Major

Graded

Student

Hemant Ramgaria

Total Points

25 / 40 pts

Question 1

Chebyshev Polynomials

1 / 6 pts

- **+ 1 pt** Orthogonal on [-1,1]
- **+ 1 pt** Degree of Chebyshev's polynomial
- **+ 1 pt** General Polynomial Form Using Roots
- **+ 1 pt** any polynomail in combination of Chebyshev's polynomial
- **+ 2 pts** Final Proof
- ✔ **+ 1 pt** Partial mark
- **+ 0 pts** Click here to replace this description.

Question 2

Lagrange Polynomial Basis

3 / 8 pts

- ✔ **+ 1 pt** Lagrange basis
- ✔ **+ 1 pt** Basis property
- **+ 2 pts** $\sum_{k=0}^{n} l_k(x) = 1$
- **+ 2 pts** First proof
- **+ 2 pts** Second proof
- ✔ **+ 1 pt** Partial mark
- **+ 0 pts** Wrong

1 How?

Question 3

Simpson's Rule

Resolved **4** / 6 pts

✔ **+ 1 pt** For taking the correct node points and correct function values at the node points.

✔ **+ 1 pt** For writing correct formula for composite Simpson’s rule.

✔ **+ 1 pt** For finding the correct approximate value of the integral.

✔ **+ 1 pt** For finding true error.

+ 1 pt For writing correct expression for theoretical error.

+ 1 pt For finding the correct bound of theoretical error.

+ 0 pts Completely wrong.

Regrade Request — **Submitted on: Nov 26**

Sir my format for theoretical error is correct I found wrong value of upper bound please provide partial marks for that. I am just at boundary so this 1 partial marks is also matters very much to me.

No, it is not correct. It should be multiplied by 2 and not by 4.

Reviewed on: Nov 27

Question 4

QR Decomposition

Resolved **5** / 6 pts

+ 0 pts completely wrong or not attempted

+ 6 pts Completely correct

✔ **+ 1 pt** q_1 is correct

✔ **+ 1 pt** q_2 is correct

✔ **+ 1 pt** q_3 is correct

✔ **+ 1 pt** Q is correct

✔ **+ 1 pt** All r_{ij}'s are correct

+ 1 pt R is correct

R is 3*3 matrix

Regrade Request — **Submitted on: Nov 26**

Sir I by mistake add last row in error as clearly 3*3 matrix only multiply with matrix of row 3. Please provide partial 0.5 marks in it. Thanks.

No marks will be awarded for wrong calculations.

Reviewed on: Nov 28

Question 5

Five Point Central Difference Formula

Resolved **6** / 8 pts

+ 8 pts for totally correct

✔ **+ 4 pts** To show , five point central difference formula for approximating the derivative of smooth function is exact for all polynomial of four degree or less

+ 1 pt To write formula

+ 0 pts for totally wrong or not attempted

+ 4 pts To find optimal value of h which minimize both roundoff errors and approximation error.

− 1 pt for not showing how h is minimum.

✔ **+ 2 pts** for partially correct

you have not given right explanation for first part.

Regrade Request **Submitted on: Nov 27**

sir I have given explanation all f'''' are considered in it and the error in only in f''''' terms which will zero if polynomial is till degree 4 please please provide marks for explanation. This 1 mark also matters very much for me as course total is just at boundary(79) so have to put this again in regrade because this may miss in last time.

okay

Reviewed on: Nov 28

Regrade Request **Submitted on: Nov 26**

Sir please check error and provide partial marks in it there is only slight calculation mistake and I get complete zero in that part please provide some partial marks in that and I have given explanation all f'''' are considered in it and the error in only in f''''' terms which will zero in errors please provide marks in it.
Thanks for consideration.

done.

Reviewed on: Nov 27

Question 6

Newton's Interpolation

6 / 6 pts

✔ **+ 6 pts** Completely Correct

+ 0 pts Completely incorrect

+ 5 pts 10*0.5 for each correct value of entry in table

+ 1 pt polynomial completely correct

+ 0.5 pts polynomial partially correct

Major Exam of MTL 107: Numerical Methods and Computation

Total Marks: 40 Time:100 Minute

Name: Hemant Ramgaria **Entry No:** 2022MT11854

1. (6 Marks) Let $\{x_k\}_{k=1}^{j}$ be the roots of the j-th degree Chebyshev's polynomials. Prove that for any polynomial $p(x)$ of degree less than or equal to $(j-1)$, we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} p(x)(x-x_1)\cdots(x-x_j)dx = 0.$$

~~$x_k = \cos\left(\frac{2j-1}{2j}\right)$~~ for Chebyshev's polynomial.

$$x_k = \cos\left(\frac{2j-1}{2j}\pi\right) \qquad w = \frac{1}{\sqrt{1-x^2}}$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} p(x)(x-x_1) \cdots (x-x_j)dx$$

$$p(x) * \phi(j)\ \phi(x)$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}$$

if $j = i$ $\int \phi_j \phi(x) = \frac{\pi}{2}$

else 0

Question 1(cont...):

$L(x)$ $P(x)-$ 3

2. (8 Marks) Prove that

$$\ell_i'(x_i) = -\sum_{k=0,k\neq i}^{n} \ell_k'(x_i)$$

for any integer $n \geq 1$, and any set of distinct points $x_0, x_1, \ldots, x_n$. Here, for any real x, $\ell_k(x)$, $k = 0 \ldots n$ denotes the Lagrange basis polynomials.

Further, prove or disprove the following $\ell_i^{(m)}(x_i) = -\sum_{k=0,k\neq i}^{n} \ell_k^{(m)}(x_i)$, where $m > 1$ is an integer and $\ell_i^{(m)}$ denotes the m-th derivative of ℓ_i.

$$l_i(x_i) = \cancel{(x - x_i)} - \prod_{\substack{j=0\\ j\neq i}}^{m} \frac{(x - x_j)}{(x_j - x_j)}$$

$(x-a)(x-b)(x-c)\cdots$

$$l_i'(x_i) = \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{m} (x_j - x_j)} \left(\sum_{j=0, j\neq i}^{m} \frac{l_k(x)}{(x - x_i)} \right.$$

as deta each fa each fact $(x-a)$ ith removed or all remain same

$\frac{l_k(x)}{(x-x_i)}$ is that same fact at upside

$\Rightarrow \sum$

$\Rightarrow$ $\cancel{l_j(x) = l_{j-1} \frac{(x - x_{i+1})}{(x - x_{j+1})}}$

Ans

$$l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{m} \frac{(x - x_j)}{(x_j - x_j)}$$

$$l_i'(x) = \frac{\sum_{\substack{j=0\\ j\neq i}}^{m} \left(\prod_{\substack{t=0\\ t\neq i\neq j}}^{m} (x - x_t) \right)}{\prod_{\substack{j=0\\ j\neq i}} (x_j - x_j)}$$

$f(i)$

$$\Rightarrow \sum_{\substack{j=0\\ j\neq i}}^{m} \left[\left(\prod_{\substack{j=0\\ t\neq i\neq j}}^{m} \frac{(x - x_i)}{(x_i - x_j)} \right) \times \frac{1}{(x_i - x_j)} \right]$$

Question 2(cont...):

$\Rightarrow$ ~~$\emptyset$~~

$$\left[\prod_{\substack{j=0\\ j\neq i}}^{m} \frac{(x - x_j)}{(x_i - x_j)}\right] \times \frac{1}{(x_i - x_j)}$$

~~But~~ put $x = x_i$

$\Rightarrow$ $-l_k'(x_j)$

By eq (i)

$$\Rightarrow \sum_{\substack{k=0\\ k\neq i}}^{m} \left[l_k(x_j) \right)\bigg)$$

Hence proved.

~~Next part is not true~~

~~Let us disprove by eq~~

Question 2(cont...):

3. (6 Marks) Use the Simpson's rule with $n = 4$ to find the approximate value of $F(2\pi)$ where the function $F(x)$ is defined by

$$F(x) = \int_0^x t\ \sin(t)\ dt.$$

Also, compute the true error and the error in the approximation (theoretical error).

so apply simpson in 4 intervals $(0, \frac{\pi}{2})$ $(\frac{\pi}{2}, \pi)$ $(\pi, \frac{3\pi}{2})$ $(\frac{3\pi}{2}, \pi)$

$f(x) = t\sin(t)$

$h = (b-a)$

Simpsons rule $\Rightarrow \frac{h}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$

(simpson on first interval)

$I_1 \Rightarrow \frac{h}{6}\left(0 + 4\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{2}\right)$

$b = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{12}\left(\frac{4\times\pi}{4\sqrt{2}} + \frac{\pi}{2}\right)$

$a = 0$

$f(x) = t\sin(t)$

$h = \cancel{b}\left(\frac{\pi}{2}\right)$

$I_2 = \frac{h}{6}\left(\frac{\pi}{2} + 4\times\frac{3\pi}{4}\sin\left(\frac{3\pi}{4}\right) + \pi(\sin(\pi))\right)$

$b = \pi$

$a = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{12}\left(\frac{\pi}{2} \cancel{6} + \frac{3\pi\times 4}{4\sqrt{2}}\right)$

$I_3 = \frac{h}{6}\left(0 + 4\frac{5\pi}{4}\sin\left(\frac{5\pi}{4}\right) + \frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right)\right)$

$b = \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{12}\left(\frac{-5\pi\times 4}{4\sqrt{2}} - \frac{3\pi}{2}\right)$

$a = \pi$

$I_4 = \frac{h}{6}\left(\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right) + 4\times\frac{7\pi}{8}\sin\left(\frac{7\pi}{8}\right) + 2\pi\sin(2\pi)\right)$

$b = 2\pi$

$a = \frac{3\pi}{2}$

$I_4 = \frac{\pi}{12}\left(\frac{-3\pi}{2} - \frac{7\pi\times 4}{8\sqrt{2}}\right)$

Question 3(cont...):

$$I = I_1 + I_2 + I_3 + I_4$$

$$= \frac{\pi}{12}\left(0 + 4 \times \frac{\pi}{4} \times \frac{1}{\sqrt{2}} + 2 \times \frac{\pi}{2} + 4 \times \frac{3\pi}{4} \times \frac{1}{\sqrt{2}} + \cancel{2 \times 0} + 4 \times \frac{5\pi}{4} \times \left(\frac{-1}{\sqrt{2}}\right) + 2 \times \frac{3\pi}{2} \times \left(\frac{-1}{1}\right) + 4 \times \frac{7\pi}{4}\left(-\frac{1}{\sqrt{2}}\right) + 0\right)$$

$$\Rightarrow \frac{\pi}{12}\left(-\frac{8\pi}{\sqrt{2}} - 2\pi\right)$$

$$\Rightarrow -\frac{\pi^2}{6}(1 + 2\sqrt{2})$$

Actual integr $\Rightarrow \int_0^{2\pi} t \sin t \Rightarrow t\int \sin t - \int\int \sin t \Big|_0^{2\pi}$

$$\Rightarrow t(-\cos t) + \sin t \Big|_0^{2\pi}$$

$$\Rightarrow 2\pi(-1) - (0)$$

$$\Rightarrow -2\pi$$

Actual error $\Rightarrow \left| -2\pi - \left(-\frac{\pi^2}{6}(1+2\sqrt{2})\right) \right|$

Actu error $\Rightarrow \left| -2\pi + \frac{\pi^2}{6}(1+2\sqrt{2}) \right|$

$f'(x) = \sin(t) + t\cos(t)$

$f''(x) = 2\cos(t) - t\sin(t)$

$f'''(x) = -3\sin(t) - \sin(t) - t\cos t$

$f''''(t) = 4\cos(t) - \cos(t) + t\sin(t)$

$f''''(2\pi) = -5$

Theoritical error $\Rightarrow -\frac{f^4(x)}{90}\left(\frac{b-a}{4}\right)^5 \times 4 \Rightarrow$

$f^4(x) \Rightarrow \cancel{\sin(x)}\ \frac{5}{90}(\cancel{2\pi})\ \left(\frac{\pi}{2}\right)^5 \times 4$

$f^4(x) = \frac{5}{90}\left(\frac{\pi}{2}\right)^5 \times 4$

$x \to \cos \xrightarrow{2} -\sin x \xrightarrow{3} -\cos x \xrightarrow{4} \cancel{\sin x}$ $\quad t\sin t \to t\cos t + \sin t \to -t\sin t + \cos t\ -\sin + \cos(t)$ $\quad -t\cos t - \sin t$

4. **(6 Marks)** Find the QR decomposition of the following matrix using Gram-Schmidt orthogonalization:

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

we do it by Gram-Schmidt

$[a_1 | a_2 | a_3]$ $\qquad a_1 = (-1, 1, -1, 1)$

$a_2 = (-1, 3, -1, 3)$

$a_3 = (1, 3, 5, 7)$

$u_1 = a_1 \Rightarrow e_1 = \frac{u_1}{\|u_1\|_2}$

$u_2 = a_2 - (a_2 \cdot e_1) e_1 \qquad e_2 = \frac{u_2}{\|u_2\|_2}$

$u_3 = a_3 - (a_3 \cdot e_1) e_1 - (a_3 \cdot e_2) e_2 \qquad e_3 = \frac{u_3}{\|u_3\|_2}$

$u = (-1, 1, -1, 1)$ $\qquad \boxed{e_1 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)}$ $\quad \|u_1\| = 2$

$u_2 = (-1, 3, -1, 3) - (4)\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$ $\qquad \boxed{e_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$

$u_2 = (1, 1, 1, 1)$ $\qquad \|u_2\| = 2$

$u_3 = (1, 3, 5, 7) - 2(e_1) - 8(e_2)$

$= (1, 3, 5, 7) - (-1, 1, -1, 1) - 4(1, 1, 1, 1)$

$\Rightarrow (-2, -2, 2, 2)$ $\qquad e_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

so $Q = [e_1 | e_2 | e_3]$

$$Q = \begin{bmatrix} -\frac{1}{2} & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$Q^T Q = I$

Question 4(cont...):

$$R = \begin{bmatrix} a_1 e_1 & a_2 e_1 & a_3 e_1 \\ 0 & a_2 e_2 & a_3 e_2 \\ 0 & 0 & a_3 e_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{QR = A}$$

$\frac{16}{6}$

5. (8 Marks) Consider the five-point central difference formula for approximating the derivative of a *smooth* function f at x_0 using the given values $f(x_0-2h)$, $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$ and $f(x_0+2h)$ for $h>0$. Show that the formula is exact for all polynomials of degree four or less. Furthermore, carry out the Round-Off error stability analysis of the formula to find the optimal value of h which minimizes both roundoff errors and the approximation error.

$\frac{1}{6}$

$\underline{dr}$ (i) $f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^{4}(x_0) + \frac{h^5}{120} f^5(\xi)$

(ii) $f(x_0-h) = f(x_0) - h f'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^4(x_0) - \frac{h^5}{120} f^5(\xi)$

(iii) $f(x_0+2h) = f(x_0) + 2h f'(x) + \frac{4h^2}{2} f''(x_0) + \frac{8h^3}{6} f'''(x_0) + \frac{16h^4}{24} f^4(x_0) + \frac{(2h)^5}{120} f^5$

(iv) $f(x_0-2h) = f(x_0) - 2h f'(x) + \frac{4h}{2} f''(x_0) - \frac{8h^3}{6} f'''(x_0) + \frac{16}{24} h^4 f^4(x_0) - \frac{(2h)^5}{120} f^5$

$8(i) - 8(ii) - (iii) + (iv)$

$\cancel{1 (iii)}$

$8(i) - 8(ii) - (iii) + (iv)$

this is multiply (i) and (ii) by 8 and add 8 (i) and (iv) and subst

8 (ii) and (iii)

$\Rightarrow$ LHS $= 8 f(x_0+h) - 8 f(x_0-h) - f(x_0+2h) + f(x_0-2h)$

RHS $=$ $0 \cdot f(x_0)$ $12 h f'(x) + 0 \cdot \frac{h^2}{2} f''(x_0) + \frac{16\times 2}{6} f'''(x_0)\times 0$

$+ 0 \cdot f'''(x_0)$

+ error of are $f'''(x)$

RHS $= 12h f'(x) + \left\{ \frac{16h^5}{120} f(\xi) - \frac{(2h)^5}{120} f(\xi) \right\}$

$\cancel{12 f'(x)}$

Question 5(cont...):

equat LHS or RHS

$$f'(x) = \frac{8f(x_0+h) - 8f(x_0-h) - f(x_0+2h) + f(x_0-2h)}{12h}$$

there is no factor till $f^4(x)$ so. formula is exact for all polynomials of degree four or less there is only error of ~~deg~~ $f^5(x)$ [approximate error]

Approximate error $\Rightarrow$ $a\,\frac{\cancel{16}^{4} h^4 f(\varepsilon)}{\cancel{120}_{30} \times \cancel{12}_{3} h}$

$$\Rightarrow \frac{h^4 f(\varepsilon)}{90}$$

and if there is round off error then it will

$$8\,\frac{e + 8e + e + e}{12h} \Rightarrow \frac{18e}{12h}$$

so Total error $e(h)$ $\frac{\cancel{18}^{3}e}{\cancel{12}_{2}h} + \frac{h^4 f(\varepsilon)}{90} \Rightarrow \frac{3e}{2h} + h^4\frac{f(\varepsilon)}{90}$

e to mi erra $e'(h)=0 \Rightarrow \frac{a3e}{2h^2} = \frac{4h^3 f(\varepsilon)}{90}$

$$\Rightarrow h = \sqrt[5]{\frac{3e \times \cancel{90}^{45}}{8\,4\,f(\varepsilon)}}$$

$$\Rightarrow h = \sqrt[5]{\frac{135e}{4f(\varepsilon)}}$$

Question 5(cont...):

6. (6 Marks) Using Newton's forward difference table, first find the interpolating polynomial $p(x)$ for the following data:

x_i	-1	1	3	4	6
y_i	3	5	3	-1	3

3, 2
4 1

$$F[x_k \cdots x_{k+j}] = \frac{F[x_k \cdots x_{k+j-1}] - F[x_{k+1} \cdots x_{k+j}]}{x_{k+j} - x_k}$$

$$p(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) \cdots c_m(x-x_0)\cdots(x-x_{m-1})$$

where $c_m = F[x_0 \cdots x_m]$

x_i	c_0	c_1	c_2	c_3
-1	3			
		1		
			$-2/4 = -1/2$	
1	5			$-1/10$
		-1		
3	3		-1	
		-4		3/5
4	-1		2	
		2		
6	3			

$\frac{(3/5 + 1/10)}{7} \Rightarrow \frac{1}{10}$

$p(x) = 3 + 1\cdot(x+1) - \frac{1}{2}(x+1)(x-1) - \frac{1}{10}(x+1)(x-1)(x-3) + \frac{1}{10}(x+1)(x-1)(x-3)(x-4)$

$p(x) = 3 + (x+1) - \frac{1}{2}(x+1)(x-1) - \frac{1}{10}(x+1)(x-1)(x-3) + \frac{1}{10}(x+1)(x-1)(x-3)(x-4)$

Question 6(cont...):

$3+5 \quad \frac{-15}{2}$

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$