Lecture-21

MTL-122

Real and Complex Analysis.

Series of functions

$$\sum_{n=1}^{\infty} f_n \quad , \quad f_n : A \to \mathbb{R}$$

$\forall n$

- $S_n(x) = \sum_{i=1}^{n} f_n(x) \leftarrow$ exist.

$S_n : A \to \mathbb{R}$.

sequence of n^{th} partial sums.

- $\sum_{n=1}^{\infty} f_n$ converges at x_0, $x_0 \in A$, if $\{S_n(x_0)\}$ converges.

Real Analysis

- $\sum_{n=1}^{\infty} f_n$ converges pointwise (uniformly) if $\{S_n\}$ converges pointwise.

(uniformly)

Riemann Integrability

$$\sum_{n=1}^{\infty} f_n$$

$$\int \sum \overset{?}{=} \sum \int$$

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n \quad ?$$

$$\frac{d}{dx} \sum = \sum \frac{d}{dx}$$

(sequences ?)

$S_n \rightrightarrows S$
uniform.

$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$

- $(f_n) \to$ continuous fns.

$$\sum_{n=1}^{\infty} f_n \xrightarrow{S_n \longrightarrow f} f, \quad [a,b] \subset A$$

uniformly

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \int_a^b f_i(x)\,dx \right)$$

$$\int_a^b \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\int_a^b S_n(x)\,dx \right)$$

Differentiation

- $(f_n) \to$ continuously diff fns.

- $\sum_{n=1}^{\infty} f_n'$ → converges uniformly.

- $\sum_{n=1}^{\infty} f_n$ → pointwise converges

Conclude

$$\sum_{n=1}^{\infty} f_n \longrightarrow f \quad \text{differentiables}$$

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty} f_n'(x)$$

pointwise

Proof are similar to the seq of f_n.

$$S_n \xrightarrow{\text{pointw}} S$$

$$S_n' \xrightarrow[\text{uni}]{} g$$

$$S' = g$$

Theo.

If $\sum_{n=1}^{\infty} f_n$,

- $|f_n(x)| \leq \alpha_n$, $\alpha_n \in \mathbb{R}^+$

s.t

- $\sum_{n=1}^{\infty} \alpha_n$ converges $\Longrightarrow$

$\sum_{n=1}^{\infty} f_n$ is uniformly converges.

Proof: $S_n(x) = \sum_{i=1}^{n} f_i(x)$, $n \in \mathbb{N}$.

$n > m$

$|S_n(x) - S_m(x)|$

$= \left| \sum_{i=m+1}^{n} f_i(x) \right| \leq \sum_{i=m+1}^{n} |f_i(x)|$

$$\leq \sum_{i=m+1}^{n} \alpha_i = (\sigma_n - \sigma_m)$$

where $\sigma_n = \sum_{k=1}^{n} \alpha_k$

$\Rightarrow$ σ_n is Cauchy

$\epsilon > 0$, $\forall n\ m > N$

$$|\sigma_n - \sigma_m| < \epsilon$$

$$\Rightarrow |S_n(x) - S_m(x)| < \epsilon$$

$\forall n > m$

$m > N$.

$\forall x \in A$

$\Rightarrow \{S_n(x)\} \to$ C. seq

Let $\quad f(x) = \lim_{n \to \infty} s_n(x)$

$$\left| f(x) - s_m(x) \right| = \lim_{n \to \infty} |s_n - s_m|$$

$$< \epsilon \quad \forall m \geq N. \quad \forall x \in A$$

$$\Rightarrow \sum_{n=1}^{\infty} f_n \xrightarrow[\text{unifon.}]{} f.$$

Ex 1.

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)} \text{ on } \mathbb{R}.$$

$$\frac{x}{n(1+nx^2)} \leq \frac{1}{n} \underbrace{\left(\frac{1}{2\sqrt{n}} \right)}_{= \alpha_n}$$

$$\sum \frac{1}{n^{3/2}} \rightarrow \text{con} \qquad \sum \frac{1}{n^p},\ p > 1$$

2) $\sum_{n=1}^{\infty} (x e^{-x})^n$ on $[0, \infty)$

$$(x e^{-x})^n = \frac{x^n}{e^{nx}}$$

$$e^{nx} = 1 + (nx) + \frac{(nx)^2}{2!} \cdots$$

$$\leq \frac{x^n}{\frac{(nx)^n}{n!}}$$

$$= \frac{n!}{n^n} = \alpha_n$$

$\sum_{n=1}^{\infty} \frac{n!}{n^n} \longrightarrow$ converges.

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3) $\sum_{n=1}^{\infty} x^{n-1}$ on $(0, 1)$

(not uniformly converges)

$$S_n(x) = \sum_{k=1}^{n} x^{k-1} = \frac{1-x^n}{1-x}$$

$$\longrightarrow \frac{1}{1-x} = f(x)$$

$\epsilon > 0$

$$|f(x) - S_n(x)| < \epsilon$$

$$\Longleftrightarrow \left| \frac{x^n}{1-x} \right| < \epsilon$$

Iff $\exists N \in \mathbb{N}$ s.t

$$\frac{x^n}{1-x} < \epsilon \quad \forall x \in (0,1) \quad \forall n \geq N \ ?$$

as $x \to 1$

$$\frac{x^N}{1-x} \longrightarrow \infty$$

- x^n $(0,1)$

$\longrightarrow [-a, a] \quad 0<a<1$

<u>Check</u>

$$\sum_{n=1}^{\infty} x^{n-1} \longrightarrow$$ converges uniformly $[-a, a]$

$0<a<1$.

<u>Ex</u>.

$C[0,1] \longrightarrow$ sup metric is a <u>complete</u> <u>metric</u>.

$\{f_n\}$

- $\{f_n\}_{n=1}^{\infty} \to$ Ca. Seq in $C[0,1]$.

- $\epsilon > 0 \; \exists N$ s.t

$$d_{sup}(f_n, f_m) < \epsilon \quad \forall n, m \geq N.$$

$\forall x \in [0,1]$

$$\underline{\sup_{x \in [0,1]}} |f_n(x) - f_m(x)| < \epsilon$$

$\forall m, n \geq N.$

$$\Rightarrow |\underline{f_n(x)} - f_m(x)| < \epsilon$$

$\forall m, n \geq N$

$\underline{\forall x \in [0,1]}$

Fix $\underline{x}$

$\{\underline{f_n(x)}\} \to$ C. Seq of $\mathbb{R}$.

- $\mathbb{R}$ is complete.

$$\Rightarrow \quad f(x) = \lim_{n\to\infty} f_n(x)$$

$$\Rightarrow \quad |f(x) - f_m(x)| \leq \epsilon \quad \forall n \geq N$$

$$\forall x \in [0,1]$$

$f_n \to f$ uniformly.

- $C[0,1]$ —— $\int_0^1 |f_m(x) - f_n(x)|\,dx$?

Check.

Ex.

$\{r_n\} \rightarrow$ enumeration of $\mathbb{Q} \cap [0,1]$

$$f_n(x) = \begin{cases} 1, & x = r_1, r_2, \dots, r_n \\ 0, & \text{o.w.} \end{cases}$$

$$f_1(x) = \begin{cases} 1, & x = r_1 \\ 0 & \text{o.w} \end{cases}$$

$$f_2(x) = \begin{cases} 1, & x = r_1, r_2 \\ 0, & \text{o.w.} \end{cases}$$

$\Rightarrow f_n \longrightarrow$ R. Int.

$$f_n(x) \longrightarrow f(x) \quad n \to \infty.$$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{o.w.} \end{cases}$$

Dirichlet's

not Rie. I

Real Analysis.