Lecture 27

MTL 122 - Real and Complex Analysis

Complex power

$z \in \mathbb{C}$

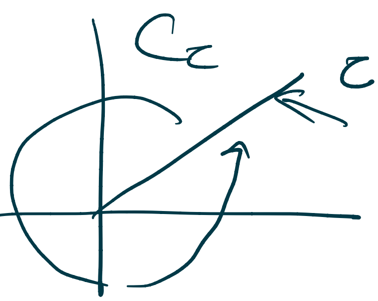
$$\log z = \ln|z| + i\underline{\arg(z)}$$

↳ multivalued function

<u>Arg z</u> → single valued

$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$

$c \in \mathbb{C}$



$$\log_c z = \ln|z| + i \arg_c z$$

holomorphic in $\mathbb{C}_c^0$

Complex powers

$z \neq 0$

Want to define

α - th power of z,

$$\boxed{z^{\alpha} \equiv e^{\alpha \underline{\log z}} = e^{\alpha(\ln|z| + i \arg z)}}$$

Since $\arg z$ is multiplevalued

$\Rightarrow$ z^{α} will have infinite values

$$z^{\alpha} = e^{\alpha(\underbrace{\ln|z| + i \operatorname{Arg} z}_{\operatorname{Log} z} + i 2\pi k)}$$

$k = 0, \pm 1, \pm 2, \ldots$

$$= e^{\alpha \operatorname{Log} z + i\alpha 2\pi k} \qquad [e^{z_1+z_2} = e^{z_1}e^{z_2}]$$

$$= e^{\alpha \operatorname{Log} z}\, e^{i\alpha 2\pi k},$$

$k = 0, \pm 1, \pm 2, \ldots$

Depending on the values of 'α'

$\exp(i2\pi\alpha k)$ will have either one, finitely many / infinitely many value.

Suppose, $\alpha \in \mathbb{R}$.

Case: α integer.

If $\alpha = n \in \mathbb{Z}$

$$\Rightarrow \alpha k = nk$$

$$\Rightarrow e^{i2\pi\alpha k} = e^{i2\pi nk} = 1$$

$\therefore$ there is one value of z^n when $n \in \mathbb{Z}$.

$$z^n = \begin{cases} 1, & n = 0 \\ \underbrace{z.z.z\cdots}_{n \text{ times}}, & n > 0 \\ \dfrac{1}{z^{-n}} & n < 0. \end{cases}$$

Case: α rational.

$$\alpha = p/q$$

Then $z^{p/q}$ will have finite number of values.

$$e^{i2\pi k p/q} , \quad k \in \mathbb{Z}$$

$$e^{i2\pi (k+q) p/q} = e^{i2\pi (k p/q + p)}$$

$$= e^{i2\pi k p/q} \cdot \underbrace{e^{i2\pi p}}_{= 1}$$

$$= e^{i2\pi k p/q}.$$

$\Rightarrow$ $z^{p/q}$ will have atmost q distinct values corresponding to the above formula with, $k = 0, 1, 2, \ldots, q-1$.

Case: α is irrational

- z^α will have infinite no of values,

Suppose $k, k' \in \mathbb{Z}$ s.t

$$e^{i\alpha 2\pi k} = e^{i\alpha 2\pi k'}$$

$$\Leftrightarrow \quad e^{i\alpha 2\pi (k-k')} = 1$$

$$\Leftrightarrow \quad \underline{\alpha (k-k')} \in \mathbb{Z}$$

This will true only when

$$k - k' = 0$$

or $k = k'$,

$\Rightarrow$ z^α must have infinite number of values.

Ex. $1^{1/q}$

$$1^{1/q} = e^{\frac{1}{q}\mathrm{Log}(1)} \cdot e^{i2\pi(k/q)}$$

$$= e^{i2\pi(k/q)}$$

$k = 0, 1, 2, \ldots, q-1$ are diff.

$1^{1/q} \rightarrow q^{th}$ root of unity.

- Every branch of the logarithm gives rise to the branch of z^{α}.

- Principal branch of z^{α}
$= e^{\alpha \mathrm{Log}(z)}$.

- $$\frac{d}{dz}\left(e^{\alpha \operatorname{Log} z}\right)\Big|_{z=z_0}$$

$$= e^{\alpha \operatorname{Log}(z_0)} \frac{\alpha}{z_0}.$$

- $$\frac{d}{dz}\left(z^{\alpha}\right)\Big|_{z=z_0}$$

$$= \alpha \underline{z_0^{\alpha} \frac{1}{z_0}}. \qquad z_0 \neq 0$$

(provided we use the same branch of z^{α} on both sides of the eqn).

Note: $\alpha z_0^{\alpha-1}$

$z^{\alpha} z^{\beta} = z^{\alpha+\beta}$ (provided they have the same branch)

Remember: One identity does not hold.

$\Rightarrow \alpha \in \mathbb{C}$, $z_1, z_2 (\neq 0) \in \mathbb{C}$

$$z_1^\alpha z_2^\alpha \neq (z_1 z_2)^\alpha$$

Complex integration

$$\int_a^b f(t) = \int_a^b u(t) + i \int_a^b v(t)$$

- Complex version of Fundamental theorem of Calculus:

$f(t) \rightarrow$ continuous in $[a, b]$

& $\exists$ $F(t)$ on $[a, b]$

such that $\dot{F}(t) = f(t)$, $a \le t \le b$

where $\dot{F}(t) = \dfrac{dF}{dt}$.

Then

$$\int_a^b f(t) = \int_a^b \frac{dF}{dt}\,dt.$$

$$= F(b) - F(a)$$

$$F(t) = U(t) + iV(t)$$

$$\Rightarrow \dot{F}(t) = \dot{U}(t) + i\dot{V}(t)$$

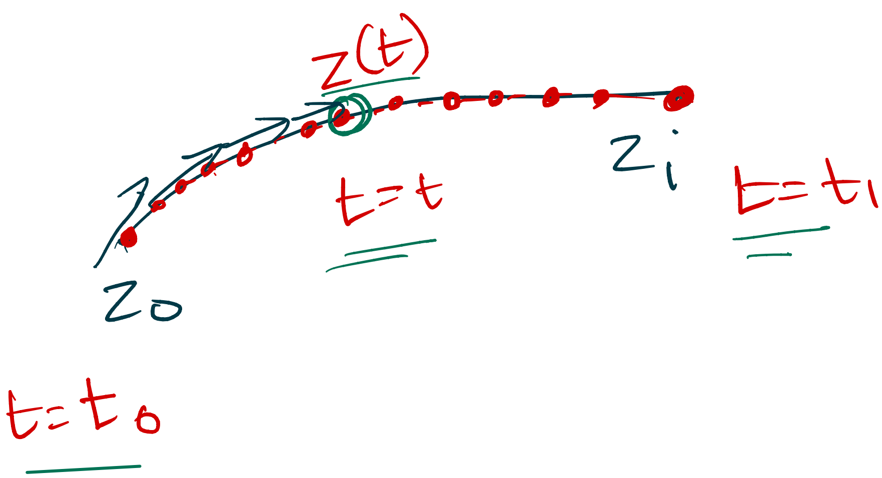
$$\Rightarrow \dot{U}(t) = u(t),$$

$$\dot{V}(t) = v(t).$$

$$\int_a^b f(t)\,dt = (U(b) - U(a)) + i(V(b) - V(a))$$

- $$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt.$$

Integral along a parametrised curve.



Curve: $t \longrightarrow z(t)$, $\quad z(t_0) = z_0$

$\uparrow \qquad\qquad\qquad\qquad\qquad\quad z(t_1) = z_1$.

$[t_0, t_1]$

Parametrised curve

A continuous fn.

$z : [t_0, t_1] \to \mathbb{C}$ s.t

$z(t_0) = z_0$ & $z(t_1) = z_1$.

$\Updownarrow$

two continuous fn.

$x : [t_0, t_1] \to \mathbb{R}$, $x(t_0) = x_0$, $x(t_1) = x_1$

$y : [t_0, t_1] \to \mathbb{R}$ $y(t_0) = y_0$, $y(t_1) = y_1$

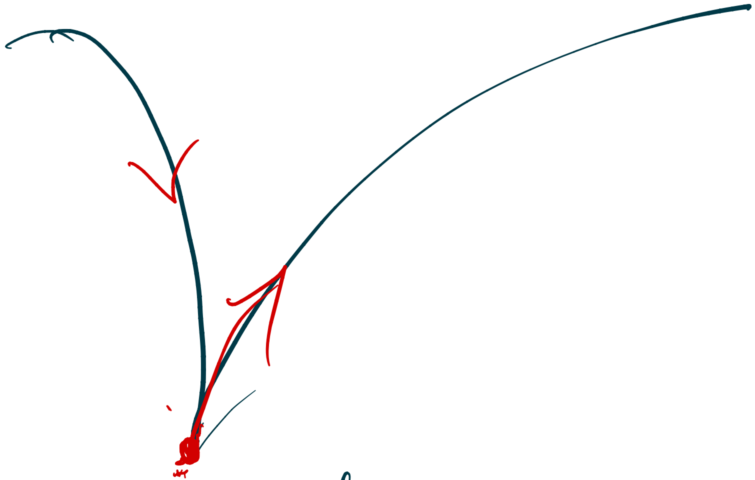
Smooth

$z(t)$ is smooth if

$\dot{z}(t)$ is a continuous fn.

& is $\neq 0$. (No cusps).

- $t \mapsto (t^3, t^2)$



Contour / piecewise smooth curve.

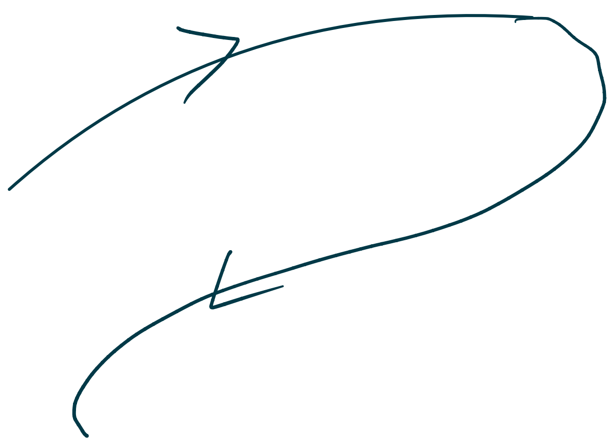
is a curve that is obtained by joining finitely many smooth curves.

- $\gamma_1(t) = e^{it}, \ t \in [0,1]$

$$\gamma_2(t) = (1-t)a + tb.$$

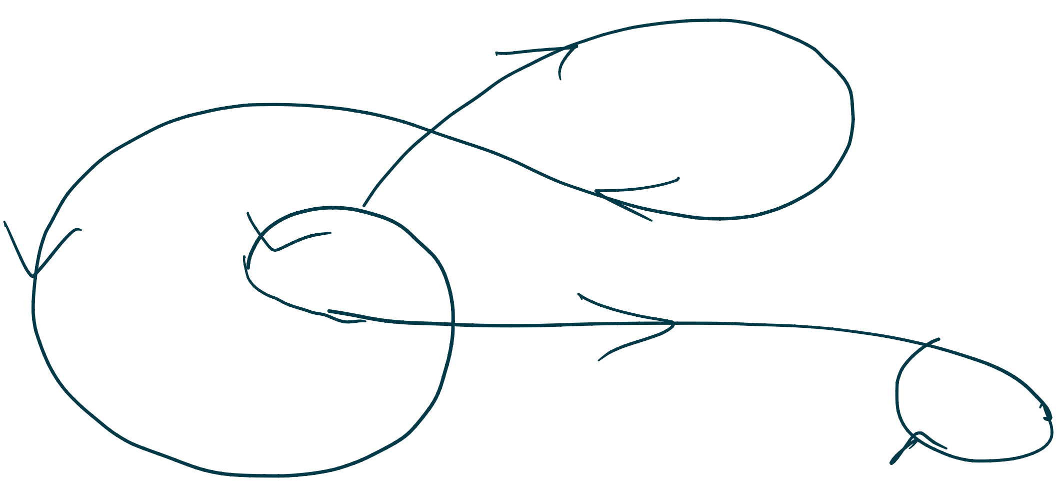
- $$z = \begin{cases} x + ix, & 0 \le x \le 1. \\ x + i, & 1 \le x \le 2. \end{cases}$$

<u>Simple</u> arc/curve

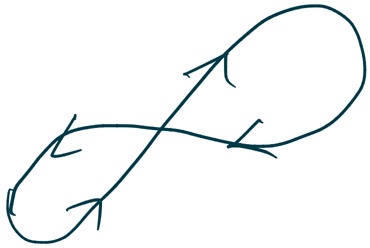


simple open

simple closed

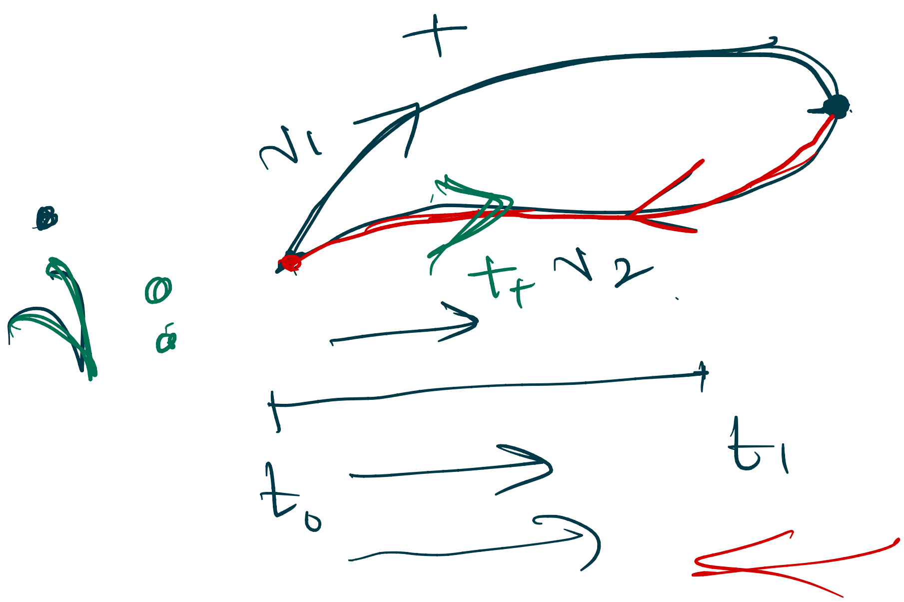


not simple



not simple.

Positive Orientation



$v_1 \to +$

$V_2 \to -$

- Contour is closed.

→ counter clockwise positive.

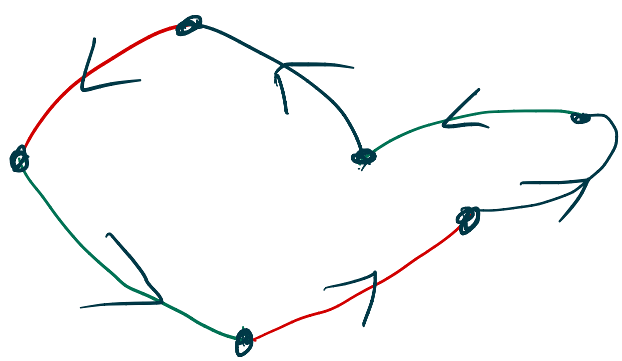
→ clockwise – negative.

Arc Length : $z(t)$

$$L = \int_a^b |z'(t)|\,dt.$$

$$= \int_a^b \sqrt{(x')^2 + (y')^2}\,dt.$$

Jordan theo.



A simple closed curve / closed contour divides the complex plane into two sets the interior is bdd

& exterior is unbdded.