Lecture 19

MTL-122

$(\mathbb{R}, d_{Eve})$ (X, d)

$(x_n)_{n \geq 1}$ $(x_n)_{n \geq 1}$ $x_n \in X$.

$x_n \in \mathbb{R}$

$(f_n)_{n \geq 1} \to f_1, f_2, f_3, \ldots$

$f_n : A \to \mathbb{R}$

$A \subseteq X$.

$(X \to \mathbb{R})$

Ex:

$f_n : (0,1) \to \mathbb{R}$

$$f_n(x) = \frac{n}{nx+1}$$

$$f_1(x) = \frac{1}{x+1}$$

$$f_2(x) = \frac{2}{2x+1}$$

$$f_3(x) = \frac{3}{3x+1}$$

$\vdots$

Ex. $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = x^n$$

Let $x \in \mathbb{R}$ (fix)

$$(f_n)_{n \geq 1}$$

$$(f_n(x))_{n \geq 1} \in \mathbb{R}$$

Defn:

(f_n), $f_n : A \to \mathbb{R}$

Let $f : A \to \mathbb{R}$.

$f_n \to f$ pointwise on A

if $f_n(x) \to f(x)$ as $n \to \infty$, $\forall x \in A$

$$\boxed{f(x) = \lim_{n\to\infty} f_n(x)}$$

Ex: $f_n : ((0,1), d_{Euc}) \to (\mathbb{R}, d_{Eucl})$

$$f_n(x) = \frac{n}{nx+1}$$

$x \neq 0$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x + 1/n}$$

$$= \frac{1}{x}$$

So $f_n \to f$ pointwise

$\forall x \in (0,1)$

$$f(x) = \frac{1}{x}$$

Boundedness

Ex $|f_n(x)| < n \quad \forall x \in (0,1)$

- f_n is bdd on $(0,1)$

Pointwise limit is it bdd ? X

$f(x) = \frac{1}{x}$. (not bdd)

- Pointwise convergence does not (in general) preserves boundedness.

- Continuity

Ex $f_n : [0,1] \to \mathbb{R}$

$f_n(x) = x^n$

$0 \le x < 1$, $x^n \to 0$ as $n \to \infty$

$x=1$ $\quad x^n \to 1$ as $n \to \infty$.

So, $\quad f_n \to f$ (pointwise)

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

- $f_n \to$ continuous on $[0,1]$
- $f \to$ not continuous.

- Pointwise convergence does not preserve continuity.

Differentiability

Ex

$$f_n : \mathbb{R} \to \mathbb{R}$$

$$f_n(x) = \frac{x^2}{\sqrt{x^2 + 1/n}}$$

Case 1

$x \neq 0$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{\sqrt{x^2+(1/n)}}$$

$$= \frac{x^2}{|x|} = |x|$$

Case 2

$x = 0$

$$f_n(0) = 0$$

$$f(x) = |x| ,$$

$f_n \to f$ (pointwise) as $n \to \infty$.

- Pointwise convergence doesnot preserve differentiability.

Suppose

$$\begin{tikzcd} f_n \arrow[r, "\text{pointwise}"] \arrow[d] & f \arrow[dr] & \\ \text{diff} \arrow[r, phantom, "\Downarrow ?"] & & \text{diff} \\ f_n' \arrow[r] & f' \; ? & \end{tikzcd}$$

No

Ex

$$f_n : \mathbb{R} \to \mathbb{R}$$

$$f_n(x) = \frac{\sin nx}{n}$$

$$f_n(x) \xrightarrow[\text{pointwise}]{} 0 \quad \text{for each } x$$

- $f_n \xrightarrow[p.w.]{} 0$ as $n \to \infty$

$f_n'(x) = \cos nx$

does not converge pointwise on $\mathbb{R}$,

$(f_n)_{n \geq 1}$ $\quad f_n : A \to \mathbb{R}$

$f : A \to \mathbb{R}$

P. C.

$\epsilon > 0 \quad \exists\, N_{(\epsilon, x)} \in \mathbb{N}$.

$|f_n(x) - f(x)| < \epsilon$

$\forall n > N_{(\epsilon, x)}$

for each $x \in A$.

U.C. (Uniform Convergence)

$\epsilon > 0 \quad \exists N(\epsilon) \in \mathbb{N}$

$$|f_n(x) - f(x)| < \epsilon$$

$\forall n \geq N_\epsilon$.

Ex $\quad f_n(x) = x^n$

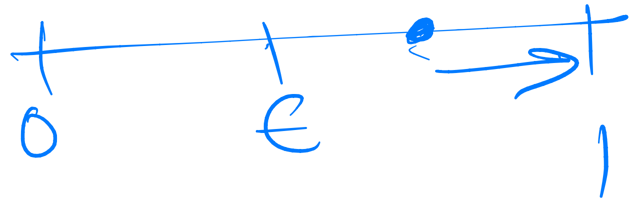
$0 \leq x < 1, \quad 0 < \epsilon < 1.$

$$|f_n(x) - f(x)| = |x^n| < \epsilon$$

iff $\quad 0 \leq x < \epsilon^{1/n}$

But $\quad \epsilon^{1/n} < 1 \quad \forall n$.

$1-\epsilon>0$



Not uniformly convergent.

Ex.

$$f_n(x) = \frac{\sin nx}{n}.$$

$$|f_n(x)| \le 1/n.$$

$$|f_n(x) - 0| < \epsilon \quad \forall x \in \mathbb{R}$$

$$n > 1/\epsilon.$$

Cauchy condition for uniform convergence.

Defn:

A seq (f_n) of fns $f_n : A \to \mathbb{R}$ is uniformly Cauchy on A if for every $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t $m, n > N$ (ind. of x)

$$\Rightarrow \quad |f_m(x) - f_n(x)| < \epsilon$$

$$\forall x \in A$$

For each x, $(f_n(x))$ is Cauchy.

Theo. A seq of fn (f_n) $(f_n : A \to \mathbb{R})$ converges uniformly on A iff it is uniformly Cauchy.

Pf. $(f_n) \xrightarrow[\text{uniformly}]{} f$ on A

$\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t

$|f_n(x) - f(x)| < \epsilon/2$ $\forall x \in A$ if $n > N$.

m, n > N then

$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)|$
$+ |f(x) - f_n(x)|$

$< \epsilon$.

$\Rightarrow$ $(f_n) \rightarrow$ uniformly Cauchy.

$\Leftarrow$ Converse.

$(f_n) \rightarrow$ uniformly Cauchy.

Then $x \in A$,

$\underbrace{(f_n(x))}_{\mathbb{R}}$ is Cauchy. (from defn)

By the completeness property

$(f_n(x))$ converges.

Define,

$$f(x) = \lim_{n \to \infty} f_n(x)$$

$$\begin{array}{ccc}
f_1(x_1) & f_1(x_2) & \\
f_2(x_1) & f_2(x_2) & \\
f_3(x_1) & f_3(x_2) & \\
\vdots & \vdots & \\
a_1 & a_2 & \cdots \\
\| & \| & \\
f(x_1) & f(x_2) & \cdots
\end{array}$$

$f_n \longrightarrow f$ pointwise.

$\epsilon > 0$

(f_n) is uniformly Cauchy

Choose $N(\epsilon)$ s.t.

$$|f_m(x) - f_n(x)| < \epsilon/2$$

$\forall x \in A$ if $m, n > N$.

Let $n > N$, $x \in A$

$\underline{\underline{m > N}}$.

$$|f_n(x) - f(x)|$$
$$\leq \underline{\underline{|f_n(x) - f_m(x)|}}$$
$$+ \underline{\underline{|f_m(x) - f(x)|}}$$

$< \epsilon/2 + |f_m(x) - f(x)|$

$\because f_m(x) \to f(x)$ as $m \to \infty$

$m > N$

dependent x

$|f_i(x) - f(x)| < \epsilon/2$ if $i > m_{(x,\epsilon)}$

$n > N$

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in A.$$

$\Rightarrow f_n \to f$ uniformly.