Lecture 25

MTL- 122

Real and Complex Analysis.

$$\log z = \ln|z| + i\,\underbrace{(\theta + 2k\pi)}_{\arg z}, \quad k = 0, \pm 1, \pm 2, \ldots$$

'θ' is any value of the branch.

<u>Defn</u> Let $\tau \in \mathbb{R}$.

Then, $z \neq 0$, define the branch $\boxed{\log_\tau z}$ of $\log z$ corresponding to 'τ' by

$$\log_\tau z = \ln|z| + i\,\boxed{\arg_\tau z}.$$

For $\tau = -\pi$

$$\arg_{-\pi} z = \operatorname{Arg} z$$

$\mathrm{Log}\, z = \log_{-\pi} z$

$$\boxed{\mathrm{Log}\, z = \ln|z| + i\, \mathrm{Arg}\, z}$$

principal logarithm of z.

Example.

Compute $\log(1+i)$, $\mathrm{Log}(1+i)$ & $\log_{\pi}(1+i)$

Soln.

$$\log z = \ln|z| + i \arg z.$$

$\arg z = \theta + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

Here

$z = 1+i$

$|z| = \sqrt{2}$

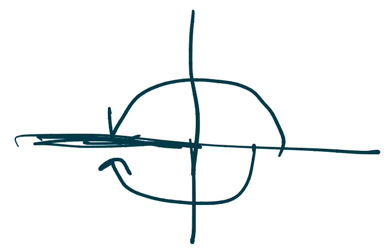
$$\boxed{\log z = \ln|z| + i\left(\frac{\pi}{4} + 2k\pi\right)}$$

$$\operatorname{Log}(1+i) = \log_{-\pi}(1+i)$$

$$= \ln\sqrt{2} + i\arg_{-\pi} z$$

$$= \ln\sqrt{2} + i\operatorname{Arg} z$$

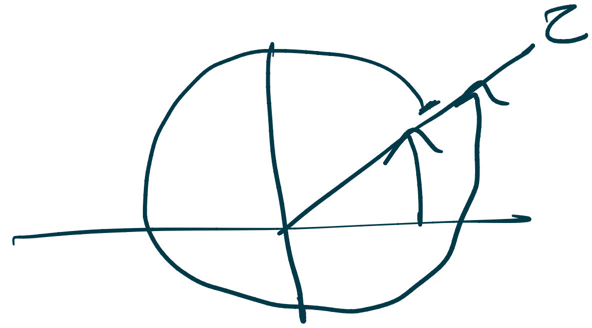
$$(-\pi < \underline{\arg z} \le \underline{\pi})$$



$$\operatorname{Log}(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4}\right)$$

'$\underline{\arg_\tau z}$' means =

Branch of $\arg z$ that lies in $\underline{(\tau, \tau + 2\pi)}$



$$\log_{\pi}(1+i) = \ln\sqrt{2} + i\,\underline{\underline{\arg_{\pi}}}(1+i)$$

$$\underline{\underline{\arg z}} = \pi/4 + 2k\pi,$$

$$k = 0, \pm 1, \pm 2, \ldots$$

$$\arg_{\pi} z \in \underline{(\pi, 3\pi]}.$$

$$\begin{array}{c} \parallel \\ \frac{9\pi}{4} \end{array}$$

$k=1$ $\quad \pi/4 + 2\pi$

$$\pi <= \underline{\underline{9\pi/4}} < \frac{12\pi}{4} \;\checkmark$$

$$= \underline{\underline{3\pi}}$$

$k=2$ $\quad \frac{17\pi}{4}$

$$\boxed{\log_{\pi}(1+i) = \ln\sqrt{2} + i\,9\pi/4.}$$

Theorem

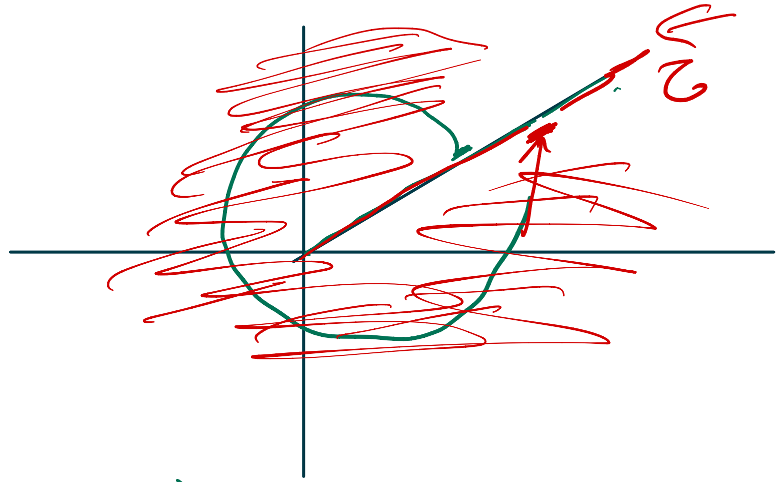
Let $\tau \in \mathbb{R}$. Then $\underline{w = f(z)}$

$= \underline{\log_\tau z}$

is analytic (holomorphic) on the domain

$\mathbb{C}_\tau^{\circ}$,

$$\mathbb{C}_\tau^{\circ} = \{ (r, \theta) : \underline{r > 0} \quad \underline{\tau < \theta < \tau + 2\pi} \}$$



&

$$\frac{d}{dz}\left(\log_\tau z\right) = \frac{1}{z},$$

$$z \in \mathbb{C}_\tau^{\circ}.$$

Proof: Let $z_0 \in C_z^0$.

- Need to prove that f is differentiable at z_0.

Let $w_0 = \log_z z_0 = f(z_0)$

We have to show,

$$\boxed{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{z_0}}$$

We know.

$$w = \log_z z \iff z = e^w$$

We know e^w is entire.

and $$e^{w_0} = \lim_{w \to w_0} \frac{e^w - e^{w_0}}{w - w_0}$$

$$= \lim_{w \to w_0} \frac{z - z_0}{w - w_0}.$$

Note that,

$$\log_{\tau}^{(w)} z = \ln|z| + i \arg_{\tau} z$$

$$\longrightarrow \ln|z_0| + i \arg_{\tau} z_0$$

$$= \log_{\tau} z_0 \quad (w_0)$$

as $z \to z_0$.

$w = \log_{\tau} z$

& $w_0 = \log_{\tau} z_0$

As $z \to z_0$ then

$w \to w_0$.

Moreover,

$z \neq z_0 \Rightarrow w \neq w_0$ $\left[w = w_0 \Rightarrow e^{w} = e^{w_0} \Rightarrow z = z_0 \right]$

$\therefore \lim_{z \to z_0} \frac{w - w_0}{z - z_0}$

$$= \lim_{w \to w_0} \frac{1}{\frac{z - z_0}{w - w_0}} = \frac{1}{e^{w_0}}$$

$$= \frac{1}{z_0} \quad \text{(as required)}.$$

<u>Example</u>

- Find the domain on which $w = f(z) = \text{Log}\,(3z - i)$ is analytic.

Comput $f'(z)$.

<u>Solution</u>:

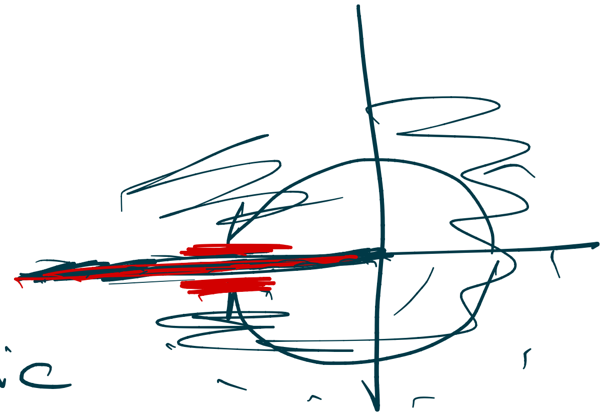
$g(z) = \text{Log}\, z$

$h(z) = 3z - i$

$$f = g \circ h.$$

$$\operatorname{Log} z = \log_{-\pi} z.$$

Here $\tau = -\pi$



$\log z$ is analytic in $\mathbb{C}^{\circ}_{-\pi} = \{(r,\theta),\ r>0,\ \theta \in (-\pi, \pi)\}$

$\operatorname{Log} z$ is analytic at points except $z \in (-\infty, 0]$

Now here

$f(z) = \operatorname{Log}(3z - i)$ is analytic at all points except

$3z - i \in (-\infty, 0]$.

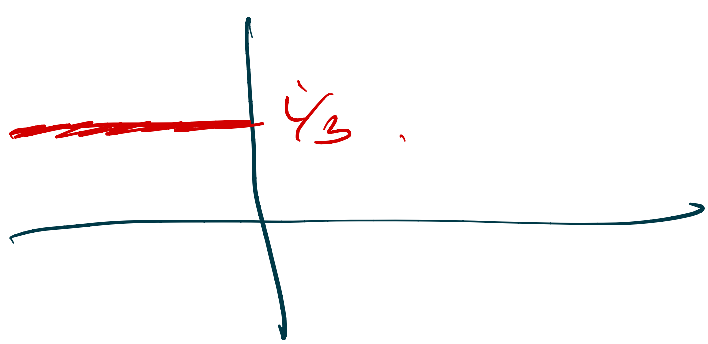
If $3z - i \in (-\infty, 0]$

$\Leftrightarrow 3z - i = x, \; x \le 0$

$\Leftrightarrow 3z = x + i, \; x \le 0$.

$\Leftrightarrow z = \frac{x}{3} + \frac{i}{3}, \; x \le 0$.

Then 'f' is analytic on the domain $\mathbb{C} \setminus \{x + i/3, x \le 0\}$



__Ex.__ Find a branch of $w = f(z) = \log(z^3 - 2)$

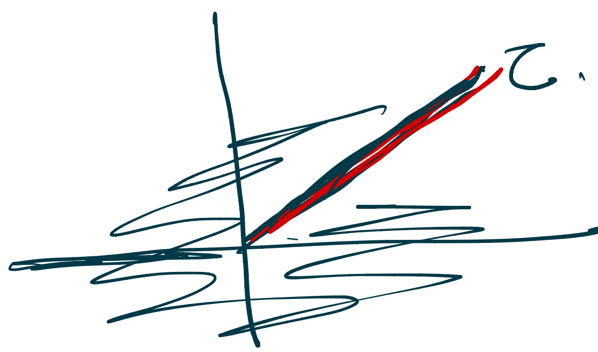
that is holomorphic at $z = 0$. Find $f(0)$ & $f'(0)$.

Soln. $z \in \mathbb{R}$,

$\log_z z$ of $\log z$

$$f(z) = \log_z (z^3 - 2)$$

$$= \log_z Z$$



$$Z = z^3 - 2$$

'We know f is holomorphic at $z = 0$'

When $z = 0 \Rightarrow Z = -2$.

f is holomorhic at $z=0$ only if $\log_z$ is holomorphic at -2

only if -2 does not belong to the branch cut $\{(r, z), \ r>0\}$

We know $\underline{\underline{-2}} \in (-\infty, 0)$

So the branch cut $((r, z); r>0)$ is not equal to $(-\infty, 0)$.

$f(0) = \log_{-z}(-2) = \ln|-2| + i\arg_z(-2)$

$f'(0) = \left.\frac{3z^2}{z^3-3}\right|_{z=0} = 0$.