Lec 3 - MTL122 -

Real and Complex Analysis :

Nested Interval Theo.

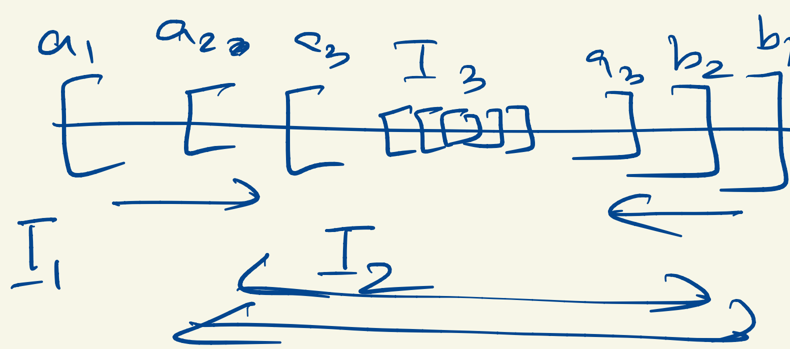
For $n \in \mathbb{N}$,

- $I_n = [a_n, b_n]$, ← closed intervals

  $-\infty < a_n < b_n < \infty$

- $I_{n+1} \subset I_n$, $\forall n \in \mathbb{N}$.

- $\lim\limits_{n \to \infty} (b_n - a_n) = 0$



Then $\bigcap\limits_{n=1}^{\infty} I_n$ consists of exactly one pt.

Ex $I_n = \left(0, \frac{1}{n+1}\right]$

$I_n = [n, \infty)$

$\bigcap I_n = \emptyset$

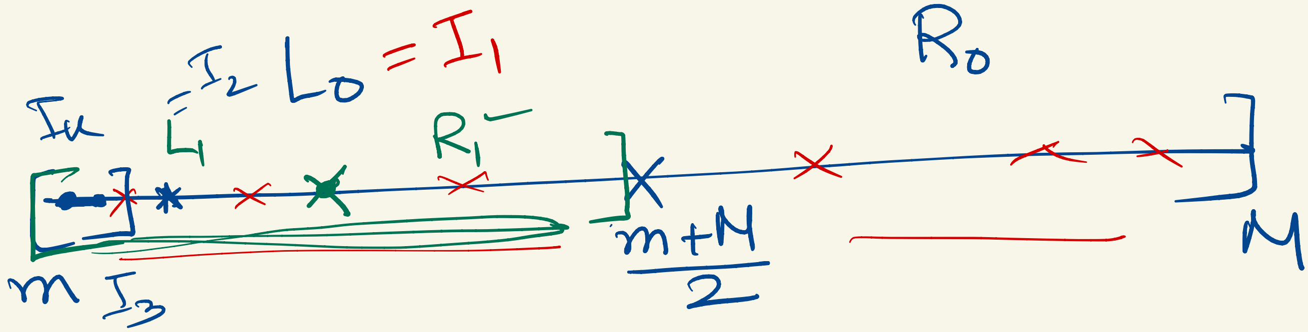
Theo: Bolzano Weierstrass theo.

Every bdd seq of real numbers has a convergent subseq.

Pf: (x_n) — bdd seq, $x_n \in \mathbb{R}$

By completeness axiom:

$$\left.\begin{aligned} M &= \sup_{n\in\mathbb{N}}\{x_n\} \\ m &= \inf_{n\in\mathbb{N}}\{x_n\} \end{aligned}\right\} \quad m \le x_n \le M \quad \forall n$$



$\boxed{I_0} = [m, M]$

$L_0 = [m, \frac{m+M}{2}]$

$R_0 = [\frac{m+M}{2}, M]$

$\boxed{I_1} \rightarrow$ int. that has $\rightarrow x_{n_1} \in I$

infinitely many terms

of seq. (x_n)

$$\boxed{I_1 \subset I_0}$$

$\boxed{I_2} \rightarrow$ sub int. that

infinitely many

term of the seq (x_n).

$n_2 > n_1$

s.t $x_{n_2} \in I_2$

$x_{n_3} \in I_3$, $n_3 > n_2$

$$I_1 \supset I_2 \supset I_3 \supset I_4 \cdots$$

$$\underbrace{|I_k|}_{\text{length}} = 2^{-k}(M-m) \longrightarrow (1)$$

$n_1 < n_2 < n_3 < n_4$

$(x_{n_k}) \rightarrow$ subseq of (x_n) $\longrightarrow (2)$

- $I_k \rightarrow$ closed.
- $I_1 \supset I_2 \supset I_3 \cdots$
- $|I_k| = 2^{-k}(M-m) \rightarrow 0$ as $k \rightarrow \infty$

Conclude. $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point.

$$l \in \bigcap_{n=1}^{\infty} I_n .$$

$$|x_{n_k} - l| < |I_k|$$

$$= 2^{-k}(M-m)$$

as $k \to \infty$

$$|x_{n_k} - l| \to 0$$

This implies

$$\lim_{k\to\infty} x_{n_k} = l .$$

Theo. A seq of real numbers converges iff it is a Cauchy seq. $\iff$

$\Rightarrow$

Pf: $(a_n)_{n\geq 1}$ is a convergent seq.

$a_n \to L$

$\epsilon > 0$ $\exists N \in \mathbb{N}$ st $j \geq N$

$$|a_j - L| \leq \epsilon/2$$

$j, k > N$.

$$|a_j - a_k| = |a_j - L + L - a_k|$$

$$< \epsilon$$

$\Rightarrow$ $(a_n) \to$ Cauchy.

$\Leftarrow$ $(a_n) \to$ Cauchy seq.

* Cauchy seq is bdd (Exercise).

By Bolzano Weierstress Theo

(a_{n_k}), $a_{n_k} \to l$ (say),

$\exists N_1 \quad r \geq N_1$

$|a_{n_r} - l| < \epsilon/2$

$\exists N_2$, $m, n \geq N_2$

$\Rightarrow |a_m - a_n| < \epsilon/2$

Choose $k > N_1$ s.t

$n_k > N_2$ $\quad \forall\, n > N_2$

$|a_n - l|$

$= |a_n - a_{n_k} + a_{n_k} - l|$

$< \epsilon$

$\Rightarrow \{a_n\} \to$ convergent.

In $\mathbb{R}$

* Cauchy $\Longleftrightarrow$ Convergent

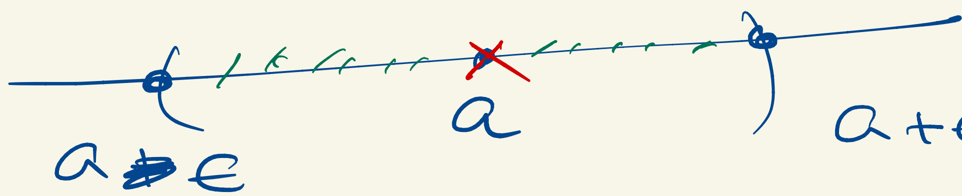
Topology on $\mathbb{R}$

Open sets

$$|x-a|<\epsilon \iff a-\epsilon<x<a+\epsilon$$

Neighbourhood

$\epsilon>0$



$$N(a,\epsilon)=\{x\in\mathbb{R} \mid \underbrace{d(x,a)<\epsilon}_{\Updownarrow \; |x-a|<\epsilon}\}$$

$\Downarrow$

neighbourhood of a

Deleted neighbourhood

$$N^*(a,\epsilon) = N(a,\epsilon)\setminus\{a\}$$

$$= \{x \in \mathbb{R} \mid 0 < |x-a| < \epsilon\}$$

$$\begin{cases} N(a,\epsilon) = (a-\epsilon,\ a+\epsilon) \\ N^*(a,\epsilon) = (a-\epsilon,\ a) \cup (a,\ a+\epsilon) \end{cases}$$

$U \subseteq \mathbb{R}$ — <u>open set</u>

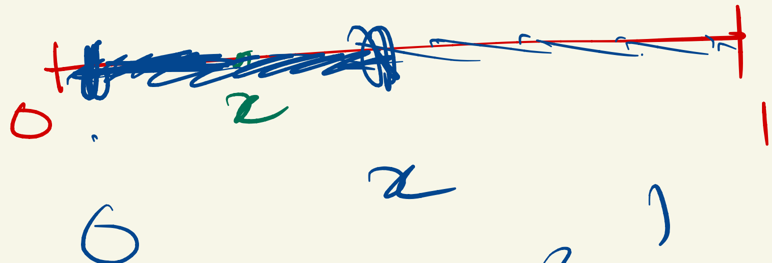
$\forall\ s \in U,\ \exists\ \epsilon > 0,$

$$(s-\epsilon,\ s+\epsilon) \subset U$$

<u>Ex.</u>

$I = (0,1) \rightarrow$ open set in $\mathbb{R}$.

$x \in I$

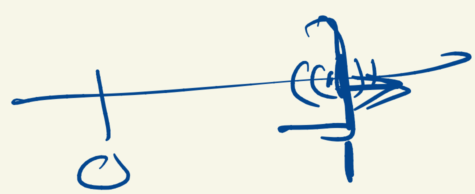


$$\delta < \min\left\{\frac{x}{2}, \frac{1-x}{2}\right\}$$

$$(x-\delta, x+\delta) \subset I$$

- $(a, b) \to$ open set
- $(-\infty, b) \to$ open set
- $(a, \infty) \to$ open.

Ex. $J = (0, 1]$



$1 \in J$

$$(1-\delta, 1+\delta) \not\subset J$$

$\Rightarrow$ J is not an open set.

$[0, 1] \to$ not open.

- Union

Intersection

Ex: $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \forall\, n \in \mathbb{N}$.

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

- Singleton sets in $\mathbb{R}$ is not open.