Homework #3

MEC 529: Introduction to Robotics

Spring 2023

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Assigned Date Friday, Feb. 24, 2023
Due Date Sunday, Mar. 5, 2023 (extended)

1. (a) (1/10) For any unit axis $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}} = 1\|$) and scalar $\theta \in \mathbb{R}$, we can always find a rotation matrix $\boldsymbol{R} \in SO(3)$ such that $\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}]\theta}$ where $[\hat{\boldsymbol{\omega}}] \in so(3)$ (recall that $\boldsymbol{R}$ is equivalent to the rotation of a frame from $\boldsymbol{I}_3 = \text{diag}\{1\} \in \mathbb{R}^{3\times 3}$ about $\hat{\boldsymbol{\omega}}$ by an angle θ). Write a MATLAB function that take $\hat{\boldsymbol{\omega}}$ and θ, and returns the corresponding rotation matrix $\boldsymbol{R}$. I recommend that you first write a function that returns a 3×3 skew-symmetric matrix corresponding to a vector (i.e., $\boldsymbol{x} \in \mathbb{R}^3 \to [\boldsymbol{x}] \in so(3)$).

   (b) (1/10) For three arbitrary pairs $(\hat{\boldsymbol{\omega}}, \theta)$, compute the rotation matrix $\boldsymbol{R}$ using your function in (a). For each pair, visualize the rotated frame along with the fixed frame $\{s\}$ and also the unit axis $\hat{\boldsymbol{\omega}}$. For visualization of the frames, you can use the attached function `triad`, and for visualization of a line corresponding the unit axis you can use the line formula as $\boldsymbol{l} = (x, y, z) = \boldsymbol{q} + t\hat{\boldsymbol{\omega}}$ where $t \in \mathbb{R}$ and $\boldsymbol{q} = \boldsymbol{0} \in \mathbb{R}^3$ (since it passes through the origin).

2. (a) (1/10) For any rotation matrix $\boldsymbol{R} \in SO(3)$, we can always find a unit axis $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}} = 1\|$) and scalar $\theta \in \mathbb{R}$ such that $\log \boldsymbol{R} = [\hat{\boldsymbol{\omega}}]\theta$ where $[\hat{\boldsymbol{\omega}}] \in so(3)$. Write a MATLAB function that take a rotation matrix $\boldsymbol{R}$ and returns the corresponding $\hat{\boldsymbol{\omega}}$ and $\theta \in [0, \pi]$ (note that your function should also support the rotation matrices corresponding $\theta = 0$ and $\theta = \pi$). I recommend that you first write a function that returns a vector corresponding to a 3×3 skew-symmetric matrix (i.e., $[\boldsymbol{x}] \in so(3) \to \boldsymbol{x} \in \mathbb{R}^3$).

   (b) (1/10) Validate your function using the three rotation matrices $\boldsymbol{R}$ computed in 1.(b).

3. (1/10) The two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ are related by

$$\boldsymbol{v}_2 = \boldsymbol{R}\boldsymbol{v}_1 = e^{[\hat{\boldsymbol{\omega}}]\theta}\boldsymbol{v}_1$$

   where $\hat{\boldsymbol{\omega}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{\omega}} = 1\|$) and $\theta \in [-\pi, \pi]$. Given $\hat{\boldsymbol{\omega}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \boldsymbol{v}_1 = (1, 0, 1), \boldsymbol{v}_2 = (0, 1, 1)$, find angle θ that satisfy the above equation.

4. (1/10) Consider a wrist mechanism with two revolute joints θ_1 and θ_2, in which the end-effector frame orientation $\boldsymbol{R} \in SO(3)$ is given by

$$\boldsymbol{R} = e^{[\hat{\boldsymbol{\omega}}_1]\theta_1} e^{[\hat{\boldsymbol{\omega}}_2]\theta_2},$$

   with $\hat{\boldsymbol{\omega}}_1 = (0, 0, 1)$ and $\hat{\boldsymbol{\omega}}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Determine whether the following orientation $\boldsymbol{R}$ is reachable.

$$\boldsymbol{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

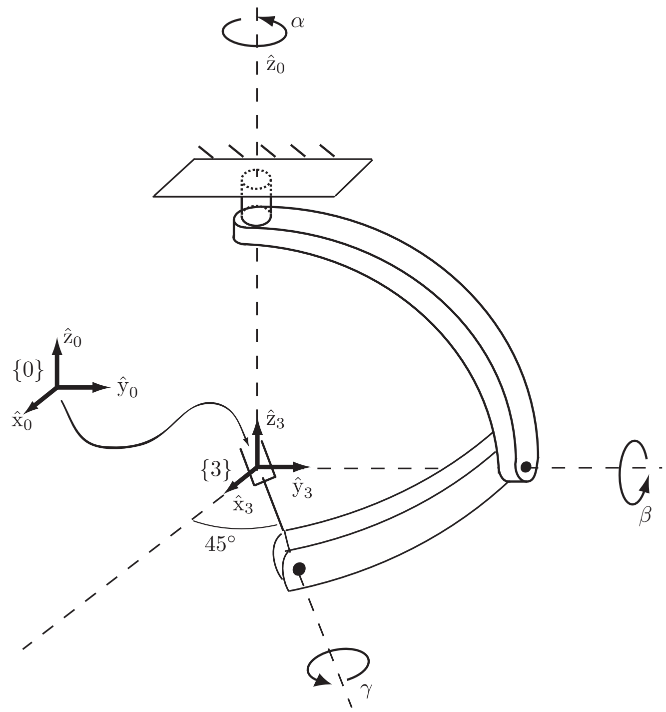

Figure 1: A 3-DOF wrist mechanism.

5. (1/10) Figure 1 shows a 3-DOF wrist mechanism in its zero position, where all joint angles α, β, γ are set to zero. Express the tool-frame orientation $\boldsymbol{R}_{03}$ at an arbitrary configuration of the wrist as a product of three rotation matrices in terms of α, β, γ, i.e., $\boldsymbol{R}_{03} = \boldsymbol{R}(\alpha, \beta, \gamma)$.

6. (a) (0.5/10) Derive a rotation matrix corresponding to the ZXZ Euler angles, with rotations about the body/current frame $\{b\}$ (i.e., rotation by α about the body z_b-axis, then by β about the body x_b-axis, and finally, by γ about the body z_b-axis).

   (b) (0.5/10) Find (α, β, γ) for any given rotation matrix $\boldsymbol{R} \in SO(3)$, when $0 \leq \beta \leq \pi$. At which configuration(s), gimbal lock occurs? Why?

   (c) (0.5/10) Find the ZXZ Euler angles (α, β, γ) for the following rotation matrix.

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

7. (a) (0.5/10) Prove that the rotation matrix $\boldsymbol{R} \in SO(3)$ corresponding to a given unit quaternion $\boldsymbol{q} = (q_0, q_1, q_2, q_3) \in S^3$ takes on the form

$$\boldsymbol{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\left(q_1 q_2 - q_0 q_3\right) & 2\left(q_0 q_2 + q_1 q_3\right) \\ 2\left(q_0 q_3 + q_1 q_2\right) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\left(q_2 q_3 - q_0 q_1\right) \\ 2\left(q_1 q_3 - q_0 q_2\right) & 2\left(q_0 q_1 + q_2 q_3\right) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

   (b) (0.5/10) Prove that the unit quaternion $\boldsymbol{q} = (q_0, q_1, q_2, q_3) \in S^3$ corresponding to a given rotation matrix $\boldsymbol{R} \in SO(3)$ takes on the form

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}\ , \quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix}, \text{ where } \boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

(c) (0.5/10) Let $\boldsymbol{p} = (p_0, p_1, p_2, p_3) \in S^3$ and $\boldsymbol{q} = (q_0, q_1, q_2, q_3) \in S^3$ denote the unit quaternions corresponding to the rotation matrices $\boldsymbol{R}_p$ and $\boldsymbol{R}_q$, respectively. Prove that the unit quaternion corresponding to the product $\boldsymbol{R}_p\boldsymbol{R}_q$ takes on the form

$$\boldsymbol{pq} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 - q_2p_3 + q_3p_2 \\ q_0p_2 + p_0q_2 + q_1p_3 - q_3p_1 \\ q_0p_3 + p_0q_3 - q_1p_2 + q_2p_1 \end{bmatrix} = \left(p_0q_0 - \boldsymbol{\epsilon}_p^T\boldsymbol{\epsilon}_q, p_0\boldsymbol{\epsilon}_q + q_0\boldsymbol{\epsilon}_p + \boldsymbol{\epsilon}_p \times \boldsymbol{\epsilon}_q\right)$$

where $\boldsymbol{\epsilon}_p = (p_1, p_2, p_3)$ and $\boldsymbol{\epsilon}_q = (q_1, q_2, q_3)$.
[*Hint*: A quaternion can be also represented in the form $\boldsymbol{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \mathbf{ij} = \mathbf{k}, \mathbf{ji} = -\mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{kj} = -\mathbf{i}, \mathbf{ki} = \mathbf{j}, \mathbf{ik} = -\mathbf{j}$.]

Notes for Questions 1, 2:

- Your report should include a brief description of your results with supporting figures which are usually output of your code. You do not have to include a screenshot of your code in the report.
- Add proper comments to your code, which detail what each part of the code is doing.
- Submit your report and code files in a single Zip file on Brightspace. Name the Zip file as `HW#N_FullName`, where `N` is the homework number and `FullName` is your full name. A report without its supporting code files and code files without a supporting report are NOT acceptable.
- Make sure to submit all the files/functions required to properly execute your code.