Lecture 9 : MTL 122 : Real and Complex analysis.

- Any open ball is an open set.

<u>Theo:</u> (X, d) be metric space

$(\mathbb{R}, d)$

1) X, ϕ are open.
2) Arbitrary union of open set is open.
3) finite intersection of open sets is open.

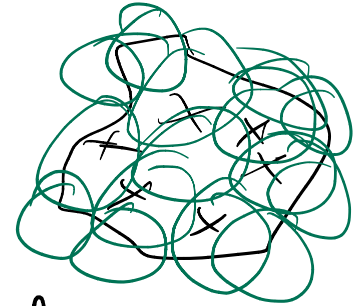
<u>Proposition:</u>

(X, d)

$A \subseteq X$ is open $\iff$ Union of open balls.

<u>Proof:</u> Suppose A is the union of open balls $\Rightarrow$ $A = \bigcup_{x \in A} B(x, r_x)$

$\Rightarrow$ A is open.



Converse

Assume A is open.

$x \in A$, $\epsilon_x > 0$, $B(x, \epsilon_x) \subset A$.

$$A = \bigcup_{x \in A} B(x, \epsilon_x)$$

- $F \subseteq X$ is closed if $X \setminus F$ is open.

Ex. (X, d_{dis})

Claim every subset of X is closed

$A \subseteq X$.

$\epsilon < 1$

Then A is open?

$B(x, \epsilon) = \{x\}$

$$A = \bigcup_{x \in A} B(x, \epsilon), \quad \epsilon < 1$$

$$= \bigcup_{x \in A} \{x\}$$

$\Rightarrow$ $X \setminus A$ is open $= \bigcup_{y \in X \setminus A} B(y, \epsilon)$

$A = X \setminus (X \setminus A) \rightarrow$ closed.

Every subset is open & closed in (X, d_{dis}).

- $\{x\} \rightarrow$ open & closed

Theo:

1) $X, \emptyset$ closed
2) Arbitrary intersection of closed sets is closed.
3) Finite union of closed sets is closed.

• Interior points (X, d)

$S \subseteq X$.

$\boxed{x \in S}$ if $\exists\, \epsilon > 0$

$B(x, \epsilon) \subset S$.

$S^o = \mathrm{int}(S) =$ set of all int pts

b) Boundary pts

$\boxed{x \in X}$

Every $\epsilon > 0$

$B(x,\epsilon) \cap S \neq \emptyset$

$B(x,\epsilon) \cap (X \setminus S) \neq \emptyset$

$\mathbb{R}$

∂S / $\mathrm{bd}(S)$ = set of all bdd points.

c) Isolated pts

$\boxed{x \in S.}$

$\exists\, \epsilon > 0$ s.t $B(x, \epsilon) \cap S = \{x\}$.

d) Accumulation point

$\boxed{x \in X}$

Every $\epsilon > 0$

$B(x, \epsilon) \setminus \{x\} \cap S \neq \emptyset$

Derived set, $S' =$ set of all accumulation point.

e) $\bar{S} = S \cup S' =$ closure of S

$(\mathbb{R}, d)$ (X, d) (Ex)

i) $A^\circ \subseteq A$.

ii) $(A^\circ)^\circ = A^\circ$

iii) $A \subseteq B$, $A^\circ \subseteq B^\circ$.

iv) $\bigcup_{i \in I} A_i^\circ \subseteq \left(\bigcup_{i \in I} A_i \right)^\circ$

v) $\left(\bigcap_{i \in I} A_i \right)^\circ \subseteq \bigcap_{i \in I} A_i^\circ$.

Ex. (X, d)

- Every finite subset of X is closed.

$\{x_1, x_2, x_3\}$

$\{x\} \rightarrow$ closed.

- $\{x\}$ is open in $X \Leftrightarrow$ if is an isolated point.

$(\mathbb{N}, d)$ $\quad X = \mathbb{N}$.

$\{1, 2, 3, \ldots\}$

$\epsilon < 1$.

$\{1\}$ $\{2\}$ ...

(X, d_{dis}) $\leftarrow$ $\{x\} \to$ clopen

$(\mathbb{N}, d_{eu})$ $\longrightarrow$ $\{x\} \to$ clopen

$\to$ discrete space.

Open.

$S \subseteq X$

i) $\partial S \cap S = \emptyset$

ii) $S = S^\circ$.

Equivalent = ?

$\Rightarrow$ S is open.

$\boxed{S^\circ = S \setminus \partial S}$ ⊛ $\rightarrow$ Please check

Closed

$\boxed{\overline{S} = S \cup \partial S.}$ ⊛ $\leftarrow$ do.

$[(\mathbb{R}, d)]$

$$\boxed{\partial S \subseteq S. \Longleftrightarrow S = \overline{S} \Longleftrightarrow S \text{ is closed}}$$

Theo. $C \subseteq X$, is closed iff $C' \subseteq C$

Pf. $\subseteq$ C closed, $\boxed{x \in C'}$

$x \notin C \Rightarrow x \in \underbrace{X \setminus C}_{\text{open}}$.

$\exists \, \epsilon > 0$ s.t

$B(x, \epsilon) \subset X \setminus C$.

$\Rightarrow B(x, \epsilon) \cap C = \phi$ ↯

$\Rightarrow x \in C$ &

$C' \subseteq C$.

$C' \subseteq C$

$X \setminus C$ open (show)

$\boxed{x \in X \setminus C}$

$x \notin C' \Rightarrow \exists \epsilon > 0$

$B(x, \epsilon) \cap C = \emptyset$

$\Rightarrow B(x, \epsilon) \subset X \setminus C$

$\Rightarrow X \setminus C$ is open.

Cor. $C \subseteq X$ C is closed iff $\overline{C} = C$.

Pf. C closed

$C' \subseteq C \Rightarrow \overline{C} = C \cup C'$

$\subseteq C$

$\Rightarrow \quad \bar{C} = C.$

Conversely

$$C = \bar{C} = C \cup C'$$

$C' \subseteq C \Rightarrow C$ is closed.

1) Bdd set.

2) diameter

3) $\operatorname{diam}(B(a, \epsilon)) \leq 2\epsilon$

$\mathbb{R}^2$

Metric Spaces.

Michael Searcôid.

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