Lecture - 14

MTL - 122 -

Real & Complex

Analysis.

- $f : X \to Y$ Lipschitz cont.

$\Rightarrow$ f is uniformly cont.

$$d_Y(f(x), f(y)) \leq K\, d_X(x, y) \quad \forall x, y \in X.$$

$\epsilon > 0$ $\quad \delta(\epsilon) = \frac{\epsilon}{K}$

Then, if $d_X(x, y) < \delta(\epsilon)$

$$\Rightarrow \quad d_Y(f(x), f(y)) < K\,\delta(\epsilon) = \epsilon.$$

$\forall x, y \in X.$

$\Rightarrow$ f is uniformly cont.

Converse not true.

$f(x) = \sqrt{x}$, $[0, \infty)$

$x = 1/n, \quad y = 0$

$$|f(1/n) - f(0)| = \sqrt{1/n}$$

$$= \sqrt{n}\,|1/n - 0|$$

$\epsilon > 0$, $\delta = \epsilon^2$ (assume)

$|x-y| < \delta$, $x, y \geq 0$

- $x, y \in [0, \delta)$

$$|f(x) - f(y)| < f(\delta) - f(0) = \sqrt{\delta} = \epsilon.$$

$x \notin [0, \delta)$ or $y \notin [0, \delta)$

$\max(x, y) \geq \delta$

$$|\sqrt{x} - \sqrt{y}| < \frac{|x-y|}{\sqrt{\max(x,y)}} < \frac{\delta}{\sqrt{\delta}} = \epsilon$$

$(X, d_{dis}) \xrightarrow[\text{continuous}]{} (Y, d)$.

$(S = \{1/n\}, d)$

$f(1/n) = n \longrightarrow$ cont. (?)

This is not u.c.

$\delta > 0$, $m, n \in \mathbb{N}$.

$m > n > 2\delta$.

$|1/m - 1/n| < \delta$ $|m-n| \geq 1$.

---x---

- $f: X \to Y \to$ uniformly

Check: Uniformly continuity does not preserve (Example bdd.

- f (u.c.) maps every C. seq of X onto a Cauchy seq of Y.

Proof: (x_n) is a C.S in X.

- $(f(x_n))$ is Cauchy in Y.

1) $\epsilon > 0$, $\exists \delta > 0$ s.t

$d_Y(f(x), f(y)) < \epsilon$ ✓

whenever $d_X(x, y) < \delta$

$\forall x, y \in X$.

- $\delta > 0$ $\exists N \in \mathbb{N}$ s.t $m, n > N$

$d_X(x_n, x_m) < \delta$ ✓

• $d_Y(f(x_n)\ f(x_m)) < \epsilon$

$\forall\, m, n > N$

$\Rightarrow \{f(x_n)\}$ is a C.S.

f map a totally bdd set of X $\Rightarrow$ t. bdd set of Y

f maps compact subspace X onto compact subspace of Y.

Compact sets.

$\mathbb{R}^n$

$[a, b] \leftarrow$

- Bolzano Weierstrass
- Heine-Borel theorem $\Big\} \rightarrow$ generalisation

Let (X, d)

Covering : collection of sets whose union is X.

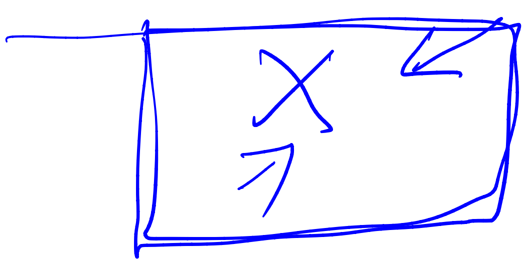
$\{U_i : i \in I\} \leftarrow U_i \rightarrow$ open sets

$Y \subseteq X$.

$'Y' \subseteq \bigcup_{i \in I} U_i$ $\qquad \cancel{X \subseteq \bigcup_{i \in I} U_i}$

$\{U_i\} \rightarrow$ open cover of Y.

$'Y' \swarrow$



Sub cover $\rightarrow$ $Y \subseteq \bigcup_{i\in I} U_i$

$\{U_i ; i \in I\}$ cover.

$\boxed{J \subset I}$

$\{U_j ; j \in J\}$ $\leftarrow$ subcover of Y.

$$Y \subseteq \bigcup_{j\in J} U_j$$

J: finite $\rightarrow$ finite subcover.

1) Compact: A metric space is compact if every open cover has a finite subcover.

2) A meteric space is <u>seq compact</u> if every seq in X has a convergent subseq.

- $(0, 1] \to$ not seq comp & not comp.

- $\mathbb{R} \leftarrow$ not compact not seq.

$(\underline{X, d}) \to$ <u>compact</u>

- $C = \{\{x\}, x \in X\} \leftarrow$

$X \subseteq \bigcup_{i=1}^{n} \{x_i\} \Rightarrow X$ has to be finite.

- Compact discrete space
  $\Longleftrightarrow$ finite space.

- Closed subspace of a compact metric space is compact. $\Big\}$ $\mathbb{R}$

- Compact $\Rightarrow$ closed & bdd.

=

$K \subseteq X$ compact

$K = \phi$

$\Rightarrow$ bdd closed.

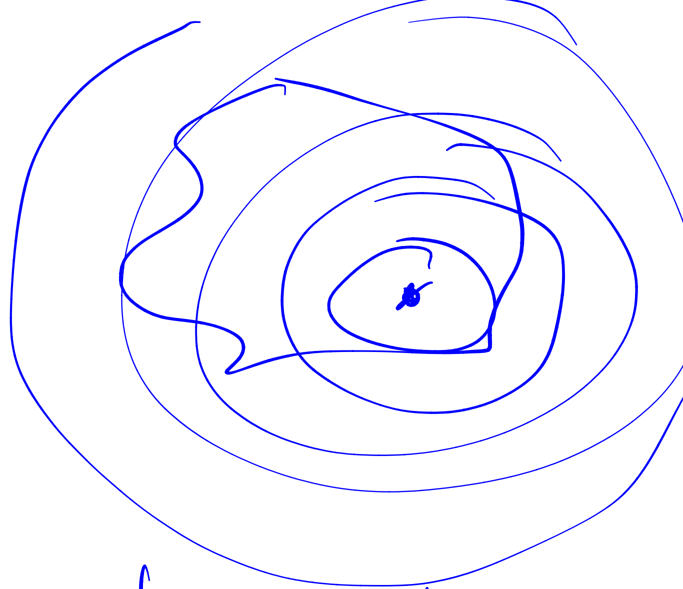
$\underline{K \neq \phi}$

$\underline{x_0} \in K$ (Let)

$\Rightarrow K \subseteq \bigcup_{\underline{n \in \mathbb{N}}} B_n(x_0)$

$F \subseteq \mathbb{N}$ s.t
finite

$K \subseteq \bigcup_{n \in F} B_n(x_0)$



'N' $\rightarrow$ largest element in F.

$K \subset B_N(x_0) \Rightarrow K$ is bdd.

$K \subseteq X$
compact.

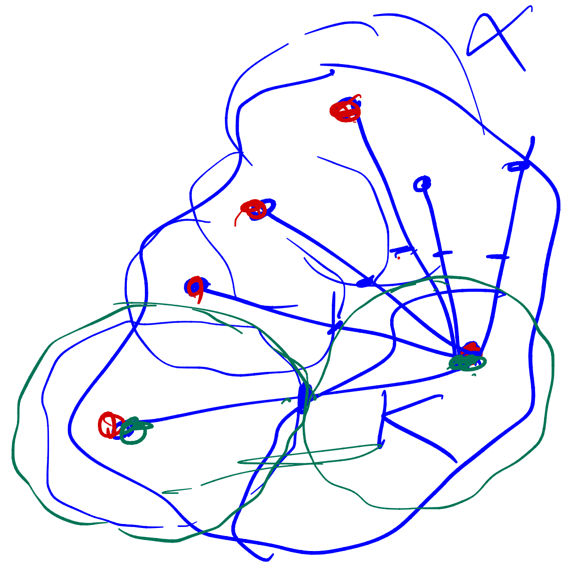
Show K is closed.

$X \setminus K$ is open.

$p \in X \setminus K$.

For each $x \in K$

$\delta_x = \frac{1}{2} d(p, x)$



$\{B_{\delta_x}(x)\}_{x \in K}$

open cover of K.

$\exists \; x_1 \quad x_2 \quad x_3 \; \dots \; x_k$

s.t. $K \subseteq \bigcup_{i=1}^{k} B_{\delta_{x_i}}(x_i)$

$V = \bigcap_{i=1}^{k} B_{\delta_{x_i}}(p) \quad \leftarrow$ open

$B_{\delta_{x_i}}(p) \cap B_{\delta_{x_i}}(x) = \emptyset \quad \forall i$

$V \subset X \setminus K$.

$\Rightarrow \quad K$ closed.

$\mathbb{R}$ ~~is~~ converse is true.

• (X, d_{dis}) ← infinite discrete metric space

$K \subseteq X$ infinite.

$d_{dis}(x,y) = 1, \; x \neq y$
$= 0, \; x = y$

closed? ✓

bdd? ✓

Totally bdd.

$A \subseteq X$ is totally bdd. if for every $\epsilon > 0 \; \exists$ $x_1, x_2 \ldots x_k \in A$ s.t

$\{B_\epsilon(x_i) \mid 1 \le i \le k\}$ is an open cover of A.

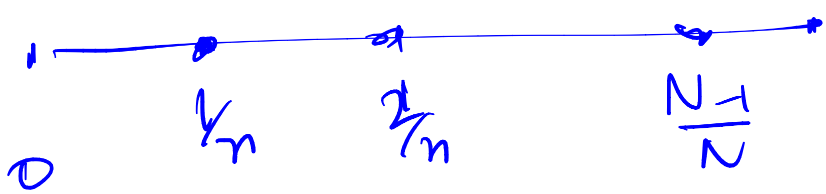
$$A \subseteq \bigcup_{i=1}^{k} B_\epsilon(x_i)$$

$$A \subseteq \bigcup_{x\in A} B_\epsilon(x)$$

$(0,1) \to$ totally bdd.

$\epsilon > 0$

$N \in \mathbb{N}$.



$N > 1/\epsilon$

$$(0,1) \subset \bigcup_{n=1}^{N} B_\epsilon(1/n)$$

$\Rightarrow$ Totally bdd $\Rightarrow$ bdd.

$B \subseteq X$.

totally bdd.

$$B \subset \bigcup_{i=1}^{N} B_1(b_i) \qquad b_i \in B.$$

$$D = \max \{ d(b_i, b_j) \}$$

$x, y \in B$

$$x, y \in B(b_I) \cup B(b_J)$$

$$d(x,y) \leq 1 + 1 + D$$

$$= 2 + D$$

Bdd $\Rightarrow$ Totally bdd.

$$U = \{ e_n \mid n \in \mathbb{N} \} \subset \underline{\underline{\ell^\infty(\mathbb{R})}}$$

$$e_n = (e_{nj})_{j \in \mathbb{N}}.$$

$$e_{nj} = 1 \quad j = n$$
$$= 0 \quad \text{o.w}$$

$$e_1 = (1, 0, 0 \ldots \text{———} 0 \ldots)$$

$$e_2 = (0, 1, 0 \ldots \text{———})$$

$$\epsilon = 1.$$