Design of FIR FILTER - Windowing Techniques

The frequency response of desired sequence $H_d(e^{j\omega})$ is given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

Then $h_d(n)$ is inverse fourier transform of $H_d(e^{j\omega})$, that is

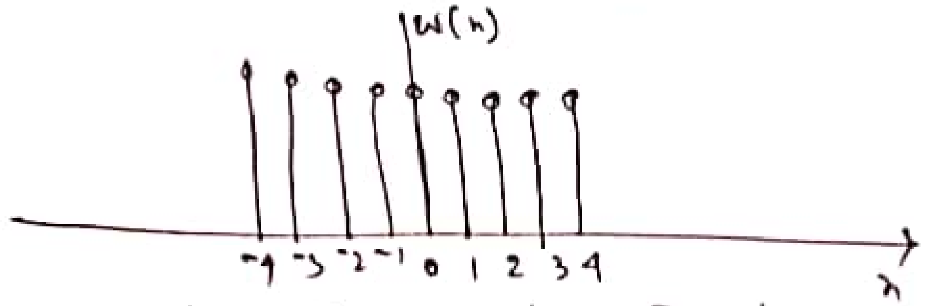
$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

The order to obtain an FIR filter of length N, The infinite length of desired impulse response $h_d(n)$ is truncated at $n = \pm\left(\frac{N-1}{2}\right)$

Instead of truncating the desired impulse response $h_d(n)$, the same result can be obtained by multiplying the desired impulse response by a rectangular window.

The rectangular window can be defined as

$$W(n) = \begin{cases} 1 & |n| \leq \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Rectangular window Function for N = 9

$$h(n) = \begin{cases} h_d(n)w(n), & |n| \le \left(\frac{N-1}{2}\right) \\ 0 & \text{therwise} \end{cases}$$

$$h_d(n) * w(n) \xleftrightarrow{FT} H_d(e^{j\omega}) \circledast W(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(\omega-\theta)})\, d\theta$$

The Fourier transform of the rectangular windows is given by

$$W(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} 1 \cdot e^{-j\omega n}$$

$$W(e^{j\omega}) = e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} e^{-j\omega n} = e^{j\omega\left(\frac{N-1}{2}\right)} \left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$W(e^{j\omega}) = e^{j\omega\left(\frac{N-1}{2}\right)} \frac{e^{-j\omega\frac{N}{2}}\left(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$W(e^{j\omega}) = \left(\frac{\sin\frac{\omega N}{2}}{\sin\omega/2}\right)$$

* Design an ideal low pass filter using rectangular window of N = 9 whose desired frequency response is

$$H_d(e^{j\omega}) = \begin{cases} 1, & \pi/3 \ge \omega \ge -\pi/3 \\ 0, & \text{otherwise}. \end{cases}$$

(i) Determine the impulse response $h(n)$

(ii) Determine $H(z)$

(iii) plot the magnitude response $|H(e^{j\omega})|$

Solution: The desired impulse response can be obtained from

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/3}^{\pi/3}$$

$$h_d(n) = \frac{1}{2\pi}\left[\frac{e^{jn\pi/3} - e^{-jn\pi/3}}{jn}\right] = \frac{\sin \frac{\pi}{3} n}{\pi n}$$

The desired filter coefficients are $\infty \geq n \geq \infty$ calculated for $n = 0, \pm 1, \pm 2, \pm 3, \pm 4,$

$$h_d(0) = \lim_{n\to 0} \frac{\sin \frac{\pi}{3} n}{n\pi} = \frac{1}{3}\lim_{n\to 0} \frac{\sin \frac{\pi}{3} n}{\frac{n\pi}{3}} = \frac{1}{3} = 0.333$$

$$h_d(1) = \frac{\sin \pi/3}{\pi} = 0.276 = h_d(-1)$$

$$h_d(2) = \frac{\sin \frac{2\pi}{3}}{2\pi} = 0.138 = h_d(-2)$$

$$h_d(3) = \frac{\sin \pi}{3\pi} = 0 = h_d(-3)$$

$$h_d(4) = \frac{\sin \frac{4\pi}{3}}{4\pi} = -0.069 = h_d(-4)$$

The FIR filter coefficient is calculated by multiplying it with rectangular window function of length $N = 9$ that is

$$h(n) = h_d(n) * w_R(n)$$

since, $w_R(n) = \begin{cases} 1 & |n| \leq 4 \\ 0 & \text{otherwise} \end{cases}$

Therefore $h(n) = \begin{cases} h_d(n) \times w_R(n) & |n| \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$h(0) = h_d(0) = 0.333$$
$$h(1) = h_d(-1) = h_d(1) = 0.276$$
$$h(2) = h_d(-2) = h_d(2) = 0.138$$
$$h(3) = h_d(-3) = h_d(3) = 0$$
$$h(4) = h_d(-4) = h_d(4) = -0.069$$

The FIR filter transfer function can be calculated from

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)(z^{n} + z^{-n})$$

$$H(z) = h(0) + \sum_{n=1}^{4} h(n)(z^{n} + z^{-n})$$

$$H(z) = 0.333 + h(1)(z + z^{-1}) + h(2)(z^{2} + z^{-2}) + h(3)(z^{3} + z^{-3}) + h(4)(z^{4} + z^{-4})$$

$$H(z) = 0.333 + 0.276(z + z^{-1}) + 0.138(z^{2} + z^{-2}) - 0.069(z^{4} + z^{-4})$$

To design a causal filter, terms z, z^2, z^3 and z^4 should not be in the equation, so multiply the entire terms by $z^{-\left(\frac{N-1}{2}\right)} = z^{-4}$ that is

$$\bar{H}(z) = z^{-4} H(z)$$

$$\bar{H}(z) = 0.333z^{-4} + 0.276z^{-3} + 0.27z^{-5} + 0.138z^{-2} + 0.138z^{-6} - 0.069 - 0.069z^{-8}$$

The frequency response for symmetric impulse response with odd length is given by

$$H(e^{jw}) = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right)\cos wn$$

For $N = 9$

$$\bar{H}(e^{j\omega}) = h(4) + 2\sum_{n=1}^{4} h(4-n)\cos\omega n$$

The symmetric condition for impulse response is

$$h(n) = h(N-1-n)$$

Therefore filter coefficients of causal filter are

$$h(0) = h(8) = -0.069$$
$$h(1) = h(7) = 0$$
$$h(2) = h(6) = 0.138$$
$$h(3) = h(5) = 0.276$$
$$h(4) = 0.333.$$

The frequency response for causal system is

$$\bar{H}(e^{j\omega}) = 0.333 + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

$$\bar{H}(e^{j\omega}) = 0.333 + 2\times(0.276)\cos\omega + 2\times(0.138)\cos 2\omega + 2\times(-0.069)\cos 4\omega$$

$$\bar{H}(e^{j\omega}) = 0.333 + 0.552\cos\omega + 0.276\cos 2\omega - 0.138\cos 4\omega$$

(Ans)

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Find the initial value ...

(1) * Design a filter with $H_d(e^{j\omega}) = e^{-j3\omega}$ $-\pi/4 \le \omega \le \pi/4$

$= 0$ $\frac{\pi}{4} < |\omega| \le \pi$

using a Hamming window with $N=7$,

Solution: Given $H_d(e^{j\omega}) = e^{-j3\omega}$

The frequency response is having a term $e^{-j\omega\left(\frac{N-1}{2}\right)}$ which gives $h(n)$ symmetrical about $n = \frac{N-1}{2} = 3$. i.e. we get a causal sequence.

We have $h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}\int_{-\pi/4}^{\frac{\pi}{4}} e^{j(n-3)\omega} d\omega$$

$$= \frac{\sin\frac{\pi}{4}(n-3)}{\pi(n-3)}$$

For $N=7$ we have

$$h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = 0.25$$

The non-causal window sequence is

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2}$$

$$= 0 \quad \text{otherwise}$$

For $N=7$

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -3 \le n \le 3$$

$$= 0 \quad \text{otherwise}$$

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$W_{Hn}(0) = 0.5 + 0.5 = 1.$

$W_{Hn}(-1) = W_{Hn}(1) = 0.5 + 0.5 \cos\frac{\pi}{3} = 0.75$

$W_{Hn}(-2) = W_{Hn}(2) = 0.5 + 0.5\cos\frac{2\pi}{3} = 0.25$

$W_{Hn}(-3) = 0.5 + 0.5\cos\pi = 0$

The causal window sequence can be obtained by shifting the sequence $W_{Hn}(n)$ to right by 3 samples i.e

$W_{Hn}(0) = W_{Hn}(6) = 0$; $W_{Hn}(1) = W_{Hn}(5) = 0.25$

$W_{Hn}(2) = W_{Hn}(4) = 0.75$ & $W_{Hn}(3) = 1$

The filter coefficients using Hanning window are

$h(n) = h_d(n) W_{Hn}(n)$ for $0 \le n \le 6$

$h(0) = h(6) = h_d(0) W_{Hn}(0) = 0.$

$h(1) = h(5) = h_d(1) W_{Hn}(1) = (0.159)(0.25)$
$= 0.03975$

$h(2) = h(4) = h_d(2) W_{Hn}(2) = (0.22)(0.75) = 0.165$

$h(3) = h_d(3) W_{Hn}(3) = (0.25)(1) = 0.25$

The transfer function of the filter is given by

$$H(Z) = h(0) + \sum_{n=1}^{3} h(n)\left(z^{-n} + z^{+n}\right)$$

$$= h(0) + h(1)(z^{-1} + z^{1}) + h(2)(z^{-2} + z^{2}) + h(3)(z^{-3} + z^{3})$$

$$= 0 + 0.03975(z^{-1} + z^{1}) + 0.165(z^{-2} + z^{2}) + 0.25(z^{-3} + z^{3})$$

The transfer function of the realizable filter is

$$H'(Z) = z^{-3} H(Z) = z^{-3}\left[0.03975(z^{-1} + z) + 0.165(z^{-2} + z^{2}) + 0.25(z^{-3} + z^{3})\right]$$ Ans.

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② Design a filter with

$$H_d(e^{j\omega}) = e^{-j5\omega} \quad -\pi/4 \le \omega < \pi/2$$
$$= 0 \quad \frac{\pi}{2} < |\omega| \le \pi$$

using Blackman window with $N = 11$

⊘ Design an ideal Hilbert transformer having frequency response

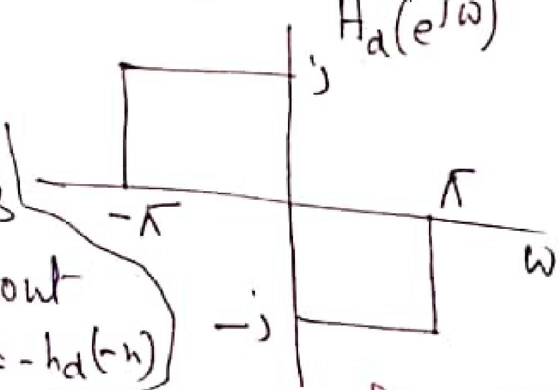
$$H(e^{j\omega}) = j \quad \text{for } -\pi \le \omega \le 0$$
$$= -j \quad \text{for } 0 \le \omega \le \pi$$

using Blackman window.

Solution The ideal frequency response is shown $h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} j e^{j\omega n} d\omega + \int_{0}^{\pi} -j e^{j\omega n} d\omega\right]$$

$$= \frac{1 - \cos \pi n}{n\pi}$$



The filter coefficients are antisymmetrical about $n=0$, satisfying $h_d(n) = -h_d(-n)$

Frequency response of ideal Hilbert transformer.

for $n = 0$

$$h_d(0) = \frac{j}{2\pi}\left[\int_{-\pi}^{0} d\omega + \int_{0}^{\pi} d\omega\right]$$

$$= \frac{j}{2\pi}(0 + \pi - \pi - 0) = 0$$

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$h_d(1) = -h_d(-1) = \frac{1-\cos\pi}{\pi} = \frac{2}{\pi}$

$h_d(2) = -h_d(-2) = \frac{1-\cos 2\pi}{2\pi} = 0$

$h_d(3) = -h_d(-3) = \frac{1-\cos 3\pi}{3\pi} = \frac{2}{3\pi}$

$h_d(4) = -h_d(-4)$

$= \frac{1-\cos 4\pi}{4\pi} = 0$

$h_d(5) = -h_d(-5) = \frac{1-\cos 5\pi}{5\pi} = \frac{2}{5\pi}$

Blackman window

The Blackman window sequence for N=11 is

$W_B(n) = 0.42 + 0.5\cos\frac{\pi n}{5} + 0.08\cos\frac{2\pi n}{5}$, for $-5 \le n \le 5$

$= 0$ otherwise

$W_B(0) = 1.$

$W_B(1) = W_B(-1) = 0.849$

$W_B(2) = W_B(-2) = 0.509$

$W_B(3) = W_B(-3) = 0.2$

$W_B(3) = W_B(-3) = 0.04$

$W_B(4) = W_B(-4) = 0$

$W_B(5) = W_B(-5)$

The coefficients of Hilbert transformer are

$h(n) = h_d(n) W_B(n)$ for $-5 \le n \le 5$

$= 0$ otherwise

$h(0) = h_d(0) W_B(0) = 0.1 = 0$

$h(1) = -h(-1) = h_d(1) W_B(1) = \left(\frac{2}{\pi}\right)(0.849) = 0.5405$

$h(2) = -h(-2) = h_d(2) W_B(2) = (0)(0.509) = 0$

Find the initial value ...

$$h(3) = -h(-3) = h_a(3)W_B(3) = \left(\frac{2}{3\pi}\right)(0.2)$$
$$= 0.0423$$

$$h(4) = -h(-4) = h_a(4)W_B(4) = 0 \cdot (0.09) = 0$$

$$h(5) = -h(-5) = h_a(5)W_B(5) = \left(\frac{2}{5\pi}\right)(0) = 0$$

The realizable transfer function is

$$H'(z) = z^{-5}H(z)$$
$$= z^{-5}\left[0.54(z-z^{-1}) + 0.0424(z^3-z^{-3})\right]$$
$$= 0.0424z^{-2} + 0.54z^{-4} - 0.54z^{-6} - 0.042z^{-8}$$