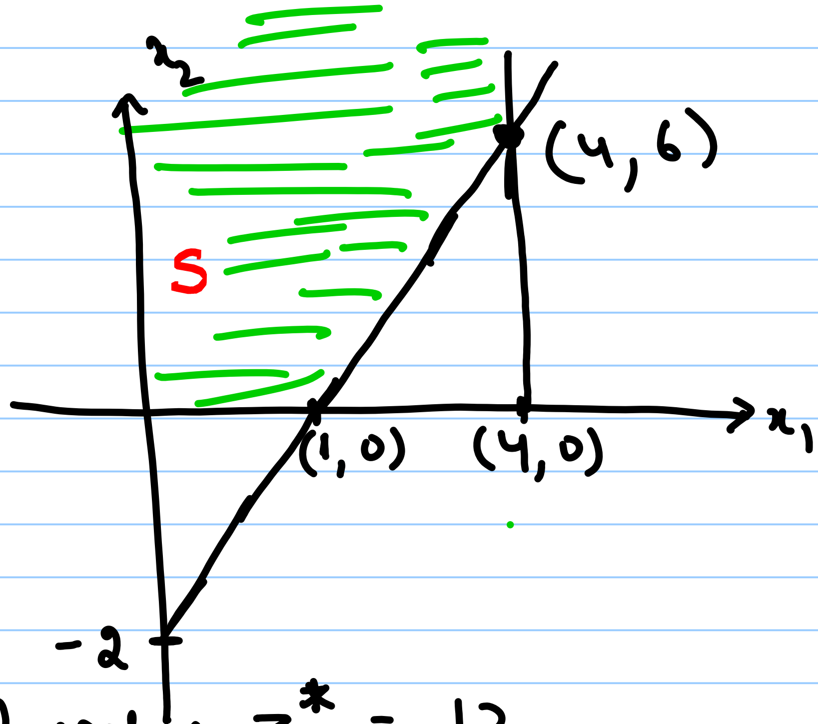
Also, S unbounded does not mean the problem is unbounded.

$$\max \quad z = 6x_1 - 2x_2$$

subject to $\quad 2x_1 - x_2 \leq 2$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$



Feasible set S is unbounded

Optimal soln is

$x_1^* = 4, \; x_2^* = 6$ and optimal value $z^* = 12$

S (feasible set)

- Φ → LPP is infeasible
- bounded → LPP has either unique optimal soln. or alternate and hence infinitely many optimal solns.
- unbounded → LPP has unique optimal soln / infinitely many optimal soln / unbounded.

In case LPP is unbounded then S must be an unbounded set.

Result: If x^* is an optimal soln. of a nonconstant LPP, then x^* can not be in interior of the feasible set S.

Proof:- Let x^* be an optimal soln of

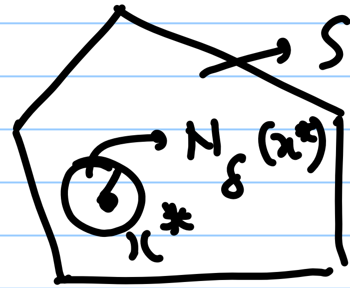
$$(P) \qquad \max \; z = c^T x$$

$$\text{s. to} \quad x \in S = \{x \in \mathbb{R}^n : Ax \leq b, \; x \geq 0\}$$

$$\Rightarrow c^T x^* \geq c^T x, \quad \forall \, x \in S.$$

Let $x^* \in$ interior of S.

$$\Rightarrow \exists \; \delta > 0 \text{ such that } N_\delta(x^*) \subset S$$



Define another vector

$$\bar{x} = x^* + \frac{\delta}{2} \frac{c}{\|c\|} \in \mathbb{R}^n$$

Then

$$\|\bar{x} - x^*\| = \frac{\delta}{2} < \delta \Rightarrow \bar{x} \in N_\delta(x^*) \subset S$$

$\Rightarrow \bar{x} \in S$ (or $\bar{x}$ is feasible soln of (P))

And

$$c^T \bar{x} = c^T x^* + \frac{\delta}{2} \frac{c^T c}{\|c\|} = c^T x^* + \frac{\delta}{2} \frac{\|c\|^2}{\|c\|}$$

$$\Rightarrow c^T \bar{x} = c^T x^* + \frac{\delta}{2} \|c\| > c^T x^*.$$

This contradicts that x^* is optimal for (P).

Result: Every local optimal solution of an LPP is its global optimal soln.

Proof:- Let x^* be a local optimal solution of

$$\max z = c^T x$$

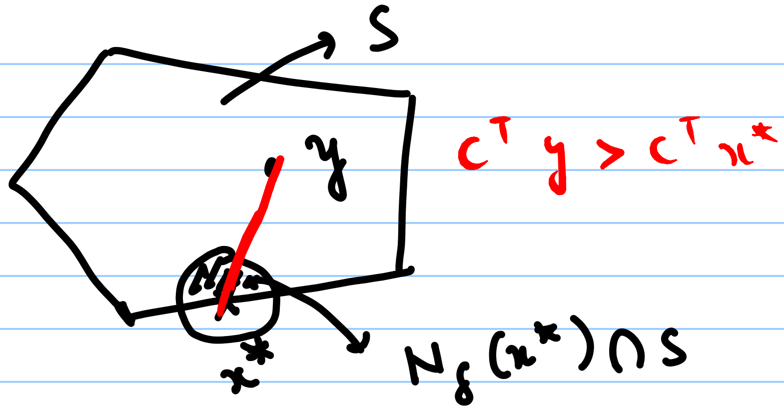
$$\text{subject to } x \in S = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$$

$\Rightarrow \exists\ \delta > 0$ such that

$$c^T x^* \geq c^T x, \quad \forall\, x \in N_\delta(x^*) \cap S. \quad \text{——} (1)$$

Claim:- $c^T x^* \geq c^T x, \ \forall\, x \in S.$

Let $\exists\ y \in S$ s.t. $c^T x^* < c^T y$ (note $y \neq x^*$)



Let us choose $\lambda \in (0,1)$, sufficiently small, so that

$(1-\lambda)x^* + \lambda y \in N_\delta(x^*) \cap S$

Note, $\forall\, \lambda \in (0,1)$,

$(1-\lambda)x^* + \lambda y \in S$ as S is a convex set.

Now $(1-\lambda)x^* + \lambda y \in N_\delta(x^*)$

$\Leftrightarrow \quad \| \lambda (y - x^*) \| < \delta \Rightarrow 0 < \lambda < \dfrac{\delta}{\|y - x^*\|}$ so choice of λ is possible.

Then $c^T((1-\lambda)x^* + \lambda y) = (1-\lambda)c^T x^* + \lambda c^T y$

$> (1-\lambda)c^T x^* + \lambda c^T x^*$

$(\because \lambda \in (0,1)$ and

$c^T y > c^T x^*)$

$\Rightarrow c^T((1-\lambda)x^* + \lambda y) > c^T x^*$

for some choice of λ such that

$(1-\lambda)x^* + \lambda y \in N_\delta(x^*) \cap S$

But this contradicts (1)

$\therefore c^T x^* \geq c^T x, \forall x \in S.$

We introduce the concept of extreme point of a convex set.

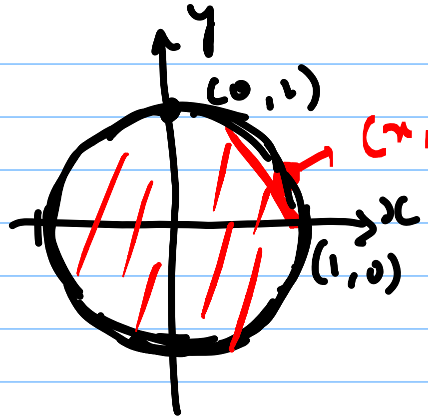
Let $A \subseteq \mathbb{R}^n$ be any non-empty convex set.

A point $x^0 \in A$ is called an extreme point of A if $\nexists$ any two distinct points $x^1, x^2 \in A$ and $\lambda \in (0,1)$ such that

$$x^0 = (1-\lambda)x^1 + \lambda x^2.$$

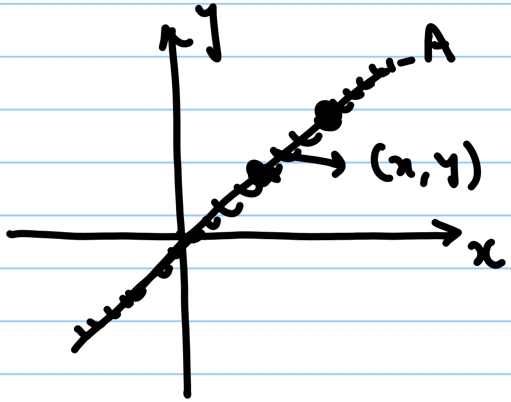
That is, x^0 can not be on any line segment connecting any two distinct pts in A.

For eg: $A = \{(x,y) : x^2 + y^2 \leq 1\}$



can not be on any chord of the circle.

Every point (x, y) on the boundary of the unit circle (i.e. $x^2+y^2=1$) is an extreme point of A

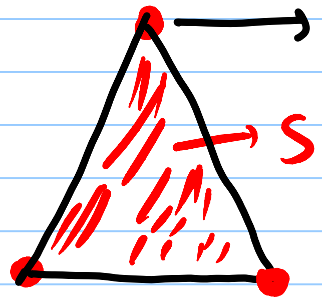


$A = \{(x, y) \in \mathbb{R}^2 : x = y\}$

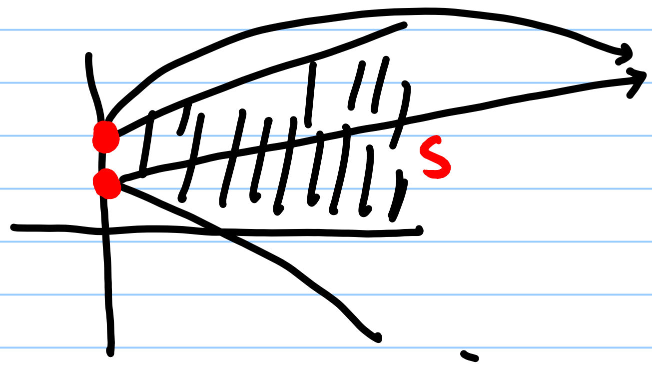
A is a convex set

but it has no extreme point as every $(x, y) \in A$ is on some line segment (part) of A

For polyhedron, the corner points are extreme points



we can not express this point on any line segment connecting any two distinct points of S.



extreme point of S.

∵ we can not find two distinct point in S, such that this red point is on the line segment connecting those two point in S.

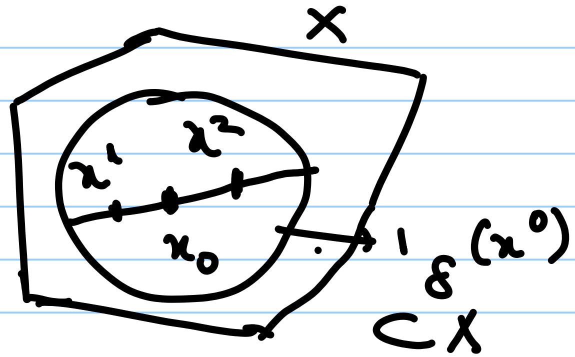
Result: An extreme point of a convex set X must be its boundary point; though the converse is not true.

Proof: Let $X \subseteq \mathbb{R}^n$ be a non-empty convex set.

Let $x^0 \in X$ be an extreme point of X

If $x^0 \in$ interior X, then $\exists$ $\delta > 0$ such that

$$N_\delta (x^0) \subset X$$



We can define $x^1 = x^0 - \frac{\delta}{2} e'$

and $x^2 = x^0 + \frac{\delta}{2} e$, $e \in \mathbb{R}^n$, $\|e\| = 1$

Then

$$\|x^0 - x^1\| = \frac{\delta}{2} < \delta \Rightarrow x^1 \in N_\delta(x^0) \Rightarrow x^1 \in X$$

$$\|x^0 - x^2\| = \frac{\delta}{2} < \delta \Rightarrow x^2 \in X$$

$x^1 \neq x^2$ as $e \neq 0$ in $\mathbb{R}^n$ $\because \|e\| = 1$

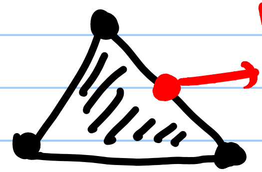
and $x^0 = \frac{1}{2}(x^1 + x^2) = (1-\alpha)x^1 + \alpha x^2$, $\alpha = \frac{1}{2} \in (0,1)$

This contradicts that x^0 is an extreme point of X.

$\therefore$ $x^0 \in X$ and $x^0 \notin$ interior of X

$\Rightarrow x^0 \in$ boundary of X.

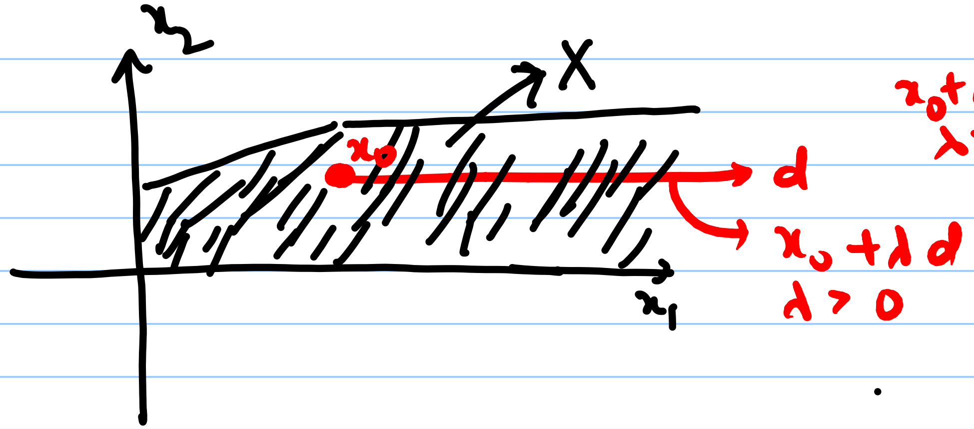
Converse is not true as X:

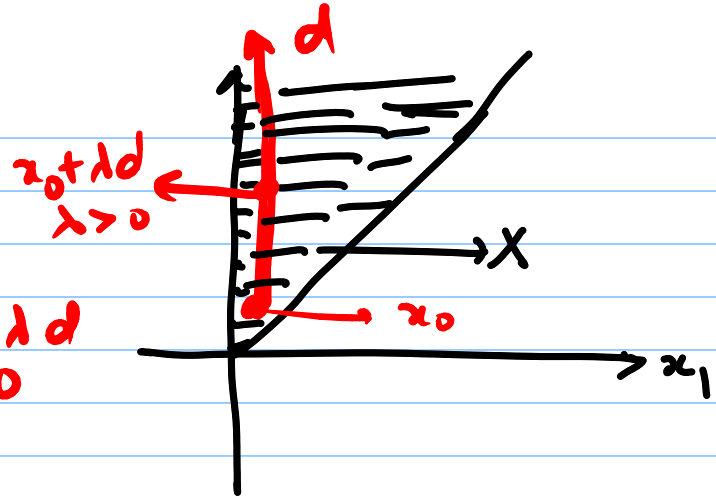


boundary pt. of X but not an extreme pt of X.

We divert slightly to define a few more concepts for a better understanding of geometry & linear algebra concepts associated with such geometries before analysing more on LPP and its optimal soln.

Direction: Let X be a convex set in $\mathbb{R}^n$. A vector $d \in \mathbb{R}^n$, $d \neq 0$, is called a direction of X if for each $x_0 \in X$, the ray $\{x_0 + \lambda d, \lambda \geq 0\} \subset X$. Or starting from $x_0 \in X$, we can recede along the vector d for any step length $\lambda \geq 0$ and remain in the set X.





If X is a bounded set, then no such d exist $\Rightarrow D_X(d) = \phi$.

Let $S = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\} \longrightarrow$ convex set

$d \ne 0$, $d \in \mathbb{R}^n$, is a direction of S iff for each $x_0 \in S$

$$A(x_0 + \lambda d) \le b, \quad x_0 + \lambda d \ge 0, \quad \forall \lambda \ge 0$$

$$\Rightarrow \quad Ad \le \frac{b - Ax_0}{\lambda}, \quad x_0 + \lambda d \ge 0, \quad \lambda > 0 \quad \text{———} \ (1)$$

As $x_0 \in S \Rightarrow b - Ax_0 \geq 0$ and $x_0 \geq 0$

$\therefore$ If ① has to hold $\forall \lambda > 0$, it is possible iff

$$Ad \leq 0, \quad d \geq 0, \quad d \neq 0$$

(if $d_j < 0$ for some j, then $\lambda > 0$, $\lambda \to \infty$, $x_{0j} + \lambda d_j < 0$)

$$D_S = \{ d \in \mathbb{R}^n : d \neq 0, \ d \geq 0, \ Ad \leq 0 \}$$

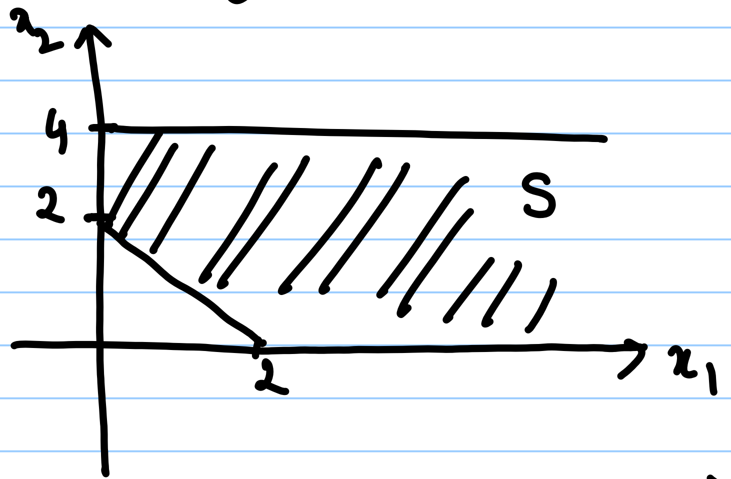
is the set of directions of S.

On the same lines, suppose, $S_1 = \{ x \in \mathbb{R}^n : Ax = b, \ x \geq 0 \}$

then $D_{S_1} = \{ d \in \mathbb{R}^n : d \neq 0, \ d \geq 0, \ Ad = 0 \}$.

The set of directions of X is a convex set in $\mathbb{R}^n$.

Eg: $S = \{ (x_1, x_2) : x_1 + x_2 \geq 2, \ x_2 \leq 4, \ x_1, x_2 \geq 0 \}$



Let $d \in D_S$.

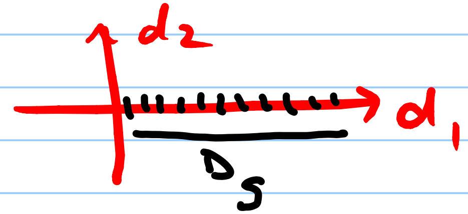
$\Rightarrow d \neq 0, \ d \geq 0$ and

$$Ad = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \leq 0$$

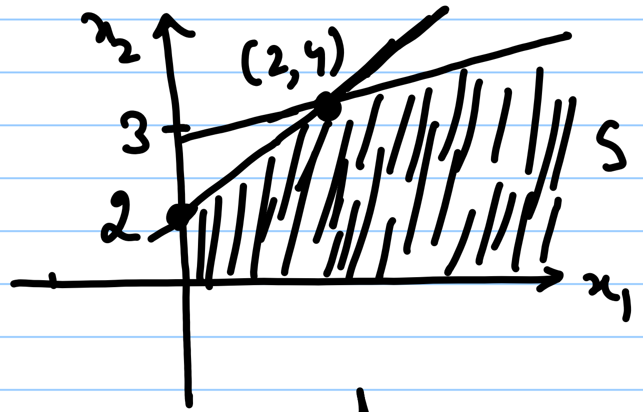
$\Rightarrow d_1 + d_2 \geq 0, \ d_2 \leq 0.$

But $d_2 \geq 0 \Rightarrow d_2 = 0 \Rightarrow d_1 \geq -d_2 \Rightarrow d_1 \geq 0$

$\therefore D_S = \{ d \in \mathbb{R}^2 : d_1 \geq 0, \ d_2 = 0 \}$



Eg: $S = \{ (x_1, x_2) : -x_1 + 2x_2 \le 6,\ -x_1 + x_2 \le 2,\ x_1, x_2 \ge 0 \}$

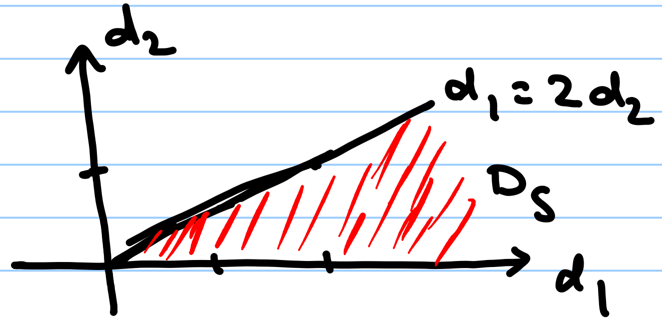


Let $d \in D_S$

$\Rightarrow d \neq 0,\ d \geq 0$ and

$$Ad = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \le 0$$

$$\Rightarrow \left.\begin{array}{l} -d_1 + 2d_2 \le 0 \\ -d_1 + d_2 \le 0 \end{array}\right\} \quad d_1 \geq 2d_2 \geq 0$$



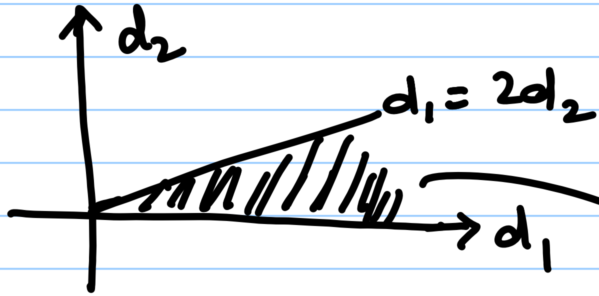
Extreme direction of a convex set X

An extreme direction of a convex set X is a vector $d \in D_X$ such that d can not be represented as a positive combination of two distinct $d_1, d_2 \in D_X$ (where by distinct d_1 and d_2 we mean that d_1 is not a positive multiple of d_2, i.e., $d_1 \neq d_2 \iff d_1 \neq \beta d_2$ for some $\beta > 0$)

$\therefore$ $d \in D_X$ is an extreme direction of X if $\nexists$ $d_1, d_2 \in D_X$ $d_1 \neq d_2$, and $\mu_1, \mu_2 > 0$ such that $d = \mu_1 d_1 + \mu_2 d_2$.

In the second eg above when

$$D_S = \{ (d_1, d_2) : d_1 \geq 0, \; d_2 \geq 0, \; d_1 \geq 2d_2 \}$$



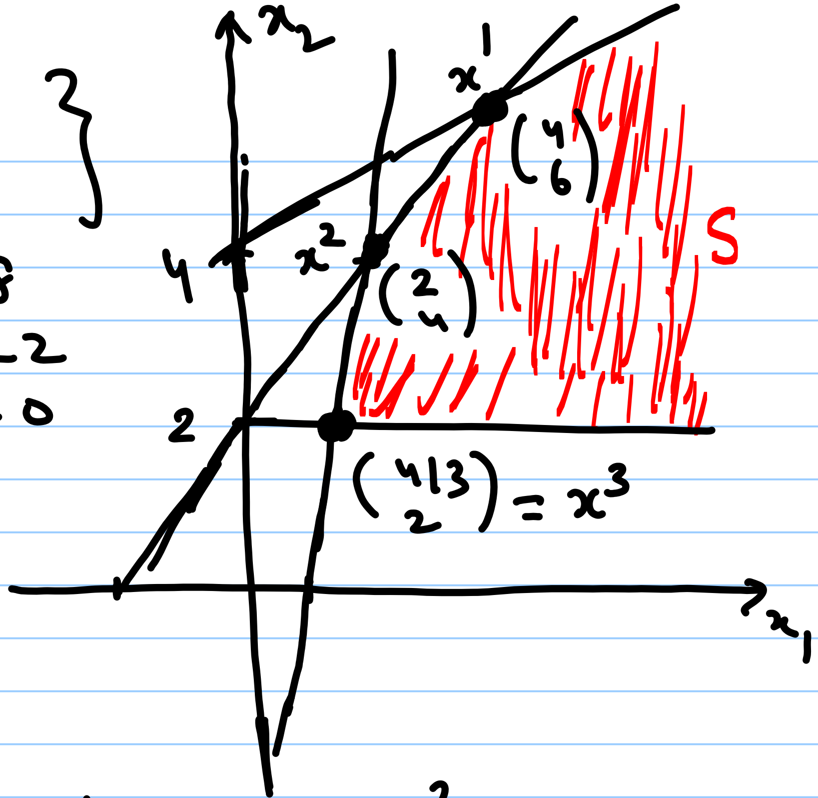
we have two extreme directions $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Any vector $d \in D_x$, can always be expressed as

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \xi_1 \geq 0, \; \xi_2 \geq 0$$

We can normalize the extreme directions to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.
using $\|d\| = |d_1| + |d_2| + \cdots + |d_n| = 1$

Eg: Let $S = \left\{ (x_1, x_2) : \begin{array}{l} -3x_1 + x_2 \le -2 \\ -x_1 + x_2 \le 2 \\ -x_1 + 2x_2 \le 8 \\ -x_2 \le -2 \\ x_1, x_2 \ge 0 \end{array} \right\}$



S has 3 extreme points

$x^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $x^2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $x^3 = \begin{pmatrix} 4/3 \\ 2 \end{pmatrix}$

and

$$D_S = \{(d_1, d_2) : d_1 \ge 2d_2,\ d_1 \ge 0,\ d_2 \ge 0,\ d \ne 0\}$$

Extreme directions of D_S are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

Result: Representation Theorem for polyhedron

Let $S = \{ x \in \mathbb{R}^n : Ax \leq b, x \geq 0 \}$ be a nonempty polyhedron.

→ Then the set of extreme points of S is nonempty finite.

→ The set of extreme directions of S is Φ iff S is bounded

→ If S is unbounded then the set of extreme directions of S is nonempty and finite. → Furthermore, any vector $\bar{x} \in S$ can be represented as a convex combination of extreme points of S and a non-negative linear combination of extreme directions of S.

i.e.

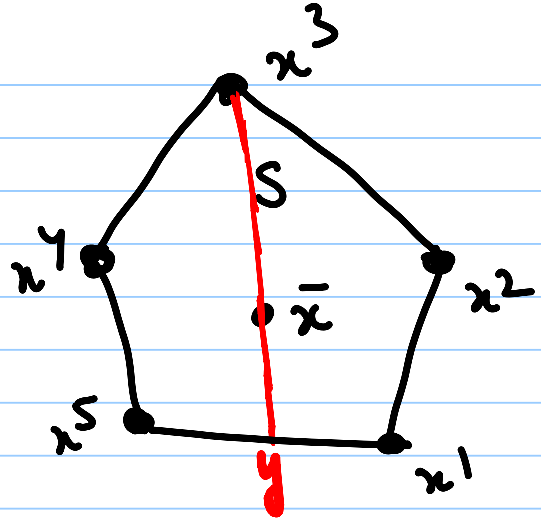
$$\bar{x} = \sum_{j=1}^{p} \alpha_j x^j + \sum_{r=1}^{k} \mu_r d^r$$

$$\alpha_j \geq 0, \quad \forall j, \quad \sum_{j=1}^{p} \alpha_j = 1$$

$$\mu_r \geq 0, \quad \forall r$$

where $x^1, x^2, \ldots, x^p$ are finite no. of extreme points of S and $d^1, d^2, \ldots, d^k$ are finite no. of extreme directions of S.

(We skip the proof — but can look at chapter 2 in LPP and network flows book by Bazaraa et al. listed in content)



Here $D_S = \Phi$

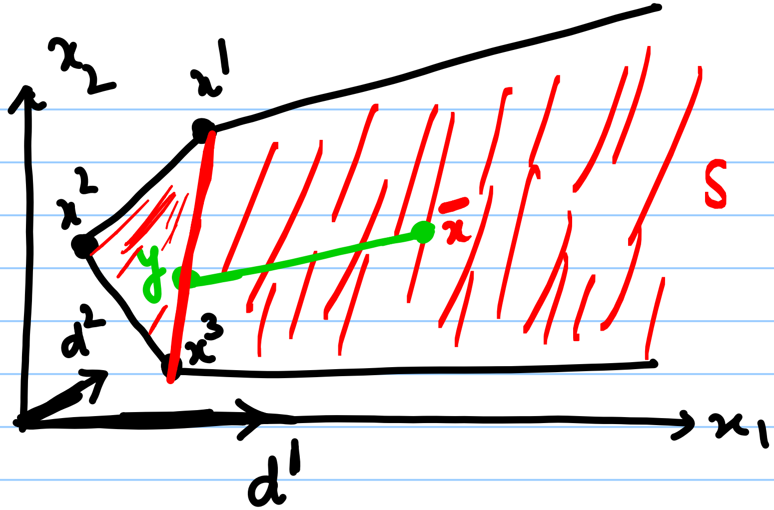
$$\bar{x} = (1-\lambda)x^3 + \lambda y, \quad \lambda \in (0,1)$$

and $y = (1-\mu)x^5 + \mu x^1, \quad \mu \in (0,1)$

$$\Rightarrow \quad \bar{x} = (1-\lambda)x^3 + \lambda(1-\mu)x^5 + \lambda\mu\, x^1,$$

$$= \xi_1 x^2 + \xi_2 x^5 + \xi_3 x^1$$

$\xi_1, \xi_2, \xi_3 \geq 0$ and $\xi_1 + \xi_2 + \xi_3 = 1-\lambda + \lambda - \lambda\mu + \lambda\mu = 1.$



$$\bar{x} = y + \mu d^2 , \quad \mu > 0$$

and

$$y = (1-\lambda)\, x^1 + \lambda x^3$$

$$\lambda \in (0,1)$$

$$\Rightarrow \quad \bar{x} = \underbrace{(1-\lambda)\, x^1 + \lambda x^3}_{\text{convex combination}} + \underbrace{\mu d^2}_{\text{non negative extreme direction combination}}$$

Result: If the LPP has an optimal solution then atleast one of the corner point of the feasible set S is an optimal soln.

Proof: Note $S \neq \Phi$ as LPP has an optimal soln.

We shall first prove the result when <u>S is a bounded polyhedron</u> (in this case LPP can not be unbounded)

We have to prove that optimal value of LPP is attained at a corner (or extreme) point of S.

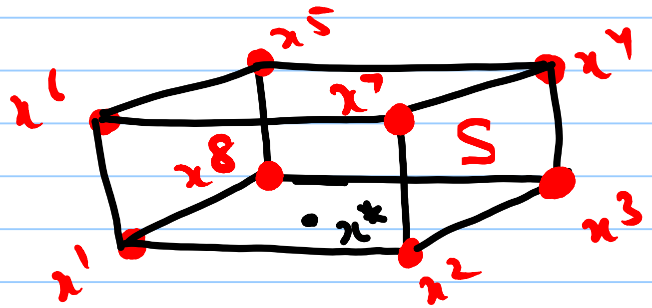
Let x^* be an optimal soln of the LPP

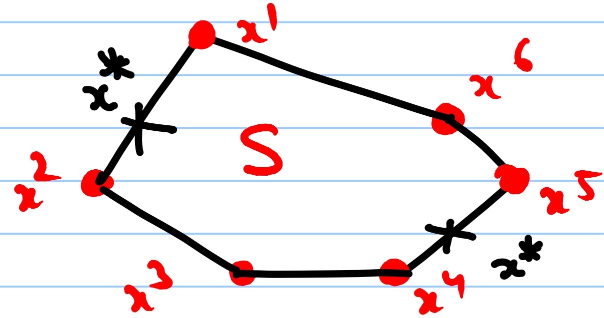
$\max\ z = c^T x$

subject to $x \in S = \{x \in \mathbb{R}^n : Ax \le b,\ x \geq 0\}$

Since S is bounded, it has only finite number of corner points say $x^1, x^2, ---, x^p$.

Let x^* be not an extreme point of S. So, x^* can be expressed as a convex combination of $x^1, x^2, ---, x^p$





$$x^* = \sum_{i=1}^{p} \lambda_i \, x^i \quad , \quad \lambda_i \geq 0 \ \forall \ i = 1, 2, \ldots, p$$

$$\text{and} \quad \sum_{i=1}^{p} \lambda_i = 1.$$

$$\Rightarrow c^T x^* = c^T \left(\sum_{i=1}^{p} \lambda_i \, x^i \right)$$

$$= \sum_{i=1}^{p} \lambda_i \left(c^T x^i \right)$$

$$\leq \left(\sum_{i=1}^{p} \lambda_i \right) \max \left\{ c^T x^1, c^T x^2, \_\_ , c^T x^p \right\} \quad (\because \lambda_i \geq 0)$$

$$= \max \left\{ c^T x^1, \ldots, c^T x^p \right\} \quad \left(\because \sum_{i=1}^{p} \lambda_i = 1 \right)$$

$$= c^T x^k \quad (\text{say})$$

$$\Rightarrow \quad c^T x^* \le c^T x^k \quad \text{——} \quad (1)$$

where x^k is that extreme point of S where $\max\{\underbrace{c^T x^1, c^T x^2, \ldots, c^T x^p}_{\text{finite set}}\}$ is attained.

so max exists though it may not be unique

But x^* is an optimal soln. of LPP and x^k is a feasible point of LPP (as $x^k \in S$), so, we must have

$$c^T x^* \ge c^T x^k \quad \text{——} \quad (2)$$

Finally, $c^T x^* = c^T x^k$ (from (1) & (2))

Thus, the optimal value of the objective fn. $z = c^T x$ is attained at $x = x^k$ also.

$\therefore$ One of the extreme point of S is surely an optimal soln of LPP.

Convex Combination : Suppose $X = \{v^1, v^2, v^3, --- v^k\}$ be a set of vectors in $\mathbb{R}^n$. Then, a convex combination of these finite number of vectors is

$$v = \sum_{i=1}^{k} \lambda_i v^i, \quad \lambda_i \geq 0, \ \forall i, \quad \sum_{i=1}^{k} \lambda_i = 1$$

We next prove the result when S is unbounded set.

Recall representation thm, any $x \in S$ can be written as

$$x = \sum_{j=1}^{p} \lambda_j x^j + \sum_{r=1}^{k} \mu_r \underbrace{d^r}_{\text{extreme directions of } S}$$

$$\sum_{j=1}^{p} \lambda_j = 1, \quad \lambda_j \geq 0 \ \forall j, \quad \mu_r \geq 0.$$

$\Rightarrow$ LPP becomes

$$\max \ c^T x = \sum_{j=1}^{p} \lambda_j c^T x^j + \sum_{r=1}^{k} \mu_r c^T d^r$$

Subject to $\sum_{j=1}^{p} \lambda_j = 1, \ \lambda_j \geq 0, \ \forall j, \ \mu_r \geq 0, \forall r$

Suppose if some index $\hat{r}$, $\hat{r} \in \{1, \dots, k\}$, $c^T d^{\hat{r}} > 0$

Then we can choose the corresponding multiplier $\mu_{\hat{r}} > 0$ sufficiently large such that $\mu_{\hat{r}} c^T d^{\hat{r}} \to +\infty$. In that case LPP is unbounded.

$\therefore$ If LPP has an optimal solution then we must have

$c^T d^r \leq 0, \ \forall r = 1, \dots, k$

$\therefore \sum_{r=1}^{k} \mu_r c^T d^r \leq 0, \ \forall \mu_r \geq 0.$

$\therefore \max c^T x = \max \sum_{j=1}^{p} \lambda_j c^T x^j$

subject to $\lambda_j \geq 0, \forall j, \ \sum_{j=1}^{p} \lambda_j = 1$

And, now we follow the same approach as in previous case, we find

$$\max\{c^T x^1, c^T x^2, \ldots, c^T x^p\}$$
$$\equiv c^T x^l \quad \text{(say attained at index } l\text{)}$$

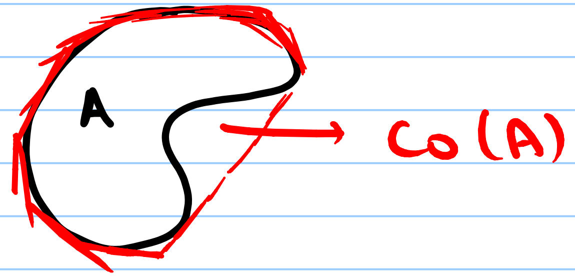
We choose $\lambda^l = 1$ and all other λ's $= 0$

$\therefore$ The optimal max of LPP is attained at an extreme point x^l of S and optimal value is finite.

This proves the result.

Let $A \subseteq \mathbb{R}^n$. We define a convex hull of A as the smallest convex set containing the set A.

eg.

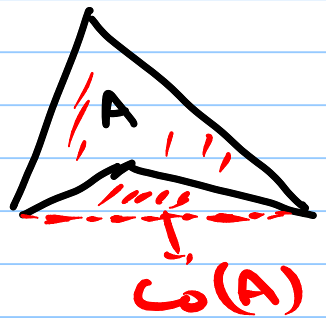


Co(A) : convex hull of the set A

(i) $A \subseteq Co(A)$

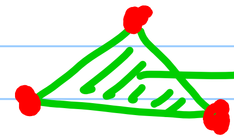
(ii) $Co(A)$ is a convex set

(iii) If $B \subseteq \mathbb{R}^n$ is any other convex set containing A, then $Co(A) \subseteq B$



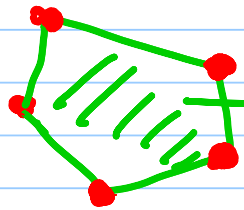
Result: If $A = \{v^1, v^2, \ldots, v^k\} \subset \mathbb{R}^n$, then

$$co(A) = \left\{ v \in \mathbb{R}^n : \exists \text{ scalars } \lambda_i \geq 0,\ i = 1, 2, \ldots, k,\ \sum_{i=1}^{k} \lambda_i = 1 \text{ and } v = \sum_{i=1}^{k} \lambda_i v^i \right\}$$



Proof: Let the RHS set be D

$$D = \left\{ v \in \mathbb{R}^n : v = \sum_{i=1}^{k} \lambda_i v^i,\ \lambda_i \geq 0\ \forall i,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}$$



(i) $A \subset D$

$\because$ any $v^p \in A$, we can take $\lambda_p = 1$, $\lambda_i = 0$, $\forall\ i = 1, \ldots, k$, $i \neq p$

and $v^p = \sum_{i=1}^{k} \lambda_i v^i \Rightarrow v^p \in S.$

$\therefore A \subset D.$

(ii) D is a convex set

Let $v \in S$, $w \in S$

$\Rightarrow v = \sum_{i=1}^{k} \lambda_i v^i$, $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$

& $w = \sum_{i=1}^{k} \mu_i v^i$, $\mu_i \geq 0$, $\sum_{i=1}^{k} \mu_i = 1$

Let $\eta \in [0,1]$.

Then, $(1-\eta) v + \eta w$

$$= \sum_{i=1}^{k} \left((1-\eta) d_i + \eta \mu_i \right) v^i$$

And

$$(1-\eta) d_i + \eta \mu_i \geq 0 \quad \forall \ i = 1, 2, \ldots, k$$

with

$$\sum_{i=1}^{k} \left((1-\eta) d_i + \eta \mu_i \right) = (1-\eta) \sum_{i=1}^{k} d_i + \eta \sum_{i=1}^{k} \mu_i = 1$$

$$\therefore \ (1-\eta) v + \eta w \in D, \quad \forall \ \eta \in [0,1], \ v, w \in D.$$

$\Rightarrow D$ is a convex set

But $co(A)$ is the smallest convex set containing A

So, $Co(A) \subseteq D$ —— 1.

To show equality b/w $Co(A)$ and S, we need to show that
$$D \subseteq Co(A)$$

This, we will prove by induction on k.

Suppose $k=1$

Then $A = \{v^1\}$ and $S = \{v^1\}$

But $v^1 \in co(A)$ also, as $A \subseteq co(A)$

$\Rightarrow$ $D \subseteq co(A)$

Let $k=2$

$A = \{v^1, v^2\}$

$D = \{ v : v = (1-\alpha)v^1 + \alpha v^2, \ \alpha \in [0,1] \}$

Now $A \subseteq Co(A) \Rightarrow v^1, v^2 \in Co(A)$ and $Co(A)$ is a convex set by definition, so $(1-\alpha)v^1 + \alpha v^2 \in Co(A)$ $\forall \ \alpha \in [0,1] \Rightarrow D \subseteq Co(A)$

Let us assume that the result holds for $k = m-1$ i.e. when $k = m-1$, $D \subseteq Co(A)$.

Take $k=m$ now $\Rightarrow A = \{v^1, v^2, --, v^{m-1}, v^m\}$

Let $v \in D$

$\Rightarrow \exists\ \lambda_i \geq 0,\ i = 1, \ldots, m$ such that

$$v = \sum_{i=1}^{m} \lambda_i v^i, \qquad \sum_{i=1}^{m} \lambda^i = 1$$

$$= \sum_{i=1}^{m-1} \lambda_i v^i + \lambda_m v^m$$

If $\lambda_m = 0$ then $\sum_{i=1}^{m} \lambda^i = 1 \Rightarrow \sum_{i=1}^{m-1} \lambda^i = 1$ and

$v = \sum_{i=1}^{m-1} \lambda_i v^i \in Co(A)$ by induction hypothesis

If $\lambda_m = 1$ then $v = v^m \in co(A)$ as $A \subseteq co(A)$

Let $0 < \lambda_m < 1$

$$\sum_{i=1}^{m-1} \frac{\lambda_i}{1-\lambda_m} v^i = \sum_{i=1}^{m-1} \mu_i v^i , \quad \mu_i = \frac{\lambda_i}{1-\lambda_m} \geq 0$$

and $\displaystyle\sum_{i=1}^{m-1} \mu_i = \frac{1}{1-\lambda_m} \sum_{i=1}^{m-1} \lambda_i = \frac{1-\lambda_m}{1-\lambda_m} = 1$

$\therefore \displaystyle\sum_{i=1}^{m-1} \mu_i v^i \in co(A)$ by induction hypothesis

$$\Rightarrow \frac{v - \lambda_m v^m}{1-\lambda_m} \in co(A).$$

And $v^m \in Co(A)$

Since $Co(A)$ is a convex set so

$$(1-\alpha_m)\left(\frac{v-\alpha_m v^m}{1-\alpha_m}\right)+\alpha_m v^m \in Co(A)$$

as $\alpha_m \in (0,1)$

$\Rightarrow v \in Co(A)$

In any case, for any $v \in S$, we showed $v \in Co(A)$

$\therefore D \subseteq Co(A)$ — (2)

(1) & (2) $\Rightarrow$ $\underline{Co(A) = D}$

- In a LPP

→ The feasible set empty ⇒ (P) is infeasible

→ The feasible set (nonempty) is a polyhedron → bounded / → unbounded

→ The feasible set is closed & convex set having a finite no. of extreme points and extreme directions.

→ The problem may be unbounded.

→ Local optimal of (P) is its global optimal.

→ Optimal value is attained at at least one of the extreme point of the feasible set. Interior pt is not an opt. soln.

Basic feasible solution

We realize that if LPP has an optimal soln then it must be at the "corner" or "extreme" point of the feasible set S of LPP.

Extreme point is a geometrical concept.

We translate it into an algebraic concept so that we can compute extreme points of S in any dimension n with utmost ease. For this, we define BFS.

→ Consider a system of linear equations

$Ax = b$

$A : m \times n, \quad x : n \times 1, \quad b : m \times 1$

Typically $m < n$. We assume $\operatorname{rank}(A) = m$.

$\Rightarrow \exists$ a set of m LI columns of A (we call these columns as basic columns).

We extract these m columns (such a choice of columns may not be unique) from A and this submatrix we denote by $B_{m \times m}$ with $|B| \neq 0$. Such a matrix B is called a basis matrix.

So $Ax = b \iff \begin{bmatrix} \overset{m\times m}{B} & \overset{m\times(n-m)}{R} \end{bmatrix} \begin{bmatrix} x_B \\ x_R \end{bmatrix} = b$, x_B: $m \times 1$, x_R: $(n-m)\times 1$

x_B: variables corresponding to the columns of B called basic variables

eg. $\begin{array}{l} 2x_1 + 3x_2 + 4x_3 = 1 \\ -x_1 - 2x_2 + x_3 = 2 \end{array} \Rightarrow \underset{A}{\begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

rank $(A) = 2$

Let $B = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$ $|B| \neq 0$.

column 1 is not included in B. so $x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$

and $x_R = [x_1]$ and the given system can be rewritten

$$\begin{array}{c} \overbrace{\hphantom{3 \quad 4}}^{B} \quad \overbrace{\hphantom{2}}^{R} \\ \begin{bmatrix} 3 & 4 & 2 \\ -2 & 1 & -1 \end{bmatrix} \end{array} \begin{bmatrix} x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \\ x_R = x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now, we put $x_R = 0$ ($n-m$ independent variables)

we get $Bx_B = b \Rightarrow \underline{x_B = B^{-1}b}$

$\therefore \quad x = \begin{pmatrix} x_B = B^{-1}b \\ x_R = 0 \end{pmatrix}$: basic solution

If we have a system

$$AX = b, \quad X \geq 0$$

we first identify B, put $x_R = 0$, compute $x_B = B^{-1}b$. If $x_B \geq 0$, then $x = \begin{pmatrix} x_B \\ x_R = 0 \end{pmatrix}$ is called a Basic Feasible soln. (BFS)

Since in A we have n columns and we have to choose m columns out of A to construct B, the maximum number of BFS of $Ax = b$, $x \geq 0$ is ${}^nC_m$. (Some of the B's submatrix of A may not be invertible so not basic).

Further, a BFS is classified into two categories:

1) Degenerate BFS
$\downarrow$
At least one of the basic variable takes value 0

2) Non-degenerate BFS
$\downarrow$
all basic variables $x_B > 0$.

We shall see the significance of this classification later on.

eg. $x_1 + 4x_2 + x_3 = 8$

$x_1 + 2x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0$

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$x \geq 0$

i) $B = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$, $x_3 = 0$, $x_4 = 0$, $x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\therefore \quad x = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ degenerate BFS

ii) $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $x_2 = 0, x_4 = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = -1\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = x_B$

$x = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix}$ non-degenerate BFS

iii) $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $x_2 = 0$, $x_3 = 0$, $\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} = x_B$

not a BFS $\because$ $x_4 < 0$

iv) $B = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}$, $x_1 = 0$, $x_4 = 0$, $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$x = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ denesagte BFS (same case as i))

v) $B = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$, $x_1 = 0$, $x_3 = 0$, $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = x_B$

same as case i)

vi) $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x_1 = 0$, $x_2 = 0$, $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = x_B \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 4 \end{pmatrix}$ non deg. BFS

A total 3 BFS $\begin{pmatrix}0\\2\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\8\\4\end{pmatrix}$, $\begin{pmatrix}4\\0\\4\\0\end{pmatrix}$

& $\begin{pmatrix}0\\2\\0\\0\end{pmatrix}$ is a degenerate BFS while other two are non-degenerate BFS.

We next relate BFS to the extreme point of the feasible set

$$S = \{x \in \mathbb{R}^n : Ax \leq b, \ x \geq 0\}$$

Note that in S we are dealing with inequalities $Ax \leq b$ while in BFS concept we work with $Ax = b$.

So, we can relate BFS with extreme pt of S if we first convert the system $Ax \leq b$ to the system $Ax = b$

Slack and Surplus variables.

Consider the inequality $x_1 - 2x_2 + x_3 \leq 7$.

We introduce a variable $x_4 \geq 0$ called slack variable and write $x_1 - 2x_2 + x_3 + x_4 = 7$

Similarly, consider the inequality $2x_1 - 3x_2 + 4x_3 \geq 5$

we can introduce a variable $x_4 \geq 0$ called surplus variable such that $2x_1 - 3x_2 + 4x_3 - x_4 = 5$

Therefore, any $\leq$ or $\geq$ constraints in the system can be converted into $=$ constraints by introducing additional variables called slack and surplus variables. These new variables are non-negative.

eg.

$$
\left.\begin{array}{rl}
x_1 + x_2 - x_3 & \leq 5 \\
3x_1 - x_2 + 2x_3 & \geq 2 \\
2x_1 - x_2 & = 3 \\
x_1, x_2, x_3 & \geq 0
\end{array}\right\}
\quad
\begin{array}{rl}
x_1 + x_2 - x_3 + \overset{\text{slack}}{x_4} & = 5 \\
3x_1 - x_2 + 2x_3 - \overset{\text{surplus}}{x_5} & = 2 \\
2x_1 - x_2 & = 3 \\
x_1, x_2, x_3, \underline{x_4, x_5} & \geq 0
\end{array}
$$

We can thus relate extreme pts of $Ax \leq b$, $x \geq 0$ and BFS of $\hat{A}\hat{x} = b$, $\hat{x} \geq 0$

eg

$$x_1 + x_2 \geq 1$$
$$-x_1 + 6x_2 \leq 3$$
$$x_1 \leq 2$$
$$x_1, x_2 \geq 0$$

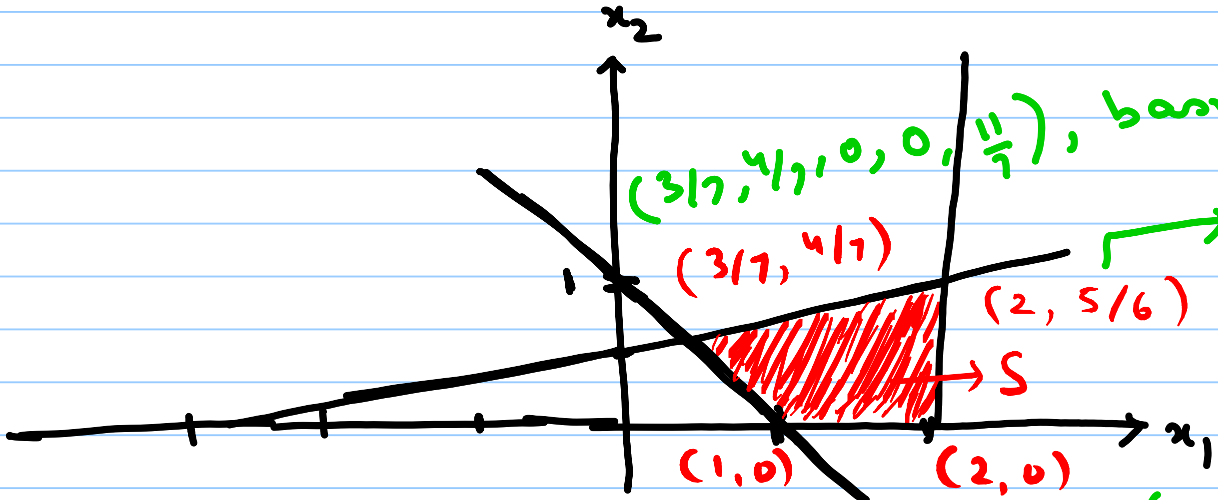
$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ -x_1 + 6x_2 + x_4 &= 3 \\ x_1 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}\right\}$$

$Ax = b$

$A: 3 \times 5$

max Basic solns

$= {}^5C_3 = 10$



$(3/7, 4/7, 0, 0, 11/7)$, basic variables x_1, x_2, x_5

$(2, \frac{5}{6}, \frac{11}{6}, 0, 0)$ BFS basic variables x_1, x_2, x_3

$(1, 0, 0, 4, 1)$ BFS, basic variables x_1, x_4, x_5

$(2, 0, 1, 5, 0)$ BFS basic variables x_1, x_3, x_4

Let us consider $\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$ or $\overbrace{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 6 & 1 \\ 1 & 0 & 0 \end{bmatrix}}^{B} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$x_B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -6 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 11 \end{bmatrix}$$ not a BFS $\quad |B| = 1$

Similarly, consider $\begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$ or $\overbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}}^{B} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$\Rightarrow x_B = -1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 5 \end{bmatrix}$$ not a BFS $\quad |B| = -1.$

$\therefore$ In 10 basic solutions possible, we have only 4 distinct BFS.

Result: Every extreme point of the feasible set of an LPP

$\max\ z = c^T x$

subject to $x \in S = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$

is a BFS of the system $Ax = b, x \geq 0$; and vice-versa.

Proof: Let x be a BFS of the system (or LPP)

i.e., $\exists$ a submatrix B of A, B is invertible, such that $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$, $x_B = B^{-1} b \in \mathbb{R}^m$, and $x_B \geq 0$.

To the contrary, let x be not an extreme point of S.

$\Rightarrow \exists\ x^1, x^2 \in S$, $x^1 \neq x^2$, and some $\hat{\lambda} \in (0,1)$ such that

$$x = (1-\hat{\lambda})x^1 + \hat{\lambda}x^2$$

$$\Rightarrow \begin{bmatrix} x_B \\ 0 \end{bmatrix} = (1-\hat{\lambda})\begin{bmatrix} u^1 \\ v^1 \end{bmatrix} + \hat{\lambda}\begin{bmatrix} u^2 \\ v^2 \end{bmatrix}, \text{ where } \begin{array}{l} u^1, u^2 \in \mathbb{R}^m \\ v^1, v^2 \in \mathbb{R}^{n-m} \end{array}$$

and u^1, u^2 correspond to the vector values in x^1, x^2, respectively

$$\Rightarrow (1-\hat{\lambda})v^1 + \hat{\lambda}v^2 = 0 \Rightarrow v^1 = 0 = v^2 \left(\because \begin{array}{l} \hat{\lambda} \in (0,1) \\ x^1, x^2 \geq 0 \end{array} \right)$$

$$\Rightarrow x^1 = \begin{bmatrix} u^1 \\ 0 \end{bmatrix}, \; x^2 = \begin{bmatrix} u^2 \\ 0 \end{bmatrix}$$

and $u^1, u^2 \geq 0$, $Bu^1 = Bu^2 = b$ ($\because x^1, x^2 \in S$)

$\Rightarrow u^1 = u^2$ ($\because B$ is invertible)

$\Rightarrow \quad x^1 = x^2$. But this is wrong

$\therefore$ x is an extreme point of S.

Conversely : Let $x \in S$ be an extreme pt of S.

Suppose the first k- component in x are non-zero, $k \le n$

That is, $x = (x_1, x_2, ---, x_k, 0, 0, --, 0)^t$ and

$$\sum_{i=1}^{k} a_i x_i = b, \quad x_i > 0 \quad \forall\, i \qquad \text{---} (1)$$

column vector of A

We claim that the set $\{a_1, a_2, --, a_k\}$ is LI in $\mathbb{R}^m$

Suppose not $\Rightarrow \{a_1, a_2, --, a_k\}$ is LD set in $\mathbb{R}^m$

$\Rightarrow \exists$ scalars $\lambda_1, \lambda_2, --\, \lambda_k$, not all zero, such that

$$\sum_{i=1}^{k} \lambda_i a_i = 0. \quad \text{--- } (2)$$

Define $\mu = \min\left\{ \frac{x_i}{|\lambda_i|} : \lambda_i \neq 0 \right\} > 0$, and choose any ε, $0 < \varepsilon < \mu$.

Set

$$\left.\begin{array}{l} \hat{x}_i = x_i + \varepsilon \lambda_i \ , \ i = 1, 2, \ldots k \\ \tilde{x}_i = x_i - \varepsilon \lambda_i \ , \ i = 1, 2, \ldots k \end{array}\right\} ; \ \hat{x}_i , \tilde{x}_i > 0 , \forall i$$

Take a vector $\hat{x} \in \mathbb{R}^n$ and $\tilde{x} \in \mathbb{R}^n$ such that

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k, 0, \ldots 0)^t \geq 0$$

$$\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k, 0, \ldots 0)^t \geq 0$$

Notice $x = \frac{1}{2}(\hat{x} + \tilde{x})$, $\hat{x} \neq \tilde{x}$, $\hat{x}, \tilde{x} \geq 0$

Also, $A\hat{x} = b$, $A\tilde{x} = b$ (by 1 and 2)

$\Rightarrow \hat{x}, \tilde{x} \in S$

This contradicts that x is an extreme point of S.

$\Rightarrow \{a_1, a_2, \ldots, a_k\}$ is a LI set in $\mathbb{R}^m$

$\Rightarrow k \leq m$

If $k = m$ then we can take a basis matrix

$B = [a_1\ a_2 \cdots a_m]$ and $Bx_B = b$, $x_B = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} > 0$

$\therefore$ x is a BFS of the system

If $k < m$ then as rank $(A) = m$, we can extend the set of columns of $A = \{a_1, a_2, \ldots, a_k\}$ by adding $k-m$ columns from matrix A to form an $m \times m$ basis matrix B.

$$B = [\, a_1 \quad a_2 \quad \cdots \quad a_k \quad \overbrace{\dot{a}_{k+1} \quad \dot{a}_{k+2} \quad \cdots \quad \dot{a}_m}^{\text{additional columns}} \,]$$

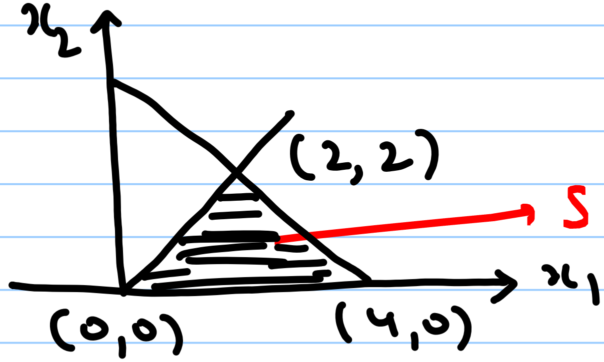
$|B| \neq 0$, and $Bx_B = b$, $x_B = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\} \rightarrow k-m$

$\therefore$ x is a BFS.

Note: In case $k < m$, we have degenerate BFS, and more than one basis corresponds to the extreme pt of S.

eg. max $x_1 + x_2$

subject to $x_1 - x_2 \geq 0$, $x_1 + x_2 \leq 4$, $x_1, x_2 \geq 0$



surplus variable

$$x_1 - x_2 - x_3 = 0$$

slack variable

$$x_1 + x_2 + x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

We can check that corresponding to the extreme point (0, 0) of S, we have 3 basis matrices, with basic variables $B_1 \leftarrow \{x_3, x_4\}$, $B_2 \leftarrow \{x_1, x_4\}$, $B_3 \leftarrow \{x_2, x_4\}$.
This is because $(0, 0, 0, 4)^t$ is a degenerate BFS.

Also, if we take $x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$, $x_1 = 0$, $x_4 = 0$, then the system becomes

$$x_2 + x_3 = 0, \quad x_2 = 4, \quad x_2, x_3 \geq 0,$$

inconsistent.

$\therefore$ it does not yield BFS.

The other two basis $\{x_1, x_2\}$ and $\{x_1, x_3\}$, yield

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_1 + x_2 &= 4 \end{aligned}\right\} \Rightarrow (2,2) \qquad \left.\begin{aligned} x_1 - x_3 &= 0 \\ x_1 &= 4 \end{aligned}\right] \Rightarrow (4,0)$$

and BFS $(2, 2, 0, 0)$ and BFS $(4, 0, 4, 0)$

In this way, all maximum 6 BFS of the system are analysed, and related to extreme pt of set S.

Adjacent extreme points

Two extreme points x^1 and x^2 of a convex polyhedron $S \subseteq \mathbb{R}^n$ are said to be adjacent iff every point $\hat{x}$ on the line segment joining them has a unique expression of extreme points of $S = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$

In other words, x^1 and x^2 are adjacent extreme point of S iff the set of the column vectors $\{a_j : j$ such that either x_j^1 or x_j^2 or both are greater than zero$\}$ is one less than its cardinality (i.e. $n-1$)

Fundamental thm of LPP

Does the LPP always have a BFS?

→ If the system $Ax = b$, $x \geq 0$, A is $m \times n$, rank$(A) = m$, has a feasible solution then it also have a BFS.

<u>Proof:</u> Let $x = (x_1, ---, x_n)^t$ be a feasible soln of the given system. Suppose k components of x are non-zero and rest $n-k$ are zero; $k \leq n$. Let

$$x = (x_1, x_2, --, x_k, 0, 0, --0)^t$$

Now, $Ax = b \Rightarrow \sum_{i=1}^{k} a_i x_i = b$, $x_i > 0$, $\forall i$ — (1)

If $\{a_1, \ldots, a_k\}$ is LI set then x is already a BFS.

Let $\{a_1, \ldots, a_k\}$ be LD.

$\Rightarrow \exists\ \lambda_i$ such that $\sum_{i=1}^{k} \lambda_i a_i = 0$ and not all λ_i's $= 0$

Suppose $\lambda_r > 0$. Then,

$$a_r = -\sum_{\substack{i=1 \\ i \neq r}}^{k} \frac{\lambda_i}{\lambda_r} a_i$$

Use it in ①, and simplify

$$\sum_{\substack{i=1 \\ i \neq r}}^{k} \left(x_i - \frac{\lambda_i}{\lambda_r} x_r\right) a_i = b \Rightarrow \sum_{\substack{i=1 \\ i \neq r}}^{k} a_i \hat{x}_i = b \quad \text{— ②}$$

where $\hat{x}_i = x_i - \lambda_i/\lambda_r \, x_r$.

Now, $\hat{x} = (\hat{x}_1, \dots \hat{x}_{r-1}, 0, \hat{x}_{r+1}, \dots \hat{x}_k, 0, \dots 0)^t$ is a soln of the system $A\hat{x} = b$.

Next, $\hat{x}_i = x_i - \frac{d_i}{d_r} x_r$, $i = 1, \dots, k$, $i \neq r$

If $\lambda_i \leq 0$, $(\lambda_r > 0, x_i > 0, x_r > 0)$, then $\hat{x}_i > 0$

and if $\lambda_i > 0$ then $\hat{x}_i > 0 \Leftrightarrow \frac{x_i}{\lambda_i} > \frac{x_r}{\lambda_r}$

or $\frac{x_r}{\lambda_r} = \min \left\{ \frac{x_i}{\lambda_i} : \lambda_i > 0 \right\}$.

With such a choice of index r, we can ensure $\hat{x} \geq 0$.

Notice $\hat{x}$ has $k-1$ components > 0, and it is a feasible soln of $Ax = b$, $x \geq 0$.

If these $k-1$ columns of A are LI then $\hat{x}$ is BFS else we can proceed on the same argument to get $\hat{\hat{x}}$ with $k-2$ positive components and $\hat{\hat{x}}$ is feasible for $Ax=b, x\geq 0$

Starting from feasible x, with k +ve components, in at most $k-1$ iterations we can obtain a BFS of $Ax=b, x\geq 0$.

eg Consider the system

$$\left.\begin{aligned} 2x_1+x_2+4x_3 &= 11 \\ 3x_1+x_2+5x_3 &= 14 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}\right\}$$

$x = (x_1=2, x_2=3, x_3=1)^t$ is a feasible soln of the system.

Note the columns of A associated with $x_i>0$ are LD.

$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$ is LD.

Can check $1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 0$, $\quad \underbrace{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1}_{*}$

Choose an index r such that

$$\frac{x_r}{\lambda_r} = \min \left\{ \frac{x_i}{\lambda_i} : \lambda_i > 0 \right\} = \min \left\{ \frac{x_1}{\lambda_1} = 2, \frac{x_2}{\lambda_2} = \frac{3}{2} \right\}$$

$$= \frac{x_2}{\lambda_2} \Rightarrow r = 2$$

$\Rightarrow a_2 = -\frac{1}{2} a_1 + \frac{1}{2} a_3$ (from $*$)

$\therefore a_1 x_1 + a_2 x_2 + a_3 x_3 = b \Rightarrow \frac{1}{2} a_1 + \frac{5}{2} a_3 = b$, so $\hat{x} = \left(\frac{1}{2}, 0, \frac{5}{2} \right)^t$

Now, $\hat{x}_1 > 0$, $\hat{x}_3 > 0$ and the corresponding columns of A are

$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$ is an LI set

$\therefore$ $\hat{x}$ is a BFS, $\hat{x}_B = \begin{bmatrix} x_1 = 1/2 \\ x_3 = 5/2 \end{bmatrix}$, $\hat{x}_R = [x_2 = 0]$

and $B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.

LPP in the standard form

(P) $\max \ z = c^T x$

s.to. $Ax \ (\leq, \geq, =) \ b$

$x \geq 0$

- First ensure $b \geq 0$.
  In case if any $b_j < 0$ then multiply the constraint by -1 and change the inequality, like,
  $$3x_1 - 2x_2 \leq -1 \iff -3x_1 + 2x_2 \geq 1.$$
- Next use the non-negative slack and surplus variables s_j to convert all inequalities into equation. Associate the cost $c_j = 0$ with the additional variables s_j in the objective function.
- Next ensure $x_i \geq 0 \ \forall \ i = 1, \dots, n$. If any x_i is allowed to be unrestricted in sign then use the transformation

$x_i = x_i' - x_i''$, $x_i' \geq 0$, $x_i'' \geq 0$, in the entire LPP.

eg. max $3x_1 + 2x_2 - x_3$

s. to $5x_1 + 2x_2 - x_3 = 15$

$2x_1 + 4x_2 + 5x_3 \geq 10$

$3x_1 - x_2 + 7x_3 \leq 2$

$x_1 \; x_2 \geq 0$

max $3x_1 + 2x_2 - x_3 + 0\,s_1 + 0\,s_2$

s. to. $5x_1 + 2x_2 - x_3 = 15$

$2x_1 + 4x_2 + 5x_3 - s_1 = 10$

$3x_1 - x_2 + 7x_3 + s_2 = 2$

$x_1, x_2, s_1, s_2 \geq 0$

Replace $x_3 = x_3' - x_3''$

$$\max \quad 3x_1 + 2x_2 - x_3' + x_4'' + 0\, s_1 + 0\, s_2$$

$$\text{s.to} \quad 5x_1 + 2x_2 - x_3' + x_4'' = 15$$

$$2x_1 + 4x_2 + 5x_3' - 5x_3'' - s_1 = 10$$

$$3x_1 - x_2 + 7x_3' - 7x_3'' + s_2 = 2$$

$$x_1, x_2, x_3', x_3'', s_1, s_2 \geq 0.$$

$\therefore$ After suitable adjustments we can always ensure, RHS in constraints ≥ 0, all constraints are $=$, all variables are ≥ 0.

We need answers of the following:

- How to find the initial BFS
- How to decide when to stop the algorithm
- How to obtain a new BFS from the current BFS to improve the objective value.

$\rightarrow$ To find initial BFS, we need a starting square submatrix B of A, $|B| \neq 0$. Then, we can find $x_B = B^{-1}b$, and check if $x_B \geq 0$; else we have to try with the another B matrix.

To ease out this choice of B, we always makes sure that the identity matrix I of order $m \times m$ is present in the system $Ax = b, x \geq 0$. In that case, to start the algo, we take

$B = I$ and then $x_B = B^{-1} b = b$ and we have already made sure that $b \geq 0$ $\therefore$ $x_B \geq 0$ is guarenteed.

If the I submatrix is already in $AX = b$, $X \geq 0$, then we have nothing to worry; however if some or all columns of I are absent then we add "artificial variables" in the system to bring in I matrix. Also, these artificial variables are non-negative. And we wish that these variables should not appear at +ve level in the last iteration of the algo so to get rid off them we associate heavy penalty in the cost coeff. say $-M$ with them in the objective fn; $M >> 0$.

eg max $z = 3x_1 + 8x_2$

subject to $3x_1 - 5x_2 \geq -10 \rightarrow -3x_1 + 5x_2 \leq 10$

$2x_1 - x_2 - x_3 \leq 20$

$x_1 + 2x_2 \leq 15$

$x_1, x_2 \geq 0$

max $z = 3x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$

subject to $-3x_1 + 5x_2 + s_1 = 10$

$2x_1 - x_2 - x_3 + s_2 = 20$

$x_1 + 2x_2 + s_3 = 15$

$x_1, x_2 \geq 0, s_1, s_2, s_3 \geq 0$

Here, we can take $x_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ and $B = I$, as initial BFS

Note, $x_B = b = \begin{bmatrix} 10 \\ 20 \\ 15 \end{bmatrix} \geq 0$.

However, if we change the second and third constraints to $\geq$ sign, and consider the LPP

$$\left.\begin{aligned} &\max\ z = 3x_1 + 8x_2 \\ &\text{subject to} \quad -3x_1 + 5x_2 \leq 10 \\ &\qquad 2x_1 - x_2 - x_3 \geq 20 \\ &\qquad x_1 + 2x_2 \geq 15 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} &\max\ 3x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3 \underline{- Ma_1 - Ma_2} \\ &\text{subject to} \\ &\quad -3x_1 + 5x_2 + s_1 = 10 \\ &\quad 2x_1 - x_2 - x_3 - s_2 + \underline{a_1} = 20 \\ &\quad x_1 + 2x_2 - s_3 + \underline{a_2} = 15 \\ &\quad x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0 \end{aligned}$$

We can now take $B = I$, $x_B = \begin{bmatrix} s_1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 15 \end{bmatrix} \geq 0$.

Here, $M >> 0$.

Note in this case we have forced I into the system by using the artificial variables a_1 and a_2.

We now see <u>how to obtain the new BFS</u> from the current BFS to improve the objective value.

Let B be the current basis and $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ be the current BFS.

Of the LPP $\max z = c^T x$ subject to $Ax = b$
$x \geq 0$

where A is $m \times n$ and $\operatorname{rank}(A) = m$.

Let $B = [\, d_1 \;\; d_2 \; \cdots \; d_m \,]$ where $d_i = a_\ell$, some column of A.

Suppose $\exists$ a column a_j of A which is not present in B.

$\Rightarrow$ $\exists$ scalars y_{ij} such that

$$a_j = \sum_{i=1}^{m} y_{ij}\, d_i \,. \qquad \left(\Rightarrow \; a_j = B y_j \Leftrightarrow y_j = B^{-1} a_j \right)$$

For this moment let some scalar $y_{rj} > 0$. Then,

$$d_r = \sum_{\substack{i=1 \\ i \neq r}}^{m} - \frac{y_{ij}}{y_{rj}}\, d_i + \frac{1}{y_{rj}}\, a_j \,.$$

Since, $Bx_B = b$ or $\sum_{i=1}^{m} x_{B_i} d_i = b$, we get

$$\sum_{i=1}^{m} x_{B_i} d_i + x_{B_r}\left(-\sum_{\substack{i=1\\ i\neq r}}^{m} \frac{y_{ij}}{y_{rj}} d_i + \frac{1}{y_{rj}} a_j\right) = b.$$

Simplifying

$$\sum_{\substack{i=1\\ i\neq r}}^{m} \hat{x}_{B_i} d_i + \hat{x}_{B_r} a_j = b, \quad \text{— } (1)$$

where, $\hat{x}_{B_r} = \dfrac{x_{B_r}}{y_{rj}}$ and $\hat{x}_{B_i} = x_{B_i} - \dfrac{y_{ij}}{y_{rj}} x_{B_r}$, $i \neq r$.

Take a new $\hat{B} = [d_1 \,---, d_{r-1}, \underline{a_j}, d_{r+1} \,---, d_m]$.

By ①, $\sum_{i=1}^{m} \hat{x}_{B_i} \hat{d}_i = b$, $\hat{d}_i = d_i \ \forall \ i \neq r$ and $\hat{d}_r = a_j$

Also, $\hat{x}_{B_i} \geq 0$ iff $\dfrac{x_{B_i}}{y_{ij}} \geq \dfrac{x_{B_r}}{y_{rj}}$ for all those indices i for which $y_{ij} > 0$.

that is, we choose index r such that

$$\frac{x_{B_r}}{y_{rj}} = \min \left\{ \frac{x_{B_i}}{y_{ij}} : y_{ij} > 0 \right\} \quad \text{— ②}$$

$\Rightarrow (\hat{x}_B, 0)$ is a new BFS with basis $\hat{B}$.

Let us compute objective value at this new BFS.

$$\hat{z} = c_{\hat{B}}^T \hat{x}_B = \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{B_i} \left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{Br} \right) + c_j \frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{B_i} \left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{Br} \right) + c_j \frac{x_{Br}}{y_{rj}}$$

$$= c_B^T x_B + \frac{x_{Br}}{y_{rj}} \left(c_j - c_B^T y_j \right)$$

$$= z + \frac{x_{Br}}{y_{rj}} \left(c_j - z_j \right) \qquad \underline{z_j = c_B^T y_j}$$

$\therefore\ \hat{z} > z$ provided $x_{Br} > 0$ and $z_j - c_j < 0$.

If $x_{Br} = 0$ or $z_j - c_j = 0$ then $\hat{z} = z$.

$\therefore$ In the absence of degeneracy, the objective value $\hat{z}$ is improved on the current objective value.

Theorem: If for some index j, $z_j - c_j < 0$ and for that j, $\exists$ index i such that $y_{ij} > 0$, then a new BFS $(\hat{x}_{\hat{B}}, 0)^t$ can be obtained such that $z(\hat{x}_{\hat{B}}) \geq z(x_B)$, where $z(\hat{x}_{\hat{B}})$ and $z(x_B)$ are the objective fn. values of LPP at new BFS $(\hat{x}_{\hat{B}}, 0)^t$ and the current BFS $(x_B, 0)^t$, respectively.

Note: Any index j for which $z_j - c_j < 0$ can be selected.

for improving the current BFS, the choice of j is not unique, yet it is a convention to choose the index j such that $z_j - c_j$ is the most negative, or

$$\max\{z_j - c_j : z_j - c_j < 0\}.$$

As long as we can choose such an index j and an index r by (2), we continue to improve the objective value, till we reach one of the following situations -

(i) $z_j - c_j \geq 0 \;\forall j$ (ii) $z_j - c_j < 0$ for atleast one j but $y_{ij} \leq 0 \;\forall i$

Also, recall in the above arguments we have assumed that atleast one of the scalar $y_{rj} > 0$ in the linear combination of column a_j of A not in B in terms of the basic columns of B.

So, we have to examine a case when $\exists$ j such that $z_j - c_j < 0$ and $y_{ij} \leq 0 \ \forall \ i$ (i.e. $y_{rj} > 0$ does not exist)

Theorem: Suppose $\exists$ a column a_j in A which is not in B (i.e., a_j is a column corresponding to a nonbasic variable x_j)

such that $z_j - c_j < 0$ but the corresponding $y_j = B^{-1} a_j$ has $y_{ij} \leq 0 \ \forall i = 1, \ldots, m$, then LPP is unbounded.

<u>Proof</u>: Let $(x_B, 0)^t$ be the BFS of LPP with basis matrix $B = [d_1 \ d_2 \ldots d_m]$.

$$\Rightarrow \quad \sum_{i=1}^{m} x_{B_i} d_i = b, \quad x_{B_i} \geq 0, \ \forall i = 1, 2, \ldots, m.$$

Let $\xi > 0$ be an arbitrary real number.

$$\sum_{i=1}^{m} x_{B_i} d_i - \xi a_j + \xi a_j = b.$$

Also, $a_j = B y_j = \sum_{i=1}^{m} y_{ij} d_i$

$$\Rightarrow \sum_{i=1}^{m} (x_{B_i} - \xi y_{ij}) d_i + \xi a_j = b$$

$$\Rightarrow \sum_{i=1}^{m} \tilde{x}_{B_i} d_i + \xi a_j = b, \quad \text{where, } \tilde{x}_{B_i} = x_{B_i} - \xi y_{ij} \geq 0 \quad \forall i.$$

$$\Rightarrow (\tilde{x}_B, \xi, \underbrace{0}_{n-(m+1)\times 1 \text{ vector}})^t \text{ is a feasible soln of } Ax = b, \ x \geq 0.$$

Now the objective value at this feasible soln is

$$\tilde{z} = \tilde{c}_B^T \tilde{x}_B = \sum_{i=1}^{m} c_{B_i} (x_{B_i} - \xi y_{ij}) + c_j \xi$$

$$= z - \xi (z_j - c_j)$$

$$\Rightarrow \tilde{z} - z = -\underbrace{\xi}_{>0}\underbrace{(z_j - c_j)}_{<0}$$

$\Rightarrow \tilde{z} >>> z$, in fact $\hat{z} - z \longrightarrow +\infty$ as $\xi \rightarrow +\infty$.

$\therefore$ while remaining in the feasible set, the objective value can be made as large as possible, leading to unbounded LPP.

<u>Note</u>: The proof of the theorem is constructive in the sense that if the given LPP is unbounded then given any value of z, say z_0, we can construct a feasible vector x_0 such that $z(x_0) = z_0$.

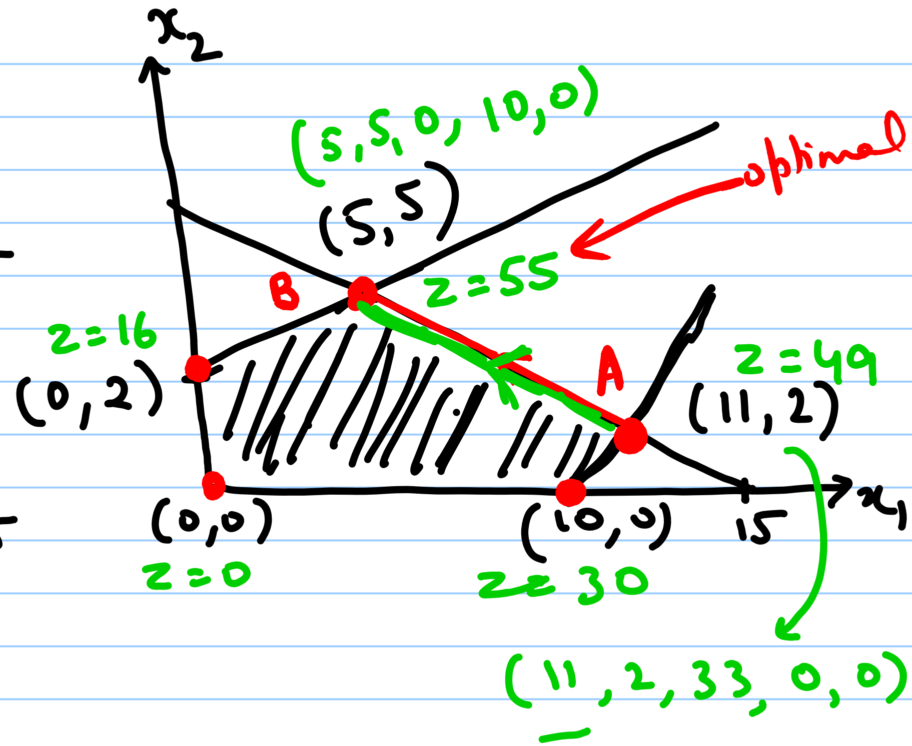
eg. $\max \; 3x_1 + 8x_2$

subject to
$$-3x_1 + 5x_2 \le 10$$
$$2x_1 - x_2 \le 20$$
$$x_1 + 2x_2 \le 15$$
$$x_1, x_2 \ge 0$$

Standard LPP

$$\max \; z = 3x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to
$$-3x_1 + 5x_2 + x_3 = 10$$
$$2x_1 - x_2 + x_4 = 20$$
$$x_1 + 2x_2 + x_5 = 15$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0$$



$\text{rank}(A) = 3 = m$

$n = 5$

Let $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so, $B = \begin{bmatrix} -3 & 5 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$, $\underbrace{x_4 = 0, \ x_5 = 0}_{\text{nonbasic}}$

$$x_B = B^{-1} b = \begin{bmatrix} 11 \\ 2 \\ 33 \end{bmatrix} \longleftrightarrow \text{extreme point } A(11, 2).$$

Now, a_4, a_5 are not in B.

$$z_4 - c_4 = c_B^t y_4 - c_4 = c_B^t B^{-1} a_4 - c_4, \quad c_B = (3, 8, 0)^t$$

$$= (3, 8, 0) \frac{1}{5} \begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = -\frac{2}{5}$$

$$z_5 - c_5 = c_B^t y_5 - c_5 = c_B^t B^{-1} a_5 - c_5 = \frac{19}{5}$$

Since, $z_4 - c_4 < 0$; so current BFS is not optimal.

$\Rightarrow$ x_4 must enters into the basis. Also, $y_4 = B^{-1} a_4 = \begin{pmatrix} 2/5 \\ -1/5 \\ 11/5 \end{pmatrix}$.

To find the outgoing variable, we use the minimum ratio criterion.

$$\frac{x_{Br}}{y_{r4}} = \min \left\{ \frac{x_{Bi}}{y_{i4}} : y_{i4} > 0 \right\}$$

not included in ratio

$$= \min \left\{ \frac{11}{2/5} , \frac{33}{11/5} \right\} = 15 \Rightarrow r = 3$$

$\Rightarrow$ x_3 leaves

$\therefore$ $x_{\hat{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix}$, $\hat{B} = \begin{bmatrix} -3 & 5 & 0 \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$. So, $x_{\hat{B}} = \hat{B}^{-1} b = \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix}$

Extreme point $(5, 5)$.

The columns of A not in $\hat{B}$ are a_3 and a_5

$\therefore$ we compute

$$z_3 - c_3 = c_{\hat{B}}^t \hat{B}^{-1} a_3 - c_3 = (3, 8, 0) \frac{1}{11} \begin{pmatrix} -2 & 0 & 5 \\ 1 & 0 & 3 \\ 5 & 11 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0 = \frac{2}{11}$$

$$z_5 - c_5 = c_{\hat{B}}^t \hat{B}^{-1} a_5 - c_5 = (3, 8, 0) \frac{1}{11} \begin{pmatrix} -2 & 0 & 5 \\ 1 & 0 & 3 \\ 5 & 11 & -7 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 = \frac{39}{11}$$

Since $z_j - c_j > 0$ $\forall$ index j : $a_j \notin \hat{B}$.

The current BFS yields optimal solution.

$\Rightarrow$ $x_1^* = 5$, $x_2^* = 5$ is an optimal soln of LPP.

The method is called the <u>Simplex</u> method eg

$$\max\ z = 3x_1 + 5x_2 + 4x_3$$

subject to

$$2x_1 + 3x_2 \le 8$$

$$3x_1 + 2x_2 + 4x_3 \le 15$$

$$2x_2 + 5x_3 \le 10$$

$$x_1, x_2, x_3 \geq 0$$

Use slack variables

$$\max\ z = 3x_1 + 5x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$$

subject to

$$2x_1 + 3x_2 + x_4 = 8$$

$$3x_1 + 2x_2 + 4x_3 + x_5 = 15$$

$$2x_2 + 5x_3 + x_6 = 10$$

$$x_i \geq 0, \ \forall\ i = 1, 2, ---, 6.$$

		$c_j \rightarrow$	3	5	4	0	0	0	
c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	ratio
0	x_4	8	2	3	0	1	0	0	$8/3 \longrightarrow$
0	x_5	15	3	2	4	0	1	0	$15/2$
0	x_6	10	0	2	5	0	0	1	5
		$z_j - c_j \rightarrow$	-3	-5	-4	0	0	0	
				$\uparrow$					

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	ratio
5	x_2	$8/3$	$2/3$	1	0	$1/3$	0	0	—
0	x_5	$29/3$	$5/3$	0	4	$-2/3$	1	0	$29/12$
0	x_6	$14/3$	$-4/3$	0	5	$-2/3$	0	1	$14/15 \longrightarrow$
		$z_j - c_j \rightarrow$	$1/3$	0	-4	$5/3$	0	0	
					$\uparrow$				

		$c_j \rightarrow$	3	5	4	0	0	0	
c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	ratio
5	x_2	8/3	2/3	1	0	1/3	0	0	4
0	x_5	89/15	41/15	0	0	−2/15	1	−4/15	89/41 $\rightarrow$
4	x_3	14/15	−4/15	0	1	−2/15	0	1/5	−
		$z_j - c_j \rightarrow$	$\frac{-11}{15}$ $\uparrow$	0	0	$\frac{17}{15}$	0	$\frac{4}{5}$	

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
5	x_2	50/41	0	1	0	15/41	−10/41	8/41
3	x_1	89/41	1	0	0	−2/41	15/41	−12/41
4	x_3	62/41	0	0	1	−6/41	4/41	5/41
		$z_j - c_j \rightarrow$	0	0	0	45/41	11/41	24/41

Since $z_j - c_j \geq 0 \ \forall j$, we reached the optimal soln.

$x_1^* = \frac{89}{41}$, $x_2^* = \frac{50}{41}$, $x_3^* = \frac{62}{41}$, and optimal value $z^* = \frac{765}{41}$

eg.

$$\begin{array}{llll}
\max & z = 2x_1 + 5x_2 - 4x_3 & \longrightarrow & 2x_1 + 5x_2 - 4x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7 \\
\text{subject to} & x_1 + 2x_2 + x_3 \leq 4 & \longrightarrow & x_4 + 2x_2 + x_3 + x_4 = 4 \\
 & -2x_1 + 3x_2 \geq 1 & \longrightarrow & -2x_1 + 3x_2 - x_5 + x_6 = 1 \\
 & 5x_1 + 4x_2 + 5x_3 = 7 & & 5x_1 + 4x_2 + 5x_3 + x_7 = 7 \\
 & x_1, x_2, x_3 \geq 0 & & x_i \geq 0 \ \forall \ i = 1, \dots, 7
\end{array}$$

$M >> 0$, x_6, x_7 are artificial variables.

C_B	V_B	$c_j \rightarrow$ / x_B	2 / y_1	5 / y_2	-4 / y_3	0 / y_4	0 / y_5	$-M$ / y_6	$-M$ / y_7	
0	x_4	4	1	2	-1	1	0	0	0	
$-M$	x_6	1	-2	3	0	0	-1	1	0	$\longrightarrow$
$-M$	x_7	7	5	4	5	0	0	0	1	
	$z_j - c_j \rightarrow$		$-3M-2$	$-7M-5$	$-5M+4$	0	M	0	0	
				$\uparrow$						

C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_7	
0	x_4	$10/3$	$7/3$	0	-1	1	$2/3$	0	
5	x_2	$1/3$	$-2/3$	1	0	0	$-1/3$	0	
$-M$	x_7	$17/3$	$23/3$	0	5	0	$4/3$	1	$\longrightarrow$
	$z_j - c_j \rightarrow$		$-\frac{23}{3}M - \frac{16}{3}$	0	$-5M+4$	0	$-\frac{4}{3}M - \frac{5}{3}$	0	
			$\uparrow$						

Continue in this way.

optimal soln is $x_1^* = 0$, $x_2^* = 7/4$, $x_3^* = 0$ and optimal value $z^* = \frac{35}{4}$.

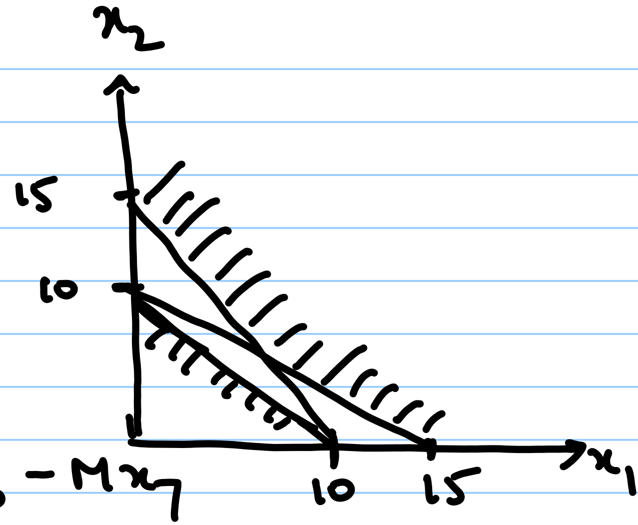
The method using M with artificial variables is called Big M- method.

We will next see examples of unbounded and infeasible LPP's. Note: If any of the artifical variable remains present in the final table of the simplex method with at +ve level then the LPP is infeasible.

eg

$$\min \; x_1 + x_2$$

subject to
$$3x_1 + 2x_2 \geq 30$$
$$2x_1 + 3x_2 \geq 30$$
$$x_1 + x_2 \leq 10$$
$$x_1, x_2 \geq 0$$



Infeasible LPP

$$\max \; -x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6 - Mx_7$$

subject to
$$3x_1 + 2x_2 - x_3 + x_6 = 30$$
$$2x_1 + 3x_2 - x_4 + x_7 = 30$$
$$x_1 + x_2 + x_5 = 10$$
$$x_i \geq 0 \;\; \forall\, i = 1, \ldots, 7 \;, \quad M >> 0$$

		$c_j \to$	-1	-1	0	0	0	$-M$	$-M$	ratio
C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	
$-M$	x_6	30	3	2	-1	0	0	1	0	$10 \to$
$-M$	x_7	30	2	3	0	-1	0	0	1	15
0	x_5	10	1	1	0	0	1	0	0	10
		$z_j - c_j \to$	$-5M+1$	$-5M+1$	M	M	0	0	0	

$\uparrow$

C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	ratio
-1	x_1	10	1	$2/3$	$-1/3$	0	0	$1/3$	0	15
$-M$	x_7	10	0	$5/3$	$2/3$	-1	0	$-2/3$	1	6
0	x_5	0	0	$1/3$	$1/3$	0	1	$-1/3$	0	$0 \to$
		$z_j - c_j \to$	0	$-\frac{5}{3}M + \frac{1}{3}$	$-\frac{2}{3}M + \frac{1}{3}$	M	0	$\frac{5}{3}M - \frac{1}{3}$	0	

$\uparrow$

		$c_j \rightarrow$	-1	-1	0	0	0	$-M$	$-M$
c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
-1	x_1	10	1	0	-1	0	-2	1	0
$-M$	x_7	10	0	0	-1	-1	-5	1	1
-1	x_2	0	0	1	1	0	3	-1	0
	$z_j - c_j \rightarrow$		0	0	M	M	$5M-1$	0	0

$z_j - c_j \geq 0 \quad \forall j$

But x_7 (artificial variable) is +ve in last basis.

$\Rightarrow$ LPP is infeasible.

Two Phase Method

Phase 1 : aim to get rid off artificial variables

Phase 2 : aim to optimize the objective fn.

Phase 1 : Instead of solving the original LPP, we modify the objective fn. and formulate a new objective fn seeking to maximize $-e^t x_a$ where $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ all 1's and x_a is the vector of artificial variables introduced in the LPP standard form. In other words, we associate cost coeff. -1 with each artificial variables.

$$\max\ z = 2x_1 - x_2 + x_3$$

subject to

$$x_1 + x_2 - 3x_3 \le 8 \quad \rightarrow \quad x_1 + x_2 - 3x_3 + x_4 = 8$$

$$4x_1 - x_2 + x_3 \geq 2 \quad \rightarrow \quad 4x_1 - x_2 + x_3 - x_5 + \underline{x_7} = 2$$

$$2x_1 + 3x_2 - x_3 \geq 4 \qquad 2x_1 + 3x_2 - x_3 - x_6 + \underline{x_8} = 4$$

$$x_1, x_2, x_3 \geq 0. \qquad x_i \geq 0 \ \forall i.$$

Phase 1 LPP is

$$\max\ -x_7 - x_8$$

subject to

$$x_1 + x_2 - 3x_3 + x_4 = 8$$

$$4x_1 - x_2 + x_3 - x_5 + x_7 = 2$$

$$2x_1 + 3x_2 - x_3 - x_6 + x_8 = 4$$

$$x_i \geq 0, \forall i..$$

Since each $x_a \geq 0$ so $-e^t x_a \leq 0$ for any feasible soln of Phase 1 LPP. $\therefore$ optimal value of Phase 1 LPP is ≤ 0.

(1) If optimal value $= 0 \Leftrightarrow x_a = 0$ $\forall$ artificial variables

(2) If optimal value $< 0 \Rightarrow \exists$ atleast one $x_{\tilde{a}} > 0$

In (2), we are not able to remove the artificial variable from the system indicating the original LPP is infeasible.

$\therefore$ Phase 2 is redundant.

In case (1), there are further two possible situations:

(1a) All artificial variables are non-basic in the final table of Phase 1 LPP. We have found a BFS of the original LPP and taking this BFS as initial BFS, we move to phase 2 of the method. We now take the original objective fn in Phase 2 to optimize, and recalculate $z_j - c_j$ and proceed with the simplex method.

(1b) One or more artificial variable is present in the basis at zero level in the final table of Phase 1 LPP. In this case we have found a feasible soln of the original LPP. we still move to phase 2, changing objective fn.

But in this case, we have to ensure that at no stage of the iterations, artificial variable becomes +ve. If we encounter such a situation then we deviate from the minimum ratio criterion to identify outgoing basic variable rather we force artificial variable to leave the basis.

c_B	v_B	x_B	y_1	----	y_k	----	y_i	----	y_p
c_{B_1}	v_{B_1}	x_{B_1}			y_{1k}		0	---	
$\vdots$	v_{B_r}	x_{B_r}			y_{rk}		$\vdots$	----	
0	$v_{B_i} = x_{a_i}$	0			y_{ik}		1	----	
$\vdots$	$\vdots$	$\vdots$			$\vdots$		$\vdots$		
c_{B_m}	v_{B_m}	x_{B_m}			y_{mk}		0	----	

$z_k - c_k < 0$

$y_{rk} > 0$

$y_{ik} < 0$

If we found $z_k - c_k < 0$ for some nonbasic variable index k, and then in y_k column vector we notice that $y_{ik} < 0$ (for artificial variable) but there is some other index $r \neq i$, for which $y_{rk} > 0$; and by minimum ratio criterion, x_{B_r} leaves the basis with y_{rk} is the pivot element for the next iteration.

Then, in the next iteration:

$$\hat{x}_{B_i} = - \frac{x_{B_r} \, y_{ik}}{y_{rk}} > 0$$ for the artificial variable a_i in the basis.

To avoid this to happen, in the current iteration, we break and do not follow minimum ratio criterion for identify the index r but force the artificial variable to leave the basis.

We illustrate Two-phase method through some examples.

eg. max $z = 2x_1 - x_2 + x_3$

subject to
$$x_1 + x_2 - 3x_3 \le 8$$
$$4x_1 - x_2 + x_3 \ge 2$$
$$2x_1 + 3x_2 - x_3 \ge 4$$
$$x_1, x_2, x_3 \ge 0.$$

Phase I LPP:

$$\max Y = -x_7 - x_8$$

subject to

$$x_1 + x_2 - 3x_3 + x_4 = 8$$
$$4x_1 - x_2 + x_3 - x_5 + x_7 = 2$$
$$2x_1 + 3x_2 - x_3 - x_6 + x_8 = 4$$
$$x_i \geq 0 \quad \forall i = 1, ---, 8.$$

Apply the simplex method, and after few iteration, the final table is as follows:

C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_4	$45/7$	0	0	$-19/7$	1	$1/14$	$5/14$
0	x_1	$5/7$	1	0	$1/7$	0	$-3/14$	$-1/14$
0	x_2	$6/7$	0	1	$-3/7$	0	$1/7$	$-2/7$

with $z_j - c_j = 0, \forall j$ and no artificial variable in the basis $\Rightarrow$ we can move to phase 2 with BFS.

Phase 2 first table:

		$c_j \rightarrow$	2	−1	1	0	0	0	
C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	ratio
0	x_4	45/7	0	0	−19/7	1	1/14	5/14	90
2	x_1	5/7	1	0	1/7	0	−3/14	−1/14	−
−1	x_2	6/7	0	1	−3/7	0	1/7	−2/7	6 →
		$z_j - c_j \rightarrow$	0	0	−2/7	0	−4/7 ↑	1/7	

Proceed with the simplex iteration hereafter.

		$c_j \rightarrow$	2	-1	1	0	0	0
C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_4	6	0	$-1/2$	$-5/2$	1	0	$1/2$
2	x_1	2	1	$3/2$	$-1/2$	0	0	$-1/2$
0	x_5	6	0	7	-3	0	1	-2
		$z_j - c_j \rightarrow$	0	4	-2	0	0	-1
					$\uparrow$			

The original LPP is unbounded. as all $y_{i3} < 0$.

$\Rightarrow$ The objective fn of LPP $\rightarrow \infty$ within feasible region

eg

$$\max\ z = -x_1 - x_2$$

subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Phase 1:

$$\max\ -x_5 - x_6$$

s.to
$$3x + x_2 + x_5 = 3$$
$$4x_1 + 3x_2 - x_3 + x_6 = 6$$
$$x_1 + 2x_2 + x_4 = 3$$
$$x_i \geq 0 \quad \forall i$$

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
-1	x_5	3	3	1	0	0	1	0	$\rightarrow$
-1	x_6	6	4	3	-1	0	0	1	
0	x_4	3	1	2	0	1	0	0	
	$z_j - c_j \rightarrow$		-7	-4	1	0	0	0	
			$\uparrow$						

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	
0	x_1	1	1	$1/3$	0	0	$1/3$	0	
-1	x_6	2	0	$5/3$	-1	0	$-4/3$	1	
0	x_4	2	0	$5/3$	0	1	$-1/3$	0	$\rightarrow$
	$z_j - c_j \longrightarrow$		0	$-5/3 \uparrow$	1	0	$7/3$	0	

		$c_j \rightarrow$	0	0	0	0	-1	-1
c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	x_1	$3/5$	1	0	0	$-1/5$	$2/5$	0
-1	x_6	0	0	0	-1	-1	-1	1
0	x_2	$6/5$	0	1	0	$3/5$	$-1/5$	0
	$z_j - c_j \longrightarrow$		0	0	1	1	2	0

$z_j - c_j \geq 0 \ \forall j$ and art. var. x_6 is in basis at 0 level.

$\therefore$ we have a feasible soln of LPP in hand (but not BFS)

<u>Phase 2:</u>

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_6
-2	x_1	$3/5$	1	0	0	$-1/5$	0
0	x_6	0	0	0	-1	-1	1
-1	x_2	$6/5$	0	1	0	$3/5$	0
	$z_j - c_j \rightarrow$		0	0	0	$-1/5$	0

Here, if we follow min. ratio criterion then x_2 leaves the basis and 3/5 is pivot. But in next iteration x_B value for x_6 is $6/5 \,/\, 3/5 = 2 > 0$ – an issue.

$\therefore$ we force x_6 to leave the basis & take -1 as pivot element. (This is the only situation when we can allow −ve pivot in the simplex method)

c_B	v_B	x_B	y_1	y_2	y_3	y_4
-2	x_1	$3/5$	1	0	$1/5$	0
0	x_4	0	0	0	1	0
-1	x_2	$6/5$	0	1	$-3/5$	1
	$z_j - c_j \longrightarrow$		0	0	$1/5$	0

$\therefore$ optimal soln is $x^* = (3/5, 6/5, 0, 0)^t$ and optimal objective value $= -12/5$.

eg. Alternate optimal soln.

$$\max z = 6x_1 + 4x_2$$

s. to $\quad 2x_1 + 3x_2 \le 30$

$\quad 3x_1 + 2x_2 \le 24$

$\quad x_1 + x_2 \geq 3$

$\quad x_1, x_2 \geq 0$

Big M-method:

$$\max 6x_1 + 4x_2 - Mx_6$$

s. to $2x_1 + 3x_2 + x_3 = 30$, $3x_1 + 2x_2 + x_4 = 24$,

$x_1 + x_2 - x_5 + x_6 = 3$, $x_i \geq 0 \ \forall i$

After 2 iterations

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	ratio
0	x_3	14	0	5/3	1	$-2/3$	0	$42/5$ →
0	x_5	5	0	$-1/3$	0	1/3	1	–
6	x_1	8	1	2/3	0	1/3	0	24/2
	$z_j - c_j$ →		0	0 ↑	0	2	0	

$z_j - c_j \geq 0 \; \forall j$ & $z_2 - c_2 = 0$ with at least one +ve entry of y_2 and $x_2 \notin B$. ⇒ Indication of alternate solu

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5
4	x_2	42/5	0	1	3/5	$-2/5$	0
0	x_5	39/5	0	0	1/5	1/5	1
6	x_1	12/5	1	0	$-2/5$	3/5	0
			0	0	0	2	0

One optimal soln is $x^* = (x_1^* = 8, x_2^* = 0)$ and another one is $\tilde{x} = (\tilde{x}_1 = 12/5, \tilde{x}_2 = 42/5)$, with optimal objective value $z^* = 48$

$\therefore$ All optimal solns of LPP are $(1-\alpha)x^* + \alpha\tilde{x}$, $\alpha \in [0,1]$, and optimal value $= 48$.

eg for cycling: Degeneracy in BFS can cause extra steps in the Simplex method while geometrically we remain at the same degenerate extreme point. This is called stalling.

In certain therotical eg, the problem becomes worse leading to cycling in the iterations.

eg. min $-\frac{3}{4}x_4 + 20x_5 - \frac{1}{2}x_6 + 6x_7$

s.to. $x_1 + \frac{1}{4}x_4 - 8x_5 - x_6 + 9x_7 = 0$

$x_2 + \frac{1}{2}x_4 - 12x_5 - \frac{1}{2}x_6 + 3x_7 = 0$

$x_3 + x_6 = 1$

$x_i \geq 0 \;\; \forall \; i = 1, --, 7.$

Here, $A = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{identity already in } A.} & \begin{matrix} 1/4 & -8 & -1 & 9 \\ 1/2 & -12 & -1/2 & 3 \\ 0 & 0 & 1 & 0 \end{matrix} \end{bmatrix}$

Table 1

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	ratio
	$c_j \rightarrow$		0	0	0	$-3/4$	20	$-1/2$	6	
0	x_1	0	1	0	0	$1/4$	-8	-1	9	0
0	x_2	0	0	1	0	$1/2$	-12	$-1/2$	3	0
0	x_3	1	0	0	1	0	0	1	0	$-$
	$z_j - c_j \rightarrow$		0	0	0	$3/4$	-20	$1/2$	-6	

} tie

Note: Degenerate BFS (tie in min ratio), and being a min problem we look for most +ve $z_j - c_j$.

Table 2

let x_1 leaves the basis (arbitrary).

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
$-3/4$	x_4	0	4	0	0	1	-32	-4	36
0	x_2	0	-2	1	0	0	4	$3/2$	-15
0	x_3	1	0	0	1	0	0	1	0
		$z_j - c_j \rightarrow$	-3	0	0	0	4	$7/2$	-33

In the next iteration, basic variables & their values

$x_4 = 0, \quad x_5 = 0, \quad x_3 = 1.$ (Table 3)

$x_6 = 0, \quad x_5 = 0, \quad x_3 = 1$ (Table 4)

$x_6 = 0, \quad x_7 = 0, \quad x_3 = 1$ (Table 5)

$x_1 = 0, \quad x_7 = 0, \quad x_3 = 1$ (Table 6)

cycling $\Big[$ $\underbrace{x_1 = 0, \quad x_2 = 0, \quad x_3 = 1}_{\text{Same as Table 1}}$ (Table 7)

$\therefore$ we get stuck up in a loop (degenerate BFS) in the method.

To avoid it, the rule of outgoing variable is suggested to change

Lexicographic ordering : Let $x = (x_1, x_2, \ldots, x_n)^t$ and $y = (y_1, y_2, \ldots, y_n)^t$ be vectors in $\mathbb{R}^n$. We say that x is lexicographically greater than y if $\exists$ an index $m < n$ so that $x_i = y_i \;\forall\; i = 1, \ldots, m$, and $x_{m+1} > y_{m+1}$.
If there is no such $m < n$, then $x_i = y_i \;\forall\; i = 1, \ldots, n$, and thus $x = y$. We write $x \succ y$ (lexicographically greater) and $x \succeq y$ (lexicographically greater than or equal)

eg $(0, 2, 3, 0)^t \succ (0, 2, 1, 4)^t$; $(1, 2, 1, 2)^t \succ (0, 4, 5, 0)^t$

We use lexicographic minimum ratio for leaving variable.

In case of non-degenerate BFS, the index r, for which

$$\frac{x_{Br}}{y_{rj}} = \min\left\{ \frac{x_{Bi}}{y_{ij}} : y_{ij} > 0 \right\} \quad ; \; j \text{ is the index of entering variable}$$

is unique, then there is no issue.

Pivot Rule

- Choose an arbitrary column a_j of A for which $z_j - c_j$ is most $-$ve. to enter in the basis.
- For each i, with $y_{ij} > 0$, divide the i^{th} row of the table by y_{ij}, and then choose the lexicographically smallest row l. The l^{th} basic variable leaves the basis.

Note: The lexicographic pivoting rule always leads to a unique choice for the exit variable $\because$ otherwise two rows of $B^{-1}A$ would be linearly dependent contradicting that the rank of A is m (full row rank).

Theorem: Suppose that the Simplex method starts with the lexicographic +ve rows $1, 2, --, m$ in the simplex table, and let the lexicographic pivoting rule is followed. Then

(i) Rows $1, --, m$, remain lexicographically +ve throughout the algo.
(ii) The row $z_j - c_j$ improves lexicographically at each iteration.
(iii) Simplex method terminates in finite number of iterations.

Going back to Table 1 of the cycling example (above),

x_4 is the variable to enter in the basis $\Rightarrow j = 4$

$y_{14} = 1/4 > 0$ and $y_{24} = 1/2 > 0$.

Take row 1 in the table & divide by $1/4$

Take row 2 in the table & divide by $1/2$

$$\left.\begin{array}{ll} \text{Row 1 :} & (4, 0, 0, 1, -32, -4, 36) \\ \text{Row 2 :} & (0, 2, 0, 1, -24, -1, 6) \end{array}\right] \begin{array}{l} \text{lexicographically} \\ \text{Row1} \succ \text{Row 2} \end{array}$$

$\therefore$ Min ratio lexicographically ordering, Row 2 or x_2 leaves the basis

						$-3/4$	20	$-1/2$	6	
C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	
0	x_1	0	1	0	0	$1/4$	-8	-1	9	
0	x_2	0	0	1	0	$1/2$	-12	$-1/2$	3	$\rightarrow$ lexico min ratio.
0	x_3	1	0	0	1	0	0	1	0	
	$z_j - c_j \rightarrow$		0	0	0	$3/4$	-20	$1/2$	-6	
						$\uparrow$				

C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	
0	x_1	0	1	$-1/2$	0	0	-2	$-3/4$	$15/2$	
$-3/4$	x_4	0	0	2	0	1	-24	1	6	
0	x_3	1	0	0	1	0	0	1	0	$\rightarrow$ lexico order
	$z_j - c_j \rightarrow$		0	$-3/2$	0	0	-2	$5/4$	$-21/2$	
								$\uparrow$		

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	x_1	$3/4$	1	$-1/2$	$3/4$	0	-2	0	$15/2$
$-3/4$	x_4	1	0	2	1	1	-24	0	6
$-1/2$	x_6	1	0	0	1	0	0	1	0
	$z_j - c_j \rightarrow$		0	$-3/2$	$-5/4$	0	-2	0	$-\frac{21}{2}$

$z_j - c_j \le 0 \quad \forall j$ (min problem)

$\therefore$ optimal soln is

$x_1^* = 3/4,\ x_2^* = 0,\ x_3^* = 0,\ x_4^* = 1,\ x_5^* = 0,\ x_6^* = 1,\ x_7^* = 0$

and optimal objective value $= -5/4$.

Duality Theory

$$\max \quad z = c^T x$$

subject to $\quad a_i^T x = b_i \quad , \; i \in M$

$$a_i^T x \le b_i \quad , \; i \in \overline{M}$$

$$x_j \geq 0, \quad j \in N$$

$$x_j \text{ unrestricted } \; j \in \overline{N}$$

where a_i^T is the i^{th} row of matrix A.

M, $\overline{M}$, N, $\overline{N}$ are finite index sets.

Convert the problem into the form slack are used in each equality in $\overline{M}$ and for each x_j, $j \in \overline{N}$, write

$x_j = x_j' - x_j''$, $x_j' \geq 0$, $x_j'' \geq 0$.

(P) $\max \ \hat{c}^T \hat{x}$ ⟶ original x, x', x'', and slack

s.to $\hat{A}\hat{x} = b$

$\hat{x} \geq 0$.

$\hat{A} = \left[A_j, \ j \in N \mid (A_j, -A_j), \ j \in \bar{N} \;\middle|\; \begin{matrix} 0, & i \in M \\ e_i, & i \in \bar{M} \end{matrix} \right]$

$\hat{c} = [c_j, \ j \in N \mid (c_j, -c_j), \ j \in \bar{N}, 0]$

$\hat{x} = \text{column}(x_j, \ j \in N \mid (x_j', x_j''), \ j \in \bar{N} \mid \underbrace{x_i^s}_{\text{slack}}, \ i \in M)$

Here, A_j denotes the j^{th} column of A.

Let $\hat{x}^*$ be the optimal soln of (P), B^* be its optimal basis. We write

$$\underset{1\times m}{w^T} = \underset{1\times m}{\hat{c}_{B^*}^T} \underset{m\times m}{(B^*)^{-1}} \quad , \quad w \in \mathbb{R}^m.$$

From optimality condition, $z_j - c_j \geq 0, \ \forall j$

$$\Rightarrow \quad \hat{c}_{B^*}^T (B^*)^{-1} \hat{A} - \hat{c}^T \geq 0 \qquad \text{—} \ (1)$$

For $j \in N$, (1) $\Rightarrow$ $w^T A_j \geq \hat{c}_j$

for $j \in \bar{N}$, (1) $\Rightarrow$ $\left.\begin{array}{l} w^T A_j \geq \hat{c}_j \\ -w^T A_j \geq -\hat{c}_j \end{array}\right\} \Rightarrow w^T A_j = \hat{c}_j.$

For $i \in \bar{M}$, (1) $\Rightarrow$ $w^T e_i \geq 0 \Rightarrow w_i \geq 0.$

The above notings help us to associate another LPP with the given LPP.

$$\min \quad b^T w$$

subject to $\quad a_j^T w \geq c_j \;, \; j \in N$

$\quad a_j^T w = c_j \;, \; j \in \bar{N}$

$\quad w_i \geq 0 \;, \; i \in \bar{M}$

$\quad w_i \in \mathbb{R} \;, \; i \in M$

In general,

(P) $\quad \max \; c^T x$ $\qquad$ (D) $\quad \text{Min} \; b^T w$

s. to $\; Ax \leq b$ $\qquad$ s. to $\; A^T w \geq c$

$x \geq 0$ $\qquad$ $w \geq 0$

We call the given problem as Primal and the other one is Dual eg.

$$\left.\begin{aligned} \max\ z &= 5x_1 + 6x_2 - 4x_3 \\ \text{subject to}\quad 2x_1 + x_2 - 3x_3 &\le 1 \\ 3x_1 - 2x_2 &\le 2 \\ 2x_1 + 5x_2 + x_3 &\le 9 \\ x_1, x_2, x_3 &\ge 0. \end{aligned}\right\} \text{Primal LPP}$$

Dual:

$$\left.\begin{aligned} \min\ w &= w_1 + 2w_2 + 9w_3 \\ \text{subject to}\quad 2w_1 + 3w_2 + 2w_3 &\ge 5 \\ w_1 - 2w_2 + 5w_3 &\ge 6 \\ -3w_1 + w_3 &\ge -4 \\ w_1, w_2, w_3 &\ge 0. \end{aligned}\right\} \text{Dual LPP}$$

Dual of the dual is the primal problem. (Reflexive Principle)

(P) $\max\ z = c^T x$

subject to $Ax \leq b$

$x \geq 0$

(D) $\min\ b^T w$

s.to $A^T w \geq c$

$w \geq 0$

write dual (D) as (D')

$-\max\ -b^T w$

s.to $(-A)^T w \leq -c$

$w \geq 0$

$\rightarrow$ (D'') $-\min\ -c^T y$

subject to $(-A^T)^T y \geq -b$

$y \geq 0$

write (D'') as : $\max\ c^T y$ subject to $A\ y \leq b,\ y \geq 0$

which is the same as (P).

If the i^{th} constraint in (P) is $=$ type then the i^{th} dual variable in (D) is unrestricted in sign.

If the j^{th} variable in (P) is unrestricted in sign then the j^{th} constraint in the dual (D) is $=$ type.

eg. $\min z = 3x_1 + 5x_2$

subject to $2x_1 + 7x_2 \geq 3$

$x_1 + 2x_2 = 4$

$x_1, x_2 \geq 0$

<u>Dual</u> $\max 3w_1 + 4w_2$

subject to

$2w_1 + w_2 \leq 3$

$7w_1 + 2w_2 \leq 5$

$w_1 \geq 0$

$w_2 \in \mathbb{R}$.

eg. (P) $\min \; 3x_1 - 2x_2 + x_3 \rightarrow -\max \; -3x_1 + 2x_2 - x_3$

s.to $x_1 + x_2 - x_3 \le 1 \rightarrow x_1 + x_2 - x_3 \le 1$

w_1 $\quad 2x_1 - x_2 + 5x_3 \ge 3 \rightarrow -2x_1 + x_2 - 5x_3 \le -3$

w_2 $\quad x_1, x_2 \ge 0 \qquad x_1, x_2 \ge 0, \; x_3 \in \mathbb{R}$

equality

Dual:

$$
\begin{aligned}
-\min \;& w_1 - 3w_2 \\
\text{s.to } \;& w_1 - 2w_2 \ge -3 \\
& w_1 + w_2 \ge 2 \\
& -w_1 - 5w_2 = -1 \\
& w_1, w_2 \ge 0
\end{aligned}
$$

or $\equiv$ (D)

$$
\left.
\begin{aligned}
\max \;& -w_1 + 3w_2 \\
\text{s.to } \;& -w_1 + 2w_2 \le 3 \\
& -w_1 - w_2 \le -2 \\
& w_1 + 5w_2 = 1 \\
& w_1, w_2 \ge 0
\end{aligned}
\right\}
$$

x_3 unrestricted

(P) $\min\ x_1 + x_2 + x_3$

subject to $x_1 - 3x_2 + 4x_3 \le 5$

$2x_1 - 2x_2 \le 3$

$2x_2 - x_3 \ge 5$

$x_1,\ x_2 \ge 0.$

Put $x_3 = x_3' - x_3''$, $x_3' \ge 0$, $x_3'' \ge 0$

$-\max(-x_1 - x_2 - x_3' + x_3'')$ $\leftarrow$ $\min\ x_1 + x_2 + x_3' - x_3''$

subject to $x_1 - 3x_2 + 4x_3' - 4x_3'' \le 5$

$2x_1 - 2x_2 \le 3$

$-2x_2 + x_3' - x_3'' \le -5$ $\leftarrow$ $2x_2 - x_3' + x_3'' \ge 5$, $x_1, x_2, x_3', x_3'' \ge 0.$

Dual is

$\max\ 5w_1 + 3w_2 - 5w_3$

subject to

$w_1 + 2w_2 \ge -1$

$3w_1 + 2w_2 + 2w_3 \le 1$

$4w_1 + w_3 = -1$

$w_1, w_2, w_3 \ge 0$

Why should we study duality theory in the context of LPP?

eg. Suppose we have a product mix problem in hand

How to utilize resources to generate maximum revenue?

Resources	Product 1	Product 2	available
A	1	1	50
B	1	2	80
C	3	2	140
Sale price / unit	4	3	

LPP: max $4x_1 + 3x_2$

subject to $x_1 + x_2 \le 50$

$x_1 + 2x_2 \le 80$

$3x_1 + 2x_2 \le 140$, $x_1, x_2 \ge 0$.

x_1: quantity of product 1 ; x_2: quantity of product 2.

$\longrightarrow$ suppose to meet the requirement of resource C for product 1, manufacturer pays Rs 0.5 / unit to resource C supplier. Now let the supplier of C think that if he himself produces 1 unit of product by using 3 units of C it costs him Rs 1.50 while the selling price is 4, so he can save more.

Instead, if manufacturer pays Rs 1.50 / unit of C, then the resource C supplier has no incentive to retain the resource

C with him and use it to manufacture product 1 on his own.

$\therefore$ It is important to find out the appropriate prices of resources/unit by the resource suppliers so as to let off their resources to be utilized by others (here manufacturer)

Note that 1.5 was not the minimum prices per unit of C

Let w_1, w_2, w_3 be the prices per unit of resources A, B, C respectively. Then the problem of resource supplier is

minimize $50w_1 + 80w_2 + 140w_3$

subject to $w_1 + w_2 + 3w_3 \geq 4$, $w_1 + 2w_2 + 2w_3 \geq 2$

$w_1, w_2 \geq 0$.

Note this problem is dual.

Manufacturer	Supplier
max profit	prices. of resources
availability of resources (constraints)	earnings are competitive enough which allows him to forgo his resources

We shall be seeing the economic interpretation of it in details. We shall also see its relationship with $z_j - c_j$ (opportunity cost).

Duality Theorems:

(P) $\max z = c^T x$
s.to $Ax \le b$
$x \ge 0,$
$x \in \mathbb{R}^n.$

(D) $\min W = b^T w$
s.to $A^T w \ge c$
$w \ge 0, \; w \in \mathbb{R}^m$

$A: m \times n, \quad b: m \times 1, \quad c: n \times 1.$

Theorem (weak duality): Let x be feasible for (P) and w be feasible for (D). Then, $c^T x \le b^T w$.

Pf: As $w \ge 0$, $Ax \le b \Rightarrow w^T \underset{1\times1}{(Ax)} \le w^T b$

$\Rightarrow (Ax)^T w \le b^T w.$

$$\Rightarrow \quad x^T (A^T w) \le b^T w. \quad — (1)$$

Next, $A^T w \geq c$, $x \geq 0 \Rightarrow x^T A^T w \geq x^T c = c^T x \quad — (2)$

(1) & (2) $\Rightarrow \quad c^T x \le x^T A^T w \le b^T w$

$$\Rightarrow \quad c^T x \le b^T w$$

The gap $b^T w - c^T x$ is called the duality gap.

As a consequence of the above theorem, we get

(i) If (P) is feasible and unbounded then (D) is infeasible

(ii) If (D) is feasible and unbounded then (P) is infeasible.

It may so happen that both (P) and its dual (D) are infeasible.

eg.

(P) $\min\ 2x_1 - 4x_2$

s. to $x_1 - x_2 = 1$

$-x_1 + x_2 = 2$

$x_1, x_2 \geq 0$

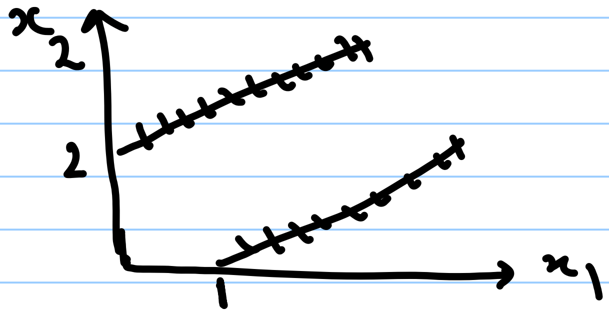
(D) $\max\ w_1 + 2w_2$

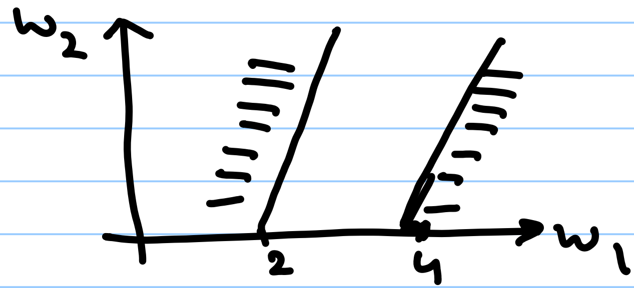
s. to $w_1 - w_2 \leq 2$

$-w_1 + w_2 \leq -4$

$w_1, w_2 \in \mathbb{R}$

Both (P) and (D) are infeasible





Strong Duality Theorem: Let $\bar{x}$ be an optimal soln of (P)
Then $\exists$ $\bar{w}$ such that $\bar{w}$ is an optimal soln of (D), and
$c^T \bar{x} = b^T \bar{w}$ (zero duality gap).

Pf: Suppose optimal $\bar{x}$ is obtained using simplex method. Let $\bar{B}$ be its basis. Then, $\bar{x}_B = \bar{B}^{-1} b$ and $\bar{x}_R = 0$. By optimality condition,

$$z_j - c_j \geq 0 \quad \forall \; j = 1, 2, \ldots, n.$$

$$\Rightarrow \quad c_B^T \bar{B}^{-1} a_j \geq c_j \quad \forall \; j$$

Let us define a vector $\bar{w}^T = c_{\bar{B}}^T \bar{B}^{-1}$

$\underset{1\times m}{\bar{w}^T} = \underset{1\times m}{c_{\bar{B}}^T} \; \underset{m\times m}{\bar{B}^{-1}}$

$\Rightarrow \bar{w}^T a_j \geq c_j \quad \forall j$

Corresponding to the identity column (slack),

$\bar{w}^T e_j \geq 0 \Rightarrow \bar{w} \geq 0$

$\boxed{\therefore \; A^T \bar{w} \geq c, \; \bar{w} \geq 0} \Rightarrow \bar{w}$ is feasible for (D).

Also, $c^T \bar{x} = c_{\bar{B}}^T \bar{x}_B = c_{\bar{B}}^T (\bar{B}^{-1} b) = \bar{w}^T b = b^T \bar{w}$

$\therefore \; \underline{c^T \bar{x} = b^T \bar{w}}$

Claim: $\bar{w}$ is optimal solution of (D).

$\because$ if w is any other feasible soln of (D) then by the weak duality thm, $c^T \bar{x} \leq b^T w$

$\Rightarrow \quad b^T \bar{w} \leq b^T w.$

$\Rightarrow \quad \bar{w}$ is optimal for the dual (D).

Note: Strong duality is a po eful and constructive, and allows to obtain the optimal solution of the second problem (if exists) without solving it and using the optimal simplex table of the first problem.

To illustrate this statement, we present an example.

eg (P) $\max \; 4x_1 + 3x_2$

s.to $x_1 + x_2 \leq 8$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0$

Optimal table

C_B	V_B	x_B	y_1	y_2	y_3	y_4
3	x_2	6	0	1	2	-1
4	x_1	2	1	0	-1	1
	$z_j - c_j \to$		0	0	2	1

$$\bar{B}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad C_B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \bar{x}_B = \begin{pmatrix} x_2 = 6 \\ x_1 = 2 \end{pmatrix}$$

$$\Rightarrow \bar{w}^T = C_B^T \bar{B}^{-1} = (2 \quad 1) \Rightarrow \bar{w}_1 = 2, \; \bar{w}_2 = 1$$

(P) $\min z = 12x_1 + 20x_2$ $\quad$ $\min z = 12x_1 + 20x_2 + 0s_1 + 0s_2 + Ma_1 + Ma_2$

s.to $6x_1 + 8x_2 \geq 100 \rightarrow 6x_1 + 8x_2 - s_1 + a_1 = 100$

$7x_1 + 12x_2 \geq 120 \rightarrow 7x_1 + 12x_2 - s_2 + a_2 = 120$

$x_1, x_2 \geq 0$ $\quad$ $x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

Optimal table of (P)

identity columns initially.

		$c_j \rightarrow$	12	20	0	0	M	M
c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
12	x_1	15	1	0	1/4	1/2	3/4	−1/2
20	x_2	5/4	0	1	7/16	−3/8	−7/16	3/8
		$z_j - c_j \rightarrow$	0	0	−1/4	−3/2	$\frac{1}{4} - M$	$\frac{3}{2} - M$

Dual

(D) $\max\ 100w_1 + 120w_2$

s.to $6w_1 + 7w_2 \leq 12$

$8w_1 + 12w_2 \leq 20$

$w_1, w_2 \geq 0$

Dual (D) optimal soln is $w_1^* = 1/4$, $w_2^* = 3/2$

optimal value of (P) and (D) = 205

Note : Optimal w^* is available in $z_j - c_j$ of the optimal table of the simplex method corresponding to the identity columns e_j as $z_j - c_j = c_B^T B^{-1} e_j - c_j = \bar{w}_j - c_j$

Complementary Slackness Theorem : Let $\bar{x}$ and $\bar{w}$ be feasible solutions of the primal-dual pair (P) and (D), respectively. Then $\bar{x}$ and $\bar{w}$ are optimal solutions to their respective problems iff $\bar{w}^T (A\bar{x} - b) = 0$, $\bar{x}^T (c - A^T \bar{w}) = 0$.

$\underbrace{\bar{w}^T (A\bar{x} - b) = 0, \; \bar{x}^T (c - A^T \bar{w}) = 0}_{}$ $\rightarrow$ CS conditions

Pf : We shall first prove the necessary part.

Let $\alpha = \bar{w}^T (A\bar{x} - b)$ and $\beta = \bar{x}^T (c - A^T \bar{w})$

$$\alpha + \beta = \underbrace{\bar{w}^T A \bar{x}}_{1\times 1} - \bar{w}^T b + \bar{x}^T c - \underbrace{\bar{x}^T A^T \bar{w}}_{1\times 1}$$

$$= c^T \bar{x} - b^T \bar{w}$$

Let $\bar{x}$ and $\bar{w}$ be optimal solns of (P) & (D) respectively.

$\Rightarrow c^T \bar{x} = b^T \bar{w}$

$\Rightarrow \alpha + \beta = 0$ —— ①

Also, by feasibility of $\bar{x}$ for (P) and $\bar{w}$ for (D), $\alpha \le 0$, and $\beta \le 0$. Therefore, ① $\Rightarrow \alpha = 0 = \beta$, proving the result.

Conversely, let $\alpha = 0$ and $\beta = 0$

$\Rightarrow \alpha + \beta = 0 \Rightarrow c^T \bar{x} = b^T \bar{w}$ (duality gap $= 0$)

Claim: $\bar{x}$ is optimal for (P).

let x be any other feasible soln of (P).

Then by the weak duality thm, $c^T x \leq b^T \bar{w}$

$\Rightarrow c^T x \leq c^T \bar{x}$, x is arbitrary feasible for (P)

$\Rightarrow \bar{x}$ is an optimal soln of (P).

Similarly, $\bar{w}$ is an optimal soln of (D).

$\therefore$ Complementary slackness conditions lead to optimality.

Note: $\bar{w}^T (A\bar{x} - b) = 0$, $\bar{w} \geq 0$, $A\bar{x} \leq b$

$\Rightarrow$ if $\sum_{j=1}^{n} a_{ij} \bar{x}_j < b_i$ for some i then the correspon-ding $\bar{w}_i = 0$

Similarly, $\bar{x}^T (c - A^T \bar{w}) = 0$, $\bar{x} \geq 0$, $A^T \bar{w} \geq c$

$\Rightarrow$ that if for some j

$$\sum_{i=1}^{m} a_{ij} \bar{w}_i > c_j \text{ , then for that } j, \ \bar{x}_j = 0.$$

Furthermore, if any $\bar{x}_k > 0$ or $\bar{w}_l > 0$ then the other problem constraints must be satisfied as equations.

Summary

(1) Both (P) & (D) are infeasible] examples

(2) (P) is infeasible & (D) is unbounded

(3) (D) is infeasible & (P) is unbounded] weak duality

(4) (P) & (D) both posses optimal solutions.] strong duality

CS conditions ensure that at optimal pair $(\bar{x}, \bar{w})$, if constraint in (P) is strict then its corresponding variable in (D) is zero, and if constraint in (D) is strict then corresponding variable in (P) is zero; if $\bar{x}_i > 0$ then

i^{th} constraint in (D) at $\bar{w}$ must be equation, and if $w_j > 0$ then the j^{th} constraint in (P) at $\bar{x}$ must be equation.

In the next example we will illustrate how dual and CS - conditions help us to solve both (P)-(D) problem pair.

eg: (P) $\max\ z = 3x_1 + 2x_2 + x_3 + 4x_4$

s.to
$$2x_1 + 2x_2 + x_3 + 3x_4 \leq 20$$
$$3x_1 + x_2 + 2x_3 + 2x_4 \leq 20$$
$$x_1,\ x_2,\ x_3,\ x_4 \geq 0$$

The dual is

(D) $\min W = 20w_1 + 20w_2$

s.to $2w_1 + 3w_2 \geq 3$

$2w_1 + w_2 \geq 2$

$w_1 + 2w_2 \geq 1$

$3w_1 + 2w_2 \geq 4$

$w_1, w_2 \geq 0$

Two variables

Can solve graphically

Optimal soln

$\bar{w}_1 = 1.2, \ \bar{w}_2 = 0.2$

Optimal value = 28

Note $2\bar{w}_1 + \bar{w}_2 > 2$ and $\bar{w}_1 + 2\bar{w}_2 > 1$

CS conditions $\Rightarrow \bar{x}_2 = 0, \ \bar{x}_3 = 0.$

Also, $\bar{w}_1 > 0, \ \bar{w}_2 > 0 \Rightarrow$ by CS conditions,

$$\left.\begin{aligned} 2\bar{x}_1 + 3\bar{x}_4 &= 20 \\ 3\bar{x}_1 + 2\bar{x}_4 &= 20 \end{aligned}\right\} \quad \bar{x}_1 = 4, \quad \bar{x}_4 = 4$$

$\therefore$ optimal soln of (P) is $(4, 0, 0, 4)$

and optimal value of (P) = 28.

eg

$$\left.\begin{aligned} \max z = 2x_1 + 4x_2 + 3x_3 + x_4 \\ \text{s.to} \quad 3x_1 + x_2 + x_3 + 4x_4 &\leq 12 \\ x_1 - 3x_2 + 2x_3 + 3x_4 &\leq 7 \\ 2x_1 + x_2 + 3x_3 - x_4 &\leq 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}\right\} \quad \text{Primal (P)}$$

optimal soln is $x_1^* = 0$, $x_2^* = 10.4$, $x_3^* = 0$, $x_4^* = 0.4$; $z^* = 42$

Dual

(D) $\min \ 12w_1 + 7w_2 + 10w_3$

s.to $3w_1 + w_2 + 2w_3 \geq 2$

$w_1 - 3w_2 + w_3 \geq 4$

$w_1 + 2w_2 + 3w_3 \geq 3$

$4w_1 + 3w_2 - w_3 \geq 1$

$w_1, w_2, w_3 \geq 0$

$$x_2^* > 0, \ x_4^* > 0 \Rightarrow \left.\begin{array}{l} w_1^* - 3w_2^* + w_3^* = 4 \\ 4w_1^* + 3w_2^* - w_3^* = 1 \end{array}\right\} \ (1)$$

Also second constraint in (P) at x^* is nonbinding

$\Rightarrow w_2^* = 0 \quad \therefore (1) \Rightarrow w_1^* = 1, \ w_3^* = 3.$

optimal soln of (D) is $w^* = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and optimal value of (D) is 42.

Note, at w^*, first and third constraints in (D) are nonbinding, in turn, leading to $x_1^* = 0$, $x_3^* = 0$ in the optimal soln of (P).

Note:

(P) $\max\ c^T x$

subject to $Ax = b$

$x \geq 0$

$x \in \mathbb{R}^n$

(D) $\min\ b^T w$

s.to $A^T w - s = c$

$s \geq 0$, $w \in \mathbb{R}^m$

$$\Omega_P = \{ x \in \mathbb{R}^n : x \text{ solves } (P) \}$$

$$\Omega_D = \{ (w, s) \in \mathbb{R}^m \times \mathbb{R}^n : (y, s) \text{ solves } (D) \}$$

Define two sets

$$N_1 = \{ i \in \{1, 2, \ldots, n\} : x_i^* > 0 \text{ for some } x^* \in \Omega_P \}$$

$$N_2 = \{ i \in \{1, 2, \ldots, n\} : s_i^* > 0 \text{ for some } (w^*, s^*) \in \Omega_D \}$$

From CS condition

$$N_1 \cap N_2 = \Phi .$$

We now state a result (without proof) also called the strict complementarity slackness conditions (SCSC).

Goldman - Tucker Theorem

For any LPP with Ω_P and Ω_D non-empty, it holds that

$N_1 \cup N_2 = \{1, 2, \ldots, n\}$, or, $\exists$ optimal solns x^* of (P) and (w^*, s^*) of (D) such that $x^* + s^* > 0$.

CS condition

$$\text{if } x_j^* > 0 \Rightarrow s_j^* = 0 \Rightarrow (A^T w^*)_j = c_j$$

$$\text{if } s_k^* > 0 \Rightarrow (A^T w^*)_k > c_k \Rightarrow x_k^* = 0$$

& both $x_j^* = 0$ and $s_j^* = 0$ is possible or $x_j^* = 0$ and $(A^T w^*)_j = c_j$ are possible.

However, SCSC means $\exists\ x^* \in \Omega_P$, $(w^*, s^*) \in \Omega_D$ such that

$x^*_j = 0 = s^*_j$ is not possible.

Note : CSC holds for all optimal pairs x^*, (w^*, s^*), while

in SCSC, $\exists$ optimal pair x^*, (w^*, s^*) to satisfy SCSC.

eg
$$\left.\begin{array}{ll} \min & 3x_1 + x_2 \\ \text{s.to} & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array}\right\} (P) \qquad \left.\begin{array}{ll} \max & w_1 \\ \text{s.to.} & w_1 + s_1 = 3 \\ & w_1 + s_2 = 1 \\ & s_1, s_2 \geq 0 \end{array}\right\} (D)$$

$x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ unique opt. soln

$w^* = 1$, $s^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ unique opt. soln

Note : $x_1^* + s_1^* > 0$ and $x_2^* + s_2^* > 0$

eg (P) $\min \; -x_1 + x_2$

subject to $x_1 + x_2 + x_3 = 1$

$x_1 + x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

unique opt. soln

$$x^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$N_1 = \{1\}$

$N_2 = \{2, 3, 4\}$

$N_1 \cap N_2 = \Phi$

(D) $\max \; w_1 + w_2$

subject to

$w_1 + w_2 + s_1 = -1$

$w_1 + s_2 = 1$

$w_1 + s_3 = 0$

$w_2 + s_4 = 0$

$s_1, s_2, s_3, s_4 \geq 0$

optimal soln

$$w^* = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad s^* = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\tilde{w}^* = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \tilde{s}^* = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

& convex combinations.

Note $\exists$ $x^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\hat{s}^* = \begin{pmatrix} 0 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \frac{1}{2} s^* + \frac{1}{2} \tilde{s}^*$

s.t.

$$x_j^* + \hat{s}_j^* > 0 , \ \forall j = 1, 2, 3, 4.$$

But note that $x_4^* + s_4^* = 0$ or $x_3^* + \tilde{s}_3^* = 0$

$\therefore$ SCSC does not hold with x^* and (w^*, s^*) or x^* and $(w^*, \tilde{s}^*)$ pair of opt. solns. but $\exists$ $x^* \in \Omega_p$ and $(\hat{w}^*, \hat{s}^*)$ satisfying SCSC; $\hat{w}^* = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$.

Economic Interpretation

Consider the LPP: (P) $\max z = c^T x$ subject to $Ax = b$, $x \geq 0$

Suppose B is the optimal basis and $\begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is an optimal soln of (P). Then, $x_B = B^{-1} b$. Let the optimal solution of the associated dual problem be $w^T = c_B^T B^{-1}$ (recall the formula from strong duality).

$$(D) \qquad \min \; b^T w$$
$$\text{s.to} \quad A^T w \geq c, \quad w \in \mathbb{R}^m$$

Let the resource vector b is slightly disturbed to $b + \Delta b$ such that the optimal basis B does not get altered. or B is the optimal basis for the LPP

$$\max \; c^T x$$

subject to $\quad Ax = b + \Delta b$

$$x \geq 0.$$

The optimal soln will see a change. Let the new optimal soln be $\begin{pmatrix} x_B + \Delta x_B \\ 0 \end{pmatrix}$, where, $x_B + \Delta x_B = B^{-1}(b + \Delta b)$

$$\Rightarrow \quad \Delta x_B = B^{-1} \Delta b$$

Also, the objective value change

$$(z + \Delta z) - z = c_B^T (x_B + \Delta x_B) - c_B^T x_B$$

$$= c_B^T B^{-1} b = w^T b$$

$$\Rightarrow \quad \Delta z = w^T \Delta b$$

$$\Rightarrow \quad \frac{\Delta z}{\Delta b_i} = w_i \,, \quad i = 1, 2, ---, m$$

$\Rightarrow$ The rate of change in the optimal objective function value w.r.to i^{th} resouych availability b_i is the i^{th} optimal dual variable w_i

In Economics, w_i are called shadow prices or implicit prices. In general calculus set up, these are called the Lagrange multipliers.

shadow price $\rightarrow$ max price that a person is willing to pay for extra unit of limited resource.

eg. min $3x_1 + 2x_2$

(P) subject to $7x_1 + 2x_2 \geq 30$

$5x_1 + 4x_2 \geq 20$

$2x_1 + 8x_2 \geq 16$

$x_1, x_2 \geq 0$

(D) max $30w_1 + 20w_2 + 16w_3$

s.to $7w_1 + 5w_2 + 2w_3 \leq 3$

$2w_1 + 4w_2 + 8w_3 \leq 2$

$w_1, w_2, w_3 \geq 0$

optimal table (D)

C_B	V_B	W_B	y_1	y_2	y_3	y_4	y_5
30	w_1	5/13	1	8/13	0	2/13	-1/26
16	w_3	2/13	0	4/26	1	-1/26	7/52
		$z_j - c_j \rightarrow$	0	4	0	4	1

$w_1^* = 5/13$, $w_2^* = 0$; $x_1^* = 4$, $x_2^* = 1$ & optimal value = 14

Suppose we change $\begin{pmatrix} 30 \\ 20 \\ 16 \end{pmatrix}$ to $\begin{pmatrix} 31 \\ 20 \\ 16 \end{pmatrix}$ or $\Delta b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in (P).

verify that current optimal basis does not change even after changing b to $b + \Delta b$.

$$\Delta z = w^T \Delta b = \frac{5}{13}$$

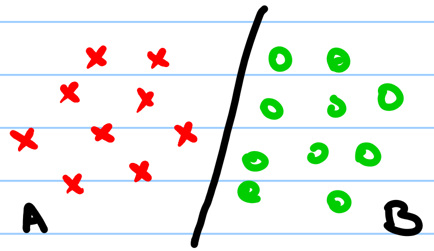
$\therefore$ optimal value change to $14 + \frac{5}{13} = 14.385$

On the other hand suppose b_2 is change from current value 20 to 19, then $w_2^* = 0$ so objective value value remains 14.

$\therefore$ The dual variable help us to study the changes in the optimal value vis a vis minor changes in the resources.

LP in ML

Binary classification : Let A comprises of m patterns with label $+1$ and B comprises of k patterns with label -1.



<u>linearly seperable sets</u> : Two sets $A, B \subset \mathbb{R}^n$ (comprising of m & k data patterns) are said to be linearly seperable if $\exists$ a hyperplane $w^T x = b$, $w \in \mathbb{R}^n$, $b \in \mathbb{R}$ such that

$$\underset{m\times n}{A}\ \underset{n\times 1}{w} > \underset{m\times 1}{e}\ \underset{1\times 1}{b} \quad \text{and} \quad \underset{k\times n}{B}\ \underset{n\times 1}{w} < \underset{k\times 1}{e}\ \underset{1\times 1}{b}$$

$e = (1, 1, \ldots 1)^T$ of appropriate dimension.

Result: Let A and B be two finite sets in $\mathbb{R}^n$. Then, A and B are linearly seperable iff $\operatorname{conv}(A) \cap \operatorname{conv}(B) = \Phi$.

(Hahn Banach thm in $\mathbb{R}^n \leftrightarrow$ strict seperation)

Note: $Aw > eb \iff \sum_{j=1}^{n} a_{ij} w_j > b, \; i = 1, 2, \ldots, m$

$\Rightarrow \sum_{j=1}^{n} a_{ij} w_j \geq b + \alpha_i, \quad \alpha_i > 0, \quad i = 1, 2, \ldots, m$

Take $\bar{\alpha} = \min_i \{\alpha_i\}$

$\Rightarrow \sum_{j=1}^{n} a_{ij} w_j \geq b + \bar{\alpha}, \quad \bar{\alpha} > 0$

Similarly, $Bw < eb \iff \sum_{j=1}^{n} b_{rj} w_j \leq b - \bar{\beta}, \; r = 1, 2, \ldots, k, \; \bar{\beta} > 0$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} \bar{w}_j \geq \bar{b} + 1 \qquad \bar{w}_j = w_j / \bar{\alpha}, \quad \bar{b} = b / \bar{\alpha}$$

and $$\sum_{j=1}^{n} b_{\ell j} \bar{w}_j \leq \bar{b} - 1, \quad \ell = 1, \ldots, k, \quad i = 1, 2, \ldots, m.$$

$\therefore$ Seperation of two sets linearly means finding a linear hyperplane $w^T x = b$ such that

$$\left.\begin{array}{l} \\ Aw \geq eb + e, \quad Bw \leq eb - e \end{array}\right\} \rightarrow \text{canonical hyperplane}$$

Equivalently

$$\min_{(w,b)} \frac{1}{m} \| \overbrace{(-Aw + eb + e)}^{m \times 1}_+ \|_1 + \frac{1}{k} \| \overbrace{(Bw - eb + e)}^{k \times 1}_+ \|_1$$

where $\|a_+\|_1 = \sum_{i=1}^{p} \max(a_i, 0)$ (L_1-norm)

Equivalently

$$\min_{w,b,y,z} \quad \frac{e^T y}{m} + \frac{e^T z}{k}$$

subject to

$$Aw - eb + y \geq e$$

$$-Bw + eb + z \geq e$$

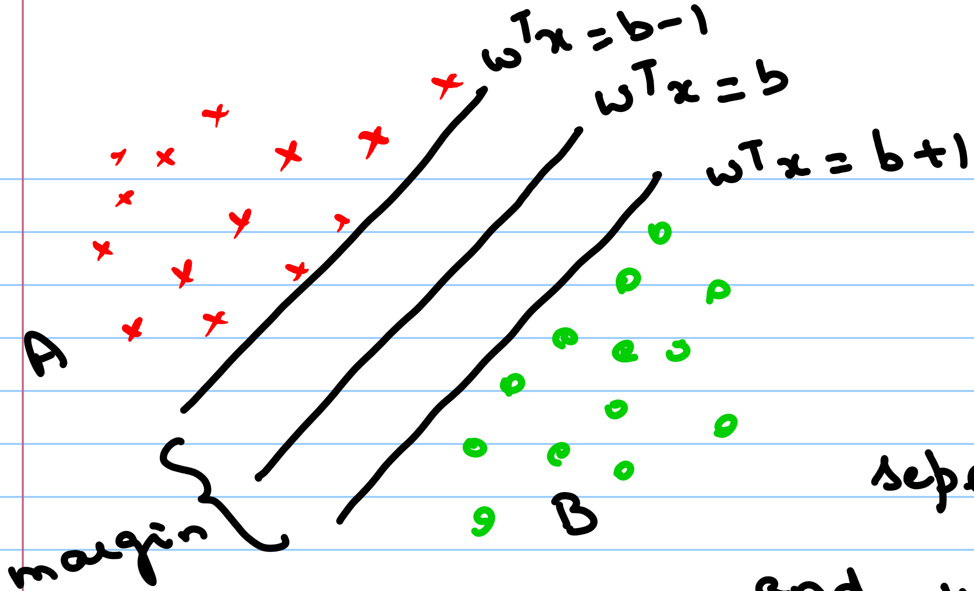
$$y \geq 0, \quad z \geq 0$$

$$w \in \mathbb{R}^n, \; b \in \mathbb{R}, \; y \in \mathbb{R}^m, \; z \in \mathbb{R}^k$$

$$y = (-Aw + eb + e)_+$$

$$z = (Bw - eb + e)_+$$

It is an LPP with $n+m+k+1$ variables and $m+k$ constraints beside $y \geq 0$, $z \geq 0$.



Margin

Let $A, B \subset \mathbb{R}^n$ be linearly seperable and $w^Tx = b$ be a seperating hyperplane. Let $w^Tx = b+1$ and $w^Tx = b-1$ be the bounding planes defining the dead zone. The distance b/w the two bounding hyperplanes is called margin $= 2/\|w\|$

$\|w\| = w_1^2 + \dots + w_n^2$

Among many such planes, the optimal one $w^Tx = b$ is the one which gives maximum margin. or

$$\min_{(w,b)} \frac{1}{2} w^T w$$

subject to $y_i (w^T x^i - b) \geq 1, \; i = 1, 2, \ldots, p$

where $y_i \in \{\underset{A}{\underset{\downarrow}{+1}}, \underset{B}{\underset{\downarrow}{-1}}\}$, and $x^i \in \mathbb{R}^n$ is the i^{th} pattern data point.

p : # of data points.

Note: $\begin{cases} w^T x^i - b > 0, \; \forall i = 1, 2, \ldots, p, & \text{when } y_i = +1 \text{ (or A)} \\ w^T x^i - b < 0, \; \forall i = 1, \ldots, p, & \text{when } y_i = -1 \text{ (or B)} \end{cases}$

This problem (although not LP) have many constraints, one for each data point. So, in this case, it is better to write Lagrange fn. and use dual with duality theory (but for quadratic problem) to get the optimal $w \in \mathbb{R}^n$, $b \in \mathbb{R}$ for the primal problem.

There are several problems of classification, clustering, regression in ML which can be casted into optimization models, and many a time duality theory is used to solve the modeled problems.

Dual Simplex Method

In the simplex method, we always maintain the feasibility of the primal problem and tries to achieve the feasibility of the dual problem.

Now, we look at it by rotating it by 90°. That means, we think of an algorithm maintaining the dual feasibility and tries to obtain the feasibility of the primal. This variant is called the dual simplex method.

1) It is similar to the simplex method applied on the

primal with appropriate modifications

2) Each basis in every iteration is dual feasible (means $z_j - c_j$ row ≥ 0) by primal infeasible (means some $x_{B_i} < 0$)

3) At the final optimal table, both primal and dual feasibility attained (assuming the problem has optimal soln)

<u>Step 1:</u> The problem is redrafted in a form that at the initial iteration we make sure to have B such that $z_j - c_j \geq 0 \ \forall j$ (for maximization), $z_j = c_B^T B^{-1} a_j$

Although in this method we may allow some $x_{B_i} < 0$

where $x_B = B^{-1} b$.

Step 2: Select the leaving variable x_{B_r} as the one with the most negative among all those $x_{B_i} < 0$.

Step 3: Determine the entering variable x_k by the following ratio criterion

$$\frac{z_k - c_k}{y_{rk}} = \max \left\{ \frac{z_j - c_j}{y_{rj}} : y_{rj} < 0 \right\}$$

Thus, k^{th} column is the pivot column & r^{th} row is the pivot row in the table. $\therefore$ $y_{rk} < 0$ is the pivot element.

Step 3: Create a new table by standard procedure of transformations & proceed.

→ If all $x_{B_i} \geq 0$, the solution is reached, so stop the algorithm, else continue.

→ If $\nexists$ any $y_{rj} < 0$ in the r^{th} pivot row (or $y_{rj} \geq 0 \ \forall \ j = 1, 2, --, n$), then the given problem is infeasible.

eg $\max\ z = -5x_1 - 35x_2 - 20x_3$

s.to. $x_1 - x_2 - x_3 \le -2 \rightarrow x_1 - x_2 - x_3 + x_4 = -2$ slack

$-x_1 - 3x_2 \le -3 \rightarrow -x_1 - 3x_2 + x_5 = -3$ slack

$x_1, x_2, x_3 \ge 0 \qquad x_1, x_2, x_3, x_4, x_5 \ge 0$

Note: $B = I$ and $z_j - c_j = c_B^T B^{-1} a_j - c_j = -c_j \ge 0$ (to start) or all $c_j \le 0$.

note $x_{Bi} < 0$.

c_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5
0	x_4	-2	1	-1	-1	1	0
0	x_5	-3	-1	-3	0	0	1 $\rightarrow$
	$z_j - c_j \longrightarrow$		5 $\uparrow$	35	20	0	0 $\leftarrow$ note $z_j - c_j \ge 0$

x_5 is most $-$ve so it leaves the basis, and row 2 in the table is the pivot row. Notice we do have $-$ve value in y_{2j}'s, $j = 1, 2, \ldots, 5$

To find the entering variable, we use the max. ratio criterion

$$\max\left\{ \frac{z_j - c_j}{y_{2j}} : y_{2j} < 0 \right\}$$

$$= \max\left\{ \frac{5}{-1}, \frac{35}{-3} \right\} = -5$$

$\therefore$ -1 is the pivot element

In the dual simplex method, the pivot element is < 0.

negative value.

C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	
0	x_4	-5	0	-4	-1	1	1	$\rightarrow$
-5	x_1	3	1	3	0	0	-1	
		$z_j - c_j \rightarrow$	0	20	20	0	5	$\rightarrow \geq 0$
				$\uparrow$				

x_4 leaves the basis &

$$\max\left\{\frac{z_j - c_j}{y_{1j}} : y_{1j} < 0\right\} = \max\left\{\frac{20}{-4}, \frac{20}{-1}\right\} = -5$$

$\Rightarrow x_2 \uparrow$

C_B	V_B	x_B	y_1	y_2	y_3	y_4	y_5	
-35	x_2	$5/4$	0	1	$1/4$	$-1/4$	$-1/4$	
-5	x_1	$-3/4$	1	0	$-3/4$	$3/4$	$-1/4$	$\rightarrow$
		$z_j - c_j \rightarrow$	0	0	15	5	10	

$$\max\left\{\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0\right\} = \max\left\{-\frac{60}{3}, -40\right\} = -20$$

$\therefore$ $-3/4$ is the pivot element.

C_B	v_B	x_B	y_1	y_2	y_3	y_4	y_5
-35	x_2	1	$1/3$	1	0	0	$-1/3$
-20	x_3	1	$-4/3$	0	1	-1	$4/3$
	$z_j - c_j \rightarrow$		20	0	0	20	5

As $z_j - c_j \geq 0 \ \forall j$ and $x_{B_i} \geq 0 \ \forall i \Rightarrow$ stop.

optimal soln is $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 1$ & opt. value $= -55$.

The dual simplex method is most suitable for problems for which an initial dual feasible soln is easily available.

The method does not require $b \geq 0$ in LPP and hence not require any artificial variable

But we shall be seeing the importance of this method with more clarity while developing the cutting plane technique to solve integer linear programs.