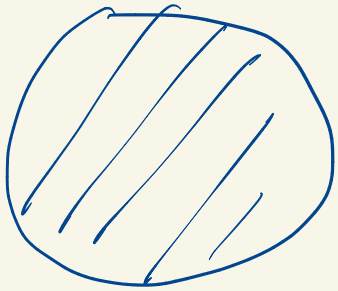
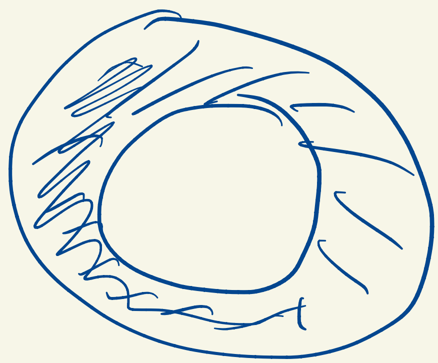
Lecture 32

MTL122



$\rightarrow$ holomorphic fn can be represented by a power series.

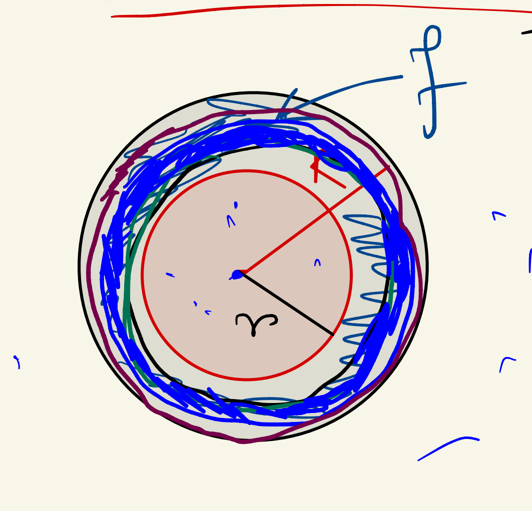
Question



Theorem:

$f \rightarrow$ holomorphic fn. on an annulus

$$D = \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$$



Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

converge on D

converges uniformly on any closed sub annulus

i.e. $\{ z \in \mathbb{C} \mid \rho_1 \le |z-z_0| \le \rho_2 \}$

Moreover,

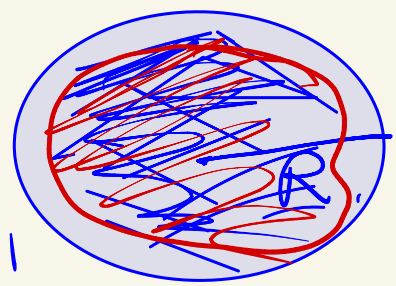
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \qquad n = 0, \pm 1, \pm 2 \ldots$$

simple ~~closed~~ contour in

D centering z_0.

Case 1

Suppose f is holomorphic in $|z-z_0|<R$



$$a_{-n} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{-n+1}}$$

$$= \frac{1}{2\pi i}\int_C \underbrace{f(\zeta)(\zeta-z_0)^{n-1}}_{\text{holomorphic } |z-z_0|<R}$$

$$= 0$$

(Cauchy-Goursat)

2) $a_n = \dfrac{f^n(z_0)}{n!}$, $n = 0, 1, 2 \ldots$

We allow, $r=0$ or $R=\infty$.

'a_{-1}' of $(z-z_0)^{-1}$
$\hookrightarrow$ <u>Residue of $f(z)$ at z_0</u>

Laurent series.

$$\underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{\text{Singular part}}}$$

Theo. $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$

a) $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on $\{z \in \mathbb{C} \mid |z-z_0|<R\}$

b) $\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$ converges on $\{z \in \mathbb{C} \mid |z-z_0|>r\}$

Then,

$\exists!$ f holomorphic on $D = \{z \in \mathbb{C} \mid r < |z-z_0| < R\}$

such that Laurent series of f

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

Example

$$f(z) = \frac{z^2 - 2z + 3}{z - 2} \text{ on}$$

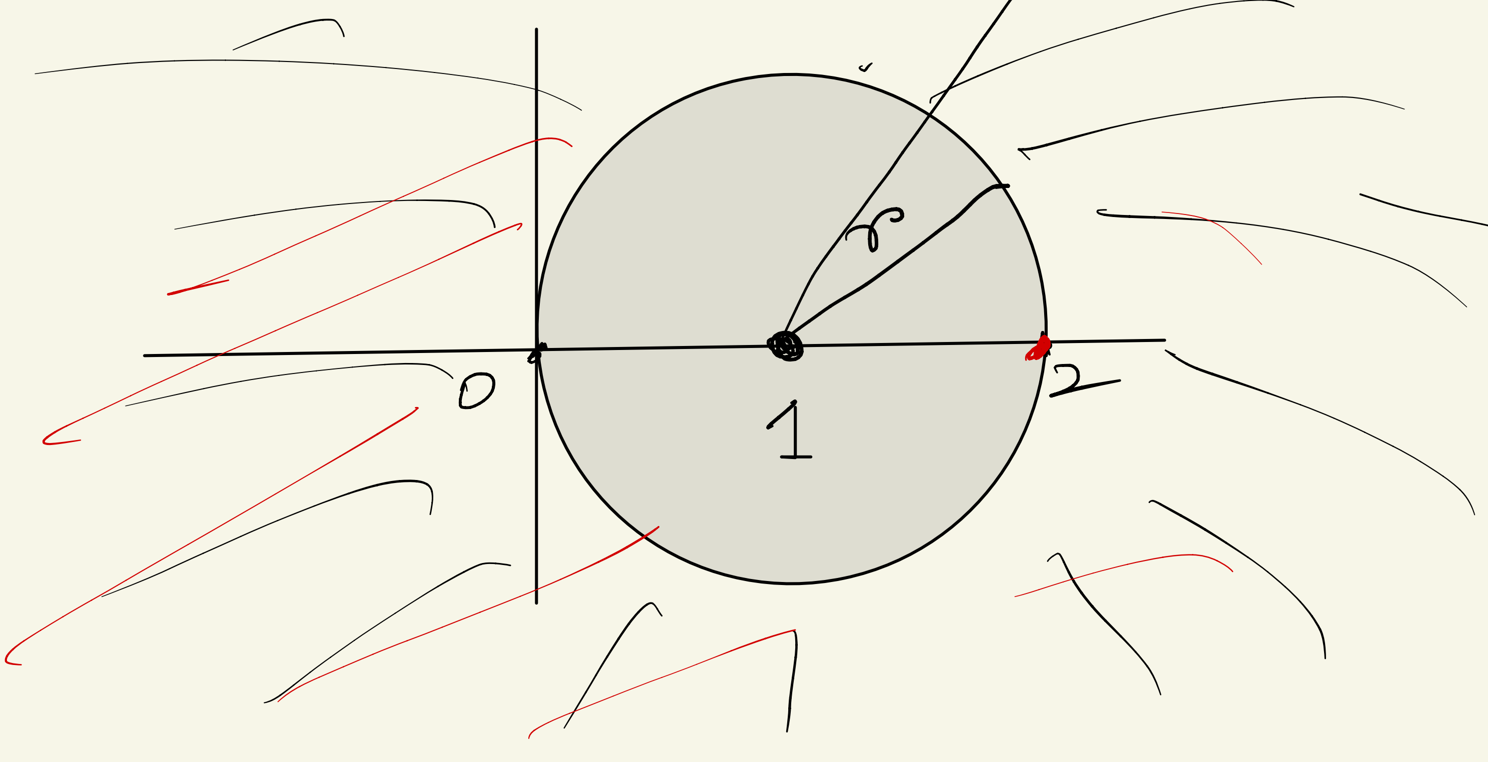
$$\{ z \in \mathbb{C} \mid |z-1| > 1 \}$$

Soln:

$$1 < |z-1|$$

$$z_0 = 1, \quad r = 1, \quad R = \infty,$$

$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$



f is holomorphic on D.

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1}\,\underbrace{\frac{1}{1-\frac{1}{z-1}}}$$

For,

$|z-1|>1$ $\left[\frac{1}{|z-1|}<1\right.$

$$\frac{1}{z-2} = \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}$$

$$z^2-2z+3 = z^2-2z+1$$

$$= (z-1)^2+2$$

$$f(z) = \left\{(z-1)^2+2\right\}\left\{\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right\}$$

$$= \underline{(z-1)+1} + \sum_{n=1}^{\infty} \frac{3}{(z-1)^n},$$

$$|z-1|>1.$$

Residue $=$ '$\underline{a_{-1}}$' $= 3$.

"Alhfors"

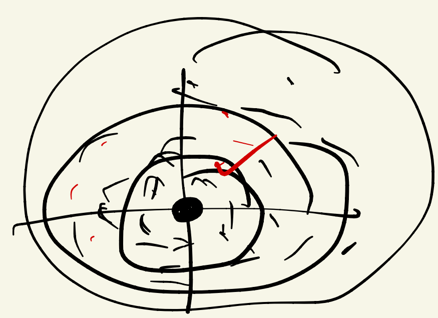
"Conway"

'Singular point'

'z_0' $\longrightarrow f$.

i.e. 'f fails to be analytic at 'z_0'

but is analytic at some point



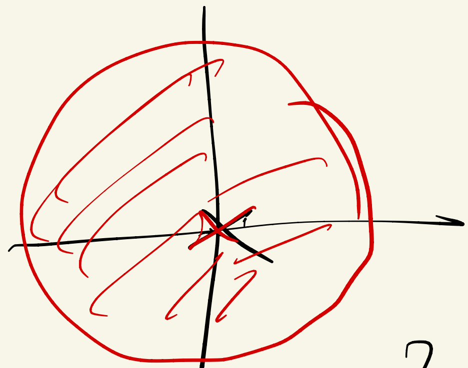
in every neighbourhood of 'z_0' $z_0 \in \mathbb{C}$ (say)

Isolated singularity

$z_0 \in \mathbb{C}$

1) f is not analytic at z_0.

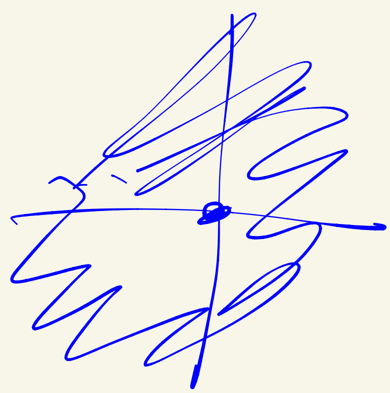
2) f is holomorphic on some punctured disk $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$.



Ex.

1) $f(z) = 1/z$.

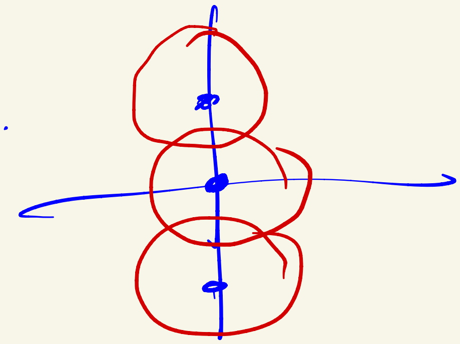
$z = 0 \rightarrow$ isolated singularity.



2) $f(z) = \dfrac{z+1}{z^3(z^2+1)}$

2) $z=0$, $z=\pm i$

isolated singularities.



'z_0' , f is holomorphic on $D=\{z\in\mathbb{C} \mid 0<|z-z_0|<r\}$

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

1) $a_{-n}=0$, $n=1,2,\dots$

'z_0' is a removable singularity.

2) $a_{-m}\neq 0$, for some int m.

but $a_{-n}=0$ $\forall n>m$.

'z_0' is pole of order 'm',

Simple pole = pole of order 1.

3) If $a_{-n} \neq 0 \; \forall n$ then 'z_0' essential singularity.

Ex:

1) $w = f(z) = e^{1/z}$

2) $f(z) = \frac{\sin z}{z}$.

3) $f(z) = \frac{e^z}{z^2}$.

4) $f(z) = \frac{e^z - 1}{z}$.

'0' is the isolated singularity

Soln.

1) $f(z) = e^{1/z}$.

$$e^{1/z} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{(z-0)^n}, \quad |z| > 0.$$

$$a_{-n} = \frac{1}{n!} \neq 0$$

'0' is an essential singularity.

2) $$\frac{\sin z}{z} = \frac{1}{z}\left\{ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right\}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$a_{-n} = 0, \ \forall n$.

'0' $\longrightarrow$ removable sing.

3) $$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$= \underbrace{\frac{1}{z^2} + \frac{1}{z}}_{a_{-2} \neq 0} + \frac{1}{2!} + \frac{1}{3!}z + \cdots$$

$a_{-3} = a_{-4} = a_{\ldots} = 0$

$\Rightarrow$ '0' pole of order 2

Uptil this is the major syllabus.