Example 3.15 Given the sequences $x_1(n) = \{1,2,3,4\}; x_2(n) = \{1,1,2,2\}$. Find $x_3(n)$ such that $X_3(k) = X_1(k)X_2(k)$.

(AU'05)

Solution

We know

$$x_3(n) = IDFT\,[X_3(k)] = IDFT\,[X_1(k)X_2(k)]$$
$$= x_1(n) \; \text{N} \; x_2(n)$$

So we find $x_3(n)$ by circular convolving $x_1(n)$ and $x_2(n)$

Given

$$x_1(n) = \{1,2,3,4\}$$
$$x_2(n) = \{1,1,2,2\}$$

Representing $x_2(n)$ as $N \times N$ matrix form and $x_1(n)$ as column matrix and multiplying we have

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 15 \\ 13 \end{bmatrix}$$

$$y(n) = \{15, 17, 15, 13\}$$

Example 3.16 Perform the circular convolution of the following sequences

$$x(n) = \{1,1,2,1\}$$
$$h(n) = \{1,2,3,4\}$$

using DFT and IDFT method.

Solution

We know $X_3(k) = X_1(k)X_2(k)$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1$$

Given $x_1(n) = \{1,1,2,1\}$ and $N = 4$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) = 1+1+2+1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n)e^{-j\pi n/2} = 1 - j - 2 + j = -1$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n)e^{-j\pi n} = 1 - 1 + 2 - 1 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n)e^{-j3\pi n/2} = 1 + j - 2 - j = -1$$

$$X_1(k) = (5, -1, 1, -1)$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N-1$$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) = 1 + 2 + 3 + 4 = 10$$

$$X_2(1) = \sum_{n=0}^{3} x_2(n)e^{-j\pi n/2} = 1 + 2(-j) + 3(-1) + 4(j) = -2 + j2$$

$$X_2(2) = \sum_{n=0}^{3} x_2(n)e^{-j\pi n} = 1 + 2(-1) + 3(1) + 4(-1) = -2$$

$$X_2(3) = \sum_{n=0}^{3} x_1(n)e^{-j3\pi n/2} = 1 + 2(j) + 3(-1) + 4(-j) = -2 - j2$$

$$X_2(k) = \{10, -2 + j2, -2, -2, -j2\}$$

$$X_3(k) = X_1(k)X_2(k) = \{50, 2 - j2, -2, 2 + j2\}$$

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi nk/N} \quad n = 0, 1, \ldots, N-1$$

$$x_3(0) = \frac{1}{4}\sum_{k=0}^{3} X_3(k) = \frac{1}{4}(50 + 2 - j2 - 2 + 2 + j2) = 13$$

$$x_3(1) = \frac{1}{4}\left[\sum_{k=0}^{4} X_3(k)e^{j\pi k/2}\right]$$

$$= \frac{1}{4}[50 + (2 - j2)j + (-2)(-1) + (2 + j2)(-j)] = 14$$

$$x_3(2) = \frac{1}{4}\left[\sum_{k=0}^{4} X_3(k)e^{j\pi k}\right]$$

$$= \frac{1}{4}[50 + (2 - j2)(-1) + (-2)(1) + (2 + j2)(-1)] = 11$$

$$x_3(3) = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k)e^{j3\pi k/2}\right]$$

$$= \frac{1}{4}[50 + (2 - j2)(-j) + (-2)(-1) + (2 + j2)(j)] = 12$$

$$x_3(n) = \{13, 14, 11, 12\}$$

Example 3.17 Consider the sequences $x_1(n) = \{0, 1, 2, 3, 4\}; x_2(n) = \{0, 1, 0, 0, 0\}$. Determine a sequence $y(n)$ so that $Y(k) = X_1(k)X_2(k)$.

Solution

The sequences $y(n)$ can be obtained by circular convolution of $x_1(n)$ and $x_2(n)$
Using Matrix approach we have

$$\begin{bmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$y(n) = \{4, 0, 1, 2, 3\}$$

Practice Problem 3.7 Perform the circular convolution of the following sequences

(i) $x_1(n) = \{1, 1, 2, 1\}; x_2(n) = \{1, 2, 3, 4\}$

Ans: $\{13, 14, 11, 12\}$

(ii) $x_1(n) = \{1, 2, 3, 1\}; x_2(n) = \{4, 3, 2, 2\}$

Ans: $\{17, 19, 22, 19\}$

Practice Problem 3.8 Find circular convolution of the following sequences $x(n) = \{1, 1, 1, 2\}; y(n) = \{1, 2, 3, 2\}$ using DFT and IDFT method.

Ans: $\{10, 11, 10, 9\}$

Practice Problem 3.9 Show that the circular convolution is commutative.

3.9 Linear Convolution From Circular Convolution

In signal processing applications we are interested in the linear convolution of two finite duration sequences. For example, one of the sequences could be a signal to be filtered, and other sequence could represent the impulse response of the system. We know multiplying the DFT of both sequences and taking inverse is equivalent to circular convolution of two sequences. In order to obtain the result of linear convolution from a circular convolution we must make certain modifications.

Let us consider two finite duration sequences $x(n)$ and $h(n)$. The duration of $x(n)$ is L samples and that of $h(n)$ is M samples. The linear convolution of $x(n)$ and $h(n)$ is given by the formula

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{3.56}$$

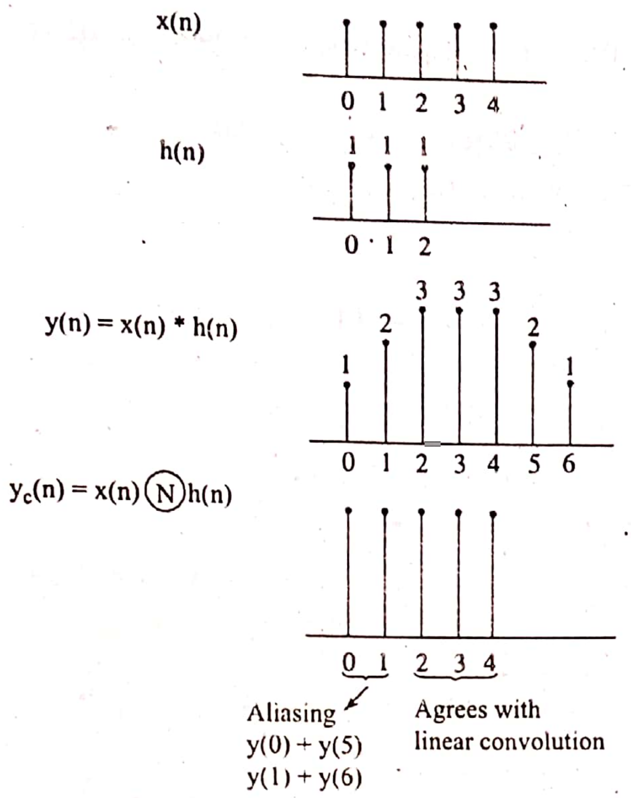


Fig. 3.15 Comparison between Linear Convolution and Circular Convolution

where $y(n)$ is a finite duration sequence of $L + M - 1$ samples. The circular convolution of $x(n)$ and $h(n)$ give N samples where $N = \text{Max}(L, M)$. If $M < L$, in order to find circular convolution, it is necessary to add $L - M$ zeros to the sequence $h(n)$. The circular convolution then results in an L-point sequence that is $M - 1$ points shorter than that given by linear convolution. In order to obtain the number of samples in circular convolution equal to $L + M - 1$, both $x(n)$ and $h(n)$ must be $L + M - 1$ point sequences. This can be achieved by appending appropriate number of zero valued samples to both $x(n)$ and $h(n)$. Now we take $L+M-1$ point DFTs of $x(n)$ and $h(n)$ and multiply the DFTs to get $Y(k)$. Then by finding inverse transform we obtain $y(n)$. In otherwords, by increasing the length of the sequences $x(n)$ and $h(n)$ to $L + M - 1$ points and then circularly convolving the resulting sequences we obtain the same result as would have been obtained with linear convolution. This is illustrated using a simple example.

Consider two sequences $x(n)$ and $h(n)$ having sequence length $L = 5$ and $M = 3$ respectively as shown in Fig. 3.15. The linear convolution of $x(n)$ and $h(n)$ shown in Fig. 3.15 consist of $5+3-1 = 7$ points. The circular convolution of $x(n)$ and $h(n)$ consist of 5 points short of $M - 1 = 2$ points. Therefore the circular convolution will contain corrupted points due to time domain aliasing. These points are first $(M - 1)$ points as shown in Fig. 3.15. If we increase the length of the sequences $x(n)$ and $h(n)$ to $L + M - 1 = 7$ by appending $M - 1 = 2$ zeros to $x(n)$ and $L - 1 = 4$ zeros to $h(n)$, then the 7-point circular convolution of $x(n)$ and $h(n)$ is identical to their linear convolution.

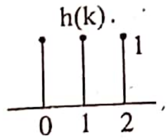
Example 3.18 Determine the output response $y(n)$ if $h(n) = \{1,1,1\}; x(n) = \{1,2,3,1\}$ by using (EIE AU'03)

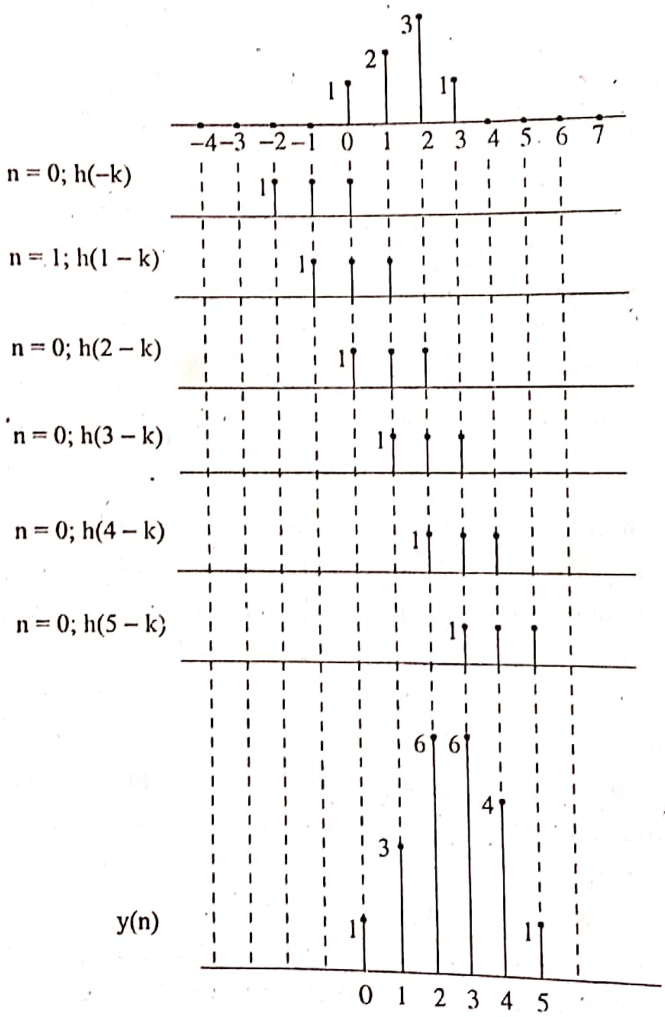
(i) Linear convolution (ii) Circular convolution

(iii) Circular convolution with zero padding

Solution

Given $x(n) = \{1,2,3,1\}, h(n) = \{1,1,1\}$





We know $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) = 1$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) = 1+2 = 3$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k) = 1+2+3 = 6$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k) = 2+3+1 = 6$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k) = 3+1 = 4$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)\, h(5-k) = 1$$

$y(n) = \{1, 3, 6, 6, 4, 1\}$

Fig. 3.16

The number of samples in linear convolution is $L + M - 1 = 4 + 3 - 1 = 6$.

(ii) ***Circular Convolution***

$$x(n) = \{1,2,3,1\}; h(n) = \{1,1,1,0\}$$

Using matrix approach we can write $h(n)$ as $N \times N$ matrix form and $x(n)$ as column matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}$$

$$y(n) = x(n) \; \textcircled{N} \; h(n) = \{5,4,6,6\}$$

Comparing the circular convolution output with that of linear convolution we find that the first 2 points ($\because M-1$) points are aliased. That is, the last two data points in linear convolution are added to first two data points as shown below

$$1+4=5 \quad \text{and} \quad 3+1=4$$

(iii) ***Circular Convolution with Zero padding***

To get the result of linear convolution with circular convolution we have to add appropriate number of zeros to both sequences. Now

$$x(n) = \{1,2,3,1,\overbrace{0,0}^{(M-1)\text{ zeros appended}}\}$$
$$h(n) = \{1,1,1,\underbrace{0,0,0}_{(L-1)\text{ zeros appended}}\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$y(n) = \{1,3,6,6,4,1\}$$

Example 3.19 We have two-50 point sequences $x_1(n)$ and $x_2(n)$ which are non-zero for the interval $0 \le n \le 49$. The two sequences are circularly convolved to form a new sequence $y(n)$.

If $x_1(n)$ is nonzero only for $10 \le n \le 24$ determine the set of values of n for which $y(n)$ is identical to the linear convolution of $x_1(n)$ and $x_2(n)$.

Solution The sequences $x_1(n)$ and $x_2(n)$ are zero outside the interval $0 \le n \le 49$. The circular convolution of two sequences is

$$y(n) = x_1(n) \; 50 \; x_2(n) = \sum_{k=0}^{49} x_1(k)x_2((n-k))_{50}; \quad 0 \le n \le 49$$

If $x_1(n)$ is nonzero only for $10 \le n \le 24$, the linear convolution has a sequence of length $25+50-1=74$, which is nonzero for the range $10 \le n \le 74$.

The 50-point circular convolution is equivalent to linear convolution with first 25 points aliased by the values in the range $50 \le n \le 74$. Therefore the linear convolution and circular convolution is equivalent in the range $25 \le n \le 49$.

Practice Problem 3.10 Obtain the output response $y(n)$ if $h(n) = \{1,2,2,1\}$; $x(n) = \{1,-1,1,-1\}$ without using linear convolution.

Ans: $\{1,1,1,0,-1,-1,-1\}$

3.10 Filtering Long Duration Sequences

Suppose an input sequence $x(n)$ of long duration is to be processed with a system having impulse response of finite duration by convolving the two sequences. Because of the length of the input sequence, it would not be practical to store it all before performing linear convolution. Therefore, the input sequence must be divided into blocks. The successive blocks are processed separately one at a time and the results are combined later to yield the desired output sequence which is identical to the sequence obtained by linear convolution. Two methods that are commonly used for filtering the sectioned data and combining the results are the overlap-save method and the overlap-add method.

3.10.1 *Overlap-Save Method*

Let the length of an input sequence be L_S and the length of an impulse response is M. In this method the input sequence is divided into blocks of data of size $N = L+M-1$. Each block consists of last $(M-1)$ data points of previous block followed by L new data points to form a data sequence of length $N = L + M - 1$. For first block of data the first $M - 1$ points are set to zero. Thus the blocks of data sequence are

$$x_1(n) = \{\underbrace{0, 0, 0, \ldots 0}_{(M-1)\text{ Zeros}}, x(0), x(n) \ldots, x(L-1)\}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots, x(L-1)}_{\text{Last }(M-1)\text{ data points from } x_1(n)}, \underbrace{x(L) \ldots, x(2L-1)}_{L\text{ new data points}}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots, x(2L-1)}_{\text{Last }(M-1)\text{ data points from } x_2(n)}, \underbrace{x(2L) \ldots, x(3L-1)}_{L\text{ new data points}}\}$$

and so on.

Now the impulse response of the FIR filter is increased in length by appending $L-1$ zeros and an N-point circular convolution of $x_i(n)$ with $h(n)$ is computed.

$$\text{i.e.,}\quad y_i(n) = x_i(n) \; \textcircled{N} \; h(n)$$

In $y_i(n)$, the first $(M-1)$ points will not agree with the linear convolution of $x_i(n)$ and $h(n)$ because of aliasing, while the remaining points are identical to the linear convolution. Hence we discard the first $M-1$ points of the filtered section $x_i(n) \; \textcircled{N} \; h(n)$. The remaining points from successive sections are then abutted to construct the final filtered output.

For example, let the total length of the sequence $L_S = 15$ and the length of the impulse response is 3. Let the length of each block is 5.

Now the input sequence can be divided into blocks as

$$x_1(n) = \{\underbrace{0, 0,}_{M-1=2 \text{ zeros}} x(0), x(1), x(2)\}$$

$$x_2(n) = \{\underbrace{x(1), x(2)}, x(3), x(4), x(5)\}$$

$\downarrow$ Last two data points from previous block
$\uparrow$

$$x_3(n) = \{\overbrace{x(4), x(5)}, x(6), x(7), x(8)\}$$

$$x_4(n) = \{x(7), x(8), x(9), x(10), x(11)\}$$

$$x_5(n) = \{x(10), x(11), x(12), x(13), x(14)\}$$

$$x_6(n) = \{x(13), x(14), 0, 0, 0\}$$

Now we perform 5 point circular convolution of $x_i(n)$ and $h(n)$ by appending two zeros to the sequence $h(n)$. In the output block $y_i(n)$, first $M-1$ points are corrupted and must be discarded.

$$y_1(n) = x_1(n) \text{ (N) } h(n) = \{\underbrace{y_1(0), y_1(1)}_{\text{discard}}, y_1(2), y_1(3), y_1(4)\}$$

$$y_2(n) = x_2(n) \text{ (N) } h(n) = \{\underbrace{y_2(0), y_2(1)}_{\text{discard}}, y_2(2), y_2(3), y_2(4)\}$$

$$y_3(n) = x_3(n) \text{ (N) } h(n) = \{\underbrace{y_3(0), y_3(1)}_{\text{discard}}, y_3(2), y_3(3), y_3(4)\}$$

$$y_4(n) = x_4(n) \text{ (N) } h(n) = \{\underbrace{y_4(0), y_4(1)}_{\text{discard}}, y_4(2), y_4(3), y_4(4)\}$$

$$y_5(n) = x_5(n) \text{ (N) } h(n) = \{\underbrace{y_5(0), y_5(1)}_{\text{discard}}, y_5(2), y_5(3), y_5(4)\}$$

$$y_6(n) = x_6(n) \text{ (N) } h(n) = \{\overbrace{y_6(0), y_6(1)}, y_6(2), y_6(3), 0\}$$

The output blocks are abutted together to get

$$y(n) = \{y_1(2), y_1(3), y_1(4), y_2(2), y_2(3), y_2(4), y_3(2), y_3(3), y_3(4), y_4(2), y_4(3),$$
$$y_4(4), y_5(2), y_5(3), y_5(4), y_6(2), y_6(3), \}$$

3.10.2 *Overlap-Add Method*

Let the length of the sequence be L_S and the length of the impulse response is M. The sequence is divided into blocks of data size having length L and $M-1$ zeros are appended to it to make the data size of $L+M-1$.

Thus the data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \ldots, x(L-1), \underbrace{0, 0, \ldots}_{M-1 \text{ zeros appended}}\}$$

$$x_2(n) = \{x(L), x(L+1), \ldots, x(2L-1), \overbrace{0, 0, \ldots}\}$$
$$x_3(n) = \{x(2L), x(2L+1), \ldots, x(3L-1), \underbrace{0, 0, \ldots}_{M-1 \text{ zeros appended}}\}$$

Now $L-1$ zeros are added to the impulse response $h(n)$ and N-point circular convolution is performed. Since each data block is terminated with $M-1$ zeros, the last $M-1$ points from each output block must be overlapped and added to the first $M-1$ points of the succeeding block. Hence this method is called overlap-add method.

Let the output blocks are of the form

$$y_1(n) = \{y_1(0), y_1(1), \ldots, y_1(L-1), y_1(L), \ldots, y_1(N-1)\}$$
$$y_2(n) = \{y_2(0), y_2(1), \ldots, y_2(L-1), y_2(L), \ldots, y_2(N-1)\}$$
$$y_3(n) = \{y_3(0), y_3(1), \ldots, y_3(L-1), y_3(L), \ldots, y_3(N-1)\}$$

The output sequence is

$$y(n) = \{y_1 0, y_1(1), \ldots, y_1(L-1), y_1(L) + y_2(0), \ldots, y_1(N-1) + y_2(M-2),$$
$$y_2(M), \ldots, y_2(L) + y_3(0), y_2(L+1) + y_3(1), \ldots, y_3(N-1)\}$$

Example 3.20 Find the output $y(n)$ of a filter whose impulse response is $h(n) = \{1, 1, 1\}$ and input signal $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ using (i) overlap-save method (ii) overlap-add method. (Annamalai Unverisity Apr' 03)

Solution

(i) Overlap-save Method

The input sequence can be divided into blocks of data as follows.

$$x_1(n) = \{\underbrace{0, 0}_{M-1=2 \text{ Zeros}} \quad \underbrace{3, -1, 0}_{L=3 \text{ data points}}\}$$

$$x_2(n) = \{\underbrace{-1, 0,}_{\text{Two datas from previous block}} \quad \underbrace{1, 3, 2}_{\text{3new data points}}\}$$

$$x_3(n) = \{3, 2, 0, 1, 2\} \text{ and } x_4(n) = \{1, 2, 1, 0, 0\}$$

given $h(n) = \{1,1,1\}$

Increase the length of the sequence to $L + M - 1 = 5$ by adding two zeros.

$$\text{i.e. } h(n) = \{1,1,1,0,0\}$$

$$y_1(n) = x_1(n) \, \textcircled{\scriptsize N} \, h(n) = \{-1,0,3,2,2\}$$

$$y_2(n) = x_2(n) \, \textcircled{\scriptsize N} \, h(n) = \{4,1,0,4,6\}$$

$$y_3(n) = x_3(n) \, \textcircled{\scriptsize N} \, h(n) = \{6,7,5,3,3\}$$

$$y_4(n) = x_4(n) \, \textcircled{\scriptsize N} \, h(n) = \{1,3,4,3,1\}$$

Note: Circular convolution of the sequences left as an exercise to the students.

$$\boxed{\underbrace{-1,0}_{\text{discard}},3,2,2}$$

$$\boxed{\underbrace{4,1}_{\text{discard}},0,4,6}$$

$$\boxed{\underbrace{6,7}_{\text{discard}},5,3,3}$$

$$\boxed{\underbrace{1,3}_{\text{discard}},4,3,1}$$

$$y(n) = \{3,2,2,0,4,6,5,3,3,4,3,1\}$$

(ii) Overlap-Add method

Let the length of data block be 3. Two zeros are added to bring the length to five $(L + M - 1 = 5)$.

Therefore,

$$x_1(n) = \{3,-1,0,0,0\}$$
$$x_2(n) = \{1,3,2,0,0\}$$
$$x_3(n) = \{0,1,2,0,0\}$$
$$x_4(n) = \{1,0,0,0,0\}$$

$$y_1(n) = x_1(n) \, \textcircled{\scriptsize N} \, h(n) = \{3,2,2,-1,0\}$$

$$y_2(n) = x_2(n) \, \textcircled{\scriptsize N} \, h(n) = \{1,4,6,5,2\}$$