Project-MATH 652
Coupled Spring Equations

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Spring 2019

Abstract

Coupled spring equations, which are ordinary differential equations(ODEs), for modeling the motion of two springs with weights attached and fixed, in series, to a vertical wall and moving on a horizontal ground are described. Assuming Hooke's Law and no external forces, the motion of each weight is described as a linear model. Under the influence of external damping and driving forces, a nonlinear model for the motion of each weight is described. Both linear and nonlinear models are solved numerically using a 2-stage explicit Runge-Kutta method, and implicit midpoint rule. The convergence, exactness, and linear stability with an investigation of the limitations of the step size are analyzed.

1 Introduction

The study of periodic motion has been an interesting field of dynamics, and furthermore, oscillatory motion such as the simple harmonic motion of a pendulum, mass attached to a spring illustrates useful and elementary characteristics of periodic motion. In general, coupled springs with attached masses produce complex periodic motions that can be used to illustrate atomic vibrations and complex dynamical systems in engineering and physics. In this study, I investigate numerical solutions to the coupled spring equations, which can be reduced to a system of 4^{th} order decoupled ODEs[2], of two springs with weights attached, fixed in series from a vertical wall and moving on a horizontal ground and simulate the motion. Moreover, I analyze the convergence, exactness, and linear stability with an investigation of the limitations of the step size.

2 Formulation of the problem

This problem consists of two springs and two masses. One spring, having spring constant k_1 and natural length L_1, is attached to the wall and a mass m_1 is attached to the other end.To mass m_1, a second spring natural length L_2 and spring constant k_2 is attached. To the other end of the second spring, a mass m_2 is attached and the entire system is shown in figure 1.

Allowing the system to be at rest and equilibrium at the beginning, we give each mass a displacement with an initial velocity and we measure the displacement of the center of each mass from equilibrium, as a function of time(t) and denote these measurements by $x_1(t)$ and $x_2(t)$ respectively.

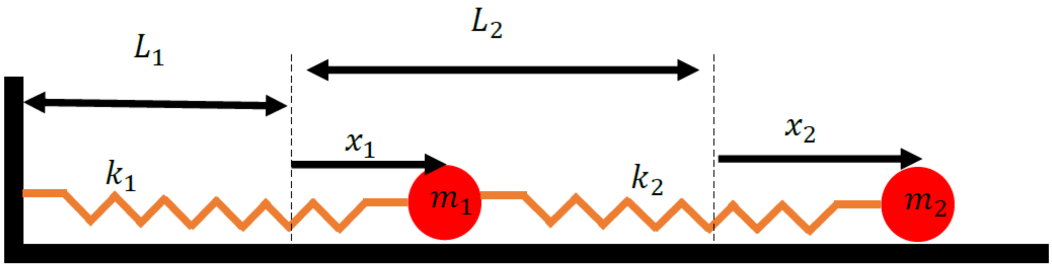

Figure 1: Two series springs with attached weights

2.1 Linear case

Under the assumption of small oscillations and Hooke's Law, the restoring forces which act on the mass m_1 are of the form $-k_1x_1$ and $-k_2(x_1 - x_2)$, and force act on mass m_2 is of the form $-k_2(x_2 - x_1)$ where x_1 and $x_2 - x_1$ are the elongations(or compressions) of the two springs.

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) \end{aligned} \tag{1}$$

Equation (1) can be written as a linear model

$$y' = Ay \tag{2}$$

where $y = \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$, $y' = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & 0 & 0 \end{pmatrix}$

2.2 Nonlinear case

If we assume that the restoring forces are nonlinear, which they most certainly are for larger vibrations, we can modify the model accordingly. Instead assuming the restoring force of the form $-kx$ by Hooke's Law, assume the force of the form $-kx + \mu x^3$. More generally, there is no disadvantage of adding the damping forces due to friction, air resistance, and driving forces, which do not change the non-linearity to the system. Then the model becomes:

$$\begin{aligned} m_1\ddot{x}_1 &= -\alpha_1\dot{x}_1 + \beta_1x_1^3 - k_1x_1 + \beta_2(x_1 - x_2)^3 - k_2(x_1 - x_2) - \mu_1m_1g + F_1\cos\omega_1 t \\ m_2\ddot{x}_2 &= -\alpha_2\dot{x}_2 + \beta_2(x_1 - x_2)^3 - k_2(x_2 - x_1) - \mu_2m_2g + F_2\cos\omega_2 t \end{aligned} \tag{3}$$

3 Methods of solving

In this study, we use two methods, namely, explicit 2-stage Runge-Kutta method and Implicit Midpoint Rule to solve each model(ODE). For the exact solutions, for the purpose of comparison and

calculating errors, we use ODE45 as it is precise.
Let's consider the initial value problem (IVP)

$$\begin{cases} y'(t) &= f(t, y(t)) \\ y(t_0) &= y_0; \quad \forall t \in [t_0, T] \end{cases}$$

3.1 2-Stage Explicit Runge-Kutta method

In general, a 2−stage explicit Runge−Kutta method is given by

$$\begin{aligned} \xi_1 &= y_n \\ \xi_2 &= y_n + h a_{21} f(t_n, \xi_1) \\ y_{n+1} &= y_n + h b_1 f(t_n, \xi_1) + h b_2 f(t_n + h c_2, \xi_2) \end{aligned}$$

This method can be represented by a tableau as follows.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$$

where b_1 and b_2 are RK weights, $c_1 (= 0)$ and c_2 are RK nodes, and ξ_1 and ξ_2 are RK stages. Furthermore, this method is of order 2 if

$$\begin{aligned} b_1 + b_2 &= 1 \\ b_2 c_2 &= \frac{1}{2} \\ c_2 &= a_{21} \end{aligned}$$

We choose values for RK method as follows

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

Thus the formula for the 2-stage RK method,

$$\begin{aligned} \xi_1 &= y_n \\ \xi_2 &= y_n + \frac{2}{3} h f(t_n, \xi_1) \\ y_{n+1} &= y_n + \frac{1}{4} h f(t_n, \xi_1) + \frac{3}{4} h f(t_n + \frac{2}{3} h, \xi_2) \end{aligned}$$

This method is not A-stable as any explicit RK method is not A-stable. The stability domain, by considering the ODE $y(t) = \lambda y(t), \lambda \in \mathbb{C}, Re(\lambda) < 0$, is shown in the following figure2.

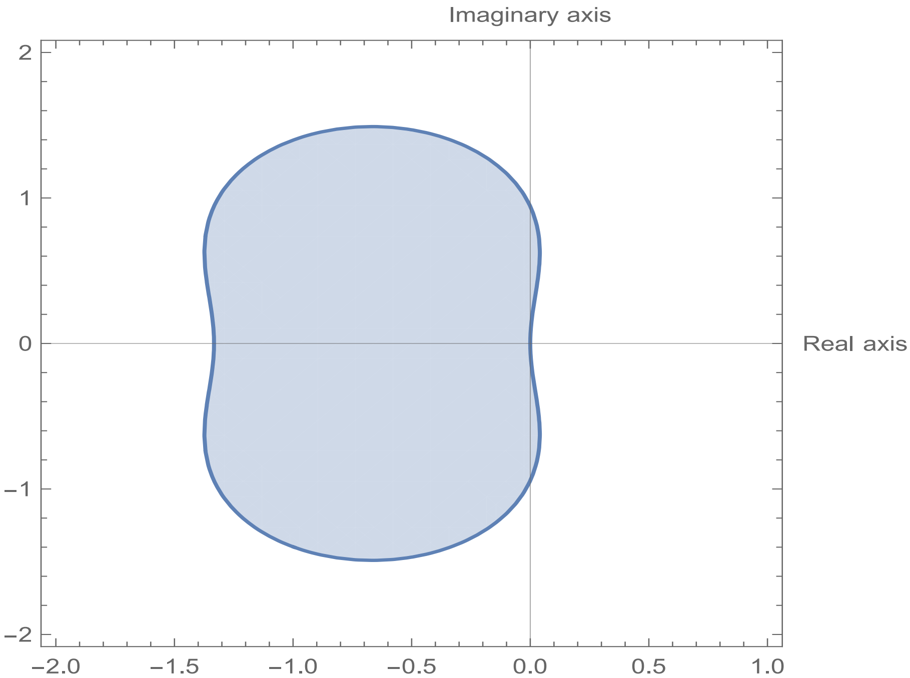

Figure 2: stability domain for 2-stage RK method

3.2 Implicit Mid-point Rule(IMR)

For a given initial value problem(IVP),

$$\begin{cases} y'(t) & = f(t, y(t)) \\ y(t_0) & = y_0; \quad \forall t \in [t_0, T] \end{cases}$$

the Implicit Mid-point Rule is given by

$$y_{n+1} = y_n + hf\left(\frac{t_n + t_{n+1}}{2}, \frac{y_n + y_{n+1}}{2}\right)$$

Implicit Mid−point rule is of order 2 and, by considering the ODE $y(t) = \lambda y(t), \lambda \in \mathbb{C}, Re(\lambda) < 0$, the stability domain D_{IMR} for the IMR is,

$$D_{IMR} = \{z : Re(z) < 0, z = \lambda h\}$$

Thus D_{IMR} contains $\mathbb{C}^-$and hence the Implicit Mid−point Rule is A-stable

4 Numerical experiments and results

In this section, we examine the behaviour of the the errors and order of convergence of the numerical methods that are used in this study. Moreover, the stability of the linear case is discussed and one unstable case is illustrated.

4.1 Linear case

Figure 3-6 show the graphical representation of the solutions for $x_1(t), x_2(t), \dot{x}_1(t)$ and $\dot{x}_2(t)$ of the linear case using 2-stage Runge-Kutta method. Figure 7-10 show the same results as mention above using Implicit Mid-point Rule. For all the solutions, $m_1 = m_2 = 1, k_1 = 6, k_2 = 4$ and $y_0 = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.5 \\ 0.4 \end{pmatrix}$ are used in equation (1).

4.1.1 Graphical results for explicit method

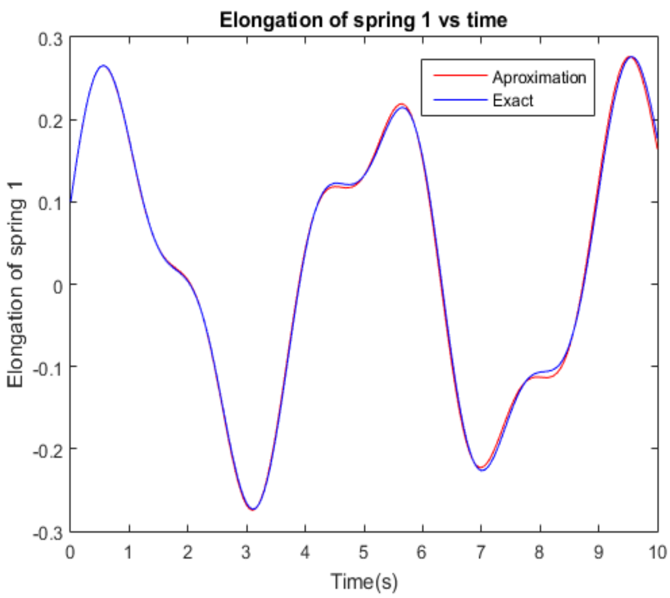

Figure 3: The graph of elongation $(x_1(t))$ vs time$(t)$for step size $(h) = 0.05$

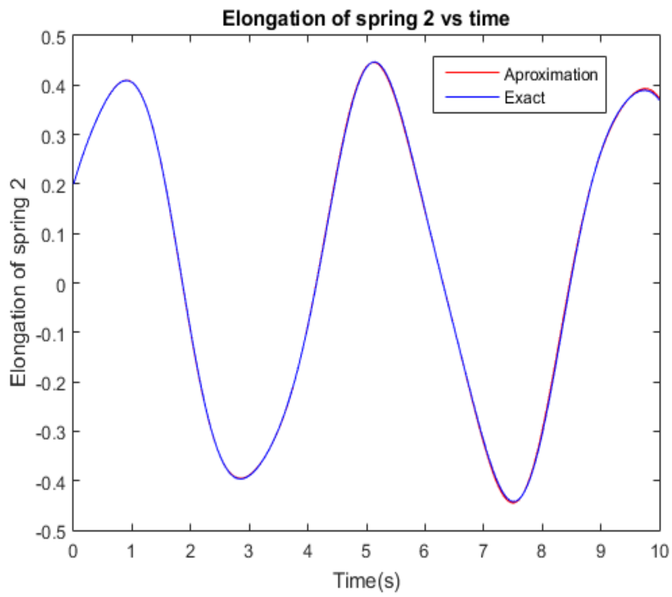

Figure 4: The graph of elongation ($x_2(t)$) vs time(t)for step size (h) = 0.05

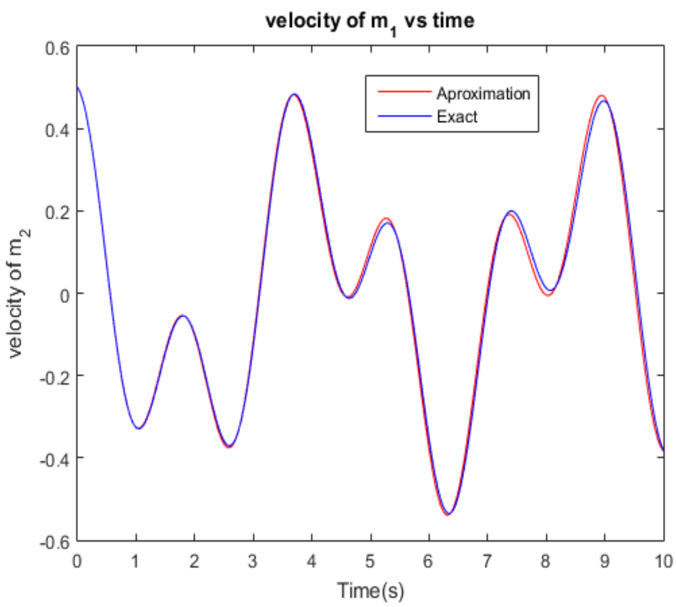

Figure 5: The graph of velocity ($\dot{x}_1(t)$) of m_1 vs time(t)for step size (h) = 0.05

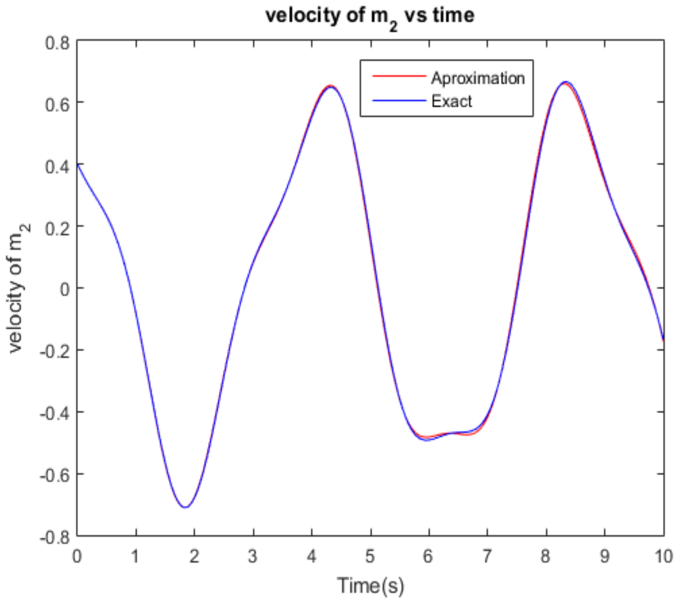

Figure 6: The graph of velocity $(\dot{x}_2(t))$ of m_2 vs time(t) for step size $(h) = 0.05$

4.1.2 Graphical results for implicit method

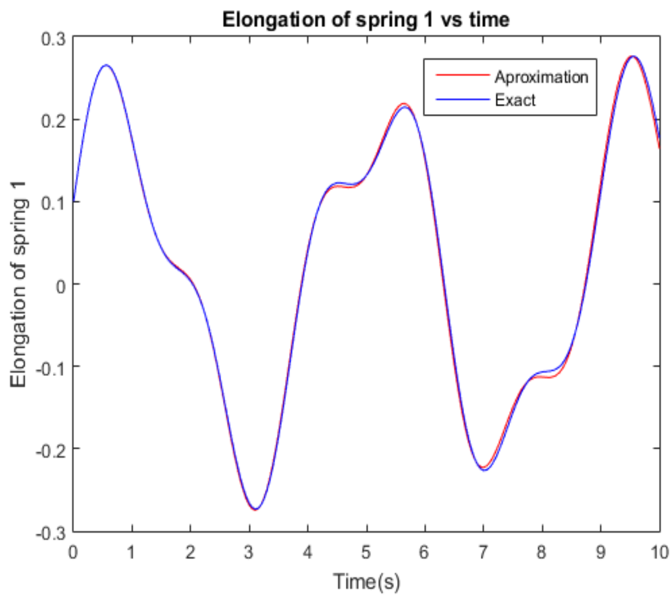

Figure 7: The graph of elongation $(x_1(t))$ vs time$(t)$for step size $(h) = 0.05$

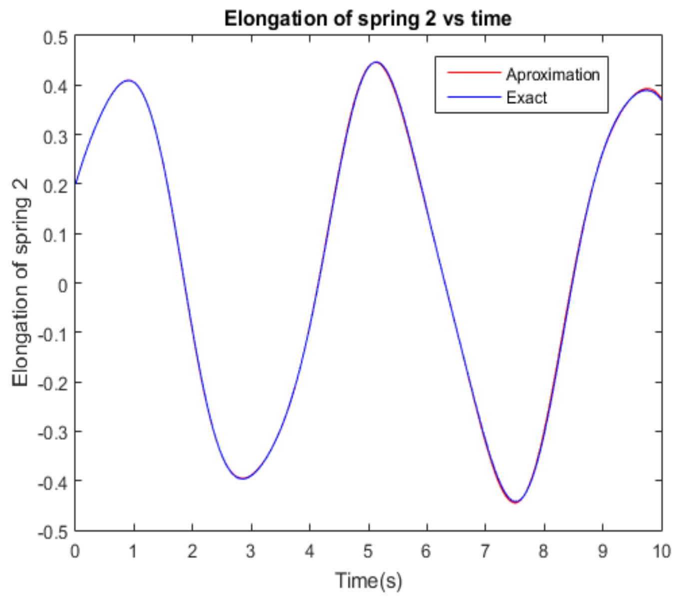

Figure 8: The graph of elongation $(x_2(t))$ vs time$(t)$for step size $(h) = 0.05$

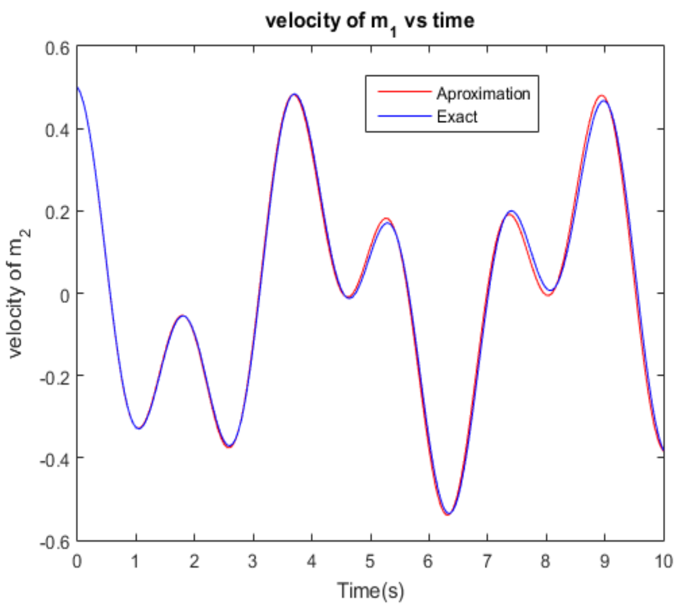

Figure 9: The graph of velocity $(\dot{x}_1(t))$ of m_1 vs time$(t)$for step size $(h) = 0.05$

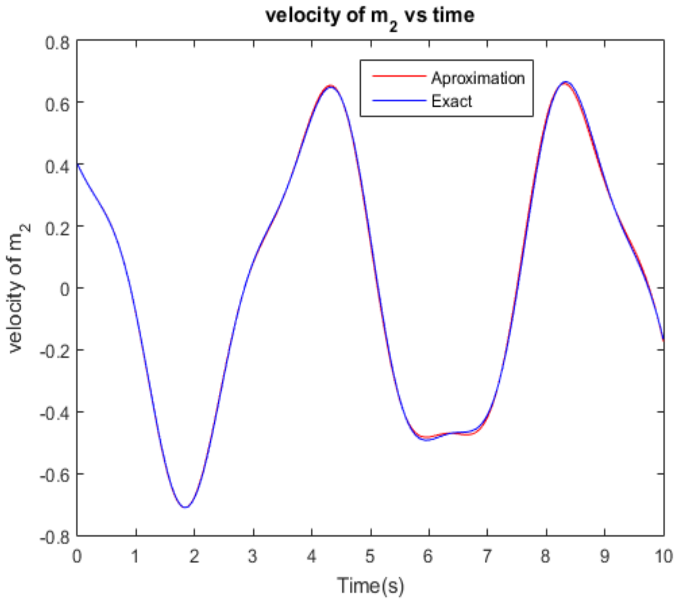

Figure 10: The graph of velocity ($\dot{x}_2(t)$) of m_2 vs time(t) for step size (h) = 0.05

4.2 Nonlinear case

Figure 11-14 shows the graphical representation of the solutions for $x_1(t), x_2(t), \dot{x}_1(t)$ and $\dot{x}_2(t)$ of the nonlinear case using 2-stage Runge-Kutta method. Figure 15-18 shows the same results as mentioned above using Implicit Mid-point Rule. For all the solutions, $m_1 = m_2 = 1, k_1 = 6, k_2 = 4, g = 9.81, \mu_1 = 0.2, \mu_2 = 0.2, \alpha_1 = 1, \alpha_2 = 1, \beta_1 = 0.5, \beta_2 = 0.5, F_1 = 1, F_2 = 1, \omega_1 = 2\pi, \omega_2 = 2\pi$ and $y_0 = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.5 \\ 0.4 \end{pmatrix}$ are used in equation (3).

4.2.1 Graphical results for explicit method

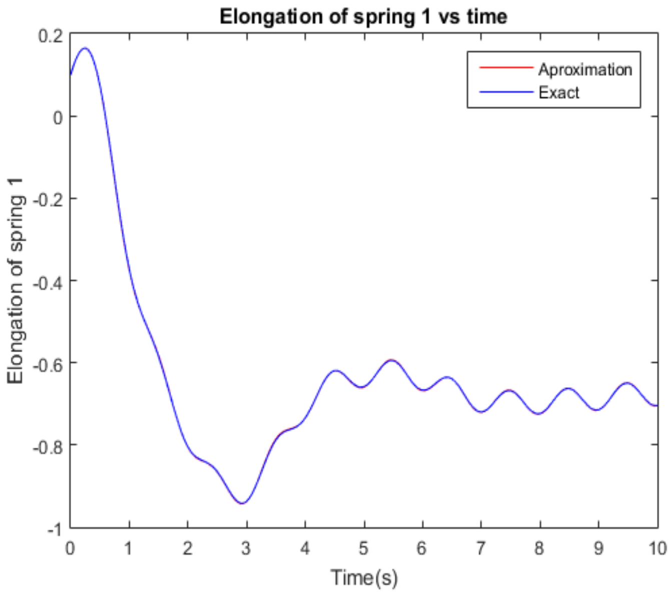

Figure 11: The graph of elongation ($x_1(t)$) vs time(t)for step size (h) = 0.05

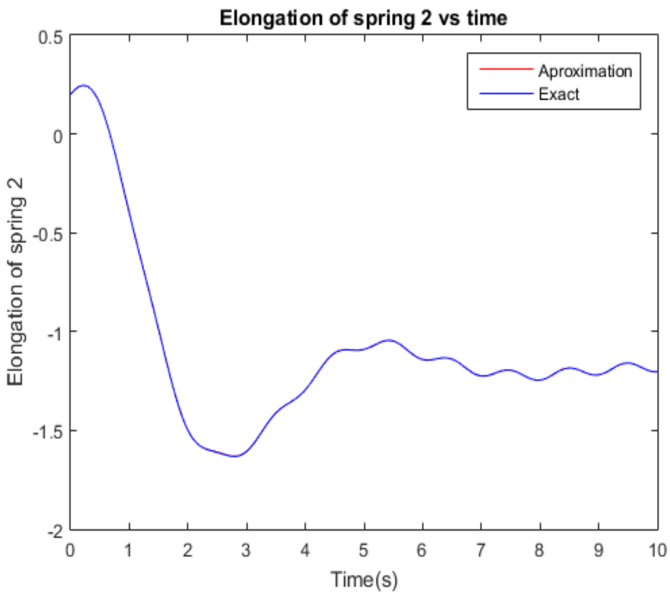

Figure 12: The graph of elongation ($x_2(t)$) vs time(t)for step size (h) = 0.05

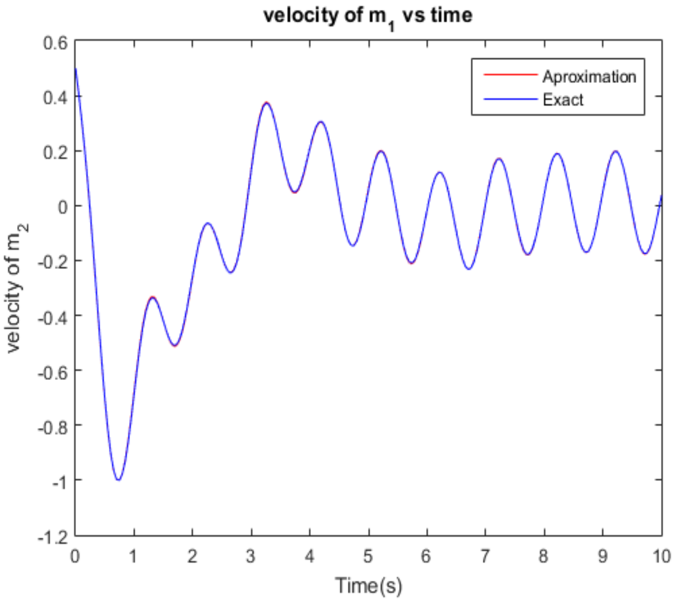

Figure 13: The graph of velocity $(\dot{x}_1(t))$ of m_1 vs time$(t)$for step size $(h) = 0.05$

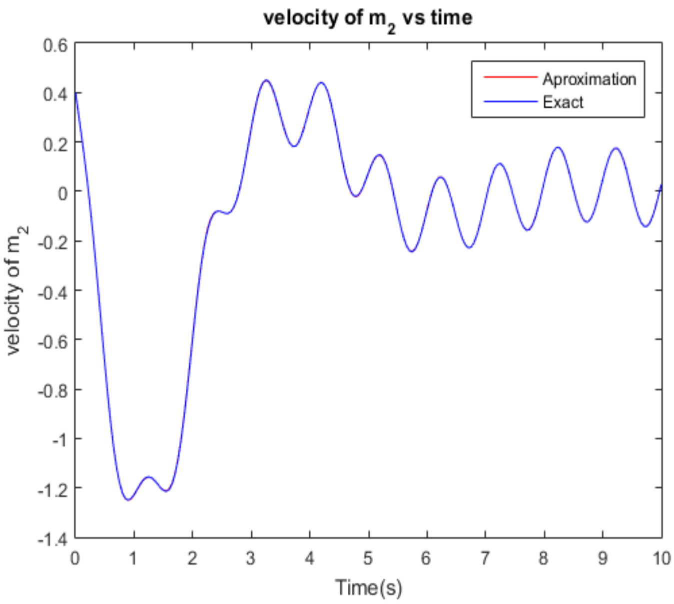

Figure 14: The graph of velocity $(\dot{x}_2(t))$ of m_2 vs time(t) for step size $(h) = 0.05$

4.2.2 Graphical results for implicit method

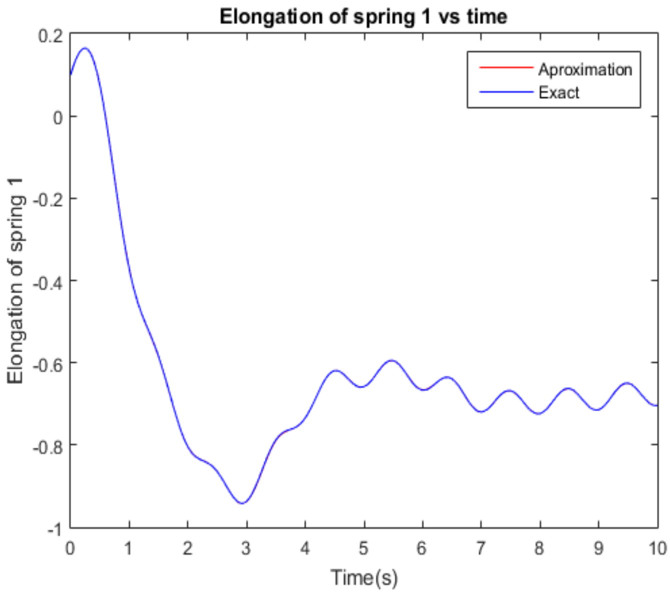

Figure 15: The graph of elongation ($x_1(t)$) vs time(t)for step size (h) = 0.05

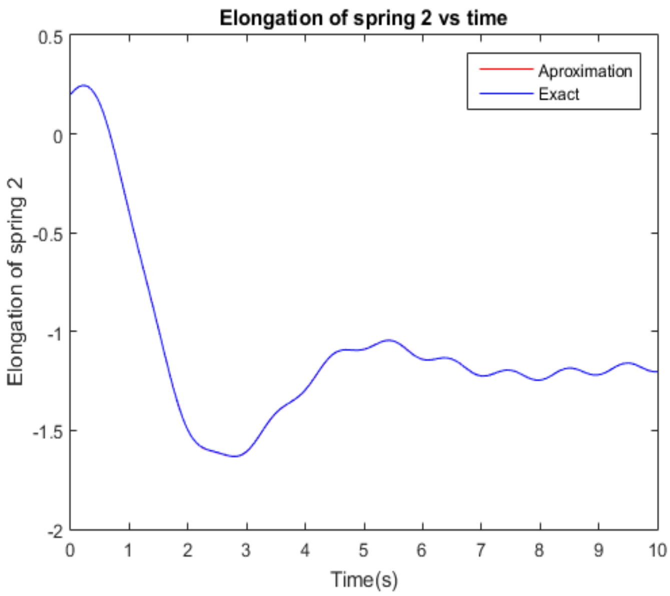

Figure 16: The graph of elongation ($x_2(t)$) vs time(t)for step size (h) = 0.05

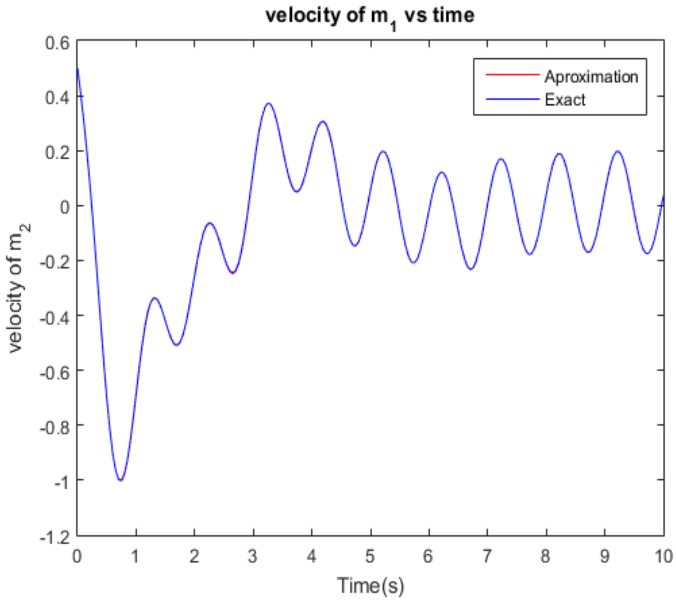

Figure 17: The graph of velocity $(\dot{x}_1(t))$ of m_1 vs time$(t)$for step size $(h) = 0.05$

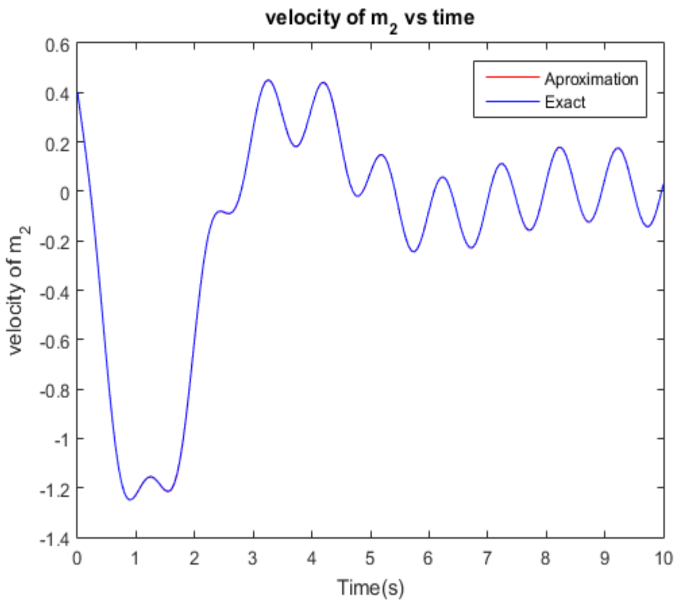

Figure 18: The graph of velocity $(\dot{x}_2(t))$ of m_2 vs time(t) for step size $(h) = 0.05$

4.3 Error and order of convergence

Errors for linear case using 2-stage explicit RK method				
Step size(h)	error of $x_1(t)$	error of $x_2(t)$	error of $\dot{x}_1(t)$	error of $\dot{x}_2(t)$
$h_1 = 0.2$	0.0205	0.0089	0.0446	0.0142
$h_2 = 0.1$	0.0050	0.0019	0.0105	0.0035
$h_3 = 0.05$	0.0012	0.0005	0.0025	0.0011
$h_4 = 0.025$	0.0003	0.0001	0.0006	0.0003
Errors for linear case using implicit mid-point rule				
Step size(h)	error of $x_1(t)$	error of $x_2(t)$	error of $\dot{x}_1(t)$	error of $\dot{x}_2(t)$
$h_1 = 0.2$	0.0093	0.0031	0.0182	0.0092
$h_2 = 0.1$	0.0024	0.0009	0.0047	0.0025
$h_3 = 0.05$	0.0006	0.0002	0.0012	0.0007
$h_4 = 0.025$	0.0002	0.0001	0.0003	0.0002

Table 1: Errors for linear case

Ratio of errors for linear case using 2-stage explicit RK method				
e_{i+1}/e_i	For $x_1(t)$	For $x_2(t)$	For $\dot{x}_1(t)$	For $\dot{x}_2(t)$
e_2/e_1	4.0745	4.6005	4.2434	4.0848
e_3/e_2	4.0415	4.1341	4.1716	3.2192
e_4/e_3	4.0350	4.0714	4.1031	3.6250
Ratio of errors for linear case using implicit mid-point rule				
e_{i+1}/e_i	For $x_1(t)$	For $x_2(t)$	For $\dot{x}_1(t)$	For $\dot{x}_2(t)$
e_2/e_1	3.8568	3.4997	3.8668	3.6068
e_3/e_2	3.9640	3.9378	3.9644	3.8918
e_4/e_3	3.9894	3.9796	3.9889	3.9634

Table 2: Ratio of errors for linear case

Errors for nonlinear case using 2-stage explicit RK method				
Step size(h)	error of $x_1(t)$	error of $x_2(t)$	error of $\dot{x}_1(t)$	error of $\dot{x}_2(t)$
$h_1 = 0.2$	0.0365	0.0209	0.0807	0.0236
$h_2 = 0.1$	0.0087	0.0055	0.0223	0.0061
$h_3 = 0.05$	0.0021	0.0014	0.0058	0.0016
$h_4 = 0.025$	0.0005	0.0004	0.0015	0.0004
Errors for nonlinear case using implicit mid-point rule				
Step size(h)	error of $x_1(t)$	error of $x_2(t)$	error of $\dot{x}_1(t)$	error of $\dot{x}_2(t)$
$h_1 = 0.2$	0.0139	0.0099	0.0425	0.0106
$h_2 = 0.1$	0.0035	0.0025	0.0111	0.0028
$h_3 = 0.05$	0.0009	0.0006	0.0028	0.0007
$h_4 = 0.025$	0.0002	0.0002	0.0007	0.0002

Table 3: Errors for nonlinear case

Ratio of errors for nonlinear case using 2-stage explicit RK method				
e_{i+1}/e_i	For $x_1(t)$	For $x_2(t)$	For $\dot{x}_1(t)$	For $\dot{x}_2(t)$
e_2/e_1	4.1759	3.8170	3.6230	3.8605
e_3/e_2	4.1975	3.9018	3.8329	3.9415
e_4/e_3	4.0889	3.9445	3.9320	3.9912

Ratio of errors for nonlinear case using implicit mid-point rule				
e_{i+1}/e_i	For $x_1(t)$	For $x_2(t)$	For $\dot{x}_1(t)$	For $\dot{x}_2(t)$
e_2/e_1	3.9725	3.9707	3.8335	3.7219
e_3/e_2	3.9970	3.9915	3.9604	3.9350
e_4/e_3	4.0016	3.9952	3.9958	3.9778

Table 4: Ratio of errors for nonlinear case

4.4 Illustration of linear instability

In order to illustrate an unstable solution for linear case using 2-stage explicit RK method, let $m_1 = m_2 = 1$ in equation (2), then $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2) & k_2 & 0 & 0 \\ k_2 & -k_2 & 0 & 0 \end{pmatrix}$ and eigenvalues of the matrix A are $\{\lambda_1, \lambda_1, \lambda_2, \lambda_2\}$, where

$$\lambda_1 = \frac{\sqrt{-k_1 - 2k_2 - \sqrt{k_1^2 + 4k_2^2}}}{\sqrt{2}}$$

$$\lambda_2 = \frac{\sqrt{-k_1 - 2k_2 + \sqrt{k_1^2 + 4k_2^2}}}{\sqrt{2}}$$

If all eigenvalues are not in the stability domain shown in figure 2 then 2-stage explicit RK method is unstable. Let's consider λ_1 for instance and $\lambda_1 = 2\sqrt{2}i$. This implies $64 - 8(k_1 + 2k_2) + k_1 k_2 = 0$ and by letting $k_1 = 1$, $k_2 = 56/15$. Now choosing a suitable value of step size(h), we can illustrate the linear instability of 2-stage explicit RK method. Let $h = 0.3$ and $y_0 = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.5 \\ 0.4 \end{pmatrix}$

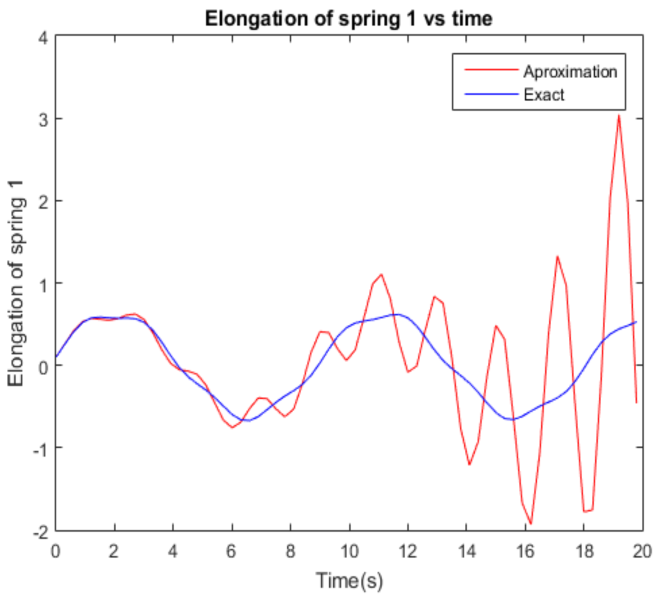

Figure 19: The graph of elongation $(x_1(t))$ vs time(t) for step size $(h) = 0.3$

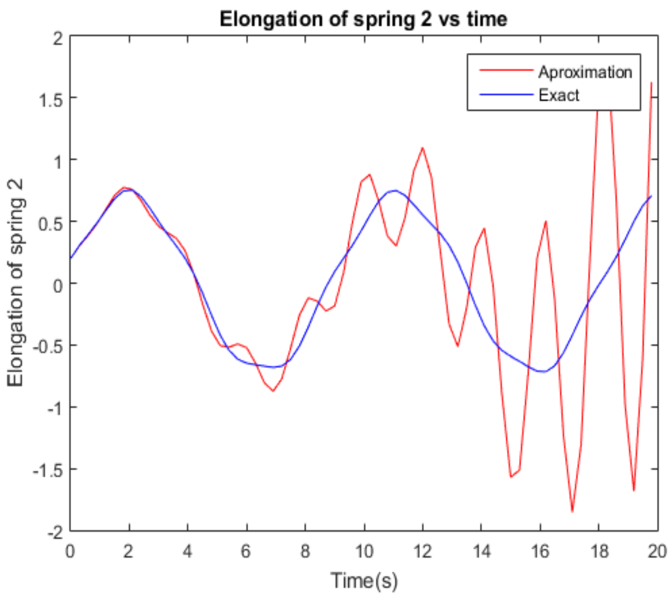

Figure 20: The graph of elongation $(x_2(t))$ vs time(t) for step size $(h) = 0.3$

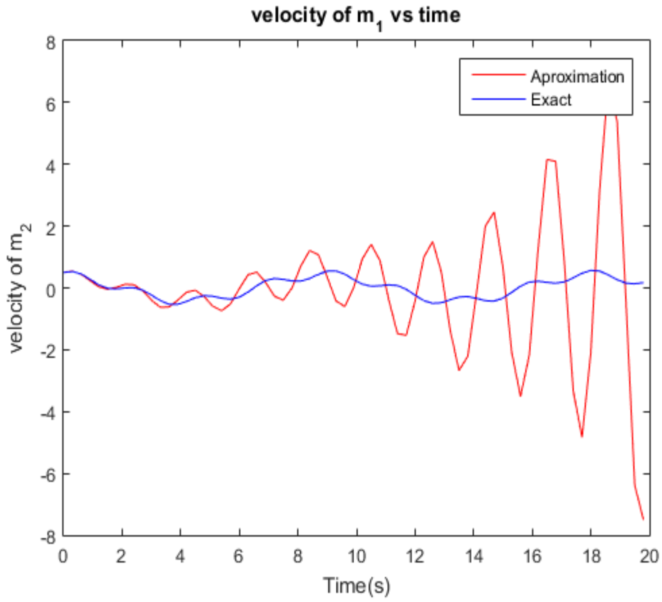

Figure 21: The graph of velocity ($\dot{x}_1(t)$) vs time(t) for step size (h) = 0.3

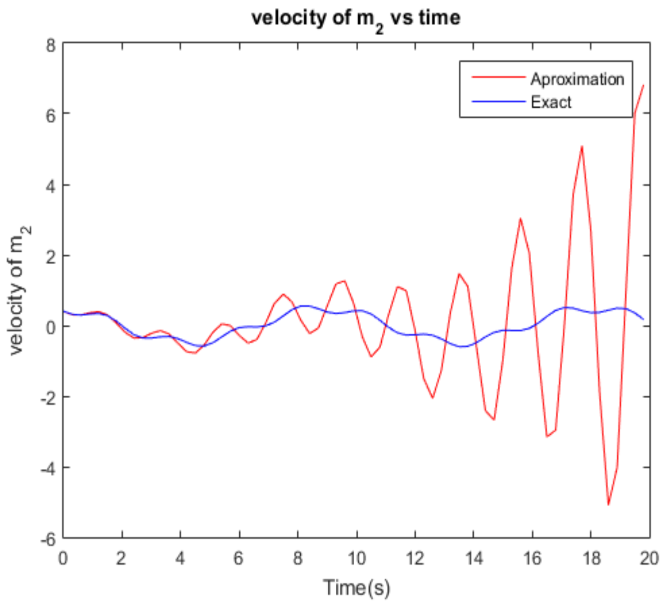

Figure 22: The graph of velocity ($\dot{x}_2(t)$) vs time(t) for step size (h) = 0.3

5 Discussion and Conclusion

Table 1 and 3 show that the errors decrease when the step size decreases and hence the more accuracy with smaller step sizes for both 2-stage explicit RK method and implicit mid-point rule. Table 2 and 4 above illustrates the fact that the ratios of errors, for both 2-stage explicit RK method and implicit mid-point rule, approach 2^2 when $h \longrightarrow 0$. Hence the verification of the order of convergence is 2 for both methods. Furthermore, section 4.4 figure 19-22 illustrates the fact that 2-stage explicit RK method is not linear stable. But in general, as long as stability criteria is satisfied for 2-stage explicit RK method, both methods are accurate enough according to the all figures from figure 1 to figure 18, and have a significant efficient. Moreover, the explicit method is more efficient than the implicit method.

References

[1] Arieh Iserles. A First Course in the Numerical Analysis of Differential Equations.

[2] TEMPLE H. FAY and SARAH DUNCAN GRAHAM. Coupled spring equations. *int. j. math. educ. sci. technol., 2003,vol. 34, no. 1, 65–79*