MTL103: Tutorial Sheet-I

1. If $S = \{(x, y) : -x + y \leq 4,\ 3x - 2y \leq 9,\ x,\ y \geq 0\}$, solve graphically the following problems:

   (a) max $x - 5y$ over S

   (b) max $6x + 4y + 18$ over S

2. Solve the following problems graphically:

   (a)
   $$\begin{aligned} \max\ & 2x_1 + x_2 \\ \text{subject to } 0 \leq\ & x_1 \leq 2, \\ & x_1 + x_2 \leq 3, \\ & x_1 + 2x_2 \leq 5, \\ & x_2 \geq 0 \end{aligned}$$

   (b)
   $$\begin{aligned} \min\ & 5x_1 + 2x_2 \\ \text{subject to } & x_1 + 4x_2 \geq 4 \\ & 5x_1 + 2x_2 \geq 10 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

3. Solve graphically:
   max $F(x, y) = x + 10y$ over x where y is a solution of
   min $f(x, y) = y$
   subject to$-25x + 20y \leq 30;\ x + 2y \leq 10;\ 2x - y \leq 15;\ 2x + 10y \geq 15;\ x,\ y \geq 0.$

4. Solve max $z = \min\ (3x - 10, -5x + 5)$ subject to $0 \leq x \leq 5$.

5. Suppose we want to show that all solutions of
   $x + y \leq 4;\ 2x - 3y \leq 6;\ x, y \geq 0$
   also satisfy $x + 2y \leq 8$. Formulate this problem as a LPP and verify your result graphically.

6. If $Ax = b,\ A:\ m \times n,\ \rho(A) = m,$ has a solution which involves precisely m non-zero variables and if this solution is unique, then prove that it must be a basic solution.

7. Find all basic solutions of

   (a)
   $$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 7 \\ 2x_1 + x_2 + x_3 + 2x_4 &= 3 \end{aligned}$$

   (b)
   $$\begin{aligned} 8x_1 + 6x_2 + 12x_3 + x_4 + x_5 &= 6 \\ 9x_1 + x_2 + 2x_3 + 6x_4 + 10x_5 &= 11 \end{aligned}$$

8. Find the number of degenerate and non-degenerate basic feasible solutions for the system graphically $2x + 3y \leq 21;\ 3x - y \leq 15;\ x + y \geq 5;\ y \leq 5;\ x,\ y \geq 0.$

9. Prove that the set of all convex combinations of finite number of L.I. vectors is a convex set.

10. Show that if LPP has more than one optimal solution then it has infinitely many solutions.

11. Examine convexity of the following sets:

    (a) $\{(x, y, z) : x^2 + y^2 \leq 4,\ 0 \leq z \leq 3\}$

    (b) $\{(x, y) : | x | \leq 1,\ | y | \leq 1\}$

    (c) $\{(x, y) : y \geq 3 - x^2\}$

    (d) $\{(x, y) : y \leq e^{-x}\}$

12. Examine convexity of the following functions:

    (a) e^{-x}

    (b) $| x |$

    (c) $f(x) = -{x_1}^2 + 2x_1x_2 - 4{x_2}^2 + 3x_1x_3 + 6x_2x_3 - 9{x_3}^2$

    (d) $f(x) = \begin{array}{l} x^2 \text{ if} - 1 \leq x < 1 \\ 2 \text{ if } x\ =\ 1 \end{array}$

    What if $f(x) = 1/2$ at $x = 1$?

13. If f is convex on R^n, show that $f(Ax+b)$ is also convex on R^n.

14. If $\{f_i : 1 \leq i \leq m\}$ is a family of convex functions on a convex set S, then show that $g(x) = \sup\{f_i(x) : 1 \leq i \leq m\}$ is a convex function on S.

15. Prove that a function f is convex if and only if its epigraph is a convex set. Use it to show that
$$f(x) = \max\{(-x/2),\ \max\{x,\ x^2\}\}$$
is a convex function on [-1,1].

16. If f and g are convex functions, show through an example that fg is not necessarily a convex function.

17. Let h be a non-decreasing convex function on R and f be a convex function on $T \subset R^n$. Then prove that the composite function hof is a convex function on T. Hence or otherwise show that $e^{(2x_1^2 + x_2^2)}$ is a convex function on R^2.

18. Let g be a positive concave function on T then show that $1/g$ is a convex function on T. What can we say about $1/g$, if g is positive convex function on T ?

19. Prove that the function f is a convex function on a set $T \subset R^n$ if and only if for every $x_1 \in T,\ x_2 \in T$, the one-dimensional function $g(y) = f(yx_1 + (1-y)x_2)$ is a convex function for all $0 \leq y \leq 1$.

20. Consider a LPP
$$\max\ z = c^t x$$
$$\text{subject to } Ax = b,\ x \geq 0.$$
Treating b as a parameter, $z^\star$ (optimal value of LPP) will be a function of b. What is the nature of function $z^\star(b)$?