MTL103: Optimization Methods and Applications

Major(2023)

Time: 2 hrs, Total marks: 40.

Read the Honour Code: "As a student of IIT Delhi, I will not give or receive aid in examinations. I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code."

Instructions

- Write your answers neatly and to the point.
- Remember that you will be graded on what you write and not what you intend to write.

Questions:

1. Consider the following graph G where the capacity of each edge is given.

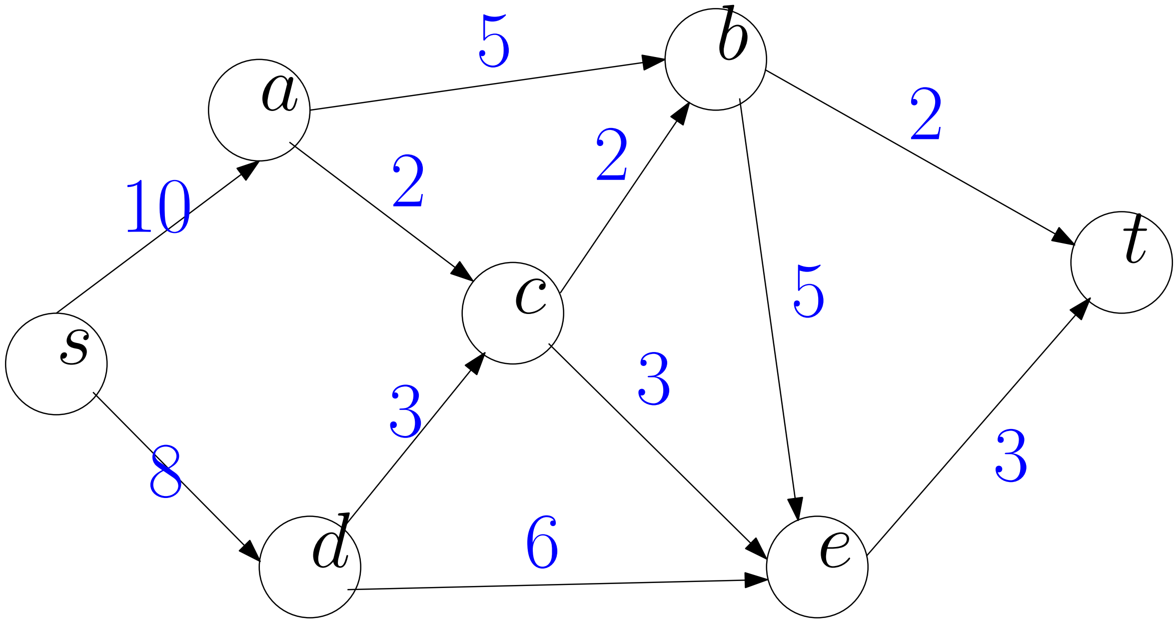

Figure 1: Directed Graph G

   a. Write down the incidence matrix of this graph.

   b. Write down the LP formulation for finding the maximum flow from s to t.

(3+5=8)

2. Suppose we have a polynomial time subroutine **EM** that takes an ellipsoid $E = E(z, D)$ in $\mathbb{R}^n$ and a halfspace $H = \{x \in \mathbb{R}^n | a^T x \geq a^T z\}$ (where a is a non-zero vector) as input; and returns an ellipsoid $E' = E(\overline{z}, \overline{D})$ satisfying the following:

- $\overline{z} = z + \frac{1}{n+1}\frac{Da}{\sqrt{a^T Da}}$,
- $\overline{D} = \frac{n^2}{n^2-1}(D - \frac{2}{n+1}\frac{Daa^T D}{a^T Da})$ is symmetric positive definite,
- $E \cap H \subset E'$,
- $Vol(E') < e^{-1/(2(n+1))} Vol(E)$.

Consider the following LP:

Maximize $c^T x$
Subject to $Ax \leq b$,

where A, b have integer entries with magnitude bounded by some U. Using **EM** as a subroutine, propose an algorithm to find out the optimum solution of the given LP. What is the time complexity of your algorithm?

(10)

3. For a given set of points $x_0, x_1, , x_m \in \mathbb{R}^n$, the Voronoi cell $V(x_i)$ associated with the point x_i is defined as follows: $V(x_i) = \{x \in \mathbb{R}^n | d(x_i, x) \leq d(x_j, x) \forall j \neq i\}$. Here, $d(x, y)$ is the Euclidean distance between the points x and y. Prove that the set $V(x_i)$ is a convex set. *(4)*

4. Let $A = [a_{ij}]$ be a pay-off matrix in a two-person zero-sum game. Let us assume that two elements a_{ij} and a_{hk} are saddle points. Prove or disprove that a_{ik} and a_{hj} are also saddle points. *(6)*

5. Consider the following LP for some fixed $0 < \epsilon < 1/2$.

Minimize x_3
Subject to $x_1 - r_1 = \epsilon$,
$x_1 + s_1 = 1$,
$x_j - \epsilon x_{j-1} - r_j = 0,\ j = 2, 3$
$x_j + \epsilon x_{j-1} + s_j = 1,\ j = 2, 3$,
$x_j, r_j, s_j \geq 0,\ j = 1, 2, 3$.

(a). How many BFS can it have? Are all of them non-degenerate? Justify your answer.
(b) Let $N \in \mathbb{Z}$ be the answer to the part (a). Enumerate these BFS in an order $S_1, S_2, \ldots S_i, \ldots, S_N$ such that the cost of the objective function is strictly decreasing and for each $i \in [N-1]$, the basic feasible solutions S_i and S_{i+1} are adjacent BFS. Here each BFS S_i is a vector of the form $(x_1, x_2, x_3, r_1, r_2, r_3, s_1, s_2, s_3)$. You need to specify their values to distinguish one from the other. *(4+8=12)*