Lecture 4 – MTL 122 – Real and Complex Analysis.

- $\underline{\text{Neighbourhood}}$ ($\mathbb{R}$) $\longrightarrow$ $a \in \mathbb{R}$ $\underline{(a-\epsilon, a+\epsilon)}$

- Deleted ngb. — $(a-\epsilon, a+\epsilon) \setminus \{a\}$

- <u>Open set</u>.

$G \subseteq \mathbb{R}$ is open if every $x \in G$ has a ngb, U, s.t $U \subseteq G$.

- <u>Interior points</u>

$x \in S$ if $\exists \epsilon > 0$ s.t

$(x-\epsilon, x+\epsilon) \subset S$

$\Rightarrow$ x is an int point of S.

<u>Set of int pts of 'S' $= S^{\circ}$</u>

$= \text{int}(S)$

Ex $(0,1)$

- Every point in open set is an int point.

- $\{x\} \rightarrow$ doesnot have int point.

- $[0,1]$, $(0,1)$

  $0, 1$ are not int points

Theo: Any arbitrary union of open sets is open.

Pf: $\{A_i : i \in I\}$: open

$$A = \bigcup_{i \in I} A_i$$

Show: A is open.

Let $x \in A = \bigcup A_i$

$\Rightarrow x \in A_i$ for some $i \in I$.

We can have an $\epsilon > 0$ s.t

$(x-\epsilon, x+\epsilon) \subset A_i$ (open)

$\Rightarrow \underline{(x-\epsilon, x+\epsilon)} \subset \bigcup_i \underline{A_i} = A$

$\Rightarrow$ A is open.

- Finite intersection of ✓ open sets is open.

- $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $\qquad \bigcap_{n \in \mathbb{N}} A_n = \underline{\{0\}}$

- $\underline{\mathbb{R} = ?}$

$r \in \mathbb{R}.$

$(r-\epsilon, r+\epsilon) \subset \underline{\mathbb{R}}$

$\mathbb{R} = \bigcup_{r \in \mathbb{R}} (-r, r)$

open set

- $\underline{\emptyset}$: $\underline{\quad}$ open set ; tautology

Closed set

$\mathbb{R}^c = \emptyset$.

$F \subseteq \mathbb{R}$ is closed if $F^c = \mathbb{R} \setminus F$ is open.

- $\mathbb{R}$, $\emptyset$. : $\to$ $\mathbb{R}$ & $\emptyset$ are both open & closed.
- $I = [a, b] \longrightarrow$ closed set

  $I^c = (-\infty, a) \cup (b, \infty)$

  $\hookrightarrow$ open.

- $(-\infty, b] \to$ Ex ?

  $[a, \infty) \to$ closed set $\Big\}$ Ex.

Et.
- Any arbitrary intersection of closed sets is closed
- Finite union of closed sets is closed.

$I_n = [\frac{1}{n}, 1 - \frac{1}{n}] \Rightarrow \bigcup I_n = (0, 1)$

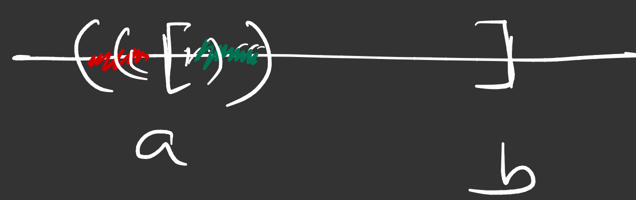
- (a, b) — open set

  $[a, b] \rightarrow$ closed set.

- $\text{int}(a, b) = (a, b)$

  $\text{int}([a, b]) = (a, b)$

'a', 'b' : Boundary pt.



$x \in S$ is a boundary point if for every $\epsilon > 0$

$N_\epsilon(x) = (x - \epsilon, x + \epsilon)$ contains points of S and also S^c.

Bd(S)

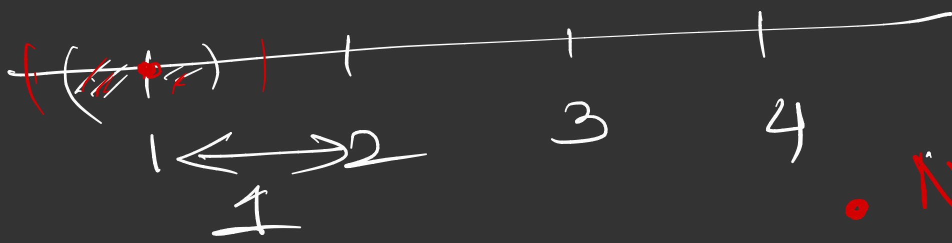
- <u>Isolated points</u>

  $x \in S$ if $\exists\, \epsilon > 0$

  $N_\epsilon(x) = (x - \epsilon, x + \epsilon) \cap S = \{x\}$

• $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{N}$.



• None are interior pts.

$\forall\, \epsilon < 1$

$$(1-\epsilon, 1+\epsilon) \cap \mathbb{N} = \{1\}$$

• Connection between isolated pts and bdd pts & int points.

$$\boxed{\mathbb{N}^\circ = \emptyset}$$

$\mathbb{N} \subseteq \mathbb{R}$.

$$\text{bdd}(\mathbb{N}) = \{1, 2, \ldots\}$$

Q: 1 Can an open set have an isolated pt?

No.

2) Can a closed set have one?

$\{x\} \rightarrow$ Yes.

3) Countable set with no isolated points.

$\mathbb{Q}$.

$q \in \mathbb{Q}$, $\delta > 0$

$(q-\delta, q+\delta)$

4) ~~A~~ Set whose all the

points are isolated

$= \mathbb{N}$.

5) $\mathbb{Q}$; $\text{int}(\mathbb{Q}) = ?$ Ex.

Prop. A set $F \subseteq \mathbb{R}$ is closed iff ($\Leftrightarrow$) the limit of every convergent seq in $\mathbb{F}$ belongs to F.

Proof: $\boxed{F \text{ is closed}}$ ($\mathbb{F}^c$ is open).

$\{x_n\} \to$ convergent seq.

$x_n \in \mathbb{F}$, $x_n \to x$.

$x \in F^c$. $-$? $\times$

Suppose $x \in F^c$.

$\exists\, N_\epsilon(x) \subseteq F^c$ (open)

for some $\epsilon > 0$.

$N_\epsilon(x) \cap F = \{\emptyset\}$

since $x_n \to x$

$x_n \in F$

$(x-\epsilon,\ x+\epsilon)$

$\forall\ \epsilon > 0$

$\exists\ n \in \mathbb{N}$ s.t

$x_n \in (x-\epsilon,\ x+\epsilon)$

def of seq convergence.

$\Rightarrow\ \underline{x \in F}$

$\Leftarrow$ <u>Converse</u>

<u>Limit of every convergent seq of points in F belongs to F.</u>

Show: F closed

$\Updownarrow$

F^c is open.

$\boxed{x \in F^c} \Rightarrow$ <u>open.</u>

$\exists\ N_\epsilon(x) \subset F^c$, for some $\epsilon > 0$ Show

$\left[\left(\exists \epsilon > 0 \text{ s.t } N_\epsilon(x) \subset F^c\right)\right.$

$= \forall \epsilon > 0 \text{ s.t } N_\epsilon(x) \not\subset F^c \left.\right]$

$$\epsilon = \frac{1}{n}$$

$$\left(x - \frac{1}{n}, \ x + \frac{1}{n}\right) \cap F \neq \emptyset$$

$$\underline{\underline{x_n \in F}}$$

$$\epsilon = 1$$

$$(x - 1, \ x + 1) \cap F \neq \emptyset$$

$$\Rightarrow \quad x_1 \in F$$

$$\epsilon = \frac{1}{2}, \qquad x_2 \in F$$

$$\vdots$$

$$\epsilon = \frac{1}{n}$$

$$\{x_n\} \quad \text{s.t}$$

$$\underline{x_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)}$$

$$\underline{\underline{x_n \in F}}$$

$$\Rightarrow \quad \underline{x_n \longrightarrow x} \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \boxed{x \in F} \; ?$$