Tutorial-1 MTL103

1. Prove that the set of all convex combinations of a finite number of L.I. vectors is a convex set.

2. Examine the convexity of the following sets:

   - $\{(x, y, z) : x^2 + y^2 \leq 4, 0 \leq z \leq 3\}$
   - $\{(x, y) : |x| \leq 1, |y| \leq 1\}$
   - $\{(x, y) : y \geq 3 - x^2\}$
   - $\{(x, y) : y \leq e^{-x}\}$

3. Examine the convexity of the following functions:

   - e^{-x}
   - $|x|$
   - $f(x) = -{x_1}^2 - 4{x_2}^2 - 9{x_3}^2 + 2x_1x_2 + 3x_1x_3 + 6x_2x_3$
   - $$f(x) = \begin{cases} x^2 & \text{for} \quad -1 \leq x < 1, \\ 2 & \text{for} \quad x = 1. \end{cases}$$

     What if $f(x) = 1/2$ at $x = 1$?

4. If f is convex on R^n, show that $f(Ax + b)$ is also convex on R^n.

5. If f and g are convex functions, show through an example that fg is not necessarily a convex function.

6. Let h be a non-decreasing convex function on R and f be a convex function on $T \subset R^n$. Then prove that the composite function hof is a convex function on T. Hence or otherwise show that $e^{2{x_1}^2 + {x_2}^2}$ is a convex function on R^2.

7. If $S = \{(x, y) : -x + y \leq 4, 3x - 2y \leq 9, x, y \geq 0\}$, solve graphically the following problems:

   - max $x - 5y$ over S
   - max $6x + 4y + 18$ over S

8. Solve the following problems graphically:

   - max $2x_1 + x_2$ subject to:
     $0 \leq x_1 \leq 2$
     $x_1 + x_2 \leq 3$
     $x_1 + 2x_2 \leq 5$
     $x_2 \geq 0$
   - min $5x_1 + 2x_2$ subject to:
     $x_1 + 4x_2 \geq 4$
     $5x_1 + 2x_2 \geq 10$
     $x_1, x_2 \geq 0$

9. Write the following minimization problem as Linear Optimization Problem.
min $||Ax - b||_\infty$ subject to:
$c^T x \leq d$
$x \geq 0$
where A = $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$, b = $\begin{bmatrix} 5 \\ -6 \end{bmatrix}$, c = $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, d = 30 In the LPP, set $x_2 = 0$ and solve the LPP with graphical methods.

10. An investor is considering investing in two securities A and B. The risk and return associated with these securities is different. Security A gives returns of 9% and a risk factor of 6 on scale of 0-10 and security B gives returns of 15% but has risk factor of 8. Mr. X wants to invest a total capital of 50,000$. He expects a minimum returns of 12% and will trade only if the combined risk factor doesn't exceed 6. Mr. X wants to find the split the capital investing in stocks A and B and maximize the returns. Formulate this as an LPP and solve it graphically.

11. A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available. Profit per unit for A is Rs. 50 and for B is Rs. 40. The factory owner wants to plan the production of A and B such that he bags maximum profit. Formulate as LPP.

MTL103

Ayush Gupta

January 2023

1 Tutorial I

1.1 Prove that the set of all convex combinations of a finite number of L.I. vectors is a convex set.

We will prove this question via induction. For k=2, its easy to see that the convex combination would lie via the property of convexity. Let the statement be true for n=k-1.

Let us take a k length combination of vectors in S

$$\lambda_1 x_1 + \lambda_2 x_2 ... + \lambda_k x_k$$

By induction hypothesis, $y = \frac{\lambda_1 x_1}{\lambda_1 + \lambda_2 + ...\lambda_{k-1}} + \frac{\lambda_2 x_2}{\lambda_1 + \lambda_2 + ...\lambda_{k-1}} + ... + \frac{\lambda_{k-1} x_{k-1}}{\lambda_1 + \lambda_2 + ...\lambda_{k-1}}$ is in S.

If $\lambda_1 + \lambda_2 + ...\lambda_{k-1} = 0$, then we would just be left with x_k which is already in S.
$\{$Note that $\lambda_1, \lambda_2 ...., \lambda_{k-1} >= 0\}$

Hence we can write the k length combination as $(\lambda_1 + \lambda_2 + ...\lambda_{k-1})$y+$x_k$ which is convex combination of two vectors y and x_k hence lies in S.

Hence by induction proved.

1.2 Examine the convexity of :

a) $\mathbf{(x, y, z) : x^2 + y^2 \leq 4, 0 \leq z \leq 3}$

Let the two vectors be (x_1, y_1, z_1) and (x_2, y_2, z_2)

A convex combination of the two vectors is: $(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2, \theta z_1 + (1-\theta)z_2)$

Now, $(\theta x_1 + (1-\theta)x_2)^2 + (\theta y_1 + (1-\theta)y_2)^2 = \theta^2(x_1^2 + y_1^2) + (1-\theta)^2(x_2^2 + y_2^2) + 2\theta(1-\theta)(x_1x_2 + y_1y_2)$ which is $\leq 4\theta^2 + 4(1-\theta)^2 + 8\theta(1-\theta) = 4$.

Also $0 \leq \theta z_1 + (1-\theta)z_2 \leq 3$.

Hence the above expression in convex.

b) $(\mathbf{x},\mathbf{y}) : |\mathbf{x}| \leq \mathbf{1}, |\mathbf{y}| \leq \mathbf{1}$

Let the two vectors be (x_1, y_1) and (x_2, y_2).

A convex combination of the two vectors is: $(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$

Now,$|\theta x_1 + (1-\theta)x_2| \leq \theta + (1-\theta) = 1$.Similarly,$|\theta y_1 + (1-\theta)y_2| \leq \theta + (1-\theta) = 1$

Hence the above expression in convex.

c) $(\mathbf{x},\mathbf{y}) : \mathbf{y} \geq \mathbf{3} - \mathbf{x^2}$

Take $(x_1, y_1) = (\sqrt{3}, 0)$ and $(x_2, y_2) = (-\sqrt{3}, 0)$ and $\theta = 0.5$.

If the convex combination holds we must have $0 \geq 3$ which is false.

d)$(\mathbf{x},\mathbf{y}) : \mathbf{y} \leq \mathbf{e^{-x}}$

Take $(x_1, y_1) = (4, 0)$ and $(x_2, y_2) = (0, 1)$ and $\theta = 1/2$.

If the convex combination also satisfies this relation then we must have $0.5 \leq e^{-2}$ which is false.

1.3 Examine the convexity of following functions :

a) $\mathbf{e^{-x}}$

We know that f is twice differentiable in I. It is convex if and only if $f'' \geq 0$.

$f'' = e^{-x} \geq 0$.Hence e^{-x} is convex.

b) $|\mathbf{x}|$

$f(\lambda x + (1-\lambda)y) = |\lambda x + (1-\lambda)y| \leq \lambda|x| + (1-\lambda)|y| = \lambda f(x) + (1-\lambda)f(y)$

Hence $|x|$ is convex.

c)$-\mathbf{x_1^2} - \mathbf{4x_2^2} - \mathbf{9x_3^2} + \mathbf{2x_1x_2} + \mathbf{3x_1x_3} + \mathbf{6x_2x_3}$

According to my interpretation this question can have two meanings,if it says that f(x) is a constant then we are done.As linear functions are convex as well as concave.

The other and most obvious interpretation is that x is a vector in $\mathbb{R}^{\not\vDash}$.For this case we could re-write the expression as:

$$-0.5 * ((x_1 - 2x_2)^2 + (2x_2 - 3x_3)^2 + (x_1 - 3x_3)^2)$$

which is not a convex function.Take $(x_1, x_2, x_3) = (1, 1, 1)$ and $(y_1, y_2, y_3) = (2, 2, 2)$ and $\theta = 1/2$

1.4 If f is convex on $\mathbb{R}^n$, show that f(Ax + b) is also convex on $\mathbb{R}^n$.

As f is convex we know that, $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$

We need to show that f(Ax+b) is also convex.Let $g(x) = f(Ax + b)$.

Hence:

$g(\theta x + (1-\theta)y) = f(A(\theta x + (1-\theta)y) + b) = f(\theta(Ax+b) + (1-\theta)(Ay+b)) \leq \theta f(Ax+b) + (1-\theta)f(Ay+b) = \theta g(x) + (1-\theta)g(y)$

$\Rightarrow$ f(Ax+b) is also convex.

1.5 If f and g are convex functions, show through an example that fg is not necessarily a convex function.

Take f(x)=x and g(x)=-x.We have that $fg = -x^2$ which is not convex.

1.6 Let h be a non-decreasing convex function on $\mathbb{R}$ and f be a convex function on $T \subset \mathbb{R}^n$. Then prove that the composite function hof is a convex function on T. Hence or otherwise show that $e^{2x_1^2+x_2^2}$ is a convex function on $\mathbb{R}^2$.

We know that h is non decreasing:

If $x \leq y \Rightarrow h(x) \leq h(y)$

Also we have f is convex:

$\Rightarrow f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$

Hence:

$hof(\theta x + (1-\theta)y) = h(f(\theta x + (1-\theta)y)) \leq h(\theta f(x) + (1-\theta)f(y)) \leq \theta * h(f(x)) + (1-\theta) * h(f(y))$

As $h(x) = e^x$ is a non-decreasing convex function and $2x_1^2 + x_2^2$ is a convex function we have that $hof = e^{2x_1^2+x_2^2}$ is a convex function.

1.7 If S=$\{(x,y) : y - x \leq 4, 3x - 2y \leq 9, x, y \geq 0\}$ solve the following equations graphically

a) **max x-5y over S**

Try it out: `https://www.desmos.com/calculator/2qzb6iw8zm`

b) **max 6x+4y+18 over S**

Try it out: `https://www.desmos.com/calculator/2qzb6iw8zm`

1.8 Solve the following problems graphically:

a) **max** $2x_1 + x_2$ **subject to** $0 \leq x_1 \leq 2, x_1 + x_2 \leq 3x_1 + 2x_2 \leq 5, x_2 \geq 0$
,
Try it out: https://www.desmos.com/calculator/ddy8ulnrht

a) **max** $5x_1 + 2x_2$ **subject to** $x_1 + 4x_2 \geq 4, 5x_1 + 2x_2 \geq 10, x_2 \geq 0, x_1 \geq 0$
,
Try it out: https://www.desmos.com/calculator/zwa1yiiiza

1.9 Write the following minimization problem as Linear Optimization Problem:

$\|Ax - b\|_\infty, c^\intercal x \leq d, x \geq 0$
where $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, c = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, d = 30$
In the LPP set $x_2 = 0$ **and solve the LPP with graphical methods.**

The infinity norm of a matrix is the maximum row sum(Taking modulus of each of row elements and summing it up).

So the question is equivalent to stating:

min $\{max(|3x_1 + 4x_2 - 5|, |4x_1 - 3x_2 - 6|)\}$ with

$2x_1 + 5x_2 \leq 30$ and $x_1, x_2 \geq 0$.

To convert it to general form we could set $max(|3x_1 + 4x_2 - 5|, |4x_1 - 3x_2 - 6|)$ =z and write two new constraints $z \geq |3x_1 + 4x_2 - 5|$ and $z \geq |4x_1 - 3x_2 - 6|$.To further simplify it, $z \geq (3x_1 + 4x_2 - 5)$,$z \geq -(3x_1 + 4x_2 - 5)$ and $z \geq (4x_1 - 3x_2 - 6)$ and $z \geq -(4x_1 - 3x_2 - 6)$.For the second part we set x_2 to be 0 and solve graphically.

1.10 An investor is considering investing in two securities A and B. The risk and return associated with these securities is different. Security A gives returns of 9% and a risk factor of 6 on scale of 0-10 and security B gives returns of 15% but has risk factor of 8. Mr. X wants to invest a total capital of 50,000$. He expects a minimum returns of 12% and will trade only if the combined risk factor does not exceed 6. Mr. X wants to find the split the capital investing in stocks A and B and maximize the returns. Formulate this as an LPP and solve it graphically.

Let x be the capital invested in A and y be the capital invested in B. The question formulated as LPP:

$max(0.09x + 0.15y)$
$x + y = 50000$
$0.09x + 0.15y \geq 6000$
$6x + 8y \leq 3,00,000$
$x, y \geq 0$

Solvable?

1.11 A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available. Profit per unit for A is Rs. 50 and for B is Rs. 40. The factory owner wants to plan the production of A and B such that he bags maximum profit. Formulate as LPP.

Let the quantity of two products A and B be x and y respectively. The question formulated as LPP:

$max(50x + 40y)$
$1.5x + 2.5y \leq 300$
$2.5x + 1.5y \leq 240$
$x, y \geq 0$

MTL103
Tutorial 2

1. Graph the convex hull of points (0, 5), (3, 5), (6, 3), (5, 0), (3, 3), (2.5, 2.5). Which of these points are extreme points of the hull? Express the non-extreme point (among given points), if any, as a convex combination of the extreme points.

2. Express the point $x = (0, 1)$ as a convex combination of the extreme points of the set $\{(x_1, x_2)^T : x_1 - x_2 \leq 3, 2x_1 + x_2 \leq 4, x_1 + 3 \geq 0\}$

3. Show that a hyperplane $H = \{x \mid Ax = b\}$ and a half-space $H^+ = \{x \mid Ax \geq k\}$ are convex sets.

4. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function and let c be some constant. Show that the set $S = \{x \in \mathbb{R}^n \mid f(x) \leq c\}$ is convex.

5. Find the number of degenerate and non-degenerate basic feasible solutions for the system graphically $2x + 3y \leq 21,\ 3x - y \leq 15,\ x + y \geq 5,\ y \leq 5,\ x, y \geq 0.$

6. Find all basic solutions of the following systems and classify them as degenerate/non-degerate

   (a) $x_1 + 2x_2 + 3x_3 + 4x_4 = 7$
   $2x_1 + x_2 + x_3 + 2x_4 = 3$

   (b) $8x_1 + 6x_2 + 12x_3 + x_4 + x_5 = 6$
   $9x_1 + x_2 + 2x_3 + 6x_4 + 10x_5 = 11$

7. Consider the following system:

$$x_1 + 2x_2 + x_3 \leq 3$$
$$-2x_1 + 2x_2 + 2x_3 \leq 3$$
$$x_1, x_2, x_3 \geq 0$$

   The point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is feasible. Verify whether it is basic.

8. Let P and Q be polyhedra in $\mathbb{R}^n$. Let $P + Q = \{x + y \mid x \in P, y \in Q\}$.

   (a) Show that $P + Q$ is a polyhedron.

   (b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

9. Let $A_1, \ldots A_n$ be a collection of vectors in $\mathbb{R}^m$. Let

$$C = \left\{ \sum_{i=1}^{n} \lambda_i A_i \mid \lambda_1, \ldots, \lambda_n \geq 0 \right\}$$

   Show that any element of C can be expressed in the form $\sum_{i=1}^{n} \lambda_i A_i$, with $\lambda_i \geq 0$, and with at most m of the coefficients A_i being nonzero.

10. Consider the standard form polyhedron $\{x \mid Ax = b, x \geq 0\}$, and assume that the rows of the matrix A are linearly independent. Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.

11. Consider the linear program: Minimize $c^T x$ subject to $Ax \leq b, x \geq 0$, where c is a nonzero vector. Suppose that the point x_0 is such that $Ax_0 < b$ and $x_0 > 0$. Show that x_0 cannot be an optimal solution.

MTL103: Optimization Methods and Applications	Date: February 27, 2023

Tutorial 3

Coordinator: *Prof Minati De* Mathematics Depertment, IIT Delhi

Question 1. *Consider the problem of minimizing $c^T x$ over a polyhedron P. Prove the following:*

1. *A feasible solution x is optimal if and only if $c^T d \geq 0$ for every feasible direction d at x.*
2. *A feasible solution x is the unique optimal solution if and only if $c^T d > 0$ for every nonzero feasible direction d at x.*

Question 2. *Let $P = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x \geq 0\}$ and consider the $x = (0, 0, 1)$. Find the set of feasible directions at x.*

Question 3. *Consider the problem*

$$\begin{aligned} \textit{Minimize } & -2x_1 - x_2 \\ \textit{Subject to } & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

1. *Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.*
2. *Carry out the full tableau implementation of the simplex method, starting with the basic feasible solution of part (1).*

Question 4. *Let x be a basic feasible solution associated with some basis matrix A_B. Prove the following:*

1. *If the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.*
2. *If x is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive*

Question 5. *Use the simplex method to find an improved solution for the linear programming problem represented in the tableau.*

variables	b	x_1	x_2	s_1	s_2	s_3
	0	-4	-6	0	0	0
s_1	11	-1	1	1	0	0
s_2	27	1	1	0	1	0
s_3	90	2	5	0	0	1

The objective function for this problem is $z = 4x_1 + 6x_2$.

Question 6. *Let x be an element of the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. Prove that a vector $d \in \mathbb{R}^n$ is a feasible direction at x if only if $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$.*

Question 7. *Raju holds two part-time jobs, Job 1 and Job 2. He never wants to work more than a total of 12 hours a week. He has determined that for every hour he works at Job 1, he needs two hours of preparation time, and for every hour he works at Job 2, he needs one hour of preparation time, and he cannot spend more than 16 hours on preparation. If he makes Rs. 40 an hour at Job 1 and Rs. 30 an hour at Job 2, how many hours should he work per week at each job to maximize his income?*

Question 8. *Find a solution using the Two-Phase method.*

Minimize $z = x_1 + x_2$
Subject to $2x_1 + x_2 \geq 4$
$x_1 + 7x_2 \geq 7$
$x_1, x_2 \geq 0.$

Tutorial-4 MTL103

1. If x' is any feasible solution of (P) and w' is feasible for (D) such that $w'(b–Ax') = 0$ and $(w'A–c)x' = 0$, then show that x' is optimal for (P) and w' is optimal for (D).

   (P) $\max z = cx$ subject to $Ax \leq b$ and $x \geq 0$

   (D) $\min y = bw$ subject to $Aw \geq c$ and $w \geq 0$

2. Write the dual. Then find the primal optimal solution from the optimal solution of the dual.

$$\begin{gathered} \min\ 3x_1 - 5x_2 - x_3 + 2x_4 - 4x_5 \\ \text{subject to:} \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \leq 6 \\ -x_1 - x_2 + 2x_3 + x_4 - x_5 \geq 3 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

3. Use duality to show that the following LP has an optimal solution.

$$\begin{gathered} \min\ 2x_1 - x_2 \\ \text{subject to:} \\ 2x_1 - x_2 - x_3 \geq 3 \\ x_1 - x_2 + x_3 \geq 2 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

4. Use complementary slackness theorem to verify that $(n, 0, 0, \ldots, 0)$ is an optimal solution of LPP

$$\begin{gathered} \min\ \textstyle\sum_{j=1}^{n} jx_j \\ \text{subject to:} \\ \textstyle\sum_{j=1}^{i} x_j \geq i, \text{ where } i = 1, 2, 3....n \\ x_j \geq 0, \forall j \end{gathered}$$

5. Are the following statements true? Give reasons for your answer.

   - The primal LP (P) and its dual LP (D), both cannot have unbounded solution.
   - The primal LP (P) and its dual LP (D), both cannot be infeasible.
   - The dual(dual(dual)) of a LPP is the primal LPP.
   - If the primal LP (P) has a unique optimal solution and the dual LP (D) is feasible, then (D) also has a unique optimal solution.

6. Solve the following by dual simplex algorithm:

$$\begin{gathered} \min\ 80x_1 + 60x_2 + 80x_3 \\ \text{subject to:} \\ x_1 + 2x_2 + 3x_3 \geq 4 \\ 2x_1 + 3x_3 \geq 3 \\ 2x_1 + 2x_2 + x_3 \geq 4 \\ 4x_1 + x_2 + x_3 \geq 6 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

7. Consider the primal problem

$$\begin{aligned} \min\ & c^T x \\ \text{subject to: } & Ax \geq 0 \\ & x \geq 0 \end{aligned}$$

Form the dual problem and convert it into an equivalent minimization problem. Derive a set of conditions on the matrix A and vectors b and c under which the dual is identical to the primal.

8. Let A be a given matrix. Show that exactly one of the following alternatives must hold -

   (a) There exists some $x \neq 0$ such that Ax = 0, x$\geq$ 0.

   (b) There exists some p such that $p^T A > 0^T$.

9. Consider the following linear programming problem of minimizing $c^T x$ subject to $Ax = b, x \geq 0$. Let x^* be an optimal solution, assumed to exist, and let p^* be the optimal solution to the dual.

   (a) Let $\bar{x}$ be the optimal solution to the primal, when c is replaced by some $\bar{c}$. Show that $(\bar{c}-c)^T(\bar{x}-x^*) \leq 0$.

   (b) Let the cost vector be fixed at c, but suppose that we now change b to $\bar{b}$, and let $\bar{x}$ be a corresponding optimal solution to the primal. Prove that $(p^*)^T(\bar{b}-b) \leq c^T(\bar{x}-x^*)$.

10. Let A be a symmetric square matrix. Consider the linear optimization problem

$$\begin{gathered} \min\ c^T x \\ \text{subject to:} \\ Ax \geq 0 \\ x \geq 0 \end{gathered}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

11. Find the minimum and maximum of $f(x, y, z) = 3x^2 + y$ subject to the constraints $4x - 3y = 9$ and $x^2 + z^2 = 9$. Formulate the lagrangian dual and solve using KKT conditions.

Tutorial - 4

① P: max $z = cx$

$Ax \leq b$

$x \geq 0$

D: min $y = bw$

$A^T w \geq c$

$w \geq 0$

$$w'b = w'Ax'$$

and $w'Ax' = cx'$

This gives $w'b = cx'$

To prove x' is optimal for P.

Let x_1 be feasible sol^n

then by weak duality we have

$$c^T x_1 \leq b^T w' = c^T x'$$

$\Rightarrow c^T x_1 \leq c^T x' \quad \forall x$ feasible

$\Rightarrow x'$ is optimal for P

$||^{ly}$ w' is optimal for D.

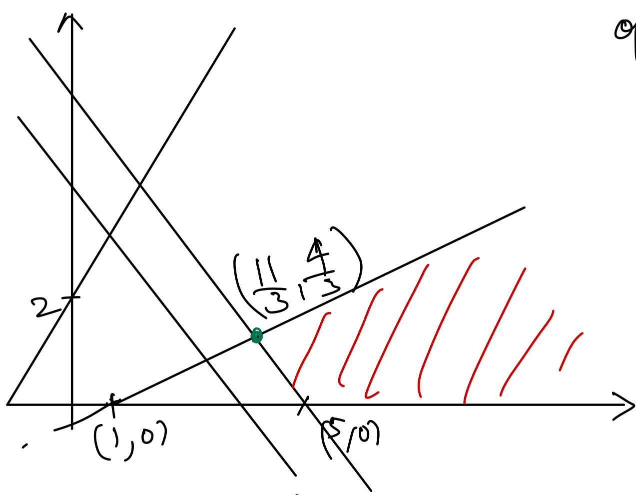
Q2

The dual of P:

$$\max \quad -6y_1 + 3y_2$$

$$-y_1 - y_2 \leq 3$$
$$-y_1 - y_2 \leq -5$$
$$-y_1 + 2y_2 \leq -1$$
$$-3y_1 + y_2 \leq 2$$
$$-y_1 - y_2 \leq -4.$$
$$y_1 \geq 0, \; y_2 \geq 0$$

Solving using graphical method



optimal value is attained at $\left(\frac{11}{3}, \frac{4}{3}\right)$

max value = -18

By strong duality, we have optimal value $= -18$ for P.

Q3 The dual of the given LPP

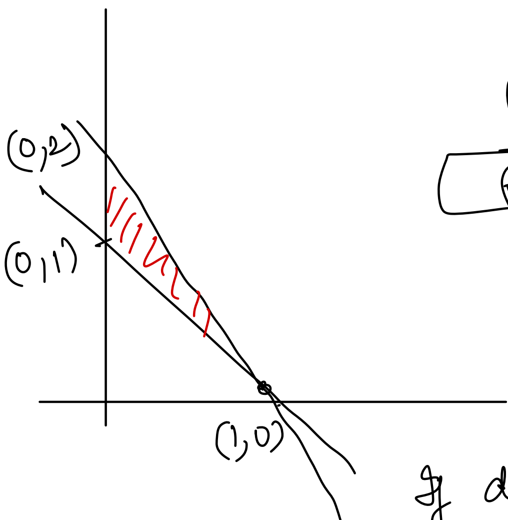
$$\max Z = 3y_1 + 2y_2$$

s.t. $2y_1 + y_2 \le 2$

$-y_1 - y_2 \le -1$

$y_1, y_2 \ge 0.$

Using graphical method:



Feasible	Z
(0, 1)	2
(0, 2)	$\boxed{4}$
(1, 0)	3

If dual has optimal, primal has optimal

Q4 Primal problem

$$\min \sum_{j=1}^{n} j x_j$$

$$\sum_{j=1}^{i} x_j \geq i \qquad \forall i = 1,2 \ldots n$$

$$x_j \geq 0 \ \forall j$$

$$A = \begin{bmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ & & & \ldots & 1 \end{bmatrix}_{n \times n}$$

$$b = [1\ 2\ 3\ \ldots\ n] = c$$

the primal can be written as:

$$\min c^T x$$
$$Ax \geq b \qquad - (P)$$
$$x \geq 0, x \in R^n$$

Dual:

$$\max b^T y$$
$$A^T y \leq c \qquad - (D)$$
$$y \geq 0, y \in R^n$$

Now, To verify

$x^* = (n, 0, 0 \cdots 0)$ is optimal for P

Let's say $y^* = (y_1^*, y_2^* \cdots y_n^*)$ are optimal for (D)

Observe that

only n^{th} constraint have 0 slack, rest all the constraints have non zero slack

$\Rightarrow y_n^* \in \mathbb{R}$ and $y_i^* = 0 \ \forall i = 1, 2 \cdots, n-1$

also, as $x_1 \neq 0 \Rightarrow 1^{st}$ const^n of dual will have zero slack

$\Rightarrow \quad y_1^* + y_2^* \cdots y_n^* = 1$

$\Rightarrow y_n^* = 1$

So the dual should have optimal sol^n

$y_1^* = y_2^* \cdots y_{n-1}^* = 0$ and $y_n = 1$.

To check for feasibility for above y (Exercise)

Primal value at $x^* = n \times 1 + 2 \times 0 \cdots + n \times 0 = n$

Dual value at y^* =: $1\times 0 + 2\times 0 \ldots n\times 1 = n$

By strong duality, x^* is optimal for P
and y^* is optimal for D

HP.

Q5 P: $\max z = c^T x$ D: $\min w = b^T y$

$Ax \leq b$ $A^T y \geq c$

$x \geq 0$ $y \geq 0$

- P and D cannot have unbdd soln:

By weak duality, we have

$z = c^T x \leq b^T y = w$ (for x & w feasible)

If P is unbounded, we can push $z \uparrow$, without any limits. Hence, there are no feasible y for dual

||ly it can be proved for unbounded soln for D.

- P and D, both can be feasible:

Counter example

$c = (1)$, $A = 0$

$b = (-1)$

both primal & dual are infeasible sets.

(Verify)

- The dual(dual(dual)) of an LPP is the primal

False

dual(dual) = primal. and dual(dual(dual)) = dual of LPP.

- Counter example: (Verify)

max: $x_1 + x_2$

$x_1 \leq 1$

$x_2 \leq 1$ $\rightarrow$ optimal $= (1,1)$

$x_1 + 2x_2 \leq 3$

$2x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

dual: $y_1 + y_2 + 3y_3 + 3y_4$

$y_1 + 2y_3 + 2y_4 \geq 1$

$y_2 + 2y_3 + y_4 \geq 1$

$y \geq 0$.

$y = (1, 1, 0, 0)$

& $\left(0, 0, \frac{1}{3}, \frac{1}{3}\right)$

are sol^n.

Q6 min $80x_1 + 60x_2 + 80x_3$

S.t. $x_1 + 2x_2 + 3x_3 \geq 4$

$2x_1 + 3x_3 \geq 3$

$2x_1 + 2x_2 + x_3 \geq 4$

$4x_1 + x_2 + x_3 \geq 6$

$x_1, x_2, x_3 \geq 0$

Introducing slack variables:

max: $-80x_1 - 60x_2 - 80x_3 + 0S_1 + 0S_2 + 0S_3 + 0S_4$

$-x_1 - 2x_2 - 3x_3 + S_1 = -4$

$-2x_1 - 3x_3 + S_2 = -3$

$-2x_1 - 2x_2 - x_3 + S_3 = -4$

$-4x_1 - x_2 - x_3 + S_4 = -6.$

$x_1, x_2, x_3, x_4, S_1, S_2, S_3, S_4 \geq 0$

Iter-1

		C_j	-80	-60	-80	0	0	0	0
B	C_B	x_B	$\boxed{x_1}$	x_2	x_3	S_1	S_2	S_3	S_4
S_1	0	-4	-1	-2	-3	1	0	0	0
S_2	0	-3	-2	0	3	0	1	0	0
S_3	0	-4	-2	-2	-1	0	0	1	0
$\boxed{S_4}$	0	-6	(-4)	-1	-1	0	0	0	1
$Z=0$		Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	80	60	80	0	0	0	0
Ratio $\frac{Z_j - C_j}{S_{4j}}$, $S_{4j} < 0$			-20 ↑	-60	-80	-	-	-	-

$R_4 = R_4 / -4$

$R_1 = R_1 + R_4$

$R_2 = R_2 + 2R_4$

$R_3 = R_3 + \cancel{2R_4}$

Leaving Basis variable: S_4, entering var: x_1.

pivot element = -4.

Iter-2

		C_j	-80	-60	-80	0	0	0	0
B	C_B	x_B	x_1	x_2	x_3	S_1	S_2	S_3	S_4
S_1	0	-2.5	0	-1.75	-2.75	1	0	0	-0.25
S_2	0	0	0	0.5	-2.5	0	1	0	-0.5
S_3	0	-1	0	-1.5	-0.5	0	0	1	-0.5
x_1	-80	1.5	1	0.25	0.25	0	0	0	-0.25
Z = -120		Z_j	-80	-20	-20	0	0	0	20
		$Z_j - C_j$	0	40	60	0	0	0	20
Ratio $\frac{Z_j - C_j}{S_{4j}}$, $S_{4j} < 0$			-	-22.85	-21.81	-	-	-	-80

Leaving : S_1 and entering : x_3.

pivot element → 2.75

Further, steps can be performed similarly

After iter - 4,

$x_1 = 1.2308 \quad x_2 = 0.4615 \quad x_3 = 0.6154$

$\max -z = -175.3846.$

$\min z = 175.3846$

Q7

$\min c^T x$

s.t. $Ax \geq b$

$x \geq 0$

Dual: $\max b^T \cancel{x}y$ $=$ $\min -b^T y$

$A^T \cancel{x}y \leq c$ $\qquad$ $-A^T y \geq -c$

$y \geq 0$ $\qquad$ $y \geq 0$

For primal to be same as dual

we need to $\forall y$

$-b^T y = c^T y \Rightarrow b = -c$

$-A^T y = Ay \Rightarrow A = -A^T$

Q8. Consider the following pair of (P-D) problem

$$\begin{aligned} \min \ & 0^T x \\ \text{s.t } \ & p^T A \geq e^T \quad -(P) \end{aligned} \qquad \begin{aligned} \max \ & e^T x \\ & Ax = 0 \quad -(D) \\ & x \geq 0. \end{aligned}$$

Suppose (b) holds

$p^T A > 0 \Rightarrow p^T A \geq c$

components of p can be scaled such that

$p^T A \geq e$ (equivalent, prove)

$\Rightarrow$ p is feasible with ~~optimal~~ cost 0.

By strong duality, the dual has optimal cost as well.

Suppose $\exists\ x \neq 0$, $Ax = 0$ and $x \neq 0$

Then x is feasible for (D) and

$e^T x > 0$ - a contradiction.

Thus only (b) holds.

Suppose (b) doesn't hold.

Then P is not feasible.

Then D can be either infeasible or unbounded.

Since 0 is always feasible for dual problem.

Thus dual problem is unbounded.

$e^T x$ is acheivable for some x, $Ax = 0$ and $x \geq 0$

$\Rightarrow \quad x \neq 0$

In other words, (a) holds.

Q9 : P: $\min c^T x$ D: $\max b^T y$

$Ax = b$ $A^T y \geq c$

$x \geq 0$ $y \in R$

x^* optimal for P and p^* optimal for D.

(a) $\tilde{x}$ optimal for (P), when c is replaced by $\tilde{c}$

$c^T x^* \leq c^T x \quad \forall x$ feasible.

$\Rightarrow c^T x^* \leq c^T \tilde{x}$

$\|^{ly}$ $\tilde{c}^T \tilde{x} \leq \tilde{c}^T x^*$

$$(\tilde{c} - c)^T (\tilde{x} - x^*) = \underbrace{(\tilde{c}^T \tilde{x} - \tilde{c}^T x^*)}_{\leq 0} + \underbrace{(c^T x^* - c^T \tilde{x})}_{\leq 0.}$$

$$\Rightarrow (\tilde{c} - c)^T (\tilde{x} - x^*) \leq 0.$$

(b) $\tilde{x}$ optimal for $\tilde{b}$ and x^* optimal for b

To prove: $p^{*T} (\tilde{b} - b) \leq c^T (\tilde{x} - x^*)$

Let $\tilde{p}^*$ denote the optimal Sol^n for modified dual (after $b \to \tilde{b}$)

by strong duality: $c^T \tilde{x} = \tilde{p}^{*T} \tilde{b}$

p^* feasible for modified dual:

$$p^{*T}\tilde{b} \leq \tilde{p}^{*T}\tilde{b} = c^T\tilde{x}$$

$$p^{*T}\tilde{b} \leq c^T\tilde{x}.$$

$$p^{*T}(\tilde{b}-b) - c^T(\tilde{x}-x^*)$$

$$= (p^{*T}\tilde{b} - c^T\tilde{x}) - (p^{*T}b - c^Tx^*)$$

$$\leq 0.$$

Q10 P: $\min c^T x$

$Ax \geq c$

$x \geq 0$

D: $\max c^T y$

$A^T y \leq c$

$y \geq 0$

given : x^* such that $Ax^* = c$ and $x^* \geq 0$

Since A is symmetric, $A^T y \leq c \Rightarrow Ay \leq c$

Dual : $\max c^T y$

$Ay \leq c$

$y \geq 0$

If we set $y_1 = x^*$, we see that y_1 is feasible for D

Also primal objective : $c^T x^* = c^T y_1 =$ Dual objective

$\Rightarrow$ x^* is optimal for (P) and y_1 is ~~optimal for~~ (D)

(Duality theorem).

MTL103
Tutorial 5 Hints

1. The following payoff matrix corresponds to a modified version of the Prisoner's Dilemma problem called the DA's brother problem. In this problem prisoner 1 is related to the District Attorney. How is this problem different? How many Nash equilibria are there? Does player 2 really have a choice?

1	2	
	NC	C
NC	$0, -2$	$-10, 1$
C	$-1, -10$	$-5, -5$

   Hint: Notice that Player 2 will never play NC no matter what player 1 does. Hence there is only one Nash equilibrium unlike the classical Prisoner's Dilemma case. Even though (NC, NC) has better payoffs for both 1 and 2, it will never be played.

2. Consider any arbitrary two player game of the following type (with a, b, c, d any arbitrary real number):

	A	B
A	a, a	b, c
B	c, b	d, d

   It is known that the game has a strongly dominant strategy equilibrium. Now prove or disprove: The above strongly dominant strategy equilibrium is the only possible mixed strategy equilibrium of the game.

   Hint: In a strongly dominant strategy equilibrium, each player plays his strictly dominant strategy. Notice only (a, a) or (d, d) can be the strongly dominant strategy equilibrium.

   If (d, d) is strongly dominant strategy equilibrium, then $d > b,\ c > a$. If (a, a) is strongly dominant strategy equilibrium, then $d < b,\ c < a$.

   If a player is randomizing between two alternatives then he must be indifferent between them. If player 2 plays A with prob q, then expected payoff of player 1 must be same if he plays pure strategy A or if he plays pure strategy B. That is $aq + b(1-q) = cq + d(1-q)$. But this is not possible if (a, a) or (d, d) is a strictly dominant strategy equilibrium because either $d > b,\ c > a$ or $d < b,\ c < a$.

3. An $m \times m$ matrix is called a latin square if each row and each column is a permutation of $(1, ..., m)$. Compute pure strategy Nash equilibria, if they exist, of a two person game for which a latin square is the payoff matrix.

   Hint: Only (m, m) can be Nash equilibria and all (m, m) are Nash equilibria.

4. Consider the following instance of the prisoners' dilemma problem.
   Find the values of x for which:

   (a) the profile (C,C) is a strongly dominant strategy equilibrium.

1	2	
	NC	C
NC	$-4, -4$	$-2, -x$
C	$-x, -2$	$-x, -x$

(b) the profile (C,C) is a weakly dominant strategy equilibrium but not a strongly dominant strategy equilibrium.

(c) the profile (C,C) is a not even a weakly dominant strategy equilibrium.

In each case, say whether it is possible to find such an x. Justify your answer in each case.

Hint: By definition of strongly/weakly dominant strategy equilibrium

(a) $x > -2$

(b) $x = -2$

(c) $x < -2$

5. Find the pure strategy Nash equilibrium of the following game.

	X	Y	Z
X	$6, 6$	$8, 20$	$0, 8$
Y	$10, 0$	$5, 5$	$2, 8$
Z	$8, 0$	$20, 0$	$4, 4$

6. Find the mixed strategy Nash equilibria for the following games:

(a) *(Matching Pennies Game)*

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Hint: Again, if a player is randomizing between two alternatives then he must be indifferent between them. If player 2 plays heads with probability q,
$q + -1(1-q) = -q + (1-q) \implies q = 1/2$

(b) *(Rock-Paper-Scissors Game)*

1	3		
	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

MTL103: Optimization Methods and Applications

Problem Set (2023)

1. Show that if any one of the matrices $A, A^T, -A, (A|A)$ or $(A|I)$ is TUM, then so are all the others.

2. Consider the integer linear programming

$$\begin{aligned}\text{Maximize } & c^T x \\ \text{Subject to } & Ax \le b \\ & x \ge 0 \\ & x \text{ integer,}\end{aligned}$$

where A, b and c are all composed of positive integers. Call the solution to ILP x_0 and the relaxed LP to x_1. Show that in ILP $\lfloor x_1 \rfloor$ is feasible, and its cost can be no farther from optimal than $\sum_{i=1}^{n} c_i$.

3. Solve the following ILP by cutting plane method:

$$\begin{aligned}\text{Minimize } & -2x_1 - x_2 \\ \text{Subject to } & x_1 + x_2 \le 5 \\ & -x_1 + x_2 \le 0 \\ & 6x_1 + 2x_2 \le 21 \\ & x_1, x_2 \ge 0. \\ & x_1, x_2 \text{ are integer.}\end{aligned}$$

(15)

4. Consider the primal problem P: Minimize $z = \mathbf{c}^T\mathbf{x}$ s.t $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}$.
a. Write down the corresponding dual D of the problem.
b. Let $\mathbf{p}$ be any feasible solution to D. Let $Q = \{j|\mathbf{p}^T\mathbf{A_j} = c_j\}$. Let RP and DRP be the corresponding restricted primal and dual of the restricted primal, respectively. Assume that $OPT(RP) > 0$ and $\mathbf{d}$ be the solution to the DRP such that $\mathbf{d}^T\mathbf{A_j} \le c_j$ for all $j \notin Q$. Prove that the primal problem P is infeasible. *(3+7)*

5. Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} \le \mathbf{b}, \mathbf{x} \ge \mathbf{o}\}$. Prove that if $\mathbf{A}$ is totally unimodular (TUM) then all the vertices of P are integers for any integer vector $\mathbf{b}$. *(7)*

6. Consider the following graph G where edge capacities are given in blue:

   a. Write down the incidence matrix of this graph.

   b. Write down the LP formulation for finding maximum flow in this graph.

   c. Find the maximum flow in this graph via the Ford Fulkerson method. Show all the iterations using figures.

(5+5+10=20)

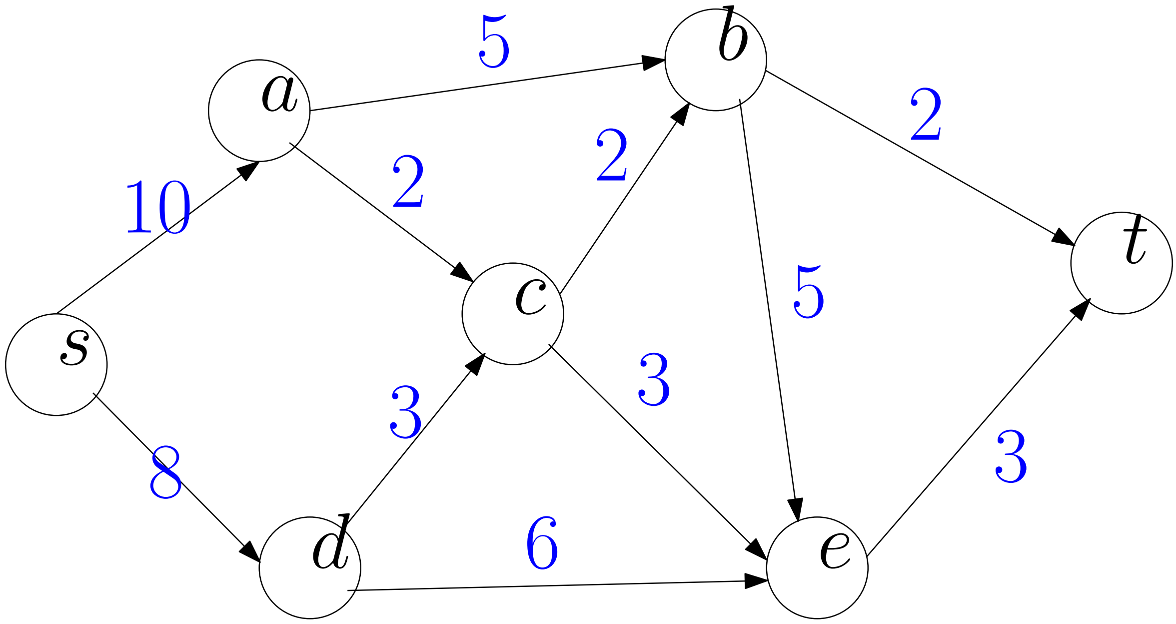


Figure 1: Directed Graph G

7. Consider the linear programming problem P: Minimize $z = \mathbf{c}^T\mathbf{x}$ s.t $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$. Let B be the set of indices corresponding to the optimal basis after solving P by simplex tableau method. Consider a new linear programming problem P' by adding the following constraint to P:

   $\sum_{j \notin B} f_j x_j \geq f_0$.

   a. Write down the linear programming P' in standard form.
   b. Assuming $f_0 > 0$, prove that if we add the new slack variable of the new constraint to the optimal basis corresponding to B to form a new basis B' for P', then the new tableaue is dual feasible but primal infeasible.
   c. Suggest an appropriate method to solve P'. *(2+5+1)*

8. Suppose there are n men, n women, and m marriage brokers. Each broker has a list of some of the men and women as clients and can arrange marriages between any pairs of men and women on that list. In addition, we restrict the number of marriages that broker i can arrange to a maximum of b_i. All marriages are heterosexual, and all men and women are monogamous. Translate the problem of finding a solution with the most marriages into one of finding the maximum flow in a capacitated flow network. *(5)*

9. Prove that any LP in standard form whose constraint matrix A is the node-arc incidence matrix of a directed graph has only integer optimal vertices. *(10)*

10. Given a directed graph $G = (V, E)$ with cost c_{ij} of each edge $(i, j) \in E$, the travelling salesman problem (TSP) is to find a tour (a cycle that visits all nodes) of minimum cost.

- Write down ILP formulation of TSP. Give a proper explanation for your formulation.

(5)

11. Problem: Given a bipartite graph $G(V, E)$, find a maximum cardinality subset $E' \subseteq E$ of edges such that no two of the edges share same vertex.

    - Write down ILP formulation of the problem. Give proper explanation for your formulation.
    - Give an algorithm to solve it. What is the running time of your algorithm?

(5+7=12)

12. Suppose that we have an algorithm A that can detect whether any optimization problem is feasible or not. Using A, explain whether it is possible to solve the optimization problem–means finding the optimum. *(8)*

13. Suppose we have a polynomial time subroutine **Ellipsoid-m1** that takes an ellipsoid $E = E(z, D)$ in $\mathbb{R}^n$ and a halfspace $H = \{x \in \mathbb{R}^n | a^T x \geq a^T z\}$ (where a is a non-zero vector) as input; and returns an ellipsoid $E' = E(\overline{z}, \overline{D})$ satisfying the following:

    - $\overline{z} = z + \frac{1}{n+1} \frac{Da}{\sqrt{a^T Da}}$,
    - $\overline{D} = \frac{n^2}{n^2-1}(D - \frac{2}{n+1} \frac{Daa^T D}{a^T Da})$ is symmetric positive definite,
    - $E \cap H \subset E'$,
    - $Vol(E') < e^{-1/(2(n+1))Vol(E)}$.

    a) Consider a polyhedron $P = \{x \in \mathbb{R}^n | Ax \geq b\}$, where A, b have integer entries with magnitude bounded by some U. Using **Ellipsoid-m1** as a subroutine, prove whether one can decide whether P is empty or not in polynomial time. What is the time complexity of your algorithm?

    b) Consider the following LP:

    Maximize $c^T x$
    Subject to $Ax \leq b$,

    where A, b have integer entries with magnitude bounded by some U. Using **Ellipsoid-m1** as a subroutine, propose an algorithm to find out the optimum solution of the given LP. What is the time complexity of your algorithm?

Problem 1 , Problem 5 , Problem 9

↘ Step 1: Prove the theorem of partitioning rows [TUM]

Step 2: For directed graph $I_2 = \phi$ submatrix of A

Problem 1

$A \Leftrightarrow A^T \Leftrightarrow -A \Leftrightarrow (A|A) \Leftrightarrow (A|I)$

$\quad a \qquad b \qquad c \qquad d \qquad e$

Theory: A is totally unimodular iff every non-singular, square is UM [$|B| = \pm 1$]

→ $R_1(A) = \{ x : Ax = b, x \geq 0 \}$

If A is TUM then vertices are integers

[Reason:- B is formed from subset of m linearly independent columns,

$x = B^{-1} b = \frac{\text{adj} B}{|B|} b$

$|B| = \pm 1$, b integer]

Problem 5

→ $R_2(A) = \{ x : Ax <= b, x \geq 0 \}$

If we can prove that if A is TUM then so is (A|I) then we can apply $R_1(A)$ & conclude every vertice is integer as we can add slack variables.

C is a square non-singular sub-matrix of (A|I) then

$$C = \left(\begin{array}{c|c} B & 0 \\ \hline D & I_R \end{array} \right)$$

B: square submatrix of A

$|C| = |B| = \pm 1$

Hence (A|I) is TUM. [This can be used to prove $a \Leftrightarrow e$]

For $e \Rightarrow a$ consider submatrix only from A of A|I

a⟺b

→ If A is TUM then ⟺ A^T is easy to prove

Proof: A has a square non-singular sub matrix B with $|B| = \pm 1$ then determinant of every non-singular submatrix of A^T will also be ± 1 as if C is a square non-singular submatrix of A^T then a_{1n} is a square submatrix of A and

$$C = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \\ a_{1n} & & a_{nn} \end{bmatrix} \text{ then } C' = \begin{bmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}$$

$|C| = |C'| = \pm 1$

similarly for opposite way

→ a⟺c If A is TUM ⟺ −A is TUM

Proof: Both A & −A will have similar square sub-matrix only with negated elements. Hence if C_R is of odd order sign of det will be flipped while it will remain same for even order

Rough

$a_1 \; a_2 \; a_3$
$b_1 \; b_2 \; b_3$
$c_1 \; c_2 \; c_3$

$a_1 \; b_1 \; c_1$
$a_2 \; b_2 \; c_2$
$a_3 \; b_3 \; c_3$

$\bullet \rightarrow \quad a \Leftrightarrow d$

$a \rightarrow d$ if A is TUM then so is (A I A)

Any square submatrix of (A I A) will be a subset of rows & colums from $\{a_1, \ldots a_m, b_1, \ldots b_n, b_1, \ldots b_n\}$

Now any subset will either have $|\,| = 0$ if two same colums

Or of form $\{a_1, \ldots a_r, b_1, \ldots b_r\}$ then square submatrices only. this is same as det of A square submatrix

$d \rightarrow$ if AIA is TUM then consider

from A $\rightarrow$ A is TUM

part of AIA

Problem 3, Problem 10

Lecture 30_31 [redacted] . [MS Teams] .

[Cutting Plane Method]

[redacted]

Slide Number $\rightarrow$ 11–17 for Problem 3

Slide Number $\rightarrow$ 29–34 for Problem 10

Problem 12

Yes, As an input to the algorithm send the primal, dual & $c'x = p'b$

[redacted]

[redacted] Conditions: $Ax = b$ $\quad p^T A <= c^T$

$x \geq 0$

& $c'x = p'b$

If yes then we have found the optimal solution as well as the point

No $\rightarrow$ infeasible

Problem 6

a) Ezz

Draw A with s, a, b, c, d, e, t as rows and $f_1, \ldots f_{11}$ as edges

b) ~~Af+dv=0~~

$$\max v$$
$$\text{s.t. } Af + dv = 0$$
$$f \le b$$
$$f \ge 0$$

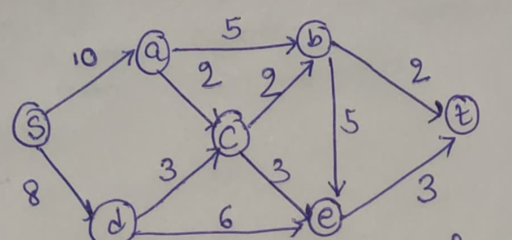
Here A is as part a

d is $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$

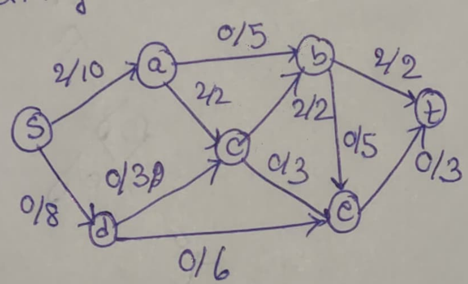
b is corresponding capacity constraints of f_i

c) Residual graph

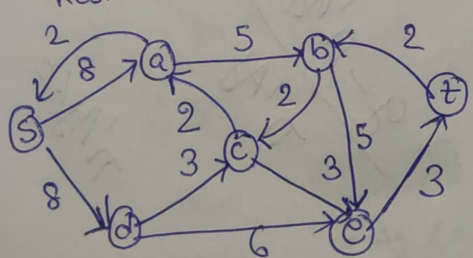
Stage 1:



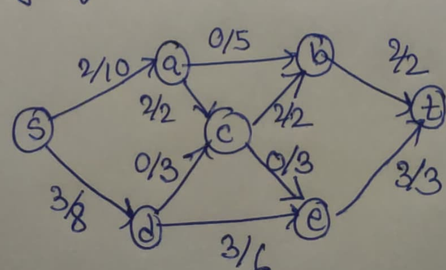
Augmenting path $s \xrightarrow{2} a \xrightarrow{2} c \xrightarrow{2} b \xrightarrow{2} t$



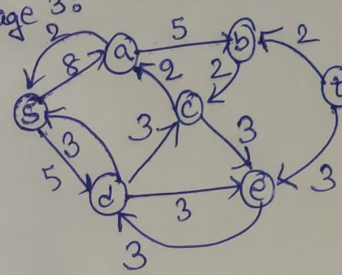
Stage 2: Residual graph



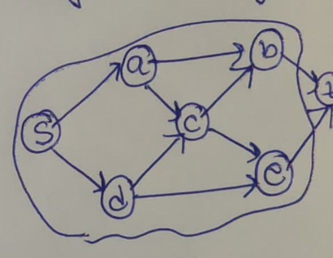
Augmenting $s \xrightarrow{3} d \xrightarrow{3} e \xrightarrow{3} t$



Stage 3:



No more paths in ~~augment~~ residual graph



Min S-T capacity 5.

Problem 2

As $A \geq 0$,

$$A \lfloor x_1 \rfloor <= A x_1$$

and As $x_1 >= 0$

$\lfloor x_1 \rfloor >= 0$

Moreover $x_1 - \lfloor x_1 \rfloor <= 1$

$$c^T (x_1 - \lfloor x_1 \rfloor) <= \sum c_i$$

Problem 7

a) We will have 'p' as

① $B^{-1} b = B^{-1} A x$

x

② $x >= 0$ $\quad 1\ 0 \cdots 0\ y_{i1}\ y_{i2}$

OR

① $(B^{-1} b)_i = x_i + \sum_{j \notin B} y_{ij} x_j$

② $x >= 0$

b) $\sum_{j \in B} f_j x_j - s = f_0$

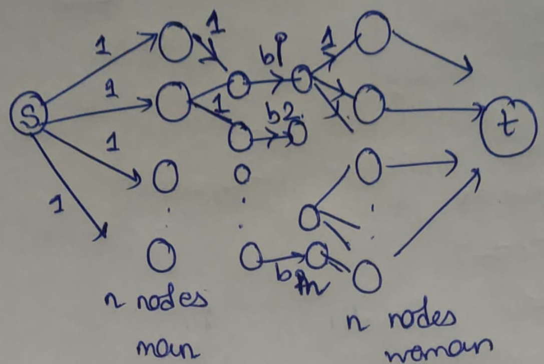
It will remain dual feasible as the cost vector will still be $>= 0$ [This needs a proof Think]

primal unfeasible as $s <= 0$

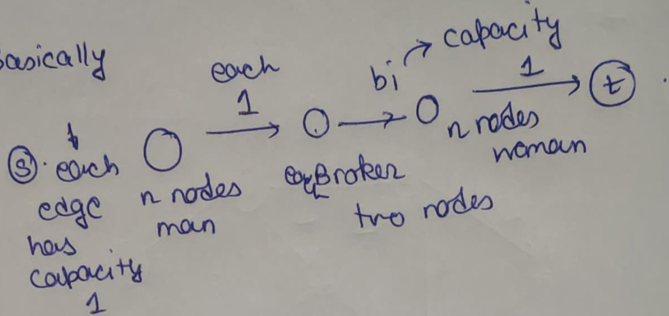
$[= -f_0]$

c) Dual simplex method

Problem 8



Basically



Problem 4

a) Min $c^T x$

$Ax = b$

$x \geq 0$

Max $p^T b$

s.t $p^T A <= c^T$.

b) Note. $\pi^* = \pi^0 + \theta \bar{\pi}$

$\pi^{*T} b = \pi^{0T} b + \theta \bar{\pi} b > 0$

$\pi^{*T} A_j = \pi^{0T} A_j + \theta \bar{\pi}^T A_j$

$\pi^{0T} A_j <= c_j$ [π is a solution of BP]

Now

Moreover $\bar{\pi}^T A_j <= 0 \ \forall j \in Q$.

Given $\bar{\pi}^T A_j <= 0 \ \forall j \notin Q$ — note question has an error it should be 0

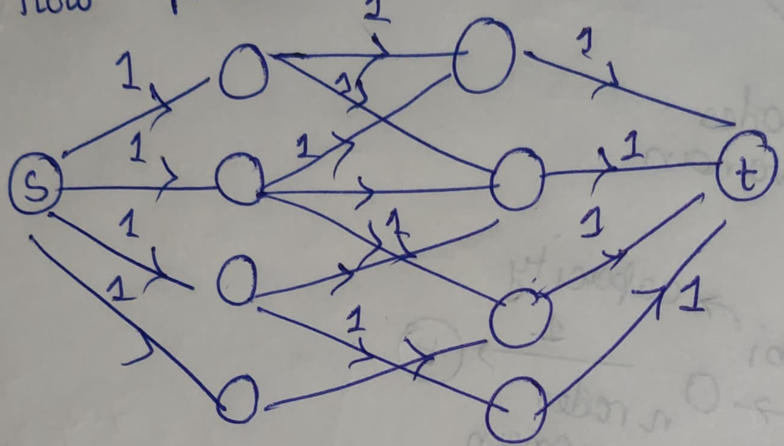
$\bar{\pi}^T A_j <= 0 \ \forall j$

$\therefore \ \pi^{*T} A_j <= c_j \ \forall j$

Hence we can increase θ indefinetely

Problem 11

Max flow problem



If |matching| = k
∃ a flow of k
If flow of k
∃ matching of k

Write LP as max flow

use Ford Fulkerson algorithm to solve it

Problem 13

a) Use the standard Ellipsoid Algorithm as stated in slides

Polynomial time

b) Send in the primal, dual as well as

$c^T x = p^T b$

MTL103: Optimization Methods and Applications	Date: 12^{th} February, 2023
Minor 1 (solution)	
Coordinator: *Prof Minati De*	Mathematics Depertment, IIT Delhi

Question 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $S \subset \mathbb{R}^n$. Let x^n be an element of S. Suppose that x^* is a local optimum for the problem of minimizing $f(x)$ over S; that is, there exists some $\epsilon > 0$ such that $f(x^*) \geq f(x)$ for all $x \in S$ for which $\|x - x^*\| \leq \epsilon$. Prove that x^* is globally optimal, that is, $f(x^*) \geq f(x)$ for all $x \in S$.*

Solution: For the sake of contradiction, say x^* is not a global optimal. Let $y \neq x^*$ be a global optimal. Clearly, $\mathsf{f(y)} < \mathsf{f(x^*)}$. Let us take $x(t) = (1-t)x^* + ty$, the convex combination of x^* and y, where $t \in [0, 1]$. Now, using the property of the convex function, we get

$$\mathsf{f(x(t)) = f((1-t)x^* + ty)} \leq (1-t)\mathsf{f(x^*)} + t\mathsf{f(y)} < (1-t)\mathsf{f(x^*)} + t\mathsf{f(x^*)} = \mathsf{f(x^*)} \cdots (1)$$

The first inequality is due property of the convex function, and for the second inequality, we use $\mathsf{f(y)} < \mathsf{f(x^*)}$.

So, for every point $x(t)$ on the line segment joining x^* and global optimal y, we get $\mathsf{f(x(t))} < \mathsf{f(x^*)}$.

Also,

$$||x(t) - x^*|| = ||(1-t)x^* + ty - x^*|| = |t| \cdot ||x^* - y||.$$

Now, if we take $t \in [0, 1]$, such that $t \leq \frac{\epsilon}{||x^* - y||}$ then the above equation becomes,

$$||x(t) - x^*|| = |t| \cdot ||x^* - y|| \leq \epsilon.$$

So, for such values of t, $||x(t) - x^*|| \leq \epsilon$. Thus, for such $x(t)$, $\mathsf{f(x^*)} \leq \mathsf{f(x(t))}$ as x^* is local optimal. Since $x(t)$ is a point on the line segment joining x^* and y, we get a contradiction to (1). Hence, by the method of contradiction, we can say that x^* is globally optimal, that is, $\mathsf{f(x^*)} \leq \mathsf{f(x)}$ for all $x \in S$.

Question 2. *Consider a linear programming problem given in the standard form*

Minimize $c^T x$
Subject to $Ax = b$
$x \geq 0.$

Let x be a feasible solution, and let $K = \{i \mid x_i = 0\}$. Show that x is an optimal solution if and only if the following linear programming problem has an optimal cost of zero:

Minimize $c^T d$
Subject to $Ad = 0$
$d_i \geq 0, i \in K.$

Solution: Since $d = 0$ is a feasible solution for problem (2), and the cost at $d = 0$ is zero, the optimal value of problem (2) is less than equal to zero.

First, we prove that if the cost of problem (2) is not equal to zero, then x is not an optimal solution of problem (1). Hence, the cost of problem (2) is strictly less than zero. Let d^* be the optimal solution of problem (2), and $x^* = x + \epsilon d^*$. Now, we need to show that x^* is a feasible solution of problem (1) and $c^T x^* < c^T x$.

Second, we prove that if x is not an optimal solution of problem (1), then the cost of problem (2) is not equal to zero. Then $\exists\ y$ such that $c^T y < c^T x$. Let $d = y - x$. Now, we need to show that d is a feasible solution of problem (2) and $c^T d < 0$.

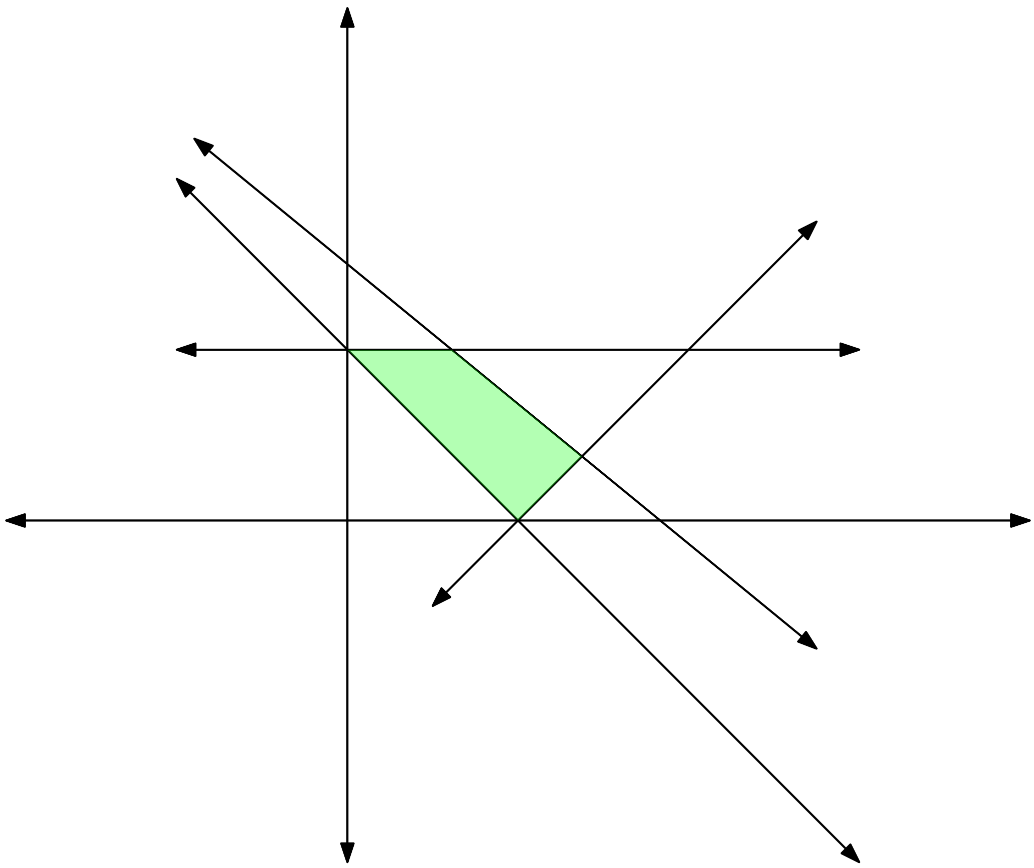


Figure 1: The shaded area is the feasible region.

Question 3. *Consider the polyhedron* $P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ *and a nondegenerate basic feasible solution* x^*. *We introduce slack variables* z *and construct a corresponding polyhedron* $P' = \{(x, z) \in \mathbb{R}^{n+m} \mid Ax + z = b, x \geq 0, z \geq 0\}$. *Show that* $(x^*, b - x^*)$ *is a nondegenerate basic feasible solution for the new polyhedron* P'.

Solution: Since x^* is a nondegenerate bfs, there exists n number of active linearly independent constraints. Let k number of constraints from $Ax \leq b$ and l number of constraints from $x \geq 0$ be active at x^* i.e. $k + l = n$.

Now it is easy to check $(x^*, b - x^*)$ is a feasible solution for the polyhedron P'. At the point $(x^*, b - x^*)$, all the equal constraints $Ax + z = b$, l number of constraints from $x \geq 0$ and k number of constraints from $z \geq 0$ are active i.e. $k + l + m = n + m$ number of active constraints. Hence $(x^*, b - x^*)$ is a nondegenerate basic feasible solution.

Question 4. *Determine all degenerate and nondegenerate basic feasible solutions of the system graphically:*

$2x + 3y \leq 21,\ 3x - y \leq 15,\ x + y \geq 5,\ y \leq 5;\ x, y \geq 0.$

Solution: The graph is given by Figure 1, and the shaded area $ABCD$ is the feasible region. Now the basic feasible solutions of the system are the vertices A $(0, 5)$, B $(5, 0)$, C $(6, 3)$, and D $(3, 5)$.

By definition, a basic solution $x \in \mathbb{R}^n$ is said to be degenerate if more than n (Here, $n = 2$) constraints are active at x. So A $(5, 0)$ and B $(0, 5)$ are degenerate bfs, and C $(3, 5)$ and D $(6, 5)$ are nondegenerate bfs.

Question 5. *Consider a district with I neighbourhoods, J schools, and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighbourhood i, the student population of grade g is S_{ig}. Finally, the distance of school j from neighbourhood i is d_{ij}. Formulate a linear programming problem that assigns all students to schools while minimizing the total distance travelled by all students.*

Solution: Let x_{ijg} = number of students of grade g in neighbourhood i are assigned to school j. Then the linear programming problem is defined by:

Minimize $\Sigma_{i,j,g}\, d_{ij}x_{ijg}$
Subject to $\Sigma_j\, x_{ijg} = S_{ig}$, for i and g,
$\Sigma_i\, x_{ijg} \leq C_{jg}$, for j and g,
$x_{ijg} \geq 0,\ x_{ijg} \in \mathbb{N}$.

MTL103: Optimization Methods and Applications	Date: April 13, 2023

Minor 2
(solution)

Coordinator: *Prof Minati De* — Mathematics Depertment, IIT Delhi

Question 1. *Consider the following problem:*

minimize $15x_1^3 + 40x_2^2$
Subject to $2x_1 + x_2 \geq 30$
$5x_1 - 6x_2 \leq 17$
$x_1, x_2 \geq 0.$

a) Write down the corresponding Lagrange dual problem.
b) Write down the KKT optimality conditions for the problem.

Solution: (a) We define the Lagrangian $L(x, \lambda, \mu)$ associated with the given problem as

$$L(x, \lambda, \mu) = (15x_1^3 + 40x_2^2) + \lambda_1(-2x_1 - x_2 + 30) + \lambda_2(5x_1 - 6x_2 - 17) + \lambda_3(-x_1) + \lambda_4(-x_2)$$

So the Lagrangian dual function $g(\lambda, \mu)$ is given by

$$\begin{aligned} g(\lambda, \mu) &= \inf_x L(x, \lambda, \mu) \\ &= \inf_x \; [(15x_1^3 + 40x_2^2) + \lambda_1(-2x_1 - x_2 + 30) + \lambda_2(5x_1 - 6x_2 - 17) + \lambda_3(-x_1) + \lambda_4(-x_2)] \end{aligned}$$

Thus, the Lagrangian dual problem is

max $g(\lambda, \mu)$

Subject to $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

(b) Let x^* and (λ^*, μ^*) be any primal and dual optimal points with zero duality gap.
Then the KKT optimality conditions for the problem are

$$\begin{aligned} 2x_1^* + x_2^* &\geq 30 \\ 5x_1^* - 6x_2^* &\leq 17 \\ x_1^*, x_2^* &\geq 0 \\ \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^* &\geq 0 \\ \lambda_1^*(-2x_1^* - x_2^* + 30) &= 0 \\ \lambda_2^*(5x_1^* - 6x_2^* - 17) &= 0 \\ \lambda_3^*(-x_1^*) &= 0 \\ \lambda_4^*(-x_2^*) &= 0 \\ 45(x_1^*)^2 - 2\lambda_1^* + 5\lambda_2^* - \lambda_3^* &= 0 \\ 80x_2^* - \lambda_1^* - 6\lambda_2^* - \lambda_4^* &= 0 \end{aligned}$$

Question 2. *Consider a linear programming problem given in the standard form*

Minimize $c^T x$
Subject to $Ax = b$
$x \geq 0.$

Let x^* *be an optimal solution, assumed to exist, and let* p^* *be an optimal solution to the dual.*
a) Let $\overline{x}$ *an optimal solution to the primal , when c is replaced by some* $\overline{c}$. *Show that* $(\overline{c}-c)^T(\overline{x}-x^*) \leq 0$.
b) Let $\overline{x}$ *an optimal solution to the primal, when b is replaced by some* $\overline{b}$. *Show that* $(p^*)^T(\overline{b}-b) \leq c^T(\overline{x}-x^*)$.

Solution: (a) The given primal LPP is

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \end{array} \tag{1}$$

So the corresponding dual problem will be

$$\begin{array}{ll} \text{maximize} & p^T b \\ \text{subject to} & \begin{array}{c} p^T A \leq c^T \\ \text{p is unrestricted in sign} \end{array} \end{array} \tag{2}$$

The new primal, when c is replaced by some $\bar{c}$ will be,

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x \\ \text{subject to} & \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \end{array} \tag{3}$$

Given that $\bar{x}$ is an optimal solution to 3 and x^* was an optimal solution to 1. Notice that, the constraints of both 1 and 3 are the same. So, both $\bar{x}$ and x^* will be feasible for both primals 1 and 3.
Now, x^* is an optimal solution for 1 and $\bar{x}$ is a feasible solution to 1. Thus,

$$c^T x^* \leq c^T \bar{x}.$$

Similarly, x^* is a feasible solution for 3 and $\bar{x}$ is an optimal solution to 3. Thus,

$$\bar{c}^T \bar{x} \leq \bar{c}^T x^*.$$

Now, adding the above two equations, we get,

$$c^T x^* + \bar{c}^T \bar{x} \leq c^T \bar{x} + \bar{c}^T x^* \implies (\bar{c} - c)^T (\bar{x} - x^*) \leq 0$$

Hence the proof of part (a).

(b) Given that p^* is an optimal solution to the dual 2. Since x^* was an optimal solution to primal 1, by Strong duality theorem, we have,

$$c^T x^* = (p^*)^T b \tag{4}$$

When b is replaced by $\bar{b}$, the new primal will be,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{array}{c} Ax = \bar{b} \\ x \geq 0 \end{array} \end{array} \tag{5}$$

So the corresponding dual problem will be,

$$\begin{array}{ll} \text{maximize} & p^T \bar{b} \\ \text{subject to} & \begin{array}{c} p^T A \leq c^T \\ \text{p is unrestricted in sign} \end{array} \end{array} \tag{6}$$

Notice that, constraints of 2 and 6 are same. Since p^* is an optimal solution to 2, p^* will be a feasible solution to 6. Also, given that $\bar{x}$ is an optimal solution to 5. Thus by the weak duality theorem, we get,

$$(p^*)^T \bar{b} \leq c^T \bar{x} \tag{7}$$

Now, from 4 and 7, we get,

$$(p^*)^T (\bar{b} - b) \leq c^T (\bar{x} - x^*).$$

Hence the proof of part (b).

Question 3. *Let us consider the following problem*

$$\begin{aligned}
&\textit{Maximize } v\\
&\textit{Subject to } Af + dv \leq 0\\
&\qquad\qquad\quad f \leq b\\
&\qquad\qquad\quad -f \leq 0.
\end{aligned}$$

*where $A_{|V|\times|E|}$ is an incidence matrix of a graph, and the $d \in \mathbb{R}^{|V|}$. Consider this to be **D** in the primal-dual method. Formulate corresponding **DRP**.*

Solution: The dual problem (**D**) is

$$\begin{aligned}
&\text{Maximize } v\\
&\text{Subject to } Af + dv \leq 0\\
&\qquad\qquad\quad f \leq b\\
&\qquad\qquad\quad -f \leq 0.
\end{aligned}$$

So the primal problem (**P**) is

$$\begin{aligned}
&\text{Minimize } \textstyle\sum_{j=1}^{|E|}(\lambda_j b_j)\\
&\text{Subject to } \mu_x - \mu_y + \lambda_e \geq 0, \text{ for all edge } e = (x, y) \in E\\
&\qquad\qquad\quad \textstyle\sum_{i=1}^{|V|}(\mu_i d_i) = 1\\
&\qquad\qquad\quad \mu \geq 0, \lambda \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{Define, } Q_1 &= \{i \mid \textstyle\sum_{j=1}^{|E|}(a_{ij} f_j) + d_i v = 0\}\\
Q_2 &= \{j \mid f_j = b_j\}\\
Q_3 &= \{k \mid f_k = 0\}
\end{aligned}$$

So the restricted primal problem (**RP**) is defined by

$$\begin{aligned}
&\text{Minimize } \textstyle\sum_{j=1}^{|E|} x_j^{(a)} + y^{(a)}\\
&\text{Subject to } \mu_x - \mu_y + \lambda_e + x_e^{(a)} \geq 0, \text{ for all edge } e = (x, y) \in E\\
&\qquad\qquad\quad \textstyle\sum_{i=1}^{|V|}(\mu_i d_i) + y^{(a)} = 1\\
&\qquad\qquad\quad x_e^{(a)} \geq 0, \text{ for all edge } e \in E\\
&\qquad\qquad\quad y^{(a)} \geq 0\\
&\qquad\qquad\quad \mu_i \geq 0, \text{ for all } i \in Q_1\\
&\qquad\qquad\quad \lambda_j \geq 0, \text{ for all } j \in Q_2\\
&\qquad\qquad\quad \mu_i = \lambda_j = 0, \text{ for all } i \notin Q_1 \text{ and } j \notin Q_2
\end{aligned}$$

Hence, the dual of the restricted primal (**DRP**) is defined by

$$\begin{aligned}
&\text{Maximize } v\\
&\text{Subject to } Af + dv \leq 0\\
&\qquad\qquad\quad f \leq 1\\
&\qquad\qquad\quad v \leq 1\\
&\qquad\qquad\quad f_j \leq 0, \text{ for all } j \in Q_2\\
&\qquad\qquad\quad f_k \geq 0, \text{ for all } k \in Q_3
\end{aligned}$$

Question 4. *For the following matrix game, formulate an appropriate LP and compute all mixed-strategy Nash equilibria.*

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \\ 0 & 0 \end{bmatrix}$$

Solution: Pay off matrix for row player is A and for column player is $-A$.
Let strategy set for row player $R = \{r_1, r_2, r_3, r_4\}$ and for column player it is $C = \{c_1, c_2\}$.

LP formulation for row player (consider this as LP1 problem):

Maximize z
Subject to $z \leq 0 \cdot x_1 + \frac{1}{2} \cdot x_2 - \frac{1}{2} \cdot x_3 + 0 \cdot x_4$
$z \leq x_1 + 0 \cdot x_2 + x_3 + 0 \cdot x_4$
$x_1 + x_2 + x_3 + x_4 = 1$
$x_i \geq 0$ for all $i = 1, 2, 3, 4$

Similarly, for column player (consider this LP2 problem):

Minimize w
Subject to $w \geq 0 \cdot y_1 + y_2$
$w \geq \frac{1}{2} \cdot y_1 + 0 \cdot y_2$
$w \geq -\frac{1}{2} \cdot y_1 + y_2$
$w \geq 0 \cdot y_1 + 0 \cdot y_2$
$y_1 + y_2 = 1$
$y_i \geq 0$ for all $i = 1, 2$

Let x^* and y^* be the optimal solution of LP1 and LP2 respectively. So by the minmax theorem, we have that (x^*, y^*) is a mixed strategy Nash equilibrium. Note that LP1 and LP2 are dual for each other, and hence we can solve any one of them and use complimentary slackness to solve the other.
Now,

$$y_1 + y_2 = 1 \Rightarrow y_2 = 1 - y_1$$

From LP2, we get

Minimize w
Subject to $w \geq 1 - y_1$
$w \geq \frac{1}{2} y_1$
$w \geq -\frac{1}{2} y_1 + y_2$
$w \geq 1 - \frac{3}{2} y_1$
$1 \geq y_1 \geq 0$

Consider the above problem as LP3. Note that LP2 and LP3 are equivalent. Solving LP3 graphically gives,

$$y_1 = \frac{2}{3}, y_2 = \frac{1}{3} \text{ and } w = \frac{1}{3}$$

.
Let $y^* = (\frac{2}{3}, \frac{1}{3})$. Now we use the complementary slackness property to obtain optimal solution $x^* = (\frac{1}{3}, \frac{2}{3}, 0, 0)$ for LP1.

Thank you for reading
Happy Majors