2.13 Relationship between s-plane and z-plane.

The Laplace transform of a continuous causal signal $x(t)$ is defined as

$$X(s) = \int_0^\infty x(t)e^{-st}dt \tag{2.45}$$

If we sample the signal with a sampling time T we get a discrete signal $x^\star(t)$ where

$$x^\star(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT) \tag{2.46}$$

The Laplace transform of $x^\star(t)$ can be written as

$$X^\star(s) = \sum_{n=0}^{\infty} x(nT)e^{-nTs} \tag{2.47}$$

Substitute $z = e^{sT}$ in Eq. (2.47) we get

$$z - \text{transform of} \quad x^\star(t) = X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n} \tag{2.48}$$

Therefore the relation between s-plane and z-plane can be described by the equation

$$z = e^{sT} \tag{2.49}$$

$$\text{Let} \quad z = re^{j\omega} \quad \text{and} \quad s = \sigma + j\Omega \tag{2.50}$$

Substituting these values in Eq. (2.49) we obtain

$$z = re^{j\omega} = e^{(\sigma+j\Omega)T} = e^{\sigma T}e^{j\Omega T} \tag{2.51}$$

From Eq.(2.51) we find $|z| = e^{\sigma T}$ and $\omega = \Omega T$. $\sigma = 0$ gives $|z| = 1; \sigma < 0$ gives $|z| < 1$ and $\sigma > 0$ gives $|z| > 1$. Since $\sigma = 0$ gives $|z| = 1$, the $j\Omega$-axis of s-plane maps into the unit circle. The left half of s-plane where $\sigma < 0$ maps into the inside the unit circle ($|z| < 1$). The RHP of s-plane where $\sigma > 0$ maps into the outside of the unit circle ($|z| > 1$). The mapping is as shown in Fig. 2.7.

From Fig. 2.7 we can observe that a single horizontal strip of width $\frac{2\pi}{T}$, from $\Omega = -\frac{\pi}{T}$ to $\Omega = \frac{\pi}{T}$ completely maps into the inside the unit circle and the horizontal strips between $\frac{\pi}{T}$ and $\frac{3\pi}{T}$, between $-\frac{\pi}{T}$ and $-\frac{3\pi}{T}$ (In general between $(2n-1)\frac{\pi}{T}$ and $(2n+1)\frac{\pi}{T}$ where $n = 0, \pm1, \pm2\ldots$) are mapped, again into the inside the unit circle. Thus many points in the s-plane are mapped to a single point in the z-plane, causing aliasing effect. That is when we sample two sinusoidal signals of the frequencies which differ by a multiple of the sampling frequency, we cannot distinguish between the results.

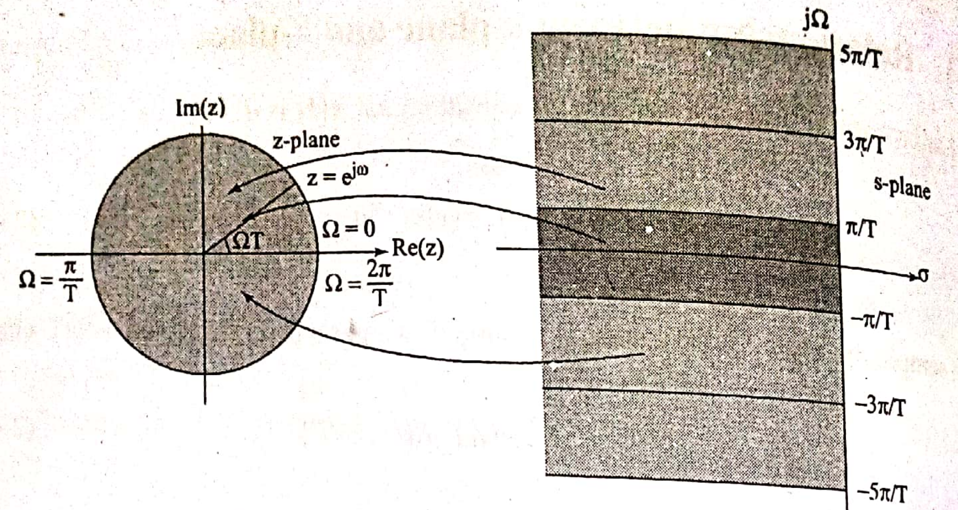


Fig. 2.7 Mapping from the s-plane to the z-plane

2.14 Inverse z-transform

There are four methods that are often used for the evaluation of the inverse z-transform

1. Long division method.
2. Partial fraction expansion method.
3. Residue method.
4. Convolution method.

2.14.1 *Long division method*

From Eq. (2.4) the one sided z-transform can be expanded into an infinite powers of z^{-1} as follows

$$X_+(z) = x(0) + x(1)z^{-1} + \ldots + x(k)z^{-k} + \ldots + x(n)z^{-n} + \cdots \qquad (2.52)$$

Thus the value of $x(n)$ at any instant of time are the coefficients of z^{-k}. If $X(z)$ is given as a ratio of two polynomials, the coefficients $x(0), x(1), \cdots, x(n)$ can be obtained by synthetic division of the numerator by the denominator as follows

$$\begin{aligned} X(z) = \frac{N(z)}{D(z)} &= \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \\ &= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \end{aligned} \qquad (2.53)$$

Therefore, the values of the sequence that represent the inverse z-transform are $x(n)$ for $n \geq 0$ is a causal sequence. The z-transform $X(z)$ can also be represented as

$$\begin{aligned} X(z) = \frac{N(z)}{D(z)} &= \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \\ &= \cdots + x(-n)z^{n} + \ldots + x(-k)z^{k} + \cdots + x(-2)z^{2} \\ &\quad + x(-1)z + x(0) \end{aligned} \qquad (2.54)$$

Therefore, the values of the sequence that represent the inverse z-transform are $x(n)$ for $n \leq 0$ is a non-causal sequence.

Example 2.20 (a) Find the inverse z-transform of $X(z) = \dfrac{z+0.2}{(z+0.5)(z-1)}$, $|z| > 1$ (b) Find the inverse z-transform of $X(z) = \dfrac{z}{(z-3)(z-4)}$, $|z| < 3$

Solution (a) From the ROC, we find that $x(n)$ is a causal sequence.

Given,

$$X(z) = \frac{z+0.2}{z^2 - 0.5z - 0.5}$$

Dividing, the numerator by denominator

$$\begin{array}{r|l}
 & z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.775z^{-4} \\
\hline
z^2 - 0.5z - 0.5 & z + 0.2 \\
 & z - 0.5 - 0.5z^{-1} \\
\cline{2-2}
 & \quad 0.7 + 0.5z^{-1} \\
 & \quad 0.7 - 0.35z^{-1} - 0.35z^{-2} \\
\cline{2-2}
 & \qquad 0.85z^{-1} + 0.35z^{-2} \\
 & \qquad 0.85z^{-1} - 0.425z^{-2} - 0.425z^{-3} \\
\cline{2-2}
 & \qquad\quad 0.775z^{-2} + 0.425z^{-3} \\
 & \qquad\quad 0.775z^{-2} - 0.387z^{-3} - 0.3875z^{-4}
\end{array}$$

$$\text{i.e.,} \quad X(z) = z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.775z^{-4} + \ldots$$
$$= \sum_{n=0}^{\infty} x(n) z^{-n}$$

where,

$$x(0) = 0, x(1) = 1, x(2) = 0.7, x(3) = 0.85, x(4) = 0.775 \quad \textit{so on.}$$

(b) From the ROC, we know $x(n)$ is a anticausal sequence

$$X(z) = \frac{z}{(z-3)(z-4)} = \frac{z}{z^2 - 7z + 12}$$

Dividing numerator by denominator

$$
\begin{array}{r|l}
 & \frac{z}{12} + \frac{7z^2}{144} + \frac{37}{1728}z^3 \\
\hline
12 - 7z + z^2 & z \\
 & z - \frac{7z^2}{12} + \frac{z^3}{12} \\
\hline
 & \frac{7z^2}{12} - \frac{z^3}{12} \\
 & \frac{7z^2}{12} - \frac{49z^3}{144} + \frac{7z^4}{144} \\
\hline
 & \frac{37z^3}{144} - \frac{7z^4}{144} \\
 & \frac{37z^3}{144} - \frac{259z^4}{1728} + \frac{37}{1728}x^5 \\
\hline
\end{array}
$$

$$
\begin{aligned}
X(z) &= \frac{1}{12}z + \frac{7}{144}z^2 + \frac{37}{1728}z^3 + \ldots \\
&= \sum_{n=-\infty}^{-1} x(n)z^{-n}
\end{aligned}
$$

Therefore,

$$x(0) = 0, x(-1) = \frac{1}{12}, x(-2) = \frac{7}{144}, x(-3) = \frac{37}{1728}, \quad \text{so on.}$$

Practice Problem 2.11 Find the inverse z-transform of $X(z) = \dfrac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$

(a) if $x(n)$ is causal **Ans:** $x(n) = \{1, 4, 7, 10, \ldots\}$

(b) if $x(n)$ is non-causal **Ans:** $x(n) = \{\ldots 11, 8, 5, 2, \underset{\uparrow}{0}\}$

2.14.2 *Partial fraction expansion method*

Consider a rational function $X(z)$ given by

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} \ldots + a_N z^{-N}} \qquad (2.55)$$

A rational function of the form Eq. (2.55) is called proper if $a_N \neq 0$ and $M < N$. An improper rational function $(M \geq N)$ can always be written as the sum of a polynomial and a proper rational function. In general any improper rational function can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + \ldots + c_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D(z)} \tag{2.56}$$

In the above Eq.(2.56) the inverse z-transform of polynomial can be easily found. We focus our attention on the inversion of proper rational functions, since any improper function can be transformed into a proper function by using Eq.(2.56). Let us consider a rational function of the form of Eq.(2.55). We eliminate negative powers of z by multiplying both numerator and denominator by z^N. This results in

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \ldots + a_N} \tag{2.57}$$

Dividing Eq. (2.57) by z we obtain

$$\begin{aligned} \frac{X(z)}{z} &= \frac{b_0 z^{N-1} + b_1 z^{N-2} + \ldots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \ldots + a_N} \\ &= \frac{b_0 z^{N-1} + b_1 z^{N-2} + \ldots + b_M z^{N-M-1}}{(z-p_1)(z-p_2)\ldots(z-p_N)} \end{aligned} \tag{2.58}$$

For distinct poles the Eq. (2.58) is expanded in the form

$$\frac{X(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + \ldots + \frac{C_N}{z-p_N} \tag{2.59}$$

We can determine the coefficients $C_1, C_2, \ldots C_N$ by using the formula

$$C_k = \left.\frac{(z-p_k)X(z)}{z}\right|_{z=p_k} \qquad k = 1, 2, \ldots, N \tag{2.60}$$

If $X(z)$ has a pole of multiplicity l, that is, it contains in denominator the factor $(z-p_k)^l$, then the expansion of the form of Eq.(2.59) is no longer valid.
Let us assume $l = 2$ and

$$\frac{X(z)}{z} = \frac{1}{(z-p_2)^2(z-p_1)} \tag{2.61}$$

The above Eq. (2.61) can be expressed in partial fraction form as

$$\frac{X(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_2}{z-p_2} + \frac{C_3}{(z-p_2)^2}$$

where

$$C_1 = (z-p_1)\left.\frac{X(z)}{z}\right|_{z=p_1} \tag{2.62}$$

$$C_2 = \frac{d}{dz}(z-p_2)^2\left.\frac{X(z)}{z}\right|_{z=p_2} \tag{2.63}$$

$$C_3 = (z-p_2)^2\left.\frac{X(z)}{z}\right|_{z=p_2} \tag{2.64}$$

To consider a general case of l-repeated roots let

$$\frac{X(z)}{z} = \frac{R(z)}{(z-p_k)^l} \tag{2.65}$$

$$= \frac{C_{1k}}{z-p_k} + \frac{C_{2k}}{(z-p_k)^2} + \ldots + \frac{C_{ik}}{(z-p_k)^i} + \ldots + \frac{C_{lk}}{(z-p_k)^l} \tag{2.66}$$

where i is any term in the partial fraction expansion and $R(z)$ is defined as

$$R(z) = \frac{X(z)}{z}(z-p_k)^l \tag{2.67}$$

Multiplying Eq.(2.66) by $(z-p_k)^l$ gives

$$R(z) = C_{1k}(z-p_k)^{l-1} + C_{2k}(z-p_k)^{l-2} + \ldots + C_{l-1k}(z-p_k) + C_{lk} \tag{2.68}$$

From this equation we can find a method to evaluate each coefficient. If $z = p_k$ all the terms in the equation vanish except C_{lk}, which can be evaluated. Next, differentiate the equation once with respect to z. The term C_{lk} will vanish but C_{l-1k} will remain without a multiplying function of z. Again C_{l-1k} can be evaluated by letting $z = p_k$. To find general term C_{ik}, differentiate Eq.(2.68) $(l-i)$ times and let $z = p_k$; then

$$C_{ik} = \frac{1}{(l-i)!}\frac{d^{l-i}}{dz^{l-i}}R(z)\Big|_{z=p_k}$$

or

$$C_{ik} = \frac{1}{(l-i)!}\frac{d^{l-i}}{dz^{l-i}}\left[(z-p_k)^l\frac{X(z)}{z}\right]\Big|_{z=p_k} \tag{2.69}$$

If $X(z)$ has complex poles then the partial fraction of the same can be expressed as

$$\frac{X(z)}{z} = \frac{C_1}{z-p_1} + \frac{C_1^\star}{z-p_1^\star} \tag{2.70}$$

where $C_1^\star$ is a complex conjugate of C_1 and $p_1^\star$ is a complex conjugate of p_1.
In other words, complex conjugate poles result in complex conjugate coefficients in the partial-fraction expansion.

Example 2.21 (a) Find the inverse z-transform of $X(z) = \dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$ $|z| > 2$ (b) Find the inverse z-transform of $X(z) = \dfrac{z(z^2-4z+5)}{(z-3)(z-1)(z-2)}$ for ROC (i) $2 < |z| < 3$ (ii) $|z| > 3$ (iii) $|z| < 1$ (Annamalai University Apr' 03)

Solution (a) Given $X(z) = \dfrac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$.

First we eliminate the negative power, by multiplying numerator and denominator by z^2

$$X(z) = \frac{z(z+3)}{z^2+3z+2} = \frac{z(z+3)}{(z+1)(z+2)}$$

Dividing $X(z)$ by z we obtain

$$\frac{X(z)}{z} = \frac{(z+3)}{(z+1)(z+2)}$$

The above equation can be written in partial fraction form as

$$\frac{X(z)}{z} = \frac{C_1}{z+1} + \frac{C_2}{z+2}$$

i.e.,

$$C_1 = (z+1)\frac{X(z)}{z}\bigg|_{z=-1}$$
$$= \cancel{(z+1)}\frac{z+3}{\cancel{(z+1)}(z+2)}\bigg|_{z=-1}$$
$$= 2$$
$$C_2 = (z+2)\frac{X(z)}{z}\bigg|_{z=-2}$$
$$= \cancel{(z+2)}\frac{z+3}{(z+1)\cancel{(z+2)}}\bigg|_{z=-2} = -1$$

Therefore,

$$\frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+2}$$
$$X(z) = 2\frac{z}{z+1} - \frac{z}{z+2}$$

As ROC is $|z| > 2$ the sequence is causal and by inspection we can find

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n)$$

(b) Given $X(z) = \dfrac{z(z^2-4z+5)}{(z-1)(z-2)(z-3)}$. Dividing, $X(z)$ by z we have

$$\frac{X(z)}{z} = \frac{z^2-4z+5}{(z-1)(z-2)(z-3)}$$
$$= \frac{C_1}{z-1} + \frac{C_2}{z-2} + \frac{C_3}{z-3}$$

where C_1, C_2, C_3 can be evaluated using Eq. (2.60).

$$C_1 = (z-1)\frac{X(z)}{z}\bigg|_{z=1} \Rightarrow C_1 = \cancel{(z-1)}\frac{z^2-4z+5}{\cancel{(z-1)}(z-2)(z-3)}\bigg|_{z=1} = 1$$

$$C_2 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} \Rightarrow C_2 = \cancel{(z-2)}\frac{z^2-4z+5}{(z-1)\cancel{(z-2)}(z-3)}\bigg|_{z=2} = -1$$

$$C_3 = (z-3)\frac{X(z)}{z}\bigg|_{z=3} \Rightarrow C_3 = \cancel{(z-3)}\frac{z^2-4z+5}{(z-1)(z-2)\cancel{(z-3)}}\bigg|_{z=3} = 1$$

Substituting C_1, C_2, C_3 values we have

$$\frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{z-3}$$

from which

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3} \tag{2.71}$$

(i) In case when the ROC is $2 < |z| < 3$, shown in Fig. 2.8, the signal $x(n)$ is two sided. The poles $z = 1$ and $z = 2$ provide the causal part and the pole $z = 3$ the anticausal part.

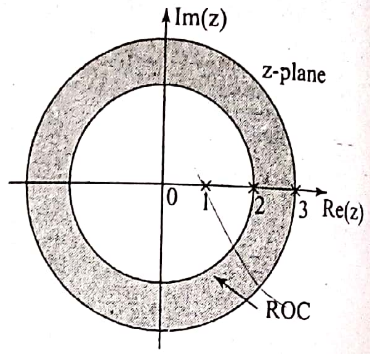


Fig. 2.8 ROC of example 2.21 b(i)

Thus, by inspection

$$x(n) = u(n) - (2)^n u(n) - (3)^n u(-n-1)$$

(ii) In case when ROC is $|z| > 3$, shown in Fig. 2.9, the signal $x(n)$ is causal and all the three terms in Eq. (2.71) are causal terms.

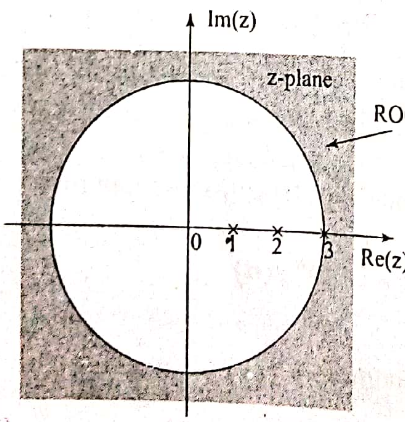


Fig. 2.9 ROC of example 2.21 b(ii).

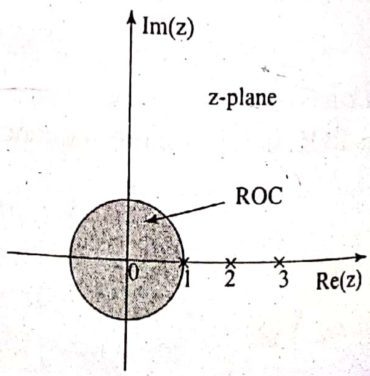


Fig. 2.10 ROC of example 2.21 b(iii).

Therefore,

$$x(n) = u(n) - (2)^n u(n) + (3)^n u(n)$$

(iii) In case when the ROC is $|z| < 1$ shown in Fig. 2.10, the signal $x(n)$ is anticausal and all the terms in Eq.(2.71) are anticausal terms. Therefore,

$$x(n) = -u(-n-1) + (2)^n u(-n-1) - (3)^n u(-n-1)$$
$$= [-1 + (2)^n - (3)^n]\, u(-n-1)$$

Example 2.22 Find the inverse z-transform of $X(z) = \dfrac{1}{2z^{-2} + 2z^{-1} + 1}$

Solution Given $X(z) = \dfrac{1}{2z^{-2} + 2z^{-1} + 1}$.

To eliminate negative powers multiply numerator and denominator with z^2. We get

$$X(z) = \frac{z^2}{z^2 + 2z + 2} \quad \text{and}$$
$$\frac{X(z)}{z} = \frac{z}{z^2 + 2z + 2}$$

The poles of $X(z)$ are complex conjugate given by $-1 \pm j$. If the poles are complex conjugate then $\frac{X(z)}{z}$ can be split into partial fraction as

$$\frac{X(z)}{z} = \frac{C_1}{z - (-1+j)} + \frac{C_1^\star}{z - (-1-j)}$$
$$= \frac{0.5 + j0.5}{z - (-1+j)} + \frac{0.5 - j0.5}{z - (-1-j)}$$
$$X(z) = \frac{(0.5 + j0.5)z}{z - (-1+j)} + \frac{(0.5 - j0.5)z}{z - (-1-j)}$$

$$\boxed{\begin{aligned} C_1 &= (z+1-j)\frac{X(z)}{z}\Big|_{z=-1+j} \\ &= (z+1-j)\frac{z}{z^2+2z+2}\Big|_{z=-1+j} \\ &= (z+1-j)\frac{z}{(z+1-j)(z+1+j)}\Big|_{z=-1+j} \\ &= \frac{-1+j}{2j} = 0.5 + j0.5 \end{aligned}}$$

Taking inverse z-transform we get

$$x(n) = (0.5 + j0.5)(-1+j)^n + (0.5 - j0.5)(-1-j)^n$$
$$= 2^{n/2}\left[(0.5 + j0.5)\left(-\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)^n + (0.5 - j0.5)\left(-\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^n\right]$$
$$= 2^{n/2}\left[(0.5 + j0.5)\left(\cos\frac{3n\pi}{4} + j\sin\frac{3n\pi}{4}\right)\right.$$
$$\left. +(0.5 - j0.5)\left(\cos\frac{3n\pi}{4} - j\sin\frac{3n\pi}{4}\right)\right]$$
$$= 2^{n/2}\left[\cos\frac{3n\pi}{4} - \sin\frac{3n\pi}{4}\right]$$
$$= 2^{n/2}2^{1/2}\left[\frac{1}{\sqrt{2}}\cos\frac{3n\pi}{4} - \frac{1}{\sqrt{2}}\sin\frac{3n\pi}{4}\right]$$
$$= 2^{(n+1)/2}\left[\cos\frac{\pi}{4}\cos\frac{3n\pi}{4} - \sin\frac{\pi}{4}\sin\frac{3n\pi}{4}\right]$$
$$= 2^{(n+1)/2}\left[\cos\left(\frac{3n\pi}{4} + \frac{\pi}{4}\right)\right] = 2^{(n+1)/2}\left[\cos(3n+1)\frac{\pi}{4}\right]$$

Example 2.23 Determine the causal signal $x(n)$ having the z-transform $X(z) = \dfrac{1}{(1-2z^{-1})(1-z^{-1})^2}$

Solution Given $X(z) = \dfrac{1}{(1-2z^{-1})(1-z^{-1})^2}$. Multiply numerator and denominator by z^3

$$X(z) = \frac{z^3}{(z-2)(z-1)^2}$$

Dividing the above result by z we have

$$\frac{X(z)}{z} = \frac{z^2}{(z-2)(z-1)^2} = \frac{C_1}{z-2} + \frac{C_2}{z-1} + \frac{C_3}{(z-1)^2}$$

$$C_1 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} = \cancel{(z-2)}\frac{z^2}{\cancel{(z-2)}(z-1)^2}\bigg|_{z=2} = 4$$

C_2 and C_3 can be found by using Eq. (2.63) and Eq. (2.64) respectively

$$C_2 = \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{X(z)}{z}\right]\bigg|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{\cancel{(z-1)^2}z^2}{(z-2)\cancel{(z-1)^2}}\right]\bigg|_{z=1} = \frac{(z-2)2z - z^2}{(z-2)^2}\bigg|_{z=1} = -3$$

$$C_3 = (z-1)^2\frac{X(z)}{z}\bigg|_{z=1} = \cancel{(z-1)^2}\frac{z^2}{(z-2)\cancel{(z-1)^2}}\bigg|_{z=1} = -1$$

Substituting C_1, C_2 and C_3 values

$$\frac{X(z)}{z} = \frac{4}{z-2} - \frac{3}{z-1} - \frac{1}{(z-1)^2} \quad \text{or} \quad X(z) = \frac{4z}{z-2} - \frac{3z}{z-1} - \frac{z}{(z-1)^2} \tag{2.72}$$

We can find inverse z-transform of the above Eq. (2.72) by referring table (2.1) which is equal to

$$x(n) = 4(2)^n u(n) - 3u(n) - nu(n)$$

Table 2.3 Inverse Transforms used for Partial Fraction Expansion Method.

S. NO.	$X(z)$	$x(n) = Z^{-1}[X(z)]$ for ROC: $\|z\| > \|a\|$
1.	$\frac{z}{z-a}$	$a^n u(n)$
2.	$\frac{z}{(z-a)^2}$	$na^{n-1}u(n)$
3.	$\frac{z}{(z-a)^3}$	$\frac{n(n-1)a^{n-2}u(n)}{2!}$
4.	$\frac{z}{(z-a)^4}$	$\frac{n(n-1)(n-2)}{3!}a^{n-3}u(n)$
.	.	.
.	.	.
.	.	.
k	$\frac{z}{(z-a)^k}$	$\frac{n(n-1)\ldots(n-(k-2))a^{n-k+1}}{(k-1)!}u(n)$

Example 2.24 Find the inverse z-transform of $X(z) = \frac{z^3+z^2}{(z-1)(z-3)}$ $ROC: |z| > 3$

Solution

$$\frac{X(z)}{z} = \frac{z^2+z}{(z-1)(z-3)}$$

Converting the above improper rational function ($\because M = N$) into sum of a constant and a proper rational function we get

$$\frac{X(z)}{z} = 1 + \frac{5z-3}{(z-1)(z-3)}$$

The rational expression can be expressed by partial fraction expansion

$$\frac{5z-3}{(z-1)(z-3)} = \frac{C_1}{z-1} + \frac{C_2}{z-3}$$

where

$$C_1 = \cancel{(z-1)}\frac{(5z-3)}{\cancel{(z-1)}(z-3)}\bigg|_{z=1} = -1$$

$$C_2 = \cancel{(z-3)}\frac{(5z-3)}{(z-1)\cancel{(z-3)}}\bigg|_{z=3} = 6$$

Therefore,

$$\frac{X(z)}{z} = 1 - \frac{1}{z-1} + \frac{6}{z-3}$$

$$X(z) = z - \frac{z}{z-1} + \frac{6z}{z-3}$$

Taking Inverse z-transform on both sides we get

$$x(n) = \delta(n+1) - u(n) + 6(3)^n u(n)$$

Practice Problem 2.12 Find inverse z-transform of

(i) $X(z) = \dfrac{1 + \frac{1}{2}z^{-1}}{1 + 3z^{-1} + 2z^{-2}}$ **Ans:** $\frac{-1}{2}(-1)^n u(n) + \frac{3}{2}(-2)^n u(n)$

(ii) $X(z) = \dfrac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}}$ **Ans:** $(n+1)\left(\frac{1}{2}\right)^n$

2.14.3 *Residue Method*

The z-transform of a sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiplying both sides by z^{k-1} and integrating with respect to z about a closed contour C in the region of convergence of $X(z)$ we have

$$\oint_C X(z) z^{k-1} dz = \oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \oint_C z^{-n+k-1} dz \tag{2.73}$$

Assume that C encloses the origin of the z-plane. Then by Cauchy residue theorem,

$$\oint_C z^{-n+k-1} dz = 2\pi j \delta_{kn} \tag{2.74}$$

where

$$\delta_{kn} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} \tag{2.75}$$

Hence $\displaystyle \oint_C X(z) z^{k-1} dz = 2\pi j \sum_{n=-\infty}^{\infty} x(n) \delta_{kn}$

$$= 2\pi j x(k)$$

or

$$x(k) = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz \tag{2.76}$$

Therefore, the inverse z-transform relation is given by contour integral

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{2.77}$$

where C is a circle in the z-plane in the ROC of $X(z)$. The Eq. (2.77) can be evaluated by finding sum of all residues of the poles that are inside the circle C. Now Eq. (2.77) can be written as

$$\begin{aligned} x(n) &= \sum [\text{residues of } \ X(z)z^{n-1} \ \text{ at the poles inside C}] \\ &= \sum_i (z - z_i) X(z) z^{n-1} \Big|_{z=z_i} \end{aligned} \tag{2.78}$$

If $X(z)z^{n-1}$ has no poles inside the contour C for one or more values of n then $x(n) = 0$ for these values. The residues of the poles can be found by using Cauchy residue theorem.

Cauchy residue theorem: Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $\frac{d}{dz} f(z)$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$ then

$$\begin{aligned} \frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz &= f(z_0) \quad \text{if} \quad z_0 \quad \text{is inside} \quad C \\ &= 0 \quad \text{if} \quad z_0 \quad \text{is outside} \quad C \end{aligned} \tag{2.79}$$

If $(k+1)^{\text{th}}$ order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$ then

$$\begin{aligned} \frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} dz &= \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} f(z) \Big|_{z=z_0} \quad \text{if} \quad z_0 \quad \text{is inside} \quad C \\ &= 0 \quad \text{if} \quad z_0 \quad \text{is outside} \quad C \end{aligned} \tag{2.80}$$

The values on the right hand side of Eq. (2.79) and Eq. (2.80) are called residues of the poles at $z = z_0$. If there are more than one pole inside the contour, the integral can be evaluated by adding up the results from each pole. For example if there are n unique poles $z_1, z_2, \ldots z_n$, and $f(z)$ contain no poles inside C, then,

$$\frac{1}{2\pi j} \oint_C \frac{f(z)dz}{(z - z_1)(z - z_2)\ldots(z - z_n)} = \sum_{i=1}^{n} \lim_{z \to z_i} [(z - z_i)\psi(z)] \tag{2.81a}$$

where $\psi(z) = \dfrac{f(z)}{(z - z_1)(z - z_2)\ldots(z - z_n)}$. (2.81b)

If any of poles z_i are of multiplicity greater than one, the formula given in Eq.(2.80) is used to determine their contribution to the integral while Eq.(2.81a) is used for simple poles.

Example 2.25 Using residue method find the inverse z-transform of $X(z) = \frac{z+1}{(z+0.2)(z-1)}$, $|z| > 1$

Solution From Eq.(2.77)

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

$$= \sum \text{ residues of } X(z)z^{n-1} \text{ at poles of } X(z)z^{n-1} \text{ within C}$$

$$= \sum \text{ residues of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at poles of same within C}$$

The closed contour C, begin in the ROC $|z| > 1$ encloses the poles at $z = -0.2$, $z = 1$ and, for $n = 0$ the pole is at $z = 0$. Therefore for $n = 0$

$$x(0) = \sum \text{ residues of } \frac{z+1}{z(z+0.2)(z+1)} \text{ at poles } z = 0, z = 1 \text{ and } z = -0.2$$

$$= (\cancel{z})\frac{z+1}{(\cancel{z})(z+0.2)(z-1)}\Big|_{z=0} + \cancel{(z+0.2)}\frac{(z+1)}{\cancel{(z+0.2)}(z-1)}\Big|_{z=0.2}$$

$$+ \cancel{(z-1)}\frac{(z+1)}{z(z+0.2)\cancel{(z-1)}}\Big|_{z=1}$$

$$= -5 + \frac{10}{3} + \frac{5}{3} = 0$$

i.e., $x(0) = 0$.

For $n \geq 1$

$$x(n) = \sum \text{ residues of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at poles } z = -0.2 \text{ and } z = 1$$

$$= \text{ residue of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at } z = -0.2$$

$$+ \text{ residue of } \frac{(z+1)z^{n-1}}{(z+0.2)(z-1)} \text{ at } z = 1$$

$$= \cancel{(z+0.2)}\frac{(z+1)z^{n-1}}{\cancel{(z+0.2)}(z-1)}\Big|_{z=-0.2} + \cancel{(z-1)}\frac{(z+1)z^{n-1}}{(z+0.2)\cancel{(z-1)}}\Big|_{z=1}$$

$$= -\frac{2}{3}(-0.2)^{n-1} + \frac{5}{3}$$

Therefore, $x(n) = -\frac{2}{3}(-0.2)^{n-1}u(n-1) + \frac{5}{3}u(n-1)$.

Example 2.26 Use the residue method to find the inverse z-transform of $X(z) = \dfrac{z}{(z-2)(z-3)}$ $\quad |z| < 2$

Solution In this case there are two poles $z = 3$ and $z = 2$ outside the ROC $|z| < 2$.

So the sequence is non-causal. For $n < 0$

$$\begin{aligned}
x(n) &= -\sum \text{ residues of } X(z)z^{n-1} \text{ at poles } z = 2 \text{ and } z = 3 \\
&= -\left[\cancel{(z-2)}\frac{z \cdot z^{n-1}}{\cancel{(z-2)}(z-3)}\bigg|_{z=2} + \cancel{(z-3)}\frac{z \cdot z^{n-1}}{(z-2)\cancel{(z-3)}}\bigg|_{z=3}\right] \\
&= -[-(2)^n + (3)^n] \\
&= (2)^n - (3)^n
\end{aligned}$$

For $n < 0$ $\quad x(n)$ can be written as

$$x(n) = [2^n - 3^n]u(-n-1)$$

Example 2.27 Using Cauchy integral method find the inverse z-transform of $X(z) = \dfrac{z}{(z-1)(z-2)}$ $\quad 1 < |z| < 2$

Solution The contour of integration C lies in the annular region of ROC, and the inverse z-transform is

$$\begin{aligned}
x(n) &= -\sum \text{ residue of } X(z)z^{n-1} \text{ at pole } z = 2 \text{ for } n < 0 \\
&= \sum \text{ residue of } X(z)z^{n-1} \text{ at pole } z = 1 \text{ for } n \geq 0
\end{aligned}$$

For $n < 0$

$$x(n) = -\cancel{(z-2)}\frac{z^n}{(z-1)\cancel{(z-2)}}\bigg|_{z=2} = -(2)^n$$

For $n \geq 0$

$$x(n) = \cancel{(z-1)}\frac{z^n}{\cancel{(z-1)}(z-2)}\bigg|_{z=1} = -(1)^n$$

Therefore

$$x(n) = -u(n) - 2^n u(-n-1)$$

Example 2.28 Find inverse Z-transform of $X(z) = \dfrac{2}{z\left(z-\frac{1}{2}\right)}$ For ROC: $|z| > \frac{1}{2}$ using residue method.

Solution We know

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

$$= \frac{1}{2\pi j}\oint_C \frac{2z^{n-1}}{z(z-\frac{1}{2})}dz$$

For $n \leq 0$ we can prove $x(n) = 0$. For $n = 0$

$$x(0) = \frac{1}{2\pi j}\oint_C \frac{2}{z^2\left(z-\frac{1}{2}\right)}dz$$

$$= \sum \text{ residues of } \frac{2}{z^2\left(z-\frac{1}{2}\right)} \text{ at poles } z=0 \text{ and } z=\frac{1}{2}$$

$$= \cancel{\left(z-\frac{1}{2}\right)}\frac{2}{z^2\cancel{\left(z-\frac{1}{2}\right)}}\Bigg|_{z=1/2} + \frac{d}{dz}\left[\cancel{z^2}\frac{2}{\cancel{z^2}\left(z-\frac{1}{2}\right)}\right]\Bigg|_{z=0}$$

$$= 8 + 2\left[-\frac{1}{\left(z-\frac{1}{2}\right)^2}\right]\Bigg|_{z=0} = 8-8=0$$

Similarly for $n=-1$

$$x(-1) = \frac{1}{2\pi j}\oint_C \frac{2}{z^3\left(z-\frac{1}{2}\right)}dz$$

$$= \cancel{\left(z-\frac{1}{2}\right)}\frac{2}{z^3\cancel{\left(z-\frac{1}{2}\right)}}\Bigg|_{z=1/2} + \frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{z^3}\frac{2}{\cancel{z^3}\left(z-\frac{1}{2}\right)}\right]\Bigg|_{z=0}$$

$$= 16 - 16 = 0$$

For $n \geq 1$

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{2z^{n-1}}{z\left(z-\frac{1}{2}\right)}dz$$

$$= \cancel{z}\frac{2z^{n-1}}{\cancel{z}\left(z-\frac{1}{2}\right)}\Bigg|_{z=0} + \cancel{\left(z-\frac{1}{2}\right)}\frac{2\cdot z^{n-1}}{z\cancel{\left(z-\frac{1}{2}\right)}}\Bigg|_{z=\frac{1}{2}}$$

$$= -4\delta(n-1) + 4\left(\frac{1}{2}\right)^{n-1} = -4\delta(n-1) + 4\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

Example 2.29 Using residue method find inverse z transform of

$$X(z) = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{9}z^{-2}} \qquad \text{ROC: } |z| > 1/3$$

Solution Given

$$X(z) = \frac{1-\dfrac{1}{4}z^{-1}}{1-\dfrac{1}{9}z^{-2}}$$

$$= \frac{z\left(z-\frac{1}{4}\right)}{z^2-\frac{1}{9}} = \frac{z\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)}$$

We know

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

For $n \geq 0$

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^n\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)}dz$$

$$= \sum \text{residues of} \frac{z^n\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)} \text{at poles } z = -\frac{1}{3} \text{ and } z = \frac{1}{3}$$

$$= \cancel{\left(z-\frac{1}{3}\right)}\frac{z^n\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\cancel{\left(z-\frac{1}{3}\right)}}\Bigg|_{z=\frac{1}{3}} + \cancel{\left(z+\frac{1}{3}\right)}\frac{z^n\left(z-\frac{1}{4}\right)}{\cancel{\left(z+\frac{1}{3}\right)}\left(z-\frac{1}{3}\right)}\Bigg|_{z=-\frac{1}{3}}$$

$$= \frac{1}{8}\left(\frac{1}{3}\right)^n + \frac{7}{8}\left(-\frac{1}{3}\right)^n$$

For $n < 0$ we can prove $x(n) = 0$. For $n = -1$

$$x(-1) = \frac{1}{2\pi j}\oint_C \frac{\left(z-\frac{1}{4}\right)}{z\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)}$$

$$= \sum \text{residues of} \frac{\left(z-\frac{1}{4}\right)}{z\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)} \text{at } z = 0, z = -\frac{1}{3} \text{ and } z = \frac{1}{3}$$

$$= \cancel{z}\frac{z-\frac{1}{4}}{\cancel{z}\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)}\Bigg|_{z=0} + \cancel{\left(z+\frac{1}{3}\right)}\frac{z-\frac{1}{4}}{z\left(z-\frac{1}{3}\right)\cancel{\left(z+\frac{1}{3}\right)}}\Bigg|_{z=-\frac{1}{3}}$$

$$+ \cancel{\left(z-\frac{1}{3}\right)}\frac{z-\frac{1}{4}}{z\left(z+\frac{1}{3}\right)\cancel{\left(z-\frac{1}{3}\right)}}\Bigg|_{z=\frac{1}{3}}$$

$$= \frac{9}{4} - \frac{63}{24} + \frac{9}{24}$$
$$= 0$$

Therefore, $x(n) = \frac{1}{8}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{8}\left(-\frac{1}{3}\right)^n u(n)$.

Practice Problem 2.13 Using residue method find the inverse z-transform of

(i) $X(z) = \dfrac{z(z^2-4z+5)}{(z-3)(z-1)(z-2)}$ **Ans:** $(3)^n u(n) + u(n) - (2)^n u(n)$

(ii) $X(z) = \dfrac{(z+1)}{z^2 - \frac{7z}{12} + \frac{1}{12}}$

Ans: $-15\left(\frac{1}{4}\right)^{n-1} u(n-1) + 16\left(\frac{1}{3}\right)^{n-1} u(n-1)$

2.14.4 *Convolution Method*

In this method the given $X(z)$ is split into $X_1(z)$ and $X_2(z)$ such that $X(z) = X_1(z)X_2(z)$. Next we find $x_1(n)$ and $x_2(n)$ by taking inverse z-transform of $X_1(z)$ and $X_2(z)$ respectively. From convolution property of z-transform we know

$$Z[x_1(n) * x_2(n)] = X_1(z)X_2(z) = X(z) \tag{2.82}$$

From the Eq. (2.82) we know that the convolution of $x_1(n)$ and $x_2(n)$ is the inverse z-transform of $X(z)$. Thus $x(n)$ can be obtained by convolving $x_1(n)$ and $x_2(n)$.

Example 2.30 Find the inverse z-transform of

$$X(z) = \frac{1}{1-3z^{-1}+2z^{-2}} \text{ using convolution method}$$

Solution Given

$$\begin{aligned} X(z) &= \frac{1}{1-3z^{-1}+2z^{-2}} \\ &= \frac{1}{(1-z^{-1})(1-2z^{-1})} = X_1(z)X_2(z) \end{aligned}$$

where $X_1(z) = \frac{1}{1-z^{-1}}$ and $X_2(z) = \frac{1}{1-2z^{-1}}$. From Table 2.1 we find $x_1(n) = u(n)$ and $x_2(n) = 2^n u(n)$. We know

$$\begin{aligned} x(n) &= x_1(n) * x_2(n) \\ &= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \\ &= \sum_{k=-\infty}^{\infty} u(k)2^{n-k}u(n-k) \\ &= \sum_{k=0}^{n} 2^{n-k} = 2^n \sum_{k=0}^{n} 2^{-k} \\ &= 2^n \left[\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} \right] u(n) \\ &= [2^{n+1}-1]u(n) \end{aligned}$$

Practice Problem 2.14 Find the inverse z-transform of $X(z) = \dfrac{z}{z^2+\frac{1}{12}z-\frac{1}{12}}$ using convolution method.

Ans: $\dfrac{-12}{7}\left(\dfrac{-1}{4}\right)^n u(n) + \dfrac{12}{7}\left(\dfrac{1}{3}\right)^n u(n)$

Example 2.31 Using Convolution method do the problem given in example 2.29.

Solution Given

$$X(z) = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{9}z^{-2}}$$

$$= \frac{z\left(z-\frac{1}{4}\right)}{z^2-\frac{1}{9}} = \frac{z\left(z-\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)}$$

$$= X_1(z)X_2(z)$$

where $X_1(z) = \frac{z}{z+\frac{1}{3}}$ and $X_2(z) = \frac{z-\frac{1}{4}}{z-\frac{1}{3}}$

$$= \frac{z}{z-\frac{1}{3}} - \frac{1}{4}z^{-1}\frac{z}{z-\frac{1}{3}}$$

Using Table (2.1) we find

$$x_1(n) = \left(-\frac{1}{3}\right)^n u(n)$$

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

We know

$$x(n) = x_1(n) * x_2(n)$$

$$= \left(-\frac{1}{3}\right)^n u(n) * \left[\left(\frac{1}{3}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{3}\right)^{n-1} u(n-1)\right]$$

$$= \left(-\frac{1}{3}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) - \frac{1}{4}\left(-\frac{1}{3}\right)^n u(n) * \left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$$= \sum_{k=0}^{n}\left(-\frac{1}{3}\right)^k\left(\frac{1}{3}\right)^{n-k} - \frac{1}{4}\sum_{k=0}^{n-1}\left(\frac{-1}{3}\right)^k\left(\frac{1}{3}\right)^{n-k-1}$$

$$= \left(\frac{1}{3}\right)^n\sum_{k=0}^{n}(-1)^k - \frac{1}{4}\left(\frac{1}{3}\right)^{n-1}\sum_{k=0}^{n-1}(-1)^k$$

$$= \left(\frac{1}{3}\right)^n\left[\frac{1+(-1)^n}{2}\right] - \frac{3}{4}\left(\frac{1}{3}\right)^n\left[\frac{1-(-1)^n}{2}\right]$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(-\frac{1}{3}\right)^n - \frac{3}{8}\left(\frac{1}{3}\right)^n + \frac{3}{8}\left(-\frac{1}{3}\right)^n$$

$$= \frac{1}{8}\left(\frac{1}{3}\right)^n + \frac{7}{8}\left(-\frac{1}{3}\right)^n \quad \text{for} \quad n \geq 0$$

or

$$x(n) = \frac{1}{8}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{8}\left(-\frac{1}{3}\right)^n u(n)$$

Example 2.32 Find inverse z-transform of $X(z) = \frac{z^2}{\left(z-\frac{1}{4}\right)^2}$ ROC: $|z| > \frac{1}{4}$ using (i) residue method (ii) convolution method.

Solution (i) Residue Method

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

For $n \geq 0$

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{\left(z - \frac{1}{4}\right)^2} dz$$

$$= \frac{1}{1!} \frac{d}{dz} \left[\left(z - \frac{1}{4}\right)^2 \frac{z^{n+1}}{\left(z - \frac{1}{4}\right)^2} \right] \Bigg|_{z=\frac{1}{4}}$$

$$= (n+1) \left(\frac{1}{4}\right)^n$$

For $n = -1$

$$x(-1) = \frac{1}{2\pi j} \oint_C \frac{1}{\left(z - \frac{1}{4}\right)^2} dz$$

$$= \frac{1}{1!} \frac{d}{dz} \left[\left(z - \frac{1}{4}\right)^2 \frac{1}{\left(z - \frac{1}{4}\right)^2} \right]$$

$$= 0$$

i.e., for $n < 0$, $x(n) = 0$.

Therefore,

$$x(n) = (n+1) \left(\frac{1}{4}\right)^n u(n)$$

(ii) Convolution Method

$$X(z) = \frac{z^2}{\left(z - \frac{1}{4}\right)^2} = \left(\frac{z}{z - \frac{1}{4}}\right) \left(\frac{z}{z - \frac{1}{4}}\right) = X_1(z) X_2(z)$$

where $X_1(z) = \frac{z}{z - \frac{1}{4}}$; $X_2(z) = \frac{z}{z - \frac{1}{4}}$. By inspection, we find $x_1(n) = x_2(n) = \left(\frac{1}{4}\right)^n u(n)$

$$x(n) = x_1(n) * x_2(n) \quad \text{for } n \geq 0$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{4}\right)^k \left(\frac{1}{4}\right)^{n-k}$$

$$\left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 1 = \left(\frac{1}{4}\right)^n (n+1) \qquad \boxed{\because \sum_{k=0}^{n} 1 = n+1}$$

Therefore, $x(n) = \left(\frac{1}{4}\right)^n (n+1)u(n)$.

Example 2.33 Find the inverse z-transform of $X(z) = \frac{z^2+z}{(z-1)(z-3)}$, ROC: $|z| > 3$. Using (a) Partial fraction expansion method (b) Residue method (c) Convolution Method.

Solution (a) Partial fraction expansion method

Given

$$X(z) = \frac{z^2+z}{(z-1)(z-3)} \quad \text{ROC: } |z| > 3$$

$$\frac{X(z)}{z} = \frac{z+1}{(z-1)(z-3)} = \frac{C_1}{z-1} + \frac{C_2}{z-3}$$

where

$$C_1 = \cancel{(z-1)} \left.\frac{(z+1)}{\cancel{(z-1)}(z-3)}\right|_{z=1} = -1$$

$$C_2 = \cancel{(z-3)} \left.\frac{(z+1)}{(z-1)\cancel{(z-3)}}\right|_{z=3} = 2$$

Therefore,

$$\frac{X(z)}{z} = \frac{-1}{z-1} + \frac{2}{z-3}$$

$$\text{or} \quad X(z) = -\frac{z}{z-1} + \frac{2z}{z-3}$$

Taking inverse z-transform we have

$$x(n) = -u(n) + 2(3)^n u(n)$$

(b) Residue Method.

We know

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

$$= \sum [\text{residues of } X(z)z^{n-1} \text{ at the poles inside } C]$$

For $n \geq 0$

$$x(n) = \sum \left[\text{residues of } \frac{(z+1)z^n}{(z-1)(z-3)} \text{ at the poles inside} |z| = 3\right]$$

$$= \cancel{(z-1)}\frac{(z+1)z^n}{\cancel{(z-1)}(z-3)}\bigg|_{z=1} + \cancel{(z-3)}\frac{(z+1)z^n}{(z-1)\cancel{(z-3)}}\bigg|_{z=3}$$

$$= -(1)^n + 2(3)^n$$

For $n < 0$ $X(z)z^{n+1}$ have poles of multiplicity 'r'.

For example, when $n = -1$, there is a pole at $z = 0$ and

$$x(-1) = \frac{1}{2\pi j}\oint_C \frac{z+1}{z(z-1)(z-3)}dz$$

$$= \frac{z(z+1)}{z(z-1)(z-3)}\bigg|_{z=0} + \frac{(z-1)(z+1)}{z(z-3)(z-1)}\bigg|_{z=1} + \frac{(z-3)(z+1)}{z(z+1)(z-3)}\bigg|_{z=3}$$

$$= \frac{1}{3} - 1 + \frac{2}{3} = 0$$

Similarly, we can prove $x(n) = 0$ for $n < -1$.

So $x(n) = -u(n) + 2(3)^n u(n)$.

(c) Convolution Method

Given $X(z) = \dfrac{z(z+1)}{(z-1)(z-3)}$. Let $X(z) = X_1(z)X_2(z)$ where

$$X_1(z) = \frac{z}{z-1}$$

$$X_2(z) = \frac{z+1}{z-3} = \frac{z}{z-3} + z^{-1}\frac{z}{z-3}$$

Then $x_1(n) = u(n)$

$$x_2(n) = (3)^n u(n) + 3^{n-1}u(n-1)$$

$$x(n) = x_1(n) * x_2(n)$$

$$= u(n) * [(3)^n u(n) + 3^{n-1}u(n-1)]$$

$$= u(n) * (3)^n u(n) + u(n) * 3^{n-1}u(n-1)$$

$$x(n) = \sum_{k=0}^{n} u(k)3^{n-k}u(n-k) + \sum_{k=0}^{n-1} u(k)3^{n-k-1}u(n-k-1)$$

For $n \geq 0$

$$x(n) = \sum_{k=0}^{n} 3^{n-k} + \sum_{k=0}^{n-1} 3^{n-k-1} = 3^n \sum_{k=0}^{n} 3^{-k} + 3^{n-1} \sum_{k=0}^{n-1} 3^{-k}$$

$$\because \sum_{k=0}^{n-1} a^{-k} = \sum_{k=0}^{n-1} \left(\frac{1}{a}\right)^k = 1 + a^{-1} + a^{-2} \cdots n \text{ terms} = \frac{1-a^{-n}}{1-a^{-1}}$$

$$= 3^n \left[\frac{1-3^{-n-1}}{2/3}\right] + 3^{n-1}\left[\frac{1-3^{-n}}{2/3}\right]$$

$$= \frac{3^{n+1}}{2}\left[1-3^{-n-1}\right] + \frac{3^n}{2}\left[1-3^{-n}\right]$$

$$= 2(3)^n u(n) - u(n)$$

2.15 Solution of difference equations using one sided z-transform

In chapter 1 we solved linear constant coefficient difference equations which involved finding the particular and homogeneous solution. Instead, we use the time shift property of one-sided z-transform to the difference equation so that we can solve the transform of the output. Then applying Inverse transform we can find the output sequence.

Example 2.34 Find the impulse response of the system described by difference equation $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$ using z-transform

Nov'05 JNTU (set 3)

Solution

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

Applying z-transform on both sides

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-3z^{-1}-4z^{-2}} = \frac{z(z+2)}{z^2-3z-4}$$

$$\frac{H(z)}{z} = \frac{z+2}{z^2-3z-4} = \frac{A}{z-4} + \frac{B}{z+1} = \frac{6}{5}\frac{1}{z-4} - \frac{1}{5}\frac{1}{z+1}$$

$$A = (z-4)\left.\frac{(z+2)}{(z-4)(z+1)}\right|_{z=4} = \frac{6}{5}$$

$$B = (z+1)\left.\frac{(z+2)}{(z-4)(z+1)}\right|_{z=-1} = -\frac{1}{5}$$

$$H(z) = \frac{6}{5}\frac{z}{z-4} - \frac{1}{5}\frac{z}{z+1}$$

$$h(n) = \frac{6}{5}(4)^n u(n) - \frac{1}{5}(-1)^n u(n)$$

Example 2.35 A causal LTI system is described by the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n-1),$$

where $x(n)$ is the input and $y(n)$ is the output. Find **May'06 JNTU**

(i) The system function $H(z) = \dfrac{Y(z)}{X(z)}$ for the system. Plot the poles and zeros of $H(z)$ and indicate the ROC.

(ii) Find the unit sample response of the system.

(iii) Is the system is stable or not?

Solution Apply z-transform on both sides

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1}$$

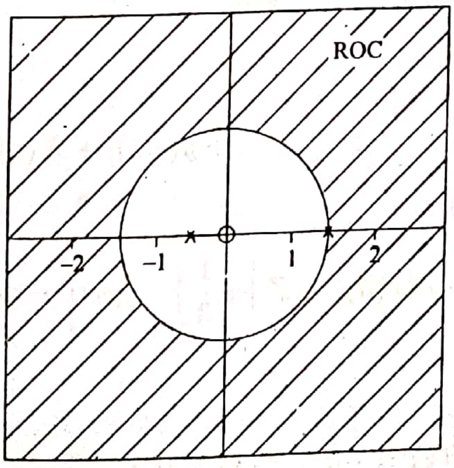


Fig. 2.11 Pole-zero pattern and ROC of example 2.35

The poles are at $\dfrac{1 \pm \sqrt{5}}{2}$ and zero at $z = 0$. The pole-zero diagram is shown in Fig. 2.11.

$$\frac{H(z)}{z} = \frac{1}{z^2 - z - 1}$$

$$\frac{H(z)}{z} = \frac{1}{(z+0.618)(z-1.618)}$$

$$= \frac{A}{z-1.618} + \frac{B}{z+0.618}$$

$$A = (z-1.618)\left.\frac{1}{(z+0.618)(z-1.618)}\right|_{z=1.618}$$

$$= 0.447$$

$$B = (z+0.618)\left.\frac{1}{(z-1.618)(z+0.618)}\right|_{z=-0.618}$$

$$= -0.447$$

$$H(z) = \frac{0.447z}{z-1.618} - \frac{0.447z}{z+0.618}$$

$$h(n) = 0.447(1.618)^n u(n) - 0.447(-0.618)^n u(n)$$

The pole $z = 1.618$ lies outside the unit circle hence the system is unstable.

Example 2.36 Use the one sided z-transform to determine $y(n), n \geq 0$ if

$$y(n) = \frac{1}{2}y(n-1) + x(n)$$

$$x(n) = \left(\frac{1}{3}\right)^n u(n); \quad y(-1) = 1$$

Solution Given $y(n) = \frac{1}{2}y(n-1) + x(n)$.

Taking z-transform on both sides

$$Y(z) = \frac{1}{2}[z^{-1}Y(z) + y(-1)] + X(z)$$

From the given $y(-1) = 1$ and

$$X(z) = Z[x(n)] = Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = \frac{z}{z - \frac{1}{3}}$$

we get

$$Y(z) = \frac{1}{2}[z^{-1}Y(z) + 1] + \frac{z}{z - \frac{1}{3}}$$

$$= 0.5z^{-1}Y(z) + 0.5 + \frac{z}{z - \frac{1}{3}}$$

$$(1-0.5z^{-1})Y(z) = 0.5 + \frac{z}{z-\frac{1}{3}}$$

$$Y(z) = \frac{0.5}{1-0.5z^{-1}} + \frac{z}{\left(z-\frac{1}{3}\right)(1-0.5z^{-1})}$$

$$= \frac{0.5z}{(z-0.5)} + \frac{z^2}{\left(z-\frac{1}{3}\right)(z-0.5)}$$

$$\frac{Y(z)}{z} = \frac{0.5}{z-0.5} + \frac{z}{\left(z-\frac{1}{3}\right)(z-0.5)}$$

$$\frac{Y(z)}{z} = \frac{0.5}{z-0.5} + \frac{3}{z-0.5} - \frac{2}{z-\frac{1}{3}}$$

$$Y(z) = \frac{0.5z}{z-0.5} + \frac{3z}{z-0.5} - \frac{2z}{z-\frac{1}{3}}$$

Taking inverse z-transform we get

$$\begin{aligned} y(n) &= [0.5(0.5)^n + 3(0.5)^n - 2(1/3)^n]u(n) \\ &= [3.5(0.5)^n - 2(1/3)^n]u(n) \end{aligned}$$

Example 2.37 Determine the unit step response of the system whose difference equation is $y(n)-0.7y(n-1)+0.12y(n-2) = x(n-1)+x(n-2)$ if $y(-1) = y(-2) = 1$.

Solution Given

$$y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$$

Taking z-transforms on both sides we have

$$\begin{aligned} &Y(z) - 0.7[z^{-1}Y(z) + y(-1)] + 0.12[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] \\ &= z^{-1}X(z) + x(-1) + z^{-2}X(z) + z^{-1}x(-1) + x(-2) \end{aligned}$$

Substituting initial conditions $y(-1) = y(-2) = 1$ and $x(-1) = x(-2) = 0$

$\boxed{\because \text{ For a step input } x(-1) = x(-2) = 0}$

$$Y(z) - 0.7[z^{-1}Y(z) + 1] + 0.12[z^{-2}Y(z) + z^{-1} + 1] = z^{-1}X(z) + z^{-2}X(z)$$

For unit step input $X(z) = \dfrac{1}{1-z^{-1}}$

$$Y(z)[1 - 0.7z^{-1} + 0.12z^{-2}] - 0.58 + 0.12z^{-1} = \frac{z^{-1}}{1-z^{-1}} + \frac{z^{-2}}{1-z^{-1}}$$

$$Y(z) = \frac{z^{-1} + z^{-2}}{(1-z^{-1})(1-0.7z^{-1}+0.12z^{-2})} + \frac{0.58 - 0.12z^{-1}}{1-0.7z^{-1}+0.12z^{-2}}$$

$$= \frac{z(z+1)}{(z-1)(z-0.4)(z-0.3)} + \frac{0.58z^2 - 0.12z}{(z-0.4)(z-0.3)}$$

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z-1)(z-0.4)(z-0.3)} + \frac{0.58z - 0.12}{(z-0.4)(z-0.3)}$$

$$= \frac{4.762}{z-1} - \frac{23.33}{z-0.4} + \frac{18.57}{z-0.3} + \frac{1.12}{z-0.4} - \frac{0.54}{z-0.3}$$

Multiplying both sides by z and taking inverse z-transform we have
$y(n) = 4.76u(n) - 22.21(0.4)^n u(n) + 18.03(0.3)^n u(n)$.

Example 2.38 Compute the response of the system $y(n) = 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$ to input $x(n) = nu(n)$. Is the system is stable?

Solution Given the difference equation

$$y(n) = 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$$

Taking z-transform on both sides and substituting initial conditions to zero

$$Y(z) = 0.7z^{-1}Y(z) - 0.12z^{-2}Y(z) + z^{-1}X(z) + z^{-2}X(z)$$

Now

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}(1+z^{-1})}{1-0.7z^{-1}+0.12z^{-2}}$$

$$= \frac{z+1}{z^2 - 0.7z + 0.12}$$

$$= \frac{z+1}{(z-0.4)(z-0.3)}$$

The poles of the system function are $z_1 = 0.4; z_2 = 0.3$ and the region of convergence is $|z| > 0.4$.

The pole-zero pattern is shown in Fig. 2.12.
The poles are lying inside the unit circle. So, the system is stable.
For the input $x(n) = nu(n)$

$$X(z) = \frac{z}{(z-1)^2}$$

$$\frac{Y(z)}{X(z)} = \frac{z+1}{(z-0.4)(z-0.3)}$$

$$Y(z) = \frac{(z+1)z}{(z-1)^2(z-0.4)(z-0.3)}$$

$$\frac{Y(z)}{z} = \frac{z+1}{(z-1)^2(z-0.4)(z-0.3)}$$

$$= \frac{c_1}{z-0.4} + \frac{c_2}{z-0.3} + \frac{c_3}{z-1} + \frac{c_4}{(z-1)^2}$$

$$= \frac{38.89}{z-0.4} - \frac{26.53}{z-0.3} - \frac{12.36}{z-1} + \frac{4.76}{(z-1)^2}$$

$$Y(z) = \frac{38.89z}{z-0.4} - \frac{26.53z}{z-0.3} - \frac{12.36z}{z-1} + \frac{4.76z}{(z-1)^2}$$

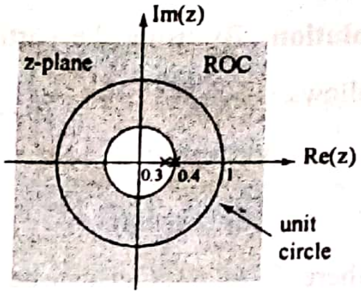


Fig. 2.12 Pole-zero pattern and ROC of example 2.38

Taking inverse z-transform we obtain

$$y(n) = 38.89(0.4)^n u(n) - 26.53(0.3)^n u(n) - 12.36u(n) + 4.76nu(n)$$

Practice Problem 2.15 Determine the impulse response of the system described by the difference equation

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Ans: $y(n) = \left[\frac{3}{5}\left(\frac{1}{2}\right)^n + \frac{2}{5}\left(-\frac{1}{3}\right)^n\right]u(n).$

Practice Problem 2.16 Find the step response of the system given in practice problem 2.15

Ans: $-\frac{3}{5}\left(\frac{1}{2}\right)^n u(n) + \frac{1}{10}\left(-\frac{1}{3}\right)^n u(n) + \frac{3}{2}u(n)$

Practice Problem 2.17 Find the step response of the following difference equation

$$y(n) - 5y(n-1) + 6y(n-2) = x(n)$$

Ans: $\left[-4(2)^n + \frac{9}{2}(3)^n + \frac{1}{2}\right]u(n)$

Example 2.39 Find the inverse z-transform of $X(z) = \frac{z(z+1)}{(z-1)^3(z-2)}$ ROC: $|z| > 2$ using partial fraction expansion method.

Solution By using the partial fraction expansion method $\frac{X(z)}{z}$ can be written as follows

$$\frac{X(z)}{z} = \frac{z+1}{(z-1)^3(z-2)} = \frac{C_{11}}{z-1} + \frac{C_{12}}{(z-1)^2} + \frac{C_{13}}{(z-1)^3} + \frac{C_4}{z-2}$$

where C_{11}, C_{12}, C_{13} can be calculated using Eq. (2.69) and C_4 can be calculated using Eq. (2.60).

$$C_{11} = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z+1}{z-2}\right)\bigg|_{z=1} = -3$$

$$C_{12} = \frac{1}{1!}\frac{d}{dz}\left(\frac{z+1}{z-2}\right)\bigg|_{z=1} = -3$$

$$C_{13} = \frac{z+1}{z-2}\bigg|_{z=1} = -2$$

$$C_4 = \frac{z+1}{(z-1)^3}\bigg|_{z=2} = 3$$

$$\frac{X(z)}{z} = \frac{-3}{z-1} - \frac{3}{(z-1)^2} - \frac{2}{(z-1)^3} + \frac{3}{z-2}$$

Therefore, $X(z) = \frac{-3z}{z-1} - \frac{3z}{(z-1)^2} - \frac{2z}{(z-1)^3} + \frac{3z}{z-2}$.

Using Table 2.3 the inverse z-transform can be found as

$$\begin{aligned} x(n) &= -3u(n) - 3nu(n) - \frac{2n(n-1)}{2}u(n) + 3(2)^n u(n) \\ &= -3u(n) - 3nu(n) - n(n-1)u(n) + 3(2)^n u(n) \end{aligned}$$

2.16 Deconvolution Using z-transform

In section 1.28 we studied about deconvolution, which is used to find the input $x(n)$ applied to the system once the impulse response $h(n)$ and the output $y(n)$ of the system are known. The z-transform also can be used for deconvolution operation. We have the relation

$$Y(z) = X(z)H(z) \tag{2.83}$$

where $Y(z), X(z)$ and $H(z)$ are the z-transforms of system's output, input and impulse response respectively. Using Eq. (2.83) we can write

$$X(z) = \frac{Y(z)}{H(z)} \tag{2.84}$$

If we know the impulse response $h(n)$ and the output $y(n)$ of the system, we can compute their z-transforms $H(z)$ and $Y(z)$. Using Eq. (2.84) we can obtain $X(z)$. Now one can easily find the input sequence $x(n)$ by finding inverse z-transform of $X(z)$. Thus, the deconvolution is reduced to the procedure of evaluating an inverse z-transform.

Example 2.40 Find the input $x(n)$ of the system, if the impulse response $h(n)$ and the output $y(n)$ are as shown below.

$$h(n) = \{\underset{\uparrow}{1}, 2, 3, 2\}$$

$$y(n) = \{\underset{\uparrow}{1}, 3, 7, 10, 10, 7, 2\}$$

Solution Given $h(n) = \{\underset{\uparrow}{1}, 2, 3, 2\} \Rightarrow h(0) = 1; \quad h(1) = 2; \quad h(2) = 3;$ $h(3) = 2$

$$\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\
&= \sum_{n=0}^{3} h(n)z^{-n} \\
&= 1 + 2z^{-1} + 3z^{-3} + 2z^{-3} \\
Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\
&= \sum_{n=0}^{6} y(n)z^{-n} \\
&= 1 + 3z^{-1} + 7z^{-2} + 10z^{-3} + 10z^{-4} + 7z^{-5} + 2z^{-6} \\
X(z) &= \frac{Y(z)}{H(z)} \\
&= \frac{1 + 3z^{-1} + 7z^{-2} + 10z^{-3} + 10z^{-4} + 7z^{-5} + 2z^{-6}}{1 + 2z^{-1} + 3z^{-2} + 2z^{-3}}
\end{aligned}$$

$$
\begin{array}{r|l}
 & 1+z^{-1}+2z^{-2}+z^{-3} \\
\hline
1+2z^{-1}+3z^{-2}+2z^{-3} & 1+3z^{-1}+7z^{-2}+10z^{-3}+10z^{-4}+7z^{-5}+2z^{-6} \\
 & 1+2z^{-1}+3z^{-2}+2z^{-3} \\
\cline{2-2}
 & z^{-1}+4z^{-2}+8z^{-3}+10z^{-4} \\
 & z^{-1}+2z^{-2}+3z^{-3}+2z^{-4} \\
\cline{2-2}
 & 2z^{-2}+5z^{-3}+8z^{-4}+7z^{-5} \\
 & 2z^{-2}+4z^{-3}+6z^{-4}+4z^{-5} \\
\cline{2-2}
 & z^{-3}+2z^{-4}+3z^{-5}+2z^{-6} \\
 & z^{-3}+2z^{-4}+3z^{-5}+2z^{-6} \\
\cline{2-2}
 & 0
\end{array}
$$

$$\Rightarrow X(z) = 1 + z^{-1} + 2z^{-2} + z^{-3}$$

$$x(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$$

Example 2.41 Determine the z-transform of the following

(i) $x(n) = n(-1)^n u(n)$ (ii) $x(n) = n^3 u(n)$

(iii) $x(n) = (-1)^n \cos\left(\frac{\pi}{3}n\right) u(n)$

Solution (i) $Z[(-1)^n u(n)] = \sum_{n=0}^{\infty}(-1)^n z^{-n} = \sum_{n=0}^{\infty}(-z^{-1})^n = \frac{z}{z+1}.$

By using differentiation property

$$Z[n(-1)^n u(n)] = -z\frac{d}{dz}\left(\frac{z}{z+1}\right) = \frac{-z}{(z+1)^2}$$

(ii) $x(n) = n^2 u(n)$

$$Z[u(n)] = \frac{z}{z-1}$$

Using differentiation property

$$Z[nu(n)] = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

Similarly

$$Z[n^2 u(n)] = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = -z\left\{\frac{(-z-1)(z-1)}{(z-1)^4}\right\} = \frac{z(z+1)}{(z-1)^3}$$

(iii) $x(n) = (-1)^n \cos\left(\frac{n\pi}{3}\right) u(n)$

$$= (-1)^n \left[\frac{e^{j(\pi/3)n} + e^{-j(\pi/3)n}}{2}\right] u(n)$$

$$X(z) = \sum_{n=0}^{\infty} \frac{(-1)^n e^{j(n\pi/3)} z^{-n}}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n e^{-jn\pi/3} z^{-n}}{2}$$

$$= \sum_{n=0}^{\infty} \frac{(-e^{j\pi/3} z^{-1})^n}{2} + \sum_{n=0}^{\infty} \frac{(-e^{-j\pi/3} z^{-1})^n}{2} \qquad \boxed{\because \sum_{n=0}^{\infty} (-a)^n = \frac{1}{1+a}}$$

$$= \frac{1}{2}\left[\frac{1}{1+e^{j\pi/3}z^{-1}} + \frac{1}{1+e^{-j\pi/3}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z(2z + e^{j\pi/3} + e^{-j\pi/3})}{z^2 + z(e^{j\pi/3} + e^{-j\pi/3}) + 1}\right] = \frac{z(z+0.5)}{z^2+z+1}$$

Example 2.42 Determine the z-transform and sketch the ROC of the following signal

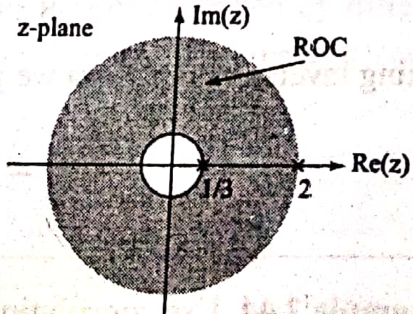
$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n, & n \geq 0 \\ \left(\frac{1}{2}\right)^{-n}, & n < 0 \end{cases}$$

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}z\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3}z^{-1}\right)^n$$

The first power series converges for $\left|\frac{z}{2}\right| < 1$, i.e., $|z| < 2$ and the second power series converges for $\left|\frac{1}{3}z^{-1}\right| < 1$, i.e., $|z| > 1/3$.



$$X(z) = \frac{-z}{z-2} + \frac{z}{z-1/3} \text{ and ROC is } \frac{1}{3} < |z| < 2$$

Fig. 2.13 ROC of example 2.42

Example 2.43 Determine the convolution of the pairs of signals by means of the z-transform $x_1(n) = \left(\frac{1}{2}\right)^n u(n); x_2(n) = \cos \pi n u(n)$

Solution From convolution theorem of z-transform, we have

$$Z[x_1(n) * x_2(n)] = X_1(z)X_2(z)$$

Therefore $x_1(n) * x_2(n) =$ Inverse z-transform of $[X_1(z)X_2(z)]$.

Given $x_1(n) = \left(\frac{1}{2}\right)^n u(n); x_2(n) = \cos \pi n u(n) = (-1)^n u(n)$.

From Table (2.1) we can find $X_1(z) = \dfrac{z}{z - \frac{1}{2}}$ and

$$X_2(z) = \frac{z}{z+1}$$

Let

$$X(z) = X_1(z)X_2(z) = \frac{z^2}{\left(z - \frac{1}{2}\right)(z+1)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-0.5)(z+1)}$$

$$= \frac{C_1}{z-0.5} + \frac{C_2}{z+1}$$

where

$$C_1 = \cancel{(z-0.5)} \frac{z}{\cancel{(z-0.5)}(z+1)}\bigg|_{z=0.5} = \frac{1}{3}$$

$$C_2 = \cancel{(z+1)} \frac{z}{(z-0.5)\cancel{(z+1)}}\bigg|_{z=-1} = \frac{2}{3}$$

$$\frac{X(z)}{z} = \frac{1/3}{z-0.5} + \frac{2/3}{z+1}$$

$$X(z) = \frac{1}{3}\frac{z}{z-0.5} + \frac{2}{3}\frac{z}{z+1}$$

Taking inverse z-transform we get

$$x(n) = \frac{1}{3}(0.5)^n u(n) + \frac{2}{3}(-1)^n u(n)$$

Example 2.44 Use convolution to find $x(n)$ if $X(z)$ is given by

$$X(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}$$

Solution Given

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$
$$= X_1(z)X_2(z)$$

where $X_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$ and $X_2(z) = \dfrac{1}{1+\frac{1}{4}z^{-1}}$.

By inspection we find that

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n) = \left(-\frac{1}{4}\right)^n u(n)$$

We know

$$Z[x(n)] = Z[x_1(n) * x_2(n)] = X_1(z)X_2(z)$$

Therefore

$$x(n) = x_1(n) * x_2(n)$$
$$= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$
$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{4}\right)^{n-k} \qquad \boxed{\sum_{k=0}^{n} a^k = \frac{a^{(n+1)}-1}{a-1}}$$
$$= \left(\frac{-1}{4}\right)^n \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{-1}{4}\right)^{-k} = \left(\frac{-1}{4}\right)^n \sum_{k=0}^{n}(-2)^k$$
$$= \left(-\frac{1}{4}\right)^n\left[\frac{(-2)^{n+1}-1}{-3}\right]u(n)$$
$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n\right]u(n)$$

Example 2.45 Determine the impulse response of the system described by difference equation $y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$. Plot the pole zero pattern and discuss on stability.

Solution Given difference equation

$$y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$$

Taking z-transform on both sides (assuming initial conditions are zero)

$$Y(z) = z^{-1}Y(z) - 0.5z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

The system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} = \frac{z(z+1)}{z^2-z+0.5}$$

For the impulse input $x(n) = \delta(n); X(z) = 1$ which gives us

$$Y(z) = \frac{z(z+1)}{z^2-z+0.5}$$

Now

$$\frac{Y(z)}{z} = \frac{z+1}{z^2-z+0.5} = \frac{A}{z-(0.5+j0.5)} + \frac{A^*}{z-(0.5-j0.5)}$$

$$= \frac{0.5-j1.5}{z-(0.5+j0.5)} + \frac{(0.5+j1.5)}{z-(0.5-j0.5)}$$

A^* is a complex conjugate of A

$$Y(z) = \frac{(0.5-j1.5)z}{z-(0.5+j0.5)} + \frac{(0.5+j1.5)z}{z-(0.5-j0.5)}$$

Taking inverse z-transform on both sides

$$\begin{aligned}
y(n) &= (0.5-j1.5)(0.5+j0.5)^n + (0.5+j1.5)(0.5-j0.5)^n \\
&= (0.5-j1.5)(0.5)^n(1+j1)^n + (0.5+j1.5)(0.5)^n(1-j1)^n \\
&= (0.5)^n(\sqrt{2})^n\left[(0.5-j1.5)\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)^n \right. \\
&\qquad \left. + (0.5+j1.5)\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)^n\right] \\
&= (0.5)^{n/2}\left[(0.5-j1.5)\left(\cos\frac{\pi}{4}+j\sin\frac{\pi}{4}\right)^n \right. \\
&\qquad \left. + (0.5+j1.5)\left(\cos\frac{\pi}{4}-j\sin\frac{\pi}{4}\right)^n\right] \\
&= (0.5)^{n/2}\left[(0.5-j1.5)\left(\cos\frac{n\pi}{4}+j\sin\frac{n\pi}{4}\right) \right. \\
&\qquad \left. + (0.5+j1.5)\left(\cos\frac{n\pi}{4}-j\sin\frac{n\pi}{4}\right)\right] \\
&= (0.5)^{n/2}\left[\cos\frac{n\pi}{4}+3\sin\frac{n\pi}{4}\right] \\
&= (0.5)^{n/2}\sqrt{10}\left[\frac{1}{\sqrt{10}}\cos\frac{n\pi}{4}+\frac{3}{\sqrt{10}}\sin\frac{n\pi}{4}\right] \\
&= (0.5)^{n/2}\sqrt{10}\left[\cos\frac{n\pi}{4}\cos\theta+\sin\frac{n\pi}{4}\sin\theta\right] \\
&= (0.5)^{n/2}\sqrt{10}\left[\cos\left(\frac{n\pi}{4}-\theta\right)\right] \qquad \theta = \tan^{-1}3 = 71.565^\circ \\
&= (0.5)^{n/2}\sqrt{10}\cos\left(\frac{n\pi}{4}-71.565^\circ\right)u(n)
\end{aligned}$$

The pole-zero pattern is shown in Fig. 2.14.

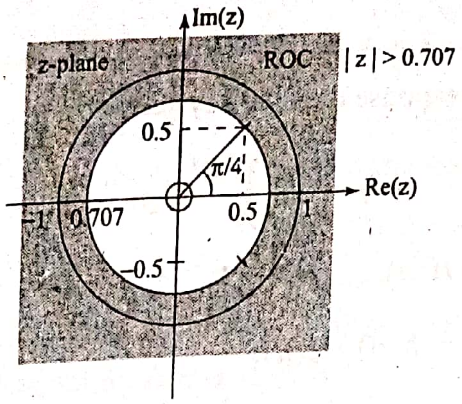


Fig. 2.14 Pole-zero pattern of Example 2.45.

There are two zeros at $z_1 = 0$ and $z_2 = -1$ and two poles at $z_3 = 0.5+j0.5$; $z_4 = 0.5 - j0.5$. Both poles are lying inside the unit circle. So the system is stable.

Example 2.46 Discuss the stability of the system described by

$$H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution The system with transfer function $H(z)$ is stable if $H(z)$ contains no poles on or outside the unit circle $|z| = 1$. In the above transfer function the poles are $z = -1/2, 1/2, 1/4$. That is all poles are inside the unit circle $|z| = 1$. So the system is stable.

Example 2.47 Find the poles of the system

$$y(n) - \frac{1}{4}y(n-1) + \frac{1}{4}y(n-2) - \frac{1}{16}y(n-3) = 2x(n) + 3x(n-1)$$

and determine whether the system is stable.

Solution Given

$$y(n) - \frac{1}{4}y(n-1) + \frac{1}{4}y(n-2) - \frac{1}{16}y(n-3) = 2x(n) + 3x(n-1)$$

Taking z-transform on both sides we get

$$Y(z) - \frac{1}{4}z^{-1}Y(z) + \frac{1}{4}z^{-2}Y(z) - \frac{1}{16}z^{-3}Y(z) = 2X(z) + 3z^{-1}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{2+3z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{1}{4}z^{-2}-\frac{1}{16}z^{-3}} = \frac{2+3z^{-1}}{\left(1+\frac{1}{4}z^{-2}\right)\left(1-\frac{1}{4}z^{-1}\right)}.$$

The poles are $z = \frac{1}{4}$ and $z = \pm\frac{j}{2}$. Both poles lie inside the unit circle $|z| = 1$. So the system is stable.

Example 2.48 The step response of an LTI system is

$$s(n) = \left(\frac{1}{3}\right)^{n-1} u(n+2)$$

Find the system function $H(z)$. JNTU April 2002

Solution We have $s(n) = h(n) * u(n)$

$$S(z) = H(z)\frac{z}{z-1} \qquad Z[u(n)] = \frac{z}{z-1}$$

Given

$$s(n) = \left(\frac{1}{3}\right)^{n-2} u(n+2)$$

$$S(z) = \sum_{n=-2}^{\infty} \left(\frac{1}{3}\right)^{n-2} z^{-n}$$

$$= 3^2 \sum_{n=-2}^{\infty} \left(\frac{1}{3z}\right)^{n}$$

$$= 3^2 \frac{\left(\frac{1}{3z}\right)^{-2}}{1 - \frac{1}{3z}} = \frac{3^4 z^2}{1 - \frac{1}{3}z^{-1}}$$

$$S(z) = \frac{81z^3}{\left(z - \frac{1}{3}\right)}$$

$$H(z) = \frac{S(z)(z-1)}{z}$$

$$= \frac{81z^3}{\left(z - \frac{1}{3}\right)} \cdot \frac{z-1}{z}$$

$$= \frac{81z^2(z-1)}{\left(z - \frac{1}{3}\right)}$$

$$= \frac{81z^3}{z - \frac{1}{3}} - \frac{81z^2}{z - \frac{1}{3}}$$

$$= 81z^2 \cdot \frac{z}{z - \frac{1}{3}} - 81z\frac{z}{z - \frac{1}{3}}$$

That is $|H(e^{j\omega})| = \frac{1}{a}$ for all values of ω. Therefore the system is an all pass system.

Example 2.50 The autocorrelation sequence $\gamma_{xx}(n)$ of a sequence $x(n)$ is defined as

$$\gamma_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)x(n+k)$$

Determine the z-transform of $\gamma_{xx}(n)$ in terms of the z-transform of $x(n)$.

Solution

$$Z[\gamma_{xx}(n)] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)x(n+k)z^{-n}$$

$$\Rightarrow \Gamma_{xx}(z) = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} x(n+k)z^{-n} \qquad \text{Let } \begin{aligned} n+k &= l \\ n &= l-k \end{aligned}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} x(l)z^{-(l-k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} x(l)z^{-l}z^{k}$$

$$= X(z) \sum_{k=-\infty}^{\infty} x(k)(z^{-1})^{-k}$$

$$= X(z)X(z^{-1})$$

Example 2.51 Consider an LTI system critically at rest described by the difference equation

$$y(n) = \frac{1}{4}y(n-2) + x(n)$$

Determine the impulse response of the system.

Solution Given $y(n) = \frac{1}{4}y(n-2) + x(n)$.

Taking z-transform on both sides

$$Y(z) = \frac{1}{4}z^{-2}Y(z) + X(z)$$

$$\Rightarrow \quad H(z) = \frac{1}{1-\frac{1}{4}z^{-2}} = \frac{z^2}{z^2 - \frac{1}{4}}$$

$$= \frac{1}{2}\frac{z}{z+\frac{1}{2}} + \frac{1}{2}\frac{z}{z-\frac{1}{2}}$$

$$h(n) = \left[\frac{1}{2}\left(-\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{2}\right)^n\right]u(n)$$

Example 2.52 Test the stability of first order IIR filter governed by the equation

$$y(n) = x(n) + by(n-1) \quad \text{where } |b| < 1$$

Solution

$$Y(z) = \frac{X(z)}{1 - bz^{-1}}$$

$$\Rightarrow \quad H(z) = \frac{z}{z-b}$$

Since $|b| < 1$, the pole lies inside the unit circle. The system is stable.

Example 2.53 Determine the z-transform of the following. PU Nov. 2002

(i) $x(n) = \{3, 1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$ (ii) $x(n) = \delta(n)$

(iii) $x(n) = \delta(n-k)$ (iv) $x(n) = \delta(n+k)$

Solution

(i) $X(z) = 3z^3 + z^2 + 2z + 5 + 7z^{-1} + z^{-3}$

(ii) $X(z) = 1$ (iii) $X(z) = z^{-k}$ (iv) $X(z) = z^k$

Example 2.54 Find the z-transform of $x(n) = \cos\omega_0 n$ for $n \geq 0$.

PU Nov. 2002 EIE, AU Apr. 2005 EEE

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos\omega_0 n z^{-n} = \sum_{n=0}^{\infty}\left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right]z^{-n}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right] = \frac{1-\cos\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1} + z^{-2}}$$

Example 2.55 Determine the inverse z-transform by the partial fraction expansion method.

PU Nov. 2002 EIE

$$X(z) = \frac{z+2}{2z^2 - 7z + 3} \quad \text{if the ROC are}$$

(i) $|z| > 3$ (ii) $|z| < \frac{1}{2}$ and $\frac{1}{2} < |z| < 3$

Solution

$$X(z) = \frac{z+2}{2(z-3)(z-\frac{1}{2})}$$
$$= \frac{1}{z-3} - \frac{1}{2}\frac{1}{z-\frac{1}{2}}$$

If ROC is $|z| > 3$

$$x(n) = (3)^{n-1}u(n-1) - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

If ROC is $|z| < \frac{1}{2}$

$$x(n) = -(3)^{n-1}u(-n) + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(-n)$$

If ROC is $\frac{1}{2} < |z| < 3$

$$x(n) = -(3)^{n-1}u(-n) - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Example 2.56 Determine the system function $H(z)$, impulse response $h(n)$ of the LTI system defined by the difference equation

$$y(n) = x(n) + 3x(n-1) + 2y(n-1) - y(n-2)$$

Solution Given

$$y(n) = x(n) + 3x(n-1) + 2y(n-1) - y(n-2)$$

Taking z-transform on both sides and finding system function

$$\frac{Y(z)}{X(z)} = \frac{1+3z^{-1}}{1-2z^{-1}+z^{-2}} = \frac{z(z+3)}{(z-1)^2}$$

$$\Rightarrow \quad H(z) = \frac{z}{z-1} + \frac{4z}{(z-1)^2}$$

$$\Rightarrow h(n) = u(n) + 4n\,u(n)$$

Example 2.57 Determine the inverse z-transform of

(Annamalai University APR 2001)

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

AU EIE' 03

Solution

$$\begin{aligned}
X(z) &= \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \\
&= \frac{z^2}{(z-1)(z-0.5)} \\
\frac{X(z)}{z} &= \frac{z}{(z-1)(z-0.5)} \\
&= \frac{2}{z-1} - \frac{1}{z-0.5} \\
x(n) &= 2u(n) - (0.5)^n u(n)
\end{aligned}$$

Example 2.58 Find the impulse response and frequency response of a second order discrete-time system described by the difference equation

$$y(n) - y(n-1) + \frac{3}{16}y(n-2) = x(n) - \frac{1}{2}x(n-1)$$

Solution Take Fourier transform on both sides of the given equation and simplify to get the frequency response

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{1 - e^{-j\omega} + \frac{3}{16}e^{-j2\omega}}$$

Take z-transform on both sides of the equation to get

$$\begin{aligned}
H(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{3}{16}z^{-2}} = \frac{z\left(z - \frac{1}{2}\right)}{z^2 - z + \frac{3}{16}} \\
\frac{H(z)}{z} &= \frac{z - \frac{1}{2}}{z^2 - z + \frac{3}{16}} = \frac{z - \frac{1}{2}}{\left(z - \frac{3}{4}\right)\left(z - \frac{1}{4}\right)} \\
&= \frac{1}{2}\left(\frac{1}{z - \frac{3}{4}}\right) + \frac{1}{2}\frac{1}{z - \frac{1}{4}} \\
H(z) &= \frac{1}{2}\frac{z}{z - \frac{3}{4}} + \frac{1}{2}\frac{z}{z - \frac{1}{4}} \\
h(n) &= 0.5\left(\frac{3}{4}\right)^n u(n) + 0.5\left(\frac{1}{4}\right)^n u(n)
\end{aligned}$$

The stability region shown in Fig.2.16 is the area covered by the lines $a_1 - a_2 < 1$; $a_1 + a_2 > -1$; $a_2 < 1$ and the curve $a_1^2 = 4a_2$.

Example 2.60 Determine the zero-response of the system $y(n) = \frac{1}{2}y(n-1) + 4x(n) + 3x(n-1)$ to the input $x(n) = e^{j\omega_0 n}u(n)$.

JNTU Apr'03 (Set 3)

Solution

$$Y(z) = \frac{(4+3z^{-1})}{\left(1-\frac{1}{2}z^{-1}\right)}X(z)$$

Given $x(n) = e^{j\omega_0 n}u(n)$

$$X(z) = \frac{1}{1-e^{j\omega_0}z^{-1}}$$

$$Y(z) = \frac{(4+3z^{-1})}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-e^{j\omega_0}z^{-1}\right)}$$

$$= \frac{z(4z+3)}{\left(z-\frac{1}{2}\right)\left(z-e^{j\omega_0}\right)}$$

$$\frac{Y(z)}{z} = \frac{4z+3}{\left(z-\frac{1}{2}\right)\left(z-e^{j\omega_0}\right)}$$

$$y(n) = \frac{5}{\frac{1}{2}-e^{j\omega_0}}\left(\frac{1}{2}\right)^n u(n) + \frac{4e^{j\omega_0}+3}{e^{j\omega_0}-\frac{1}{2}}e^{j\omega_0 n}u(n)$$

Example 2.61 A digital system is characterized by the following difference equation.

$$y(n) = x(n) + ay(n-1)$$

Assuming that the system is relaxed initially, determine its impulse response

(JNTU APR' 03):

Solution

$$y(n) = x(n) + ay(n-1)$$

$$y(n) - ay(n-1) = x(n)$$

$$Y(z) - az^{-1}Y(z) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}}$$

$$H(z) = \frac{1}{1-az^{-1}}$$

$$h(n) = a^n u(n)$$

Example 2.62 Determine the causal signal $x(n)$ having z-transform

$$X(z) = \frac{z^2+z}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{4}\right)}$$

Nov'05 (set 1, set 2, set 3)

Solution

$$\frac{X(z)}{z} = \frac{(z+1)}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{4}\right)}$$

$$X(z) = \frac{-20z}{z-\frac{1}{2}} + \frac{6z}{\left(z-\frac{1}{2}\right)^2} + \frac{20z}{z-\frac{1}{4}}$$

$$X(z) = \frac{-20z}{z-\frac{1}{2}} + \frac{6z}{\left(z-\frac{1}{2}\right)^2} + \frac{20z}{z-\frac{1}{4}}$$

$$x(n) = -20\left(\frac{1}{2}\right)^n u(n)$$

$$+ 12u(n) + 20\left(\frac{1}{4}\right)^n u(n)$$

$$A = \frac{1}{1!}\frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2 \frac{z+1}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{4}\right)}\right]\Bigg|_{z=\frac{1}{2}}$$

$$= \frac{\left(z-\frac{1}{4}\right)-(z+1)}{\left(z-\frac{1}{4}\right)^2}\Bigg|_{z=\frac{1}{2}} = -20$$

$$B = \left(z-\frac{1}{2}\right)^2 \frac{z+1}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{4}\right)}\Bigg|_{z=\frac{1}{2}} = 6$$

$$C = \left(z-\frac{1}{4}\right)\frac{z+1}{\left(z-\frac{1}{2}\right)^2\left(z-\frac{1}{4}\right)}\Bigg|_{z=\frac{1}{4}}$$

$$= \frac{5/4}{\frac{1}{16}} = 20$$

Example 2.63 A causal system is represented by the following difference equation

$$Y(z) + \frac{1}{4}z^{-1}y(z) = X(z) + \frac{1}{2}z^{-1}x(z)$$

Find the system transfer function $H(z)$, unit sample response and frequency response of the system

Solution

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}$$

$$h(n) = Z^{-1}[H(z)]$$

$$H(z) = \frac{1}{1+\frac{1}{4}z^{-1}} + \frac{\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}$$

$$h(n) = \left(-\frac{1}{4}\right)^n u(n) + \frac{1}{2}\left(\frac{-1}{4}\right)^n u(n-1)$$

The frequency response $H\left(e^{j\omega}\right) = H(z)\big|_{z=e^{j\omega}}$, since both poles are inside the unit circle

$$H\left(e^{j\omega}\right) = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$$

Example 2.64 Obtain the frequency response of a first order system with difference equation $y(n) = x(n) + 10y(n-1)$ with initial condition $y(1) = 0$ and comment about its stability. May' 05(set 3)

Solution

$$Y(z) = X(z) + 10z^{-1}X(z)$$

$$H(z) = \frac{1}{1+10z^{-1}} = \frac{z}{z+10}$$

The pole is outside the unit circle. Hence the system is unstable.

For unstable system, we cannot substitute $z = e^{j\omega}$ in the above equation to obtain the frequency response.

Example 2.65 An LTI system is described by the equation

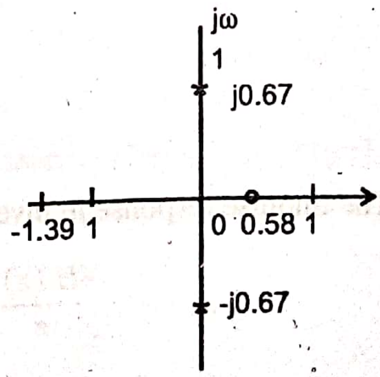
$$y(n) = x(n) + 0.81x(n-1) - 0.81x(n-2) - 0.45y(n-2).$$

Determine the transfer function. Sketch the poles and zeros on the z-plane. May'03 (set 1), May'06 (set 1)

Solution

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.81z^{-1}-0.81z^{-2}}{1+0.45z^{-2}}$$

$$= \frac{z^2+0.81z-0.81}{z^2+0.45}$$



The zeros are at 0.58 and -1.39

The poles are $\pm\ j0.67$

Example 2.66 Find the impulse and step responses for the given system

$$y(n) + y(n-1) = x(n) - 2x(n-1)$$

May' 05 (set 4), May' 06

Solution

Take z-transform on both sides

$$Y(z) + z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1+z^{-1}}$$

$$h(n) = (-1)^n u(n) - 2(-1)^{n-1} u(n-1)$$

For step input

$$X(z) = \frac{z}{z-1}$$

$$Y(z) = \frac{1-2z^{-1}}{1+z^{-1}} \cdot \frac{z}{z-1}$$

$$= \frac{z(z-2)}{(z+1)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{z-2}{(z+1)(z-1)}$$

$$y(n) = \frac{3}{2}(-1)^n u(n) - \frac{1}{2}u(n)$$

Example 2.67 Determine the impulse response of the system described by the difference equation $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$ using z-transform NOV'05 (set 3)

Solution Apply z-transform on both sides

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-3z^{-1}-4z^{-2}}$$

$$= \frac{z(z+2)}{z^2-3z-4}$$

The impulse response in inverse z-transform of $H(z)$

$$\frac{H(z)}{z} = \frac{z+2}{(z-4)(z+1)}$$

$$= \frac{A}{z-4} + \frac{B}{z+1}$$

$$H(z) = \frac{6}{5}\frac{z}{z-4} - \frac{1}{5}\frac{z}{z+1}$$

$$h(n) = \frac{6}{5}(4)^n u(n) - \frac{1}{5}(-2)^n u(n)$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

Time shift or translation:

(a) If $X(z) = Z\{x(n)\}$ and the initial conditions for $x(n)$ are zeros, then

$$Z\{x(n-m)\} = z^{-m}X(z) \tag{2.11}$$

where m is a positive or a negative integer.

Proof:

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)z^{-n}$$

$$= z^{-m}\sum_{n=-\infty}^{\infty} x(n-m)z^{-(n-m)}$$

Let $(n-m) = l$, then we have

$$Z\{x(n-m)\} = z^{-m}\sum_{l=-\infty}^{\infty} x(l)z^{-l}$$

$$= z^{-m}X(z)$$

(b) If $X_+(z) = Z\{x(n)\}$ then

$$\text{(i)}\quad Z\{x(n-m)\} = z^{-m}\left\{X_+(z) + \sum_{k=1}^{m} x(-k)z^{k}\right\} \tag{2.12}$$

$$\text{(ii)}\quad Z\{x(n+m)\} = z^{m}\left\{X_+(z) - \sum_{k=0}^{m-1} x(k)z^{-k}\right\} \tag{2.13}$$

where m is a positive integer.

(i) Proof:

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m)z^{-n}$$

$$= z^{-m}\sum_{n=0}^{\infty} x(n-m)z^{-(n-m)}$$

$$= z^{-m} \sum_{l=-m}^{\infty} x(l) z^{-l} \quad where \quad l = n - m$$

$$= z^{-m} \left\{ \sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-m}^{-1} x(l) z^{-l} \right\}$$

$$= z^{-m} \left\{ X_+(z) + \sum_{k=1}^{m} x(-k) z^{k} \right\} \quad \text{where} \quad l = -k$$

(ii) Proof:

$$Z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n}$$

$$= z^{m} \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

$$= z^{m} \sum_{l=m}^{\infty} x(l) z^{-l} \quad \text{where} \quad l = n + m$$

$$= z^{m} \left\{ \sum_{l=0}^{\infty} x(l) z^{-l} - \sum_{l=0}^{m-1} x(l) z^{-l} \right\}$$

$$= z^{m} \left\{ X_+(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right\}$$

Multiplication by an exponential sequence

If $X(z) = Z\{x(n)\}$, then

$$Z\{a^n x(n)\} = X(a^{-1} z) \tag{2.14}$$

Proof:

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n} = X(a^{-1} z)$$

where the ROC is $|a|R_1 < |z| < |a|R_2$.

Time reversal

If $X(z) = Z\{x(n)\}$, then

$$Z\{x(-n)\} = X(z^{-1}) \tag{2.15}$$