Lec - 16

Real and Complex Analysis -

MTL 122

Theo. $f : C \to f(C)$

(X, d_1) (Y, d_2)

$f : X \to Y$ continuous.

Each compact subset $C \subseteq X$

$\Rightarrow f(C) \subset Y$, $f(C)$ compact.

Proof: $\{U_i : i \in I\}$ open cover of $f(C)$.

$f : C \to f(C)$

$$\boxed{V_i = f^{-1}(U_i)}$$

open ($\because$ f is continuous).

Claim. $\{V_i : i \in I\}$ open cover of C.

$$C \subseteq \bigcup_{i\in I} V_i = \bigcup_{i\in I} f^{-1}(U_i)$$

$$f(C) \subseteq \bigcup_{i\in I} U_i \longrightarrow$$

$c \in C \qquad f(c) \in f(C)$

$f(c) \in U_j$

$$\Rightarrow c \in V_j \Rightarrow c \in \bigcup_{i\in I} V_i$$

$$\Rightarrow C \subset \bigcup_{i\in I} V_i$$

$$\{ V_1, \dots, V_n \}$$

$$C \subseteq \bigcup_{i=1}^{n} V_i$$

$$\Rightarrow f(C) \subseteq \bigcup_{i=1}^{n} U_i$$

Ex.

$\Rightarrow$ $f(C)$ is compact.

Continuity

- Bdd.
- t. bdd.
- Comp.

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cont.

- $\boxed{f : \underline{[a,b]} \to [c,d]}$ u.c.

Theo. $f : \underline{(X, d_X)} \to (Y, d_Y)$ continuous.

$\boxed{X : \text{compact}}$

$\Rightarrow$ f is uniformly cont.

Proof:

$$\begin{cases} \cdot \ (X, d_X) \text{ compact.} \\ \cdot \ f : X \to Y \text{ continuous.} \end{cases}$$

$\epsilon > 0 \quad \exists \, \delta_\epsilon$ such that

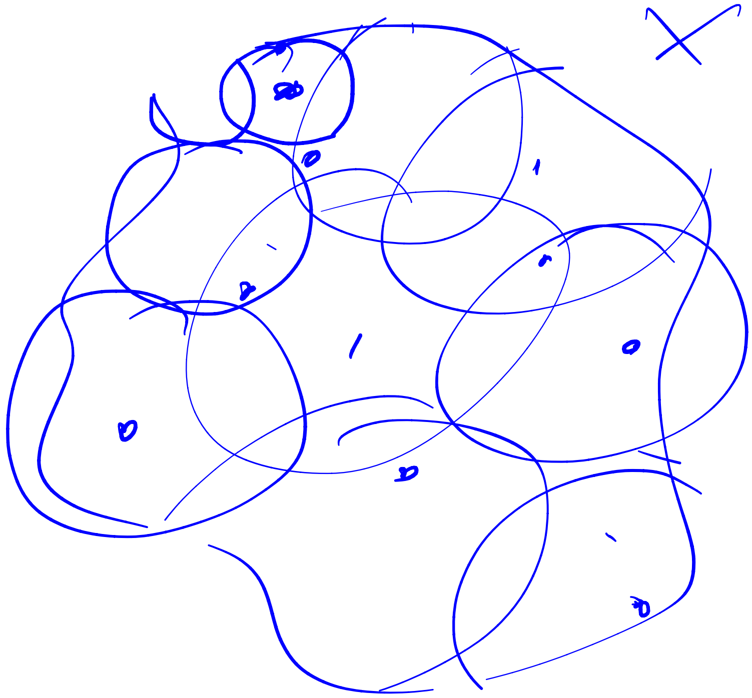
$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon. \quad \forall x, y \in X.$$

$\epsilon' = \epsilon/2 > 0$

Each $x \in X$. $\exists \, \delta_{x, \epsilon'} > 0$

- $f(B(x, \delta_x)) \subseteq B(f(x), \epsilon/2)$

- $\{B(x, \delta_x/2)\}_{x \in X}$ $\to$ open cover.



$$X = \bigcup_{x \in X} B\left(x, \frac{\delta_x}{2}\right)$$

open cover

$$\left\{ B\left(x_i, \frac{\delta_{x_i}}{2}\right) \right\}_{i=1}^{n}$$

$$\delta_\epsilon = \min_i \left(\frac{\delta_{x_i,\epsilon}}{2} \right)$$

$x, y \in X$

$d(x, y) < \delta_\epsilon$.

$x \in X$.

$\Rightarrow x \in B\left(x_i, \frac{\delta_{x_i}}{2}\right)$.

$\boxed{d(y, x_i) < d(x, y) + d(x_i, x)}$

$< \delta_\epsilon + \frac{\delta_{x_i}}{2}$

Claim, $y \in B(x_i, \delta_{x_i})$ $\leq \delta_{x_i}$

(Triangle inequality)

$d_Y(f(x), f(y))$

$\leq d_Y(f(x), f(x_i))$

$+ d_Y(f(x_i), f(y))$

$< \epsilon/2 + \epsilon/2 = \epsilon$.

$\Rightarrow$ f is u.c.

- C. Product
- F. Union

$\Big\}$ compact.

Connectedness.

"separated" set.

- $\mathbb{R} \to$ connected

$$\boxed{\mathbb{R} = (-\infty, 0] \cup (0, \infty)}$$

$\mathbb{R} \setminus \{0\}$.

$$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$$

Separated

- A, B, disjoint, nonempty
- A, B are both open in X.
- A, B are both closed.
- $\overline{A} \cap B = \overline{B} \cap A = \emptyset$.

Separation

U, V are separation of X.

$$\boxed{\begin{gathered} X = U \cup V \\ \overline{U} \cap V = \overline{V} \cap U = \emptyset \end{gathered}}$$

X is disconnected if

there exists a separation

Ex:

(X, d) is disconnected

$\Downarrow$

$\exists$ open sets. $X = U \cup V$.

$\Downarrow$

$\exists$ closed sets $X = A \cup B$.

$Y \subset X$.

connected.

Ex 1. $X \rightarrow$ discrete space.

at least two point.

discrete metric.

2) $\mathbb{R} \setminus \{0\}$.

$X = \mathbb{Q} \subseteq \mathbb{R}$.

$\sqrt{2}$

~~irrational~~ $r \in \mathbb{R} \setminus \mathbb{Q}$

$$A = \mathbb{Q} \cap (-\infty, r)$$

$$B = \mathbb{Q} \cap (r, \infty)$$

$$\mathbb{Q} = A \cup B,$$

Theo.

X is connected

$\iff$ Every 2-valued continuous fns. on X

is const.

2-valued

$X \to \{0, 1\}$

discrete topology.