Lecture - 13

MTL 122 - Real and Complex Analysis.

(X, d_X) (Y, d_Y)

Theo.

$f : X \to Y$ is continuous at 'a' $\Leftrightarrow$ it is seq. cont. at 'a'.

'$\Rightarrow$'

[$f : X \to Y$ seq cont. at 'a'

$\Rightarrow$ ('$x_n \to a$' $\Rightarrow$ $f(x_n) \to f(a)$)]
$n \to \infty$

Pf. '$\Rightarrow$'

f is continuous at 'a'.

$a \in X$.

$\epsilon > 0$ $\exists\, \delta(\epsilon, a) > 0$ s.t.

$d_Y(f(x), f(a)) < \epsilon$ for

$d_X(x, a) < \delta$. —— (1)

Let $(x_n)_{n \in} \in X$ s.t

$x_n \to a$ as $n \to \infty$.

Now for $\delta > 0$ $\exists N \in \mathbb{N}$

st

$$d(x_n, a) < \delta, \quad \forall n \geq N \qquad (2)$$

From (1) we get

$$d_Y\big(f(x_n), f(a)\big) < \epsilon$$

$$\forall n \geq N.$$

$$\Rightarrow \quad f(x_n) \to f(a) \quad \text{as } n \to \infty.$$

Converse

Suppose f is not continuous at 'a'.

$\big($ $\forall \epsilon$ $\exists \delta$ s.t $d(x, a) < \delta$ $\Rightarrow d(f(x), f(a)) < \epsilon$ $\big)$

$\big($ $\exists \epsilon$ $\forall \delta$ s.t $d(x, a) < \delta$ but $d(f(x), f(a)) \geq \epsilon$ $\big)$

$\exists\ \epsilon_0 > 0$ s.t for every $n \in \mathbb{N}$, $\exists\ x_n$.

$$\begin{cases} d(x_n, a) < 1/n \to 0 \\ \text{but } d(f(x_n), f(a)) > \epsilon_0. \end{cases}$$

Then $x_n \to a$ but $f(x_n) \not\to f(a)$.

$\Rightarrow f$ is not seq cont.

$\Rightarrow$ This gives

f is seq cont $\Rightarrow f$ is cont at 'a'.

$$0 \le d(x_n, a) < 1/n$$

$\downarrow$

$U \subseteq Y$, $\quad f : X \to Y$

$$f^{-1}(U) = \{ x : f(x) \in U \}$$

$\Rightarrow$ set of preimage.

Theo. $f : X \to Y$ is continuous iff for every open (closed) set $U \subset Y$, the set $f^{-1}(U)$ is open (closed) in X.

$$\left[f(B_\delta(a)) \subset B_\epsilon(f(a)) \right]$$

$\cancel{f}$

Pf: The inverse image under f of every open set is open. (Assume)

Show: f is continuous.

- $U \subset Y$ open $\Rightarrow$ $f^{-1}(U) \subseteq X$ open.

- Aim: $f(B_\delta(x_0)) \subseteq B_\epsilon(f(a))$

Let $y_0 \in Y$, $f(x_0) = y_0$

$\epsilon > 0$, $U = B_\epsilon(f(x_0))$

$\Rightarrow$ open.

$\Rightarrow$ $f^{-1}(B_\epsilon(f(x_0)))$ is open.

& $\underline{x_0} \in \underline{f^{-1}}(B_\epsilon(f(x_0)))$

$\Rightarrow \exists\, \delta > 0$ s.t

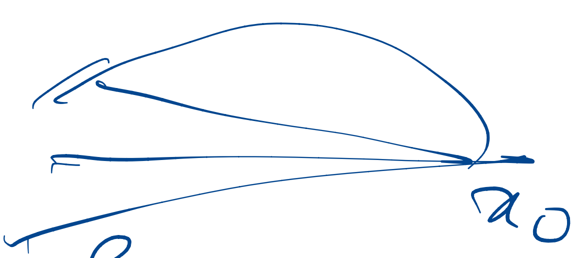
$B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$

$\Rightarrow f(B_\delta(x_0)) \subset B_\epsilon(f(x_0))$

$\Rightarrow$ f is continuous.

$$U = \bigcup_{x \in U} B_\epsilon(x)$$

'f^{-1}' is not a inverse.



$\underline{f : U \to f(U)}$

$f(U) \neq Y$

$f(U) = \delta$

Converse.

$f : X \to Y$ continuous.

$U \underset{\text{open}}{\subset} Y$.

$f^{-1}(U)$ is open.

$x_0 \in f^{-1}(U)$:

$\left[= \{ x \in X \mid f(x) \in U \} \right]$

$\Rightarrow \quad f(x_0) \in U$

$\Rightarrow \quad \exists \ \epsilon > 0$ s.t

$B_\epsilon(f(x_0)) \subset U$

$\exists \ \delta > 0$ s.t

$$f\left(B_\delta(x_0)\right) \subset B_\epsilon\left(f(x_0)\right)$$

$$\Rightarrow \quad B_\delta(x_0) \subseteq f^{-1}\left(B_\epsilon(f(x_0))\right)$$

$$\subseteq f^{-1}(U)$$

$\Rightarrow$ x_0 is an int. pt.

$\Rightarrow$ $f^{-1}(U)$ is open.

Ex. $X, \; Y, \; f: X \to Y$

(X, d_{dis})

Claim: f is continous irr of the metric Y.

$(\mathbb{N}, d) \rightarrow$ — — -
is always continuous.

Ex. cont.

Claim Open set criteria $\Rightarrow$.

$\Leftrightarrow$ Open ball criteria.

($\Leftarrow$?)

(Hint: Every open is arbitrary union of open balls).

Thm. Composition of continuous fns are conti

Pf. $\underbrace{f : X \to Y}_{\text{cont}}$, $\underbrace{g : Y \to Z}_{\text{cont}}$.

- $g \circ f : X \to Z$.

$\Rightarrow$ $\underset{\text{open}}{W \subset Z}$

Show, $(g \circ f)^{-1}(W)$ is open in X.

- $V = g^{-1}(W)$ open in Y.
- $U = f^{-1}(V)$ open in X.

$$U = f^{-1}(g^{-1}(W))$$

& also, $g \circ f(U) = W$

(set inclusion)

$\Rightarrow$ $g \circ f$ is continuous.

Uniform Continuous.

$f: X \to Y$ is uniformly continuous.

$\epsilon > 0 \quad \exists\, \delta(\epsilon)$ s.t

$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$

$\forall\, x, y \in X$.

- $f(x) = \frac{1}{x}$, $(0,1)$ not u.c.

Suppose we take $\delta > 0$

$\forall\, x, y \in (0,1)$

$|x - y| < \delta \quad \left|\frac{1}{x} - \frac{1}{y}\right| \leq 1$.

$x \in (0,1)$, $\quad x < \delta$, $\quad y = \frac{x}{2}$

$|x - y| < \delta \Rightarrow \frac{x}{2} < \frac{\delta}{2} < \delta$.

but $\left|\frac{1}{x}-\frac{1}{y}\right| = \left|\frac{1}{x}\right| > 1$.

$\Rightarrow$ $f(x) = \frac{1}{x}$ is not U.C.

— x —

(X, d_X) (Y, d_Y) $z \in X$.

$f : X \to Y$

1) $\forall \epsilon > 0$ $\exists \delta > 0$ s.t $\forall x \in X$.

$d_X(x, z) < \delta$,

$d_Y(f(x), f(z)) < \epsilon$.

2) $\forall \epsilon > 0$ $\exists \delta > 0$ s.t

$$f(B_\delta(z)) \subseteq B_\epsilon(f(z))$$

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3) Each open ball of Y that contains $f(z)$ includes the image under f of open ball of X that contain z.

4) For each open subset V of Y with $f(z) \in V$ $\exists$ an open set $U \subset X$ with $z \in U$ s.t $f(U) \subseteq V$.

5) For every seq (x_n) ~~is~~ of X that converges to z the seq $f(x_n)$ of Y converges ~~to~~ in Y to $f(z)$.