Module II : Digital Communication : EC 601

Orthogonal Representation of Signal :

It is a very useful technique to represent any arbitrary signal in terms of orthogonal basis function. Let us consider a set of functions $g_1(t), g_2(t), \ldots, g_n(t)$ defined over the interval $t_1 \leq t \leq t_2$ and which are related to one another in the very special way that satisfy the condition ;

$$\int_{t_1}^{t_2} g_i(t)\, g_j(t)\, dt = 0 \quad \text{---} (i)$$

That is, when we multiply two different functions and then integrate over the interval from t_1 to t_2 the result is zero. A set of functions which has this property is described as being Orthogonal over the interval t_1 to t_2.

Now, consider that we have some arbitrary function $f(t)$ and we are interested in only the interval t_1 to t_2, in the interval over which the set of functions $g(t)$ are Orthogonal.

We can express, $f(t) = c_1 g_1(t) + c_2 g_2(t) + \cdots + c_n g_n(t)$ --- (ii)

Now, to evaluate the co-efficients c_n, we multiply both sides by $g_n(t)$ and integrate over the interval of orthogonality.

$$\therefore \int_{t_1}^{t_2} f(t)\, g_n(t)\, dt = c_1 \int_{t_1}^{t_2} g_1(t)\, g_n(t)\, dt + c_2 \int_{t_1}^{t_2} g_2(t)\, g_n(t)\, dt$$

$$\cdots + c_n \int_{t_1}^{t_2} g_n^2(t)\, dt \quad \text{---} (iii)$$

Now, because of the orthogonality, all the terms of RHS becomes zero,

$$\therefore \int_{t_1}^{t_2} f(t)\, g_n(t)\, dt = c_n \int_{t_1}^{t_2} g_n^2(t)\, dt$$

$$\therefore C_n = \frac{\int_{t_1}^{t_2} f(t)\, g_n(t)\, dt}{\int_{t_1}^{t_2} g_n^2(t)\, dt} \quad \text{---- (iv)}$$

If, we select $g_n(t)$ in such a fashion, so that

$$\int_{t_1}^{t_2} g_n^2(t)\, dt = 1$$

$$\therefore C_n = \int_{t_1}^{t_2} f(t)\, g_n(t)\, dt \quad \text{--- (v)}$$

So, when orthogonal functions $g_n(t)$ are selected as above, they are described as being Normalized. The use of normalized functions has the merit that C_n's can be calculated from eqn. (v) and thereby avoids the need of evaluating $\int_{t_1}^{t_2} g_n^2(t)\, dt$

A set of functions which is both orthogonal and normalized is called as <u>Orthonormal Set</u>.

Gram-Schmidt Orthogonalization Procedure :

Orthogonal sets of functions that are complete for the purpose of representing an arbitrary function always have an infinite number of components. Gram-Schimdt procedure helps us to construct an Orthogonal or Orthonorm set with finite no. of elements.

Now, let us call the time functions which are to be expanded $s_1(t)$, $s_2(t)$, ..., $s_M(t)$ and the orthonormal functions $u_1(t)$, $u_2(t)$, ..., $u_N(t)$. Then we require, given the functions $s(t)$, to find the functions $u(t)$ and to evaluate s_{ij} ;

$$s_1(t) = s_{11}u_1(t) + s_{12}u_2(t) + \cdots + s_{1N}u_N(t) \quad \text{---- (1)}$$

$$s_2(t) = s_{21}u_1(t) + s_{22}u_2(t) + \cdots + s_{2N}u_N(t) \quad \text{---- (2)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$s_M(t) = s_{N1}u_1(t) + s_{N2}u_2(t) + \cdots + s_{NN}u_N(t) \quad \text{.... (3)}$$

$$\boxed{\therefore \; s_i(t) = \sum_{j=1}^{N} s_{ij}u_j(t), \quad i = 1, 2, \ldots M \text{ ....(4)}}$$

The Orthogonality of the functions $u(t)$ over the interval T is expressed by,

$$\int_T u_j(t)\,u_k(t)\,dt = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad \text{....... (4)}$$

The Gram-Schmidt method proceeds as follows :

<u>Step I</u> : In eqn. (1), except s_{11}, set all co-efficients to zero ;

$$\therefore \; s_1(t) = s_{11}u_1(t). \quad \text{... (5)}$$

Since, $u_1(t)$ is to be a normalized function we find that,

$$\boxed{\therefore \; s_{11} = \left[\int_T s_1^2(t)\,dt\right]^{1/2}} \quad \text{.... (6)}$$

Squaring both sides and integrating over T.

as $\int_T u_1(t)\,u_1(t)\,dt = 1$

and $u_1(t) = s_1(t)/s_{11}$ is now determined.

Step 2 : In eqn. (2) we set all co-efficients to zero except first two, i.e. s_{21} and s_{22}.

$$\therefore s_2(t) = s_{21} u_1(t) + s_{22} u_2(t) \quad \cdots\cdots (7)$$

Multiplying both sides by $u_1(t)$, integrating over T and taking account of the orthonormality of $u_1(t)$ and $u_2(t)$, we find that,

$$s_{21} = \int_T s_2(t)\, u_1(t)\, dt \quad \cdots\cdots (8)$$

If, s_{21} is known, then we can evaluate s_{22}

$$\therefore s_2(t) - s_{21} u_1(t) = s_{22} u_2(t) \quad \cdots\cdots (9)$$

Squaring and Integrating both sides,

$$\int_T [s_2(t) - s_{21} u_1(t)]^2\, dt = s_{22}^2 \int_T u_2^2(t)\, dt = s_{22}^2$$

$$\Rightarrow s_{22} = \left\{ \int_T [s_2(t) - s_{21} u_1(t)]^2\, dt \right\}^{1/2} \quad \cdots\cdots (10)$$

Finally, as both s_{21} and s_{22} are known, so from eqn. (7);

$$u_2(t) = \frac{1}{s_{22}} [s_2(t) - s_{21} u_1(t)]$$

$$= \frac{1}{s_{22}} \left[s_2(t) - \frac{s_{21} s_1(t)}{s_{11}} \right] \quad \cdots\cdots (11)$$

Step III : Continuing as above, we write

$$s_3(t) = s_{31} u_1(t) + s_{32} u_2(t) + s_{33} u_3(t) \quad \cdots\cdots (12)$$

We set, s_{34} & others to zero.

We can find, $s_{31} = \int_T s_3(t)\, u_1(t)\, dt$

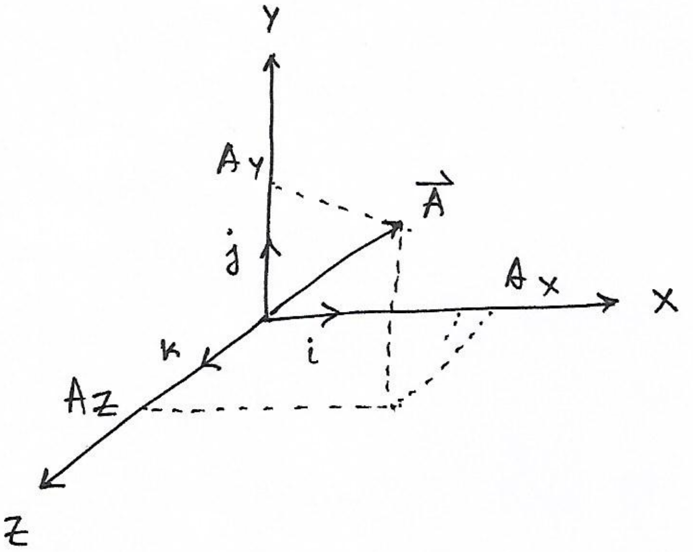
$$s_{32} = \int_T s_3(t)\, u_2(t)\, dt$$

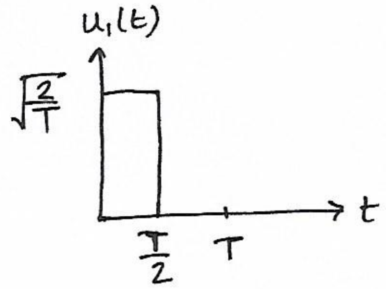
$$s_{33} = \left\{ \int_T [s_3(t) - s_{31} u_1(t) - s_{32} u_2(t)]^2 dt \right\}^{1/2}$$

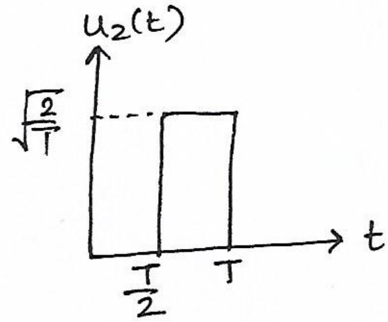
and finally, $u_3(t) = \dfrac{s_3(t) - s_{31} u_1(t) - s_{32} u_2(t)}{s_{33}}$

Step 4 : If we continue the procedure until we have used all N equations and shall finally have N orthonormal functions $u_1(t)$, $u_2(t)$, ... $u_N(t)$. We can also evaluate the co-efficients s_{ij} needed to express the functions $s_1(t)$, $s_2(t)$, ... $s_M(t)$ in terms of $u(t)$.

Analogy between Signals and Vectors :







In fig $\vec{A}$ represents a vector in XYZ co-ordinate system.

$\therefore$ $i.i = j.j = k.k = 1$. ($\because$ i, j, k are unit vectors)

Let us consider $u_1(t)$ and $u_2(t)$ are two unit vectors. and char. of orthonormal basis function is

$$\int_T u_i(t)\, u_j(t) = 1. \quad \cdots (1)$$

also, unit vectors i, j, k have the prop. that,

$$i.j = j.k = k.i = 0$$

in correspondence with the property of orthonormal functions, $\int_T u_i(t)\, u_j(t)\, dt = 0$, when $i \neq j$. $\cdots (2)$

An arbitrary vector $\vec{A}$ can be written in terms of its components as

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$

While an arbitrary function $s(t)$ can be written as,

$$s(t) = c_1 u_1(t) + c_2 u_2(t) + \cdots + c_N u_N(t) \quad \cdots (3)$$

If, we want to evaluate a vector component, say A_y, we can do so by multiplying it by $\vec{j}$,

$$\therefore \vec{j}\cdot\vec{A} = A_x \cdot \vec{i}\cdot\vec{j} + A_y \cdot \vec{j}\cdot\vec{j} + A_z \cdot \vec{z}\cdot\vec{j}$$

$$= 0 + A_y + 0$$

$$\therefore A_y = \vec{j}\cdot\vec{A}.$$

Similarly, if we want to evaluate, say c_2 in eqn. (3), we can do it by multiplying by $u_2(t)$ and integrating.

$$\therefore \int_T s(t)\, u_2(t)\, dt = c_1 \int_T u_1(t) \cdot u_2(t)\, dt + c_2 \int_T u_2(t) \cdot u_2(t)\, dt + \cdots + c_N \int_T u_N(t) \cdot u_2(t)\, dt$$

$$= 0 + c_2 + \cdots + 0$$

$$\therefore c_2 = \int_T s(t)\, u_2(t)\, dt.$$

So, the Sieve technique serves effectively in both cases.

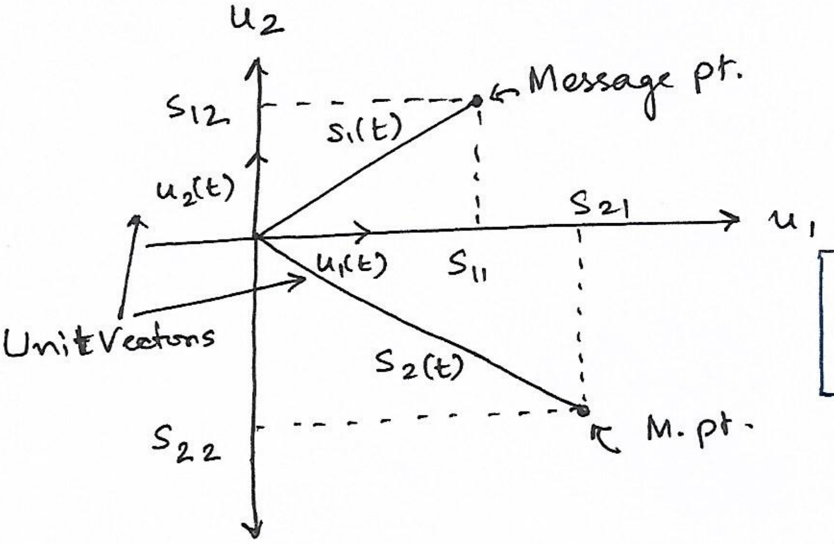
Finally, the vector, can be expressed specifically by its components in three orthonormal directions, i.e.

$$A = \{A_x, A_y, A_z\}$$

Also, for a signal $s(t)$ can be expressed by specifying its components in N orthonormal directions, i.e.

$$s(t) = \{c_1, c_2, \ldots, c_N\}$$

Distinguishability of Signals :



$S_1(t) = S_{11}\, u_1(t) + S_{12}\, u_2(t)$

$S_2(t) = S_{21}\, u_1(t) + S_{22}\, u_2(t)$

$$\boxed{\therefore\ S_i(t) = \sum_{j=1}^{N} S_{ij}\, u_j(t).}$$

$i = 1, 2\ (M)$ $\quad (N \leq M).$
$j = 1, 2\ (N).$

Decomposition of two Signal Vectors $s_1(t)$ & $s_2(t)$ in terms of the Orthonormal vectors $u_1(t)$ & $u_2(t)$.

Expansion of a function $s_i(t)$ into an orthonormal series yields the geometric representation as shown in fig. In this example the geometric representation as shown in fig. In this example for two independent functions $s_1(t)$ and $s_2(t)$ a co-ordinate system is defined by the unit orthonormal vectors $u_1(t)$ and $u_2(t)$. The functions $s_1(t)$ and $s_2(t)$ are then represented by their components, i.e. the co-ordinates (s_{11}, s_{12}) and (s_{21}, s_{22}). So, s_{11} is component of $s_1(t)$ in the direction of $u_1(t)$ ...etc.

Geometric Representation of Signals :

In geometric representation of signals, we represent any set of M no. of signals $s_i(t)$ as linear combinations of N orthonormal basis functions. $(N < M)$

This means that given a set of real-valued signals $s_1(t)$, $s_2(t)$, ..., $s_M(t)$ each of duration T sec, we may write $s_i(t)$ as follows :

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\,\phi_j(t) \quad , \begin{cases} 0 \le t \le T, \\ i = 1, 2, \ldots, M. \end{cases}$$

where, ϕ_j = Basis Function

where, co-efficients of the expansion can be defined by ;

$$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt \quad \begin{cases} i = 1, 2, \ldots M \\ j = 1, 2, \ldots N \end{cases}$$

Here, the real valued Basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are orthonormal. And orthonormality is defined by ;

$$\int_0^T \phi_i(t)\,\phi_j(t)\,dt = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad , \quad \delta = \text{Kronecker Delta.}$$

No of dimensions are always less than no. of message point.

$[M \ge N]$

Response of the noisy signal at the Receiver :

(A Baseband Signal Receiver)

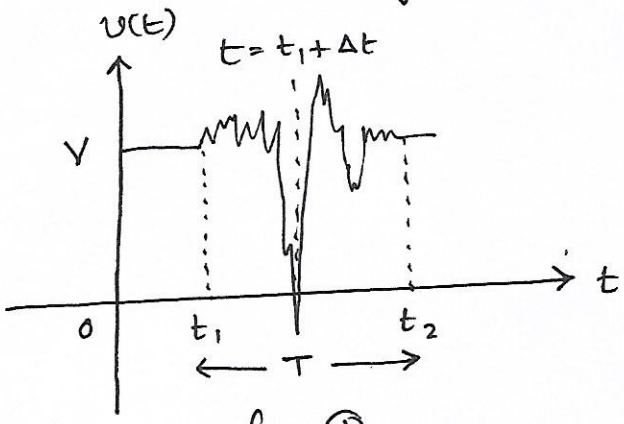


fig (i)

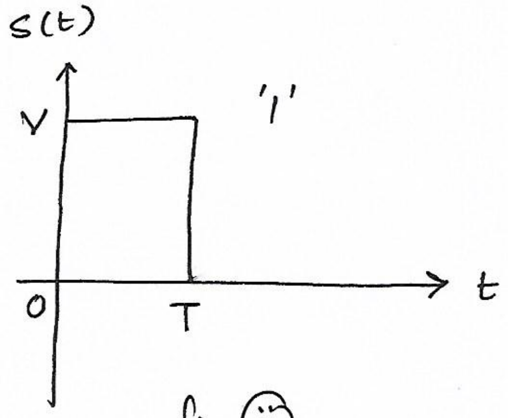


fig (ii)

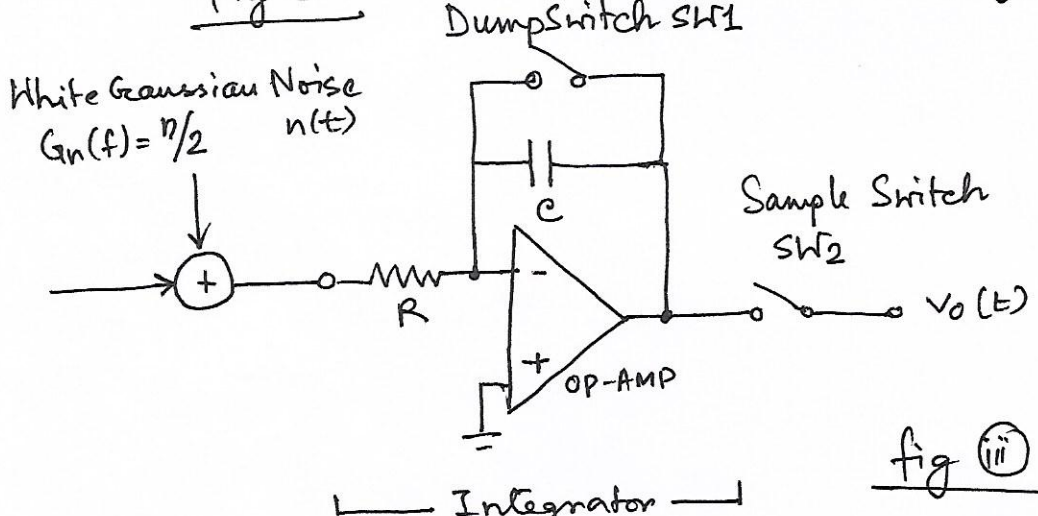


fig (iii)

Consider a binary-coded signal that consists of time sequence of voltage levels $+V$ or $-V$. But when this signal is corrupted by noise, it may look like fig (i), whereas fig (ii) is actual waveform. Now, the bit is transmitted over t_1 to t_2 i.e. for T s. But, if we perform sampling at $t_1 + \Delta t$ an error occurs. So, to reduce the prob. of error we use a different kind of receiver when the sample voltage is emphasized relative to the sample voltage due to noise.

The Signal $s(t)$ with added White Gaussian noise $n(t)$ of PSD $\eta/2$ is presented to the I/P of an Integrator. At time $t = 0+$ we require that capacitor C be uncharged. This is ensured by briefly closing switch SW1 at time, $t = 0-$. Now, the sample is taken at the O/P of integrator by closing sample switch SW2. This sample is taken after a bit interval, i.e, $t = T$. This signal processing is thus described by Integrate and Dump.

The integrator yields an o/p which is the integral of its input multiplied by $1/RC$, $[\because \tau = RC]$

$$\therefore \quad v_o(t) = \frac{1}{\tau}\int_0^T [\, s(t) + n(t)\,]\, dt$$

$$= \frac{1}{\tau}\int_0^T s(t)\, dt + \frac{1}{\tau}\int_0^T n(t)\, dt = s_o(T) + n_o(T).$$

So, sample voltage due to signal is;

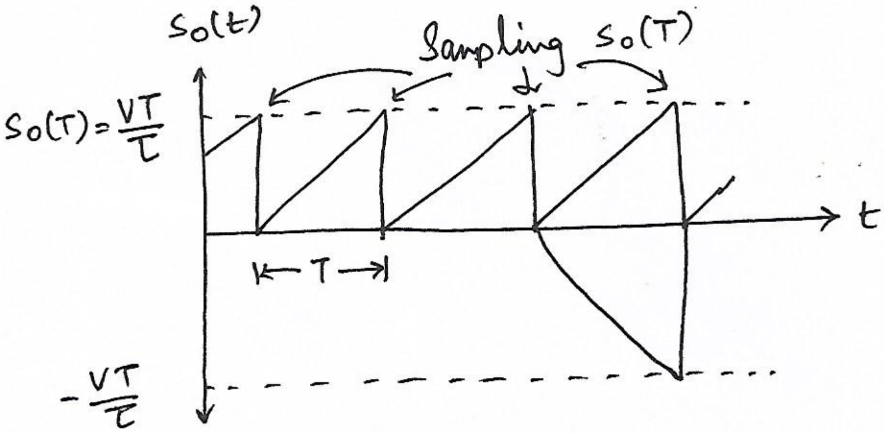
$$s_o(T) = \frac{1}{\tau}\int_0^T V\, dt = \frac{VT}{\tau}$$

The sample voltage due to noise is;

$$n_o(T) = \frac{1}{\tau}\int_0^T n(t)\, dt$$

We know the variance of $n_o(T) = \sigma_o^2 = \overline{n_o^2(T)} = \dfrac{\eta T}{2\tau^2}$

So, at the end of interval the ramp attains a voltage $s_o(T)$ which is $\pm \dfrac{VT}{\tau}$ depending on bit is 1 or 0. At end of every interval the switch SW1 is closed momentarily to discharge cap. C



$$\therefore v_o(T) = s_o(T) + n_o(T).$$

fig.(a)

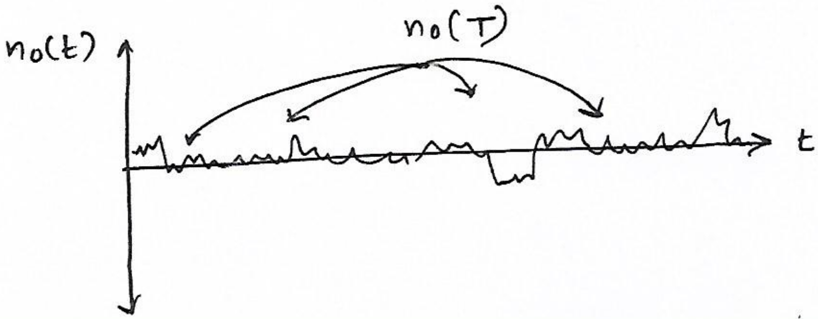


fig.(b).

So, the figure of merit is Signal to Noise ratio

$$\frac{[s_o(T)]^2}{[n_o(T)]^2} = \frac{\frac{V^2T^2}{\tau^2}}{\frac{\eta T}{2\tau^2}} = \frac{2}{\eta} V^2 T .$$

So, signal to noise ratio increases with increasing T and it depends on V^2T which is normalized energy of the bit signal. So, a bit represented by a narrow, high amplitude signal and one by a wide, low amplitude signal are equally effective, provided V^2T is kept constant.

Also, $s_o(T) = \frac{VT}{\tau}$ and $n_o(T) = \sqrt{\frac{\eta T}{2\tau^2}}$

$\therefore$ $s_o(T) \propto T$ but, $n_o(T) \propto \sqrt{T}$.

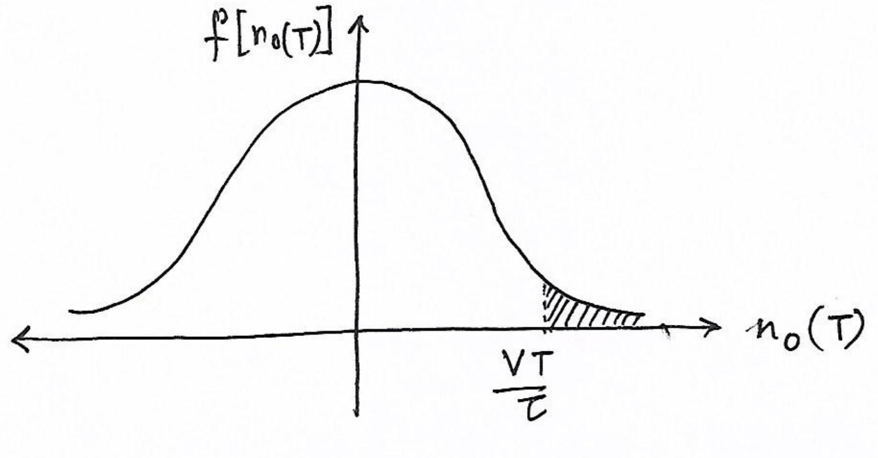
So, over the time interval T, the integrator enhances the signal relative to noise as shown in fig.

Probability of Error of Integrate and Dump filter :

As we have already assumed that prob. density of noise is Gaussian, hence,

$$f[n_o(T)] = \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}} \quad \text{-----} \quad (1)$$

Where, σ_o^2 is variance and $\sigma_o^2 = \overline{n_o^2(T)}$



~~tett~~ let us consider during some bit interval the input-signal voltage is held at, say $-V$. Then, at the sample time, the signal sample voltage is $s_o(T) = -\frac{VT}{\tau}$, while the noise sample is $n_o(T)$. If $n_o(T)$ is positive and larger in magnitude than $\frac{VT}{\tau}$ then, total sample voltage $v_o(T) = s_o(T) + n_o(T)$ will be positive, which results in error. So, the error prob. is $n_o(T) > VT/\tau$

$$\therefore P_e = \int_{VT/\tau}^{\infty} f[n_o(T)]\, dn_o(T) = \int_{\frac{VT}{\tau}}^{\infty} \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}}\, dn_o(T) \quad \text{--} (2)$$

Substituting, $x = \frac{n_o(T)}{\sqrt{2\sigma_o^2}}$, then eqn. (2) can be written as,

$$P_e = \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}}\int_{x = V\sqrt{\frac{T}{\eta}}}^{\infty} e^{-x^2}\, dx = \frac{1}{2}\,\text{erfc}\left(V\sqrt{\frac{T}{\eta}}\right)$$

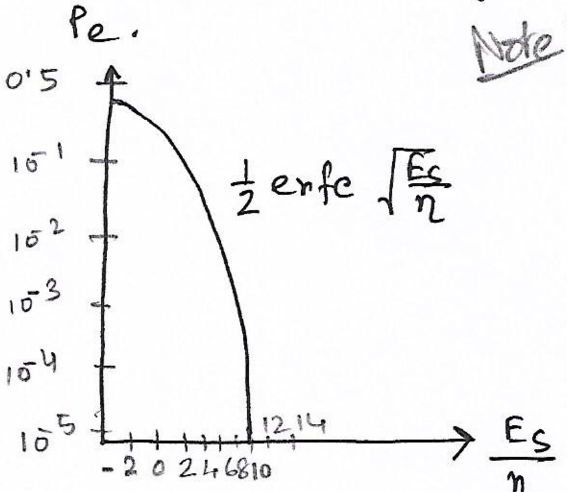
$$= \frac{1}{2}\,\text{erfc}\left(\frac{V^2T}{\eta}\right)^{1/2}$$

$$= \frac{1}{2}\,\text{erfc}.\left(\frac{E_s}{\eta}\right)^{1/2}$$

$\because P = \frac{V^2}{R} \Rightarrow \underline{P = V^2}$.

$R = 1\,\Omega$. $\quad PT = V^2T \Rightarrow$ Signal Energy $= E_s$.

So, P_e decreases as E_s/η increases. The max value of P_e is $\frac{1}{2}$. So, even the entire signal is lost in noisy environment, the receiver cannot be wrong more than half the time on avg.



Note:

$$\sigma_o^2 = \frac{\eta T}{2\tau^2}, \quad \sigma_o = \sqrt{\frac{\eta T}{2\tau^2}}$$

$$n_o(T) \simeq \frac{VT}{\tau}$$

$$* \quad \therefore x = \frac{n_o(T)}{\sqrt{2\sigma_o^2}} = \frac{VT}{\tau\sqrt{2}.\sqrt{\frac{\eta T}{2\tau^2}}}$$

$$\therefore dx = \frac{1}{\sqrt{2}\sigma_o}\cdot d[n_o(T)]$$

or, $d[n_o(T)] = \sqrt{2}.\sigma_o.dx$

$$\boxed{* \; x = V\sqrt{\frac{T}{\eta}}}$$

Optimum Threshold : Maximum Likelihood Detector

In previous discussion we considered that binary data 1 and 0 are associated with +V and −V respectively and decision threshold is 0. This make P_e minimum. Here, both symbols are equally likely and prob. densities are symmetric like Gaussian.

But, to decide decision threshold when apriori prob. are not equal, we have to follow diff^n approach.

Consider, when symbol sent is s_1, the prob. of receiving voltage v is $p(v/s_1)$ and for symbol s_2 it is $p(v|s_2)$. We define apriori prob. of presence of these symbols as $P(S_1)$ and $P(S_2)$ respectively. If the decision threshold is λ, such that $v > \lambda$, symbol s_1 is selected and for $v < \lambda$, symbol s_2 is selected. So, prob. of error,

$$P_e = P(S_1) \cdot \int_{v<\lambda} p(v|s_1)\, dv + P(S_2) \int_{v>\lambda} p(v|s_2)\, dv \quad \text{---} (1)$$

Since, some voltage v is received for certain when symbol s_1 is sent we can write

$$\int_{v>\lambda} p(v|s_1)\, dv + \int_{v<\lambda} p(v|s_1)\, dv = 1 \quad \text{---} (2)$$

So, from eqn. (1) and (2), we get,

$$P_e = P(S_1)\left[1 - \int_{v>\lambda} p(v|s_1)\, dv\right] + P(S_2) \int_{v>\lambda} p(v|s_2)\, dv$$

$$= P(S_1) + \int_{v>\lambda} \left[P(S_2)\, p(v|S_2) - P(S_1)\, p(v|S_1)\right] dv$$

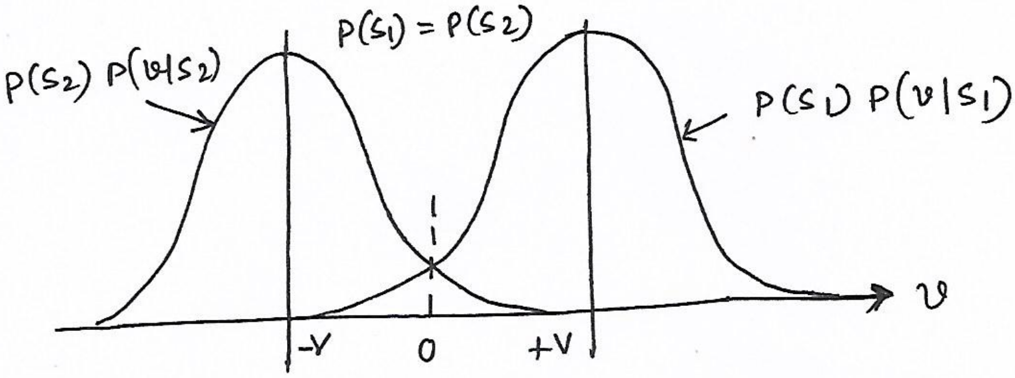
Thus, prob. of error is min, if for every $v > \lambda$.

$$P(S_1)\, p(v|S_1) > P(S_2)\, p(v|S_2)$$

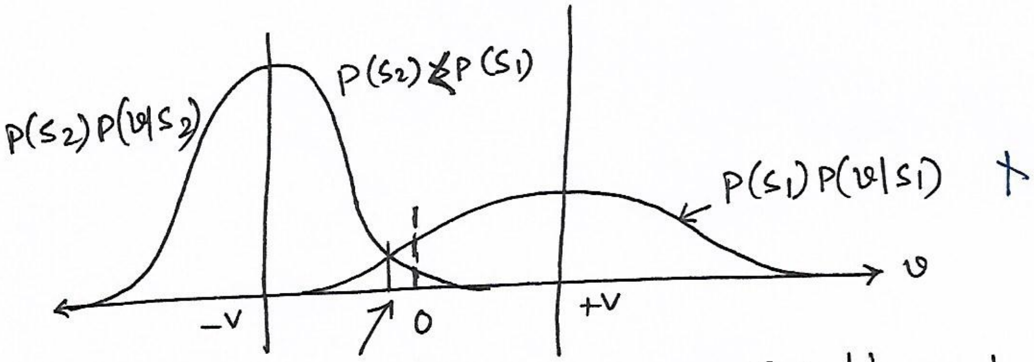
or $\dfrac{p(v|S_1)}{p(v|S_2)} > \dfrac{P(S_2)}{P(S_1)}$, at decision boundary, $v = \lambda$

$$\therefore \frac{P(\lambda|S_1)}{P(\lambda|S_2)} = \frac{P(S_2)}{P(S_1)}$$

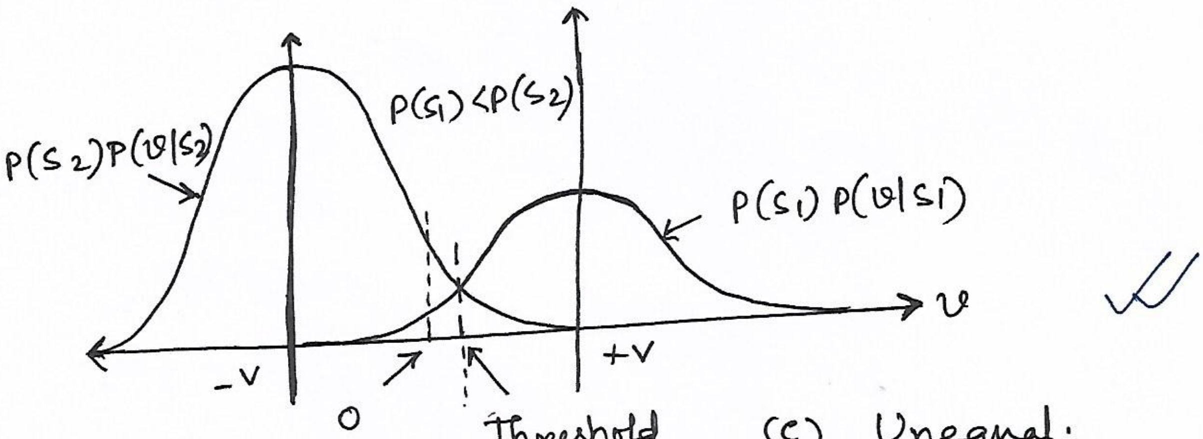
This can be treated as Max. Likelihood Detector. That means if the cond^n is valid *



(a) A priori prob. are equal.



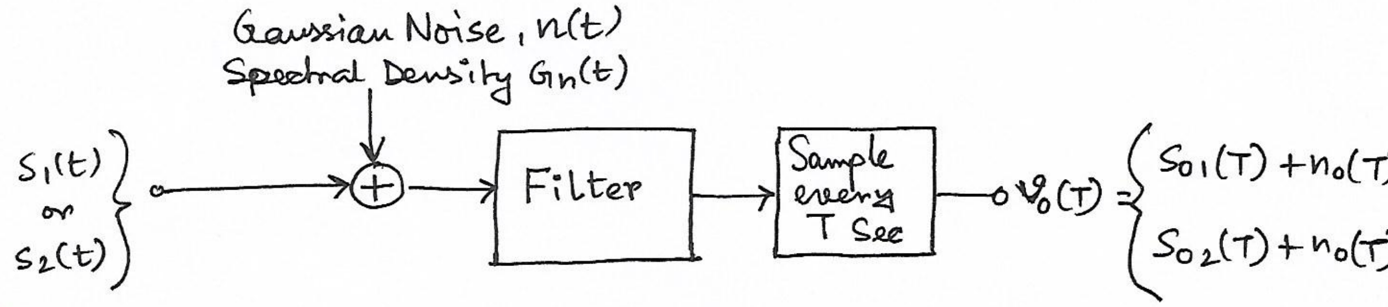
(b) Unequal



(c) Unequal.

* then symbol sent is most likely to be s_1 when received voltage is v. The reverse is also true.

Optimum Receiver :



Let us consider a receiver like above, which receives binary digits i.e. bits by two waveforms $s_1(t)$ and $s_2(t)$. Both the wave forms lasts for T s. Example of $s_1(t) = +V$ and $s_2(t) = -V$.

As shown the I/P signals are corrupted by white noise $n(t)$, which is Gaussian in nature and has spectral density $G(f) = \frac{\eta}{2}$

The signal and noise are then filtered and sampled at end of each bit interval. So o/p sample is either $v_0(T) = S_{01}(T) + n_0(T)$ or $S_{02}(T) + n_0(T)$. In absence of noise o/p will be either $S_{01}(T)$ or $S_{02}(T)$. But, in presence of noise to minimize error prob. the receiver should assume that $s_1(t)$ was transmitted if $v_0(T)$ is closer to $S_{01}(T)$ and vice versa. Hence, the decision boundary is midway between $S_{01}(T)$ and $S_{02}(T)$. Now, in the baseband system as above, $S_{01}(T) = \frac{+VT}{\tau}$ and $S_{02}(T) = -\frac{VT}{\tau}$ and decision boundary is $v_0(T) = 0$.

$$\therefore \; v_0(T) = \frac{S_{01}(T) + S_{02}(T)}{2} \quad \text{---- (1)}$$

Now, to determine error prob., let us take example of $s_2(t)$ was transmitted, but at sampling instance, $n_0(T)$ is ~~large~~ positive and larger in magnitude than the voltage difference

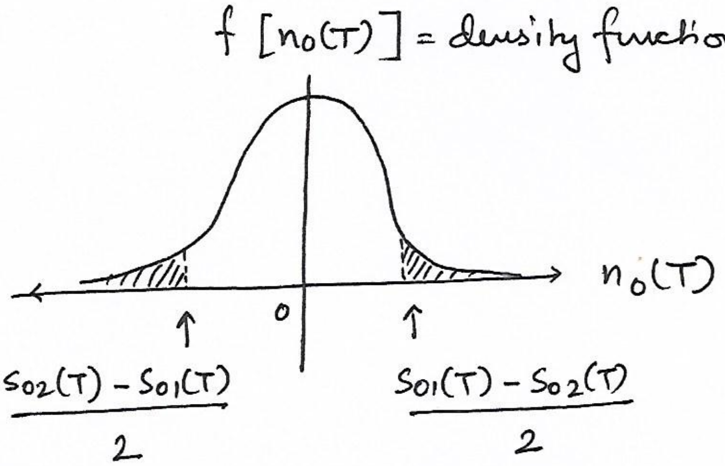
$$\frac{1}{2}\left[S_{01}(T) + S_{02}(T)\right] - S_{02}(T)$$

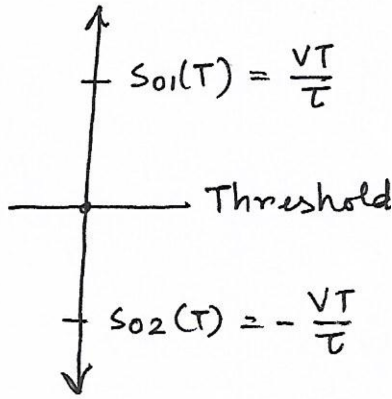
, an error occurs and receiver assumes that $s_1(t)$ was transmitted.

So, if $n_o(T) \geq \dfrac{s_{o1}(T) - s_{o2}(T)}{2}$ ----(2) an error will result.

Hence, prob. of error is

$$P_e = \int_{\frac{s_{o1}(T) - s_{o2}(T)}{2}}^{\infty} \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}}\, dn_o(T) \quad \text{----(3)}$$





Substitute, $x \equiv \dfrac{n_o(T)}{\sqrt{2}\,\sigma_o}$ or $\equiv \dfrac{n_o(T)}{\sqrt{2\sigma_o^2}} \neq \cancel{\dfrac{VT/\tau}{\sqrt{2}\cdot\sqrt{\frac{\eta T}{2\tau^2}}}} = \cancel{V\sqrt{\frac{T}{\eta}}}$

$$\Rightarrow x = \frac{s_{o1}(T) - s_{o2}(T)}{2\sqrt{2}\,\sigma_o}.$$

$$\therefore P_e = \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}} \int_{\frac{s_{o1}(T) - s_{o2}(T)}{2\sqrt{2}\,\sigma_o}}^{\infty} e^{-x^2}\, dx$$

Also; $x = \dfrac{n_o(T)}{\sqrt{2\sigma_o^2}}$ $\therefore dx = \dfrac{d[n_o(T)]}{\sqrt{2\sigma_o^2}}$

$$\therefore P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{s_{o1}(T) - s_{o2}(T)}{2\sqrt{2}\,\sigma_o}\right] \quad \text{----(4)}$$

We know, $s_{o1}(T) = \dfrac{VT}{\tau}$, $s_{o2}(T) = -\dfrac{VT}{\tau}$ and $\sigma_o^2 = \overline{n_o^2(T)} = \dfrac{\eta T}{2\tau^2}$

$$\therefore P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\frac{VT}{\tau} - \left(-\frac{VT}{\tau}\right)}{2\sqrt{2}\cdot\sqrt{\frac{\eta T}{2\tau^2}}}\right] = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\frac{\cancel{2}VT}{\tau}\cdot\tau}{\cancel{2}\sqrt{\eta T}}\right]$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(V\sqrt{\frac{T}{\eta}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{V^2T}{\eta}\right)^{1/2} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_s}{\eta}\right)^{1/2}$$

$[\because E_s = V^2T]$

The Complementary error function is a monotonically decreasing function. So, P_e decreases as the diff.^{n} $S_{o1}(T) - S_{o2}(T)$ becomes larger and rms noise voltage σ_o becomes smaller.

The Optimum Filter, then, is the filter which maximizes the ratio

$$\gamma = \left. \frac{S_{o1}(T) - S_{o2}(T)}{\sigma_o} \right|_{max}.$$

Transfer Function of an Optimum Filter :

We have seen that Prob. of error of an optimum filter is dependent on the difference $S_{o1}(T) - S_{o2}(T)$, or

$$\gamma \propto [S_{o1}(T) - S_{o2}(T)]$$

which in turn $\gamma \propto [S_1(t) - S_2(t)]$

Hence, we can conclude that error prob. is actually dependent on diff.^{n} of $S_1(t)$ and $S_2(t)$, but not on individual waveform.

Let us consider, the diff.^{n} signal is $p(t) \equiv S_1(t) - S_2(t)$ --- (1)

and to calculate minimum error prob., we shall assume that the I/P signal to optimum filter is $p(t)$. and corresponding output signal of the filter is then

$$\cancel{P_o(f) = S_{o1}(T) - S_{o2}(T)}$$

$$p_o(t) = S_{o1}(t) - S_{o2}(t). \quad \text{--- (2)}$$

Let, $P(f)$ and $P_o(f)$ be the Fourier Transforms respectively, of $p(t)$ and $p_o(t)$.

If, $H(f)$ is transfer function of the filter, then

$$P_o(f) = H(f)\,P(f) \quad \text{---- (3)}$$

and $p_o(T) = \int_{-\infty}^{\infty} P_o(f)\, e^{j2\pi f T}\, df = \int_{-\infty}^{\infty} H(f)\, P(f)\, e^{j2\pi f T}\, df$ --- (4)

The I/P noise to the optimum filter is $n(t)$. The output noise is $n_o(t)$ which has a power spectral density $G_{n_o}(f)$ and is related to the power spectral density of the input noise $G_n(f)$ by

$$G_{n_o}(f) = |H(f)|^2 G_n(f)\, df. \quad \text{----(5)}$$

Using Perseval's theorem we find the normalized O/P noise power, i.e. variance

[Normalized energy of a periodic signal is $E = \int_{-\infty}^{\infty} V(f)\, V^*(f)\, df$

$= \int_{-\infty}^{\infty} |V(f)|^2\, df$

$= \int_{-\infty}^{\infty} [v(t)]^2\, dt.$]

$$\sigma_o^2 = \int_{-\infty}^{\infty} G_{n_o}(f)\, df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df \quad \text{---(6)}$$

So, using eqn. no. (4) and (6), we find,

$$\gamma^2 = \frac{p_o^2(T)}{\sigma_o^2} = \frac{\left| \int_{-\infty}^{\infty} H(f)\, P(f)\, e^{j2\pi f T}\, df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df.} \quad \text{----(7)}$$

According to Schwartz Inequality of complex functions $X(f)$ and $Y(f)$ of a common variable f,

$$\left| \int_{-\infty}^{\infty} X(f)\, Y(f)\, df \right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2 df \cdot \int_{-\infty}^{\infty} |Y(f)|^2 df \quad \text{---(8)}$$

The equality sign applies when,

$$X(f) = K\, Y^*(f). \quad \text{----(9)}$$

where, K is an arbitrary constant and $Y^*(f)$ is complex conjugate of $Y(f)$.

So, we now apply Schwartz inequality to eqn. (7), making the identification ;

$$X(f) = \sqrt{G_n(f)}\; H(f) \quad \text{----} \; (9)$$

and
$$Y(f) = \frac{1}{\sqrt{G_n(f)}}\, P(f)\, e^{j2\pi fT} \quad \text{----} \; (10)$$

So, using eqn. (9) and (10) and Schwartz inequality, we may rewrite eqn. (7) as,

$$\gamma^2 = \frac{p_o^2(T)}{\sigma_o^2} = \frac{\left|\int_{-\infty}^{\infty} X(f)\,Y(f)\,df\right|^2}{\int_{-\infty}^{\infty} |X(f)|^2\,df} \leq \frac{\int_{-\infty}^{\infty} |X(f)|^2 df \cdot \int_{-\infty}^{\infty} |Y(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2\,df}$$

$$= \int_{-\infty}^{\infty} |Y(f)|^2\,df.$$

$$= \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)}\,df. \quad \text{---} \; (11)$$

So, the ratio $\frac{p_o^2(T)}{\sigma_o^2}$ will attain its max. value when equal sign. is employed. i.e. $X(f) = K\,Y^*(f)$ ..

So from eqn. (9) and (10), which yields a max. ratio $\frac{p_o^2(T)}{\sigma_o^2}$ has a transfer function,

$$\Rightarrow \sqrt{G_n(f)}\; H(f) = \frac{K \cdot 1}{\sqrt{G_n(f)}}\, P^*(f)\, e^{-j2\pi fT}$$

$$\Rightarrow \boxed{H(f) = K \cdot \frac{P^*(f)}{G_n(f)}\, e^{-j2\pi fT}}$$

and
$$\boxed{\left.\frac{p_o^2(T)}{\sigma_o^2}\right|_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)}\,df.}$$

~~Transfer Function of Optimum Filter H(f)~~

Impulse Response of a Matched Filter :

An optimum filter which yields a maximum ratio $p_0^2(T)/\sigma_0^2$ is called a Matched filter when the input noise is white. In this case, $G_n(f) = \frac{\eta}{2}$ and we write

$$H(f) = k \cdot \frac{P^*(f)}{\eta/2} e^{-j2\pi fT} \quad \text{---} \quad (1)$$

The impulse response of this filter, i.e. the response of the filter to a unit strength impulse applied at $t=0$, is,

$$h(t) = F^{-1}[H(f)] = \frac{2k}{\eta}\int_{-\infty}^{\infty} P^*(f)\, e^{-j2\pi fT} \cdot e^{j2\pi ft}\, df$$

$$= \frac{2k}{\eta}\int_{-\infty}^{\infty} P^*(f)\, e^{j2\pi f(t-T)}\, df \quad \text{---} \quad (2)$$

A physically realizable filter will have an impulse response which is real, i.e. not complex. Therefore $h(t) = h^*(t)$. Replacing right-hand member of eqn (2) by its complex conjugate, we have,

$$h(t) = \frac{2k}{\eta}\int_{-\infty}^{\infty} P(f)\, e^{j2\pi f(T-t)}\, df$$

$$= \frac{2k}{\eta} p(T-t). \quad \text{---} \quad (3)$$

Since, $p(t) \equiv s_1(t) - s_2(t)$, so we have,

$$h(t) = \frac{2k}{\eta}\left[s_1(T-t) - s_2(T-t)\right] \quad \text{---} \quad (4)$$

So, here the response (O/P) matches with the I/P, hence it is known as Match filter., only a delay (T) is introduced, except for reverse polarity.

Error Probability of Matched Filter :

The error prob. of a Matched filter can be evaluated by calculating max. signal to noise ratio $\left.\frac{p_o^2(T)}{\sigma_o^2}\right|_{max}$. Given,

$G_n(f) = \frac{\eta}{2}$,

$$\left.\frac{p_o^2(T)}{\sigma_o^2}\right|_{max} = \frac{2}{\eta}\int_{-\infty}^{\infty} |P(f)|^2 \, df \quad \text{----} (1)$$

From, Perseval's theorem, we have,

$$\int_{-\infty}^{\infty} |P(f)|^2 \, df = \int_{-\infty}^{\infty} p^2(t)\, dt = \int_0^T p^2(t)\, dt \quad \text{...} (2)$$

Substituting, $p(t) = s_1(t) - s_2(t)$, in eqn. (1)

$$\left.\frac{p_o^2(T)}{\sigma_o^2}\right|_{max} = \frac{2}{\eta}\int_0^T [s_1(t) - s_2(t)]^2\, dt$$

$$= \frac{2}{\eta}\left[\int_0^T s_1^2(t)\, dt + \int_0^T s_2^2(t)\, dt - 2\int_0^T s_1(t)\, s_2(t)\, dt\right]$$

$$= \frac{2}{\eta}\left[E_{s1} + E_{s2} - 2E_{s12}\right] \quad \text{----} (3)$$

E_{s1}, E_{s2} are signal energies of $s_1(t)$ and $s_2(t)$ respectively.

E_{s12} is energy due to correlation betn. $s_1(t)$ and $s_2(t)$.

If we choose, $s_2(t) = -s_1(t)$, they will have same energy, say E_{s1}

$\therefore\ E_{s1} = E_{s2} = -E_{s12} \equiv E_s$.

So, eqn. (3) becomes,

$$\boxed{\left.\frac{p_o^2(T)}{\sigma_o^2}\right|_{max} = \frac{8E_s}{\eta}} \quad \text{----} (4)$$

As, we know, output of optimum filter is $p_o(T) = s_{o1}(T) - s_{o2}(T)$

$$\therefore\ P_e = \frac{1}{2}\,\text{erfc}\left[\frac{p_o(T)}{2\sqrt{2}\,\sigma_o}\right] = \frac{1}{2}\,\text{erfc}\left[\frac{p_o^2(T)}{8\,\sigma_o^2}\right]^{1/2} \quad \text{---} (5)$$

So, combining eqn. ④ and ⑤, we have,

$$[P_e]_{min} \text{ for max. value of } \left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max}$$

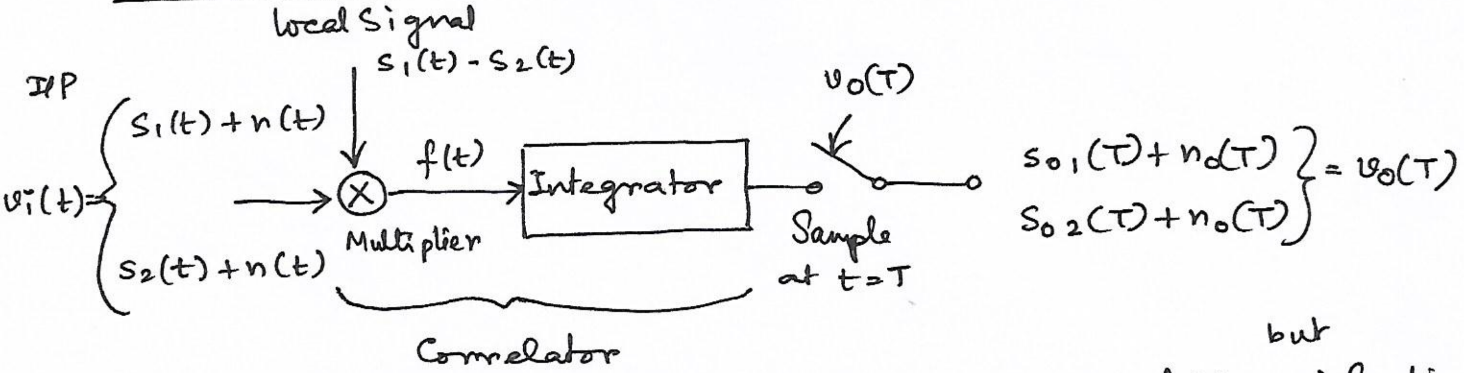
$$\therefore (P_e)_{min} = \frac{1}{2}\,\mathrm{erfc}\left\{\frac{1}{8}\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max}\right\}^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}.\left\{\frac{1}{8}\cdot\frac{8E_s}{\eta}\right\}^{1/2}$$

$$(P_e)_{min} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_s}{\eta}\right)^{1/2}$$

So, again we establish the idea that error prob. depends on signal energy and not on signal wave shape.

Realization of Optimum Filter using Correlator :



Here we see an alternative approach of Matched filter, but identical in performance. Here also, input is a binary waveform $s_1(t)$ or $s_2(t)$ corrupted by noise $n(t)$. The received signal plus noise $v_i(t)$ is first multiplier by a locally generated waveform $s_1(t) - s_2(t)$. The o/p of multiplier is then passed through an Integrator, whose output is sampled at $t = T$. As before, immediately after each sampling, at the beginning of each new bit interval, all energy-storing elements in the integrator are discharged. This type of receiver is called Correlator.

Since we are correlating the received signal and noise in waveform $s_1(t) - s_2(t)$.

So, the output of correlator for signal and noise part is,

$$s_o(T) = \frac{1}{\tau} \int_0^T s_i(t) \left[s_1(t) - s_2(t) \right] dt \quad \cdots \text{①}$$

$$n_o(T) = \frac{1}{\tau} \int_0^T n(t) \left[s_1(t) - s_2(t) \right] dt \quad \cdots \text{②}$$

Where, $s_i(t)$ is either $s_1(t)$ or $s_2(t)$ and $\tau = RC$ = time const.

Now, let us compare these o/p with <u>Matched Filter.</u>

If $h(t)$ is impulse response of matched filter, then o/p of it $v_o(t)$ can be found using Convolution integral.

$$\therefore v_o(T) = \int_{-\infty}^{\infty} v_i(\lambda)\, h(t-\lambda)\, d\lambda = \int_0^T v_i(\lambda)\, h(t-\lambda)\, d\lambda \quad \cdots \text{③}$$

Also, we have, $h(t) = \frac{2k}{\eta} \left[s_1(T-t) - s_2(T-t) \right]$

$$\therefore h(t-\lambda) = \frac{2k}{\eta} \left[s_1(T-t+\lambda) - s_2(T-t+\lambda) \right] \quad \cdots \text{④}$$

Substituting ④ in ③, we get,

$$v_o(T) = \frac{2k}{\eta} \int_0^T v_i(\lambda) \left[s_1(T-t+\lambda) - s_2(T-t+\lambda) \right] d\lambda \quad \cdots \text{⑤}$$

Now, $v_i(\lambda) = s_i(\lambda) + n(\lambda)$ and $v_o(t) = s_o(t) + n_o(t)$.

Put, $t = T$,

$\therefore$ eqn ⑤ can be written as,

$$s_o(T) = \frac{2k}{\eta} \int_0^T s_i(\lambda) \left[s_1(\lambda) - s_2(\lambda) \right] d\lambda \quad \cdots \text{⑥}$$

and $$n_o(T) = \frac{2k}{\eta} \int_0^T n(\lambda) \left[s_1(\lambda) - s_2(\lambda) \right] d\lambda \quad \cdots \text{⑦}$$

So, eqn ⑥ and ⑦ are outputs of ~~Correlator~~ Matched filter which is very similar to outputs of Correlator in eqn ① & ②.
So, they are identical, but independent techniques.