Lecture 2

MTL-122 - Real and Complex Analysis

$$\bigcup_{\lambda \in \Omega} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Omega\}$$

$$\bigcap_{\lambda \in \Omega} A_\lambda = \{x : x \in A_\lambda \ \forall \lambda \in \Omega\}$$

<u>Ex</u>

$$A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$$

$$= \left\{x \in \mathbb{R} : \tfrac{1}{n} \leq x \leq 1 - \tfrac{1}{n}\right\}$$

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = (0, 1)$$

$$B_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

$$\bigcap_{n \in \mathbb{N}} B_n = \{0\}$$

• $\{B_\lambda : \lambda \in \Omega\}$

$$A \setminus \bigcup_{\lambda\in\Omega} B_\lambda = \bigcap_{\lambda\in\Omega} (A \setminus B_\lambda)$$

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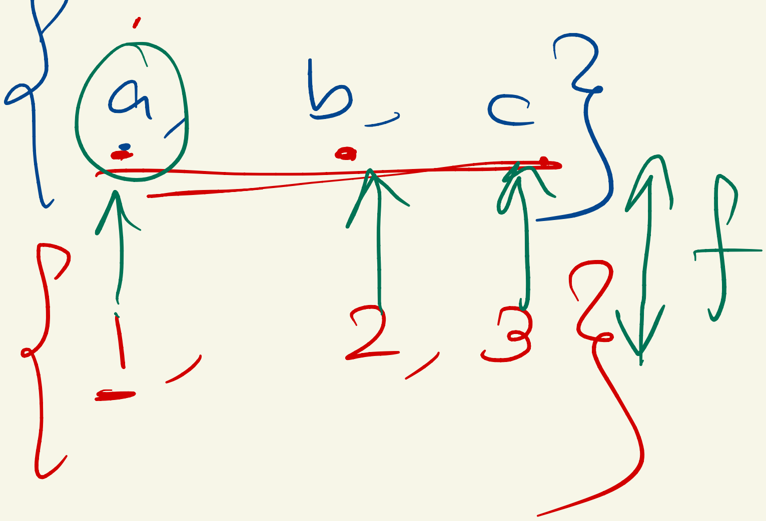


Defn $n \in \mathbb{N}$.

S cardinality n

$$f : S \longrightarrow \{1, 2, \dots, n\}$$

$\mathrm{card}(S) = n$

$$g : \underline{A} \xrightarrow{\text{bijection}} \underline{B}$$

$$\boxed{\operatorname{card}(\mathbb{N}) = \aleph_0}$$

Ex.

$$f : \underline{\mathbb{N}} \to \mathbb{Z}.$$

$$f(n) = \begin{cases} \frac{n+1}{2}, & n \text{ odd} \\ 1 - \frac{n}{2}, & n \text{ even}. \end{cases}$$

- f is a bijection.

$$\operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathbb{Z}) = \aleph_0.$$

Countable.

A set, S, is countable

S,

$f: \mathbb{N} \to S$ $\{f(1), f(2) \ldots f(n)\}$

$\text{card}(S) = \aleph_0$.

$\{S_1, S_2, S_3 \ldots\}$

- Infinite subset of a countable $\to$ countable.

- Superset of an uncountable set $\to$ uncountable

- Finite / countable union $\longrightarrow$ countable

We learn.

- Finite if empty / $A \approx \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

- Countably infinite if $A \approx \mathbb{N}$.

- Infinite if ~~its~~ not finite.

- Field.

$\mathbb{F} \; (\neq \emptyset)$ '$\underline{+}$', '$\times$'

- Associative.

$a, b, c \in \mathbb{F}$

$(a+b)+c = a+(b+c)$

$(a\times b)\times c = a\times(b\times c)$

- Commutative.

- Distributive.

- Existence of identities

$0, 1 \in \mathbb{F}$

$0 \neq 1.$

$a+0 = a,$ $a1 = a.$

- Additive inverse.

$$a \in \mathbb{F} \Rightarrow -a \in \mathbb{F}.$$

$$a + (-a) = 0$$

- Multiplicative inverse

$$a \in \mathbb{F} \setminus \{0\}$$

$$\Rightarrow a^{-1} \in \mathbb{F} \quad \text{s.t} \quad a \times a^{-1} = 1$$

Ordered Field.

'$<$' relation

a) $x, y \in \mathbb{F}$,

$$x < y, \quad x = y, \quad y < x$$

b) $x < y, \; y < z \Rightarrow x < z$.

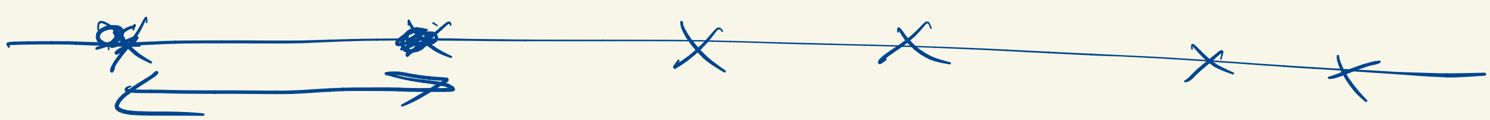
c) $y < z \Rightarrow x + y < x + z$

d) $x>0$ — $y>0 \Rightarrow xy>0$

$\mathbb{Q}$, $\underline{\mathbb{R}}$

$x \in \mathbb{R}$

$$\underline{|x|} = \begin{cases} x, & \underline{x \geq 0} \\ -x, & \underline{x < 0} \end{cases}$$



$$d(x,y) = |x-y|$$

- $d(x,y) = d(y,x)$
- $d(x,y) \geq 0$
- $d(x,y) = 0 \iff x = y$

- Triangle.

$$d(x,z) \leq d(x,y) + d(y,z)$$

- 'd' : distance fn

$\boxed{\text{metric}}$ of $\mathbb{R}$.

$(\mathbb{R}, d) \to$ metric space

$A \subseteq \mathbb{R}$.

$M \in \mathbb{R}$,
$x \leq M$, $x \in A$. $\Big\}$ bdd above

$m \in \mathbb{R}$, $x \geq m$, $\forall x \in A$ $\Big\}$ bdd below.

$A \subseteq \mathbb{R}$ bdd if.

$A \subseteq I = [m, M]$

$M \in \mathbb{R}$. s.t

$M \leq M'$ for every

bdd M' of A.

$M = \sup A$.

$m \in \mathbb{R}$.

$m \geq m'$

$m = \inf A$.

- sup / inf is unique.

(Exercise).

- $\sup A \in A \Rightarrow \max A$.
- $\inf A \in A \Rightarrow \min A$.

Example.

1) Both sup & inf of a finite set belongs to the set.

2) $A = (0, 1)$

$M \geq 1$ upper bd of A.

$\text{lub} A = \sup A = 1$

3) $\left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$

$\sup A = 1$

$\inf A = 0$

Dedekind completeness

(1872)

Axiom: Every nonempty set of real numbers that is bdd from above has a sup.

- If $x \in \mathbb{R}$, $\exists$ $n \in \mathbb{N}$, $x < n$.

- $S (\neq \emptyset) \subseteq \mathbb{R}$, $M \in \mathbb{R}$.

$M = \sup S$ iff

i) M is an ub of S.

ii) for any $\epsilon > 0$ $\exists$ $s \in S$ s.t $M - \epsilon < s$

Pf. $\epsilon > 0$.

$M - \epsilon \geq s \quad \forall s \in S$.

$\Rightarrow M - \epsilon$ is an upper bd of S. ↯

$\underline{A = \sup S}$.

$A \leq M$.

If $\underline{A < M}$, $\underline{\underline{\epsilon = M - A > 0}}$

$M - \underline{\underline{\epsilon}} < s, \; s \in S$.

$\Rightarrow M - (M - A) < s \leq \underline{\underline{A}}$.

$A < A$ ↯

$\Rightarrow \boxed{A = M}$

Check the condition for inf.

Seq.

$$f: \mathbb{N} \to \mathbb{R}.$$

$$x_n = f(n)$$

$$(x_n)_{n \geq 1}$$

- $x_n \to x \; (\in \mathbb{R})$ as $n \to \infty$.

$\epsilon > 0 \;\; \exists \; N \in \mathbb{N}$ s.t

$$|x_n - x| < \epsilon \quad \forall n \geq N$$

$$\Updownarrow$$

$$d(x_n, x) < \epsilon, \; \forall n > N$$

- limit of a seq is unique.
- $x_n \to \infty$.

$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$

$x_n > M \quad \forall n > N.$

- A convergent seq is bdd.
- Converse is not true.

$x_n = (-1)^{n+1}$

- Cauchy seq.

$|x_m - x_n| < \epsilon \quad \forall m, n > N$

- Cauchy seq $\Leftrightarrow$ convergent in $\mathbb{R}$. *

(x_n) $x_1, \quad x_2, \; \ldots \quad x_n, \ldots$

(x_{n_k})

$x_{n_1}, \quad x_{n_2}, \quad x_{n_3}, \ldots$

$\boxed{n_1 < n_2 < n_3 < \ldots}$

Ex $\left(\frac{1}{n}\right)$

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \frac{1}{5}, \ldots$

$\left(\frac{1}{k^2}\right)$

$1 \quad \frac{1}{4}, \quad \frac{1}{9}, \ldots$

$\boxed{n_k = k^2}$

$1, \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \ldots$