Lecture 17

MTL 122 -

Real & Complex analysis.

$f: X \to \{0,1\}$ ← discrete metric space

↑ discrete topology

$\boxed{D(f) = X}$

$R(f) = \{0,1\}$

$\# R(f) = 2$

Theo. X is connected iff every 2-valued continuous function on X is constant.

Proof: Let X be connected.

$f: X \to \{0,1\}$ → continuous.

Let $A = f^{-1}\{0\}$.

So then A is open set in X.

$B = f^{-1}\{1\} \rightarrow$ open set

- $A \cap B = \phi$ (?) ✓

[if not, $x \in A \cap B$

$\Rightarrow x \in A$ & $x \in B$

$\Rightarrow f(0) = x$ & $f(1) = x$]

- $X = A \cup B$

Can this be possible?

No

As X is connected.

and this implies

$A = \phi$ or $B = \phi$.

$\Rightarrow f$ is const.

Conversely

Every two-valued fn is a constant fn on X.

Suppose $X = A \cup B$, A, B are open & $A \cap B = \phi$.

Define
$$f(x) = \begin{cases} 0, & x \in A \\ 1, & x \in B \end{cases}$$

$\Rightarrow$ f is continuous.

Say $f(x) = 0 \quad \forall x \in X$.

$\Rightarrow$ $A = X$ & $B = \phi$.

$\Rightarrow$ X is connected.

Theo. Let $f: X \to Y$ be a continuous fn. If X is connected, then the image $f(X)$ is connected.

Proof. $g: f(X) \longmapsto \{0,1\}$ continuous.

$\Rightarrow$ $g \circ f: X \to \{0,1\}$

We know that X is connected

$\Rightarrow$ $g \circ f$ is constant.

$\Rightarrow$ g is const on $f(X)$

$\Rightarrow$ $f(X)$ is connected.

$A \subset X$.

Theo:

A connected

$\Rightarrow$ $\overline{A}$ is connected

[$Y \subset X$

U open set of X

then

$U \cap Y$ is open set in Y

$\hookrightarrow$ Relative open set.]

Proof: Let

$f : \overline{A} \to \{0,1\}$ continuous.

$f|_A \to$ const (?)

(Because A is connected)

Say $\boxed{f|_A = 0}$

Let $x \in \overline{A}$ s.t

Now $f(x) = 1$

$\{1\} \to$ open in $\{0,1\}$

& f is continuous.

$\Rightarrow$ $f^{-1}(1)$ is an open subset of $\overline{A}$

$f^{-1}(1) = U \cap \overline{A}$ for some open set U in X.

Then $f(y) = 1$ since

$y \in \underline{U \cap A} \subset U \cap \overline{A}$

$\Rightarrow$ $f(y) = 1$, & $y \in A$ ↯

$\Rightarrow \quad \bar{A}$ is connected.

Ex.

$(-\infty, 0) \quad (0, \infty)$

$\mathbb{R}$ connected subsets

$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

not connected.

Theo: Let $\{A_i\}_{i \in I}$ is a family of connected subsets of X s.t $\bigcap_{i \in I} A_i \neq \emptyset$

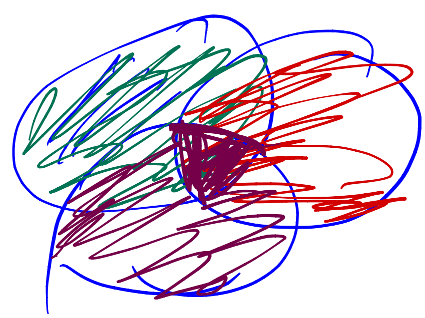
then $A = \bigcup_{i \in I} A_i$

Proof: $f : A \to \{0,1\}$.

$f|_{A_i} \to$ continuous. $\forall i$.

$\Rightarrow f|_{A_i}$ is a constant fn.

$\because \bigcap_{i \in I} A_i \neq \emptyset$

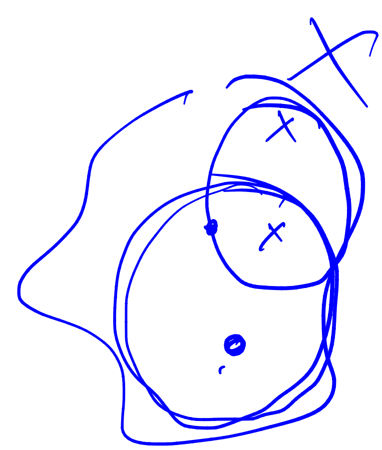


$\Rightarrow f$ is const.

$\Rightarrow A$ is connected

Theo.

Suppose for any two points in a space X there exists a connected subspace



containing these two points.
then X is connected.

<u>Proof</u>: Let $a \in X$ (fix).

For $b \in X$

$C(b) \Rightarrow$ connected set containing a & b.

Then $X = \bigcup_{b \in X} C(b)$

Now $a \in \bigcap_{b \in X} C(b)$

Previous theorem $\Rightarrow$ X connected.