Quiz

Graded

Student

Hemant Ramgaria

Total Points

19.5 / 20 pts

Question 1

Pivoting

9.5 / 10 pts

- ✔ **+ 0.5 pts** For correctly finding P_1 and evaluating P_1A
- ✔ **+ 0.5 pts** For correctly finding L_1
- ✔ **+ 0.5 pts** For evaluating L_1P_1A
- ✔ **+ 0.5 pts** For correctly finding P_2 and evaluating $P_2L_1P_1A$
- ✔ **+ 0.5 pts** For correctly finding L_2
- ✔ **+ 0.5 pts** For evaluating $L_2P_2L_1P_1A$
- ✔ **+ 0.5 pts** For correctly finding P_3 and evaluating $P_3L_2P_2L_1P_1A$
- ✔ **+ 0.5 pts** For correctly finding L_3
- ✔ **+ 0.5 pts** For evaluating $L_3P_3L_2P_2L_1P_1A$
- ✔ **+ 0.5 pts** For writing the equation $L_3P_3L_2P_3^{-1}P_3P_2L_1P_2^{-1}P_3^{-1}P_3P_2P_1A = U$
- ✔ **+ 2 pts** For showing that $P_3P_2L_1P_2^{-1}P_3^{-1}$ is lower triangular matrix
- ✔ **+ 1 pt** For showing that $P_3L_2P_3^{-1}$ is lower triangular matrix
- ✔ **+ 2 pts** For correctly finding L where $PA = LU$
- **− 0.5 pts** Calculation mistakes in L

Question 2

Iterative Method

6 / 6 pts

- ✔ **+ 3 pts** Forward Direction Correct, i.e $\rho(G) < 1 \Rightarrow Convergance$
- ✔ **+ 3 pts** Reverse Direction Correct i.e Convergance implies $Convergance \Rightarrow \rho(G) < 1$

Question 3

Newton's Method

4 / 4 pts

- ✔ **+ 1 pt** Root found correctly
- ✔ **+ 3 pts** $f'(x) = 0 \Rightarrow$ Linear Convergance

MTL 107: 1st Semester 2024-25

Numerical Methods and Computation

Quiz

Total Marks: 20

Exam Time: 10:30 AM to 11:15 AM

Name: Hemant Ramgarie

Entry No: 2022MT11854

Instructions

(1) Please write your Name and Entry Number properly. This is important as we will upload your answer sheet to Gradescope, and this will streamline the process.
(2) This is a template-based exam. So, you must write the solution to each question at the designated place only. Otherwise, it will not be mapped properly, and we will be unable to correct it.

1. **(10 Marks)** Consider the following matrix:

$$\begin{bmatrix} 2 & 3.5 & 4 & 6 \\ 4 & 2 & 1 & 0 \\ -8 & 2 & 8 & 8 \\ 2 & 2.5 & 4 & -3 \end{bmatrix}$$

Using the partial pivoting, find the LU factorization of the matrix in the following form:

$$L_3P_3L_2P_2L_1P_1A = U.$$

where P_1, P_2 and P_3 are the permutation matrices and L_1, L_2 and L_3 are the lower triangular matrices representing the row operations. Then show that, $P_3P_2L_1P_2^{-1}P_3^{-1}$ and $P_3L_2P_3^{-1}$ are lower triangular matrices. Finally find L using permutation matrices P_1, P_2 and P_3 and L_1, L_2 and L_3 such that $PA = LU$ with $P = P_3P_2P_1$. Explain all the steps clearly and write all the steps with complete details.

$$A = \begin{bmatrix} 2 & 3.5 & 4 & 6 \\ 4 & 2 & 1 & 0 \\ -8 & 2 & 8 & 8 \\ 2 & 2.5 & 4 & -3 \end{bmatrix}$$

if first colum max is at ~~colum~~ row 3 so

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_1A = \begin{bmatrix} -8 & 2 & 8 & 8 \\ 4 & 2 & 1 & 0 \\ 2 & 3.5 & 4 & 6 \\ 2 & 2.5 & 4 & -3 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \end{bmatrix}$$

$$L_1P_1A = \begin{bmatrix} -8 & 2 & 8 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 4 & 6 & 8 \\ 0 & 3 & 6 & -1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_2L_1P_1A = \begin{bmatrix} -8 & 2 & 8 & 8 \\ 0 & 4 & 6 & 8 \\ 0 & 3 & 5 & 4 \\ 0 & 3 & 6 & -1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & -3/4 & 0 & 1 \end{bmatrix}$$

$$L_2P_2L_1P_1A = \begin{bmatrix} -8 & 2 & 8 & 8 \\ 0 & 4 & 6 & 8 \\ 0 & 0 & 0.5 & -2 \\ 0 & 0 & 1.5 & -7 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Question 1(cont...):

$$P_3 L_2 P_2 L_1 P_1 A = \begin{bmatrix} 8 & 2 & 8 & 8 \\ 0 & 4 & 6 & 8 \\ 0 & 0 & 1.5 & -7 \\ 0 & 0 & 0.5 & -2 \end{bmatrix} \quad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/3 & 1 \end{bmatrix}$$

$$L_3 P_3 L_2 P_2 L_1 P_1 A = \begin{bmatrix} 8 & 2 & 8 & 8 \\ 0 & 4 & 6 & 8 \\ 0 & 0 & 1.5 & -7 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} = U$$

$$P_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_3 L_2 P_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 0 & 1 \\ 0 & -3/4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/4 & 1 & 0 \\ 0 & -3/4 & 0 & 1 \end{bmatrix}$$

so lower triangular.

$$\rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & 3/4 & 1 & 0 \\ -1/4 & 3/4 & 1/3 & 1 \end{bmatrix} \longrightarrow \cancel{PA} \; LU = PA$$

$$\cancel{PA} = \quad P_3 P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (P_3 P_2)^{-1} = P_2^{-1} P_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_3 P_2 L_1 P_2^{-1} P_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \\ 1/2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}$$

it is lower triangle matrix

Question 1(cont...):

$$L_3 P_3 L_2 P_2 L_1 P_1 A = U$$

$$P_1 A = \cancel{L_3^{-1} P_3^{-1} L_2^{-1} P_2^{-1} L_1^{-1}} \; L_1^{-1} P_2^{-1} L_2^{-1} P_3^{-1} L_3^{-1} U$$

$$P_3 P_2 P_1 A = \underbrace{P_3 P_2 L_1^{-1} P_2^{-1} L_2^{-1} P_3^{-1} L_3^{-1}}_{L} U$$

$$P = \left(P_3 P_2 L_1^{-1} P_2^{-1} P_3^{-1}\right)\left(P_3^{-1} L_2^{-1} P_3^{-1}\right) L_3^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & 3/4 & 1 & 0 \\ -1/4 & 3/4 & 1/3 & 1 \end{bmatrix}$$

2. **(6 Marks)** Show that the iterations

$$\vec{x}_{k+1} = \mathbf{G}\vec{x}_k + \vec{c}$$

converges for any initial guess $\vec{x}_0$ if and only the spectral radius of $\mathbf{G}$, $\rho(G) < 1$.

let take $\rho(G) < 1$ so we have to prove it conver.

Given $\vec{x}_{k+1} = G\vec{x}_k + \vec{c}$

$\Rightarrow x_{k+1} = G(Gx_{k-1} + c) + c$

$\Rightarrow x_{k+1} = G^{k+1}x_0 + (I + G + G^2 \cdots G^k)c$

so given $\rho(G) < 1$ so it implies $\lim_{m\to\infty} G^m(x) = 0$
G will converge.

and we also know $\lim_{m\to\infty} \sum_{j=0}^{m} G^j = (I - G)^{-1}$

we can also prove it
$(I - G)$ inverse exist otherwise $\rho(G) = 1$ becau $\lambda = 1$ will eigenval.
so let calcut
$(I - G)\left(\sum_{j=0}^{m} G^j\right) = I - G^{m+1}$
as G converses $G^{m+1} = 0$
so $(I - G)$ is Inverse of $\sum_{j=0}^{m} G^j$

$\Rightarrow \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty}\left(G^{k+1}x_0 + c\left(\sum_{j=0}^{m} G^j\right)\right)$

$\Rightarrow x_{k+1} = (I - G)^{-1}c$

so clearly when k goes to infinit.
it will become const and conver.
to $(I - G)^{-1}c$.
Hence proved.

Question 2(cont...):

let take $x_{k+1} = G x_k + c$ converges so we have to prove $\rho(G) < 1$

It also enough to show that G will converse because $\rho(G) < 1$ if and only if G will converse.

so $\lim_{m \to \infty} G^m z = 0$ for all z.

$x_{k+1} = G x_k + c$

$x^* = G x^* + c$ ← converge term

$\| x^* - x_{k+1} \| = \| G(x^* - x_k) \|$ $\lim_{k \to \infty} \| x^* - x_{k+1} \| < \varepsilon$ (converge)

$\cancel{\| x^* - x_{k+1} \| \leq \|G\| \, \| x^* - x_k \|}$

$|x^* - x_{k+1}| = |G^{k+1}(x^* - x_0)|$

$\cancel{\| x^* - x_{k+1} \| \leq \|G\|^{k+1} \| x^* - x_0 \|}$

$\lim_{k \to \infty} | G^{k+1}(x^* - x_0) | < \varepsilon$

so for any z put $x_0 = x^* - z$.

so $|G^{k+1} z| < \varepsilon$

for all z G^{k+1} give value 0

hence G^{k+1} converges

this implies $\rho(G) < 1$

3. **(4 Marks)** Consider the function

$$f(x) = 3x\frac{\sin(x)}{\tan(x)} - \frac{3}{2}\pi\frac{\sin(x)}{\tan(x)}.$$

Discuss the order of convergence of Newton's method for a root between $\frac{\pi}{4}$ and $\frac{2\pi}{3}$ (you have to find the root).

$f'(x) = 3x\cos(x) - \frac{3}{2}\pi\cos(x)$ $\Rightarrow$ $\cancel{3\left(x\cos x + \frac{\pi}{2}\cos x\right)}$ $\frac{3}{2}\cos(x)\left(2x - \pi\right)$

$|f'(x)| \neq 0$ for ~~all~~ some valu of x

it only 0 at $x = \frac{\pi}{2}$ or wh $\cos(x) = 0$.

$|f'(x)| < 1$

so it is linear convergence.

$\cancel{f\left(\frac{\pi}{4}\right) < 0}$

$\cancel{\text{or } f\left(\frac{2\pi}{3}\right)}$

Root of it is $3x\cos(x) = \frac{3}{2}\pi\cos(x)$

$\Rightarrow x = \frac{\pi}{2}$

so root is $x = \frac{\pi}{2}$

Question 3(cont...):