## 1.21 Impulse Response and Convolution Sum

A discrete-time system performs an operation on an input signal based on a prede-fined criteria to produce a modified output signal. The input signal $x(n)$ is the system excitation, and $y(n)$ is the system response. This transform operation is shown in Fig. 1.57.

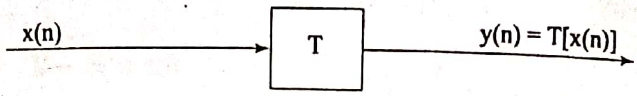



**Fig. 1.57** A Discrete-time system representation

If the input to the system is a unit impulse i.e., $x(n) = \delta(n)$ then the output of the system is known as impulse response denoted by $h(n)$ where

$$h(n) = T[\delta(n)] \tag{1.117}$$

We know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses (Eq. 1.116). Now the system response is given by

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \tag{1.118}$$

For a linear system Eq. (1.118) reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)] \tag{1.119}$$

The response to the shifted impulse sequence can be denoted by $h(n,k)$ defined as

$$h(n,k) = T[\delta(n-k)] \tag{1.120}$$

For a time-invariant system $h(n,k) = h(n-k)$ (1.121)

Substituting Eq. (1.121) in Eq. (1.120) we obtain

$$T[\delta(n-k)] = h(n-k) \tag{1.122}$$

Substituting Eq. (1.122) in Eq. (1.119) we have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{1.123}$$

For a linear time-invariant system, if the input sequence $x(n)$ and impulse response $h(n)$ are given, we can find the output $y(n)$ by using the equation

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

which is known as convolution sum and can be represented as

$$y(n) = x(n) * h(n), \quad \text{where } * \text{ denotes the convolution operation.}$$

This is an extremely powerful result that allows us to compute the system output for any input signal excitation.

The convolution sum of two sequences can be found by using the following steps.

**Step 1:** Choose an initial value of $n$, the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$ then $n = n_1 + n_2$ is a good choice.

**Step 2:** Express both sequences in terms of the index $k$.

**Step 3:** Fold $h(k)$ about $k = 0$ to obtain $h(-k)$ and shift by $n$ to the right if $n$ is positive and left if $n$ is negative to obtain $h(n-k)$.

**Step 4:** Multiply the two sequences $x(k)$ and $h(n-k)$ element by element and sum up the products to get $y(n)$.

**Step 5:** Increment the index $n$, shift the sequence $h(n-k)$ to right by one sample and do Step 4.

**Step 6:** Repeat Step 5 until the sum of products is zero for all the remaining values of $n$.

## 1.22 Properties of Convolution

(i) Commutative Law: $x(n) * h(n) = h(n) * x(n)$ (1.124)

(ii) Associative Law: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$ (1.125)

(iii) Distributive Law: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$ (1.126)

## 1.23 Causality

In section(1.19.2) we already studied that a causal system is one whose output depends on past or/and present values of the input. Using convolution sum, we have

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=-\infty}^{-1} h(k)x(n-k) + \sum_{k=0}^{\infty} h(k)x(n-k) \\ &= \underbrace{\ldots h(-2)x(n+2) + h(-1)x(n+1)}_{\text{depends on future inputs}} + \underbrace{h(0)x(n)}_{\text{present input}} + \underbrace{h(1)x(n-1) + \ldots}_{\text{depends on past inputs}} \end{aligned} \qquad (1.127)$$

From Eq. (1.127) we find that the output depends on the past and present values of the input if the index $k \geq 0$. If $k < 0$ then the output depends on the future values of the input. Therefore for a causal system whose output does not depend on future values of the input, the limits on the summation change as

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \tag{1.128}$$

From Eq. (1.128) we find that for causal system $h(k)$ should be zero for $k < 0$. That is,

$$h(k) = 0 \quad \text{for } k < 0 \tag{1.129}$$

An LTI system is causal if and only it its impulse response is zero for negative values of $n$.

The limits in the convolution sum can be modified according to the type of sequence and system. For a causal system, the impulse response $h(n) = 0$ for $n < 0$. Therefore the limit in the convolution sum is modified as

$$y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) \tag{1.130}$$

$$= \sum_{k=0}^{\infty} h(k)x(n-k) \tag{1.131}$$

If the input to the causal system is a causal sequence (i.e., $x(n) = 0$ for $n < 0$) the limit in the convolution sum is modified as

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) \tag{1.132}$$

$$= \sum_{k=0}^{n} h(k)x(n-k) \tag{1.133}$$

## 1.24 FIR and IIR Systems

Linear time invariant systems can be classified according to the type of impulse response. They are 1. FIR system 2. IIR system.

### *FIR system*

**Definition** If the impulse response of the system is of finite duration, then the system is called a Finite Impulse-Response (FIR system).

An example of a FIR system is

$$h(n) = \begin{cases} 1 & \text{for } n = -1, 2 \\ 2 & \text{for } n = 1 \\ 3 & \text{for } n = 0, 3 \\ 0 & \text{otherwise} \end{cases} \tag{1.134}$$

***IIR system***

**Definition** An Infinite Impulse Response (IIR) system has an impulse response for infinite duration. An example of an IIR system is

$$h(n) = a^n u(n) \tag{1.135}$$

## 1.25 Stable and Unstable systems

An LTI system is stable if it produces a bounded output sequence for every bounded input sequence. If, for some bounded input sequence $x(n)$, the output is unbounded (infinite), the system is classified as unstable. Let $x(n)$ be a bounded input sequence, $h(n)$ be the impulse response of the system and $y(n)$ be the output sequence. Taking the magnitude of the output

$$\text{we have} \quad |y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \tag{1.136}$$

We know that the magnitude of the sum of terms is less than or equal to the sum of the magnitudes, hence

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \tag{1.137}$$

Let the bounded value of the input is equal to $M$, the Eq. (1.137) can be written as

$$|y(n)| \leq M \sum_{k=-\infty}^{\infty} h(k) \tag{1.138}$$

The above condition will be satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{1.139}$$

So, the necessary and sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{1.140}$$

**Example 1.11** Test the stability of the system whose impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$. (EIE AU Dec' 03)

**Solution** The necessary and sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Given $h(n) = (1/2)^n u(n)$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |(1/2)^n u(n)|$$

$$= \sum_{n=0}^{\infty} (1/2)^n$$

$$= 1 + 1/2 + 1/2^2 \ldots \infty \qquad \boxed{\left(\because\ 1 + a + a^2 \ldots \infty = \frac{1}{1-a}\right)}$$

$$= \frac{1}{1 - 1/2} = 2 < \infty$$

Hence the system is stable.

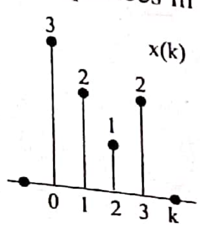
---

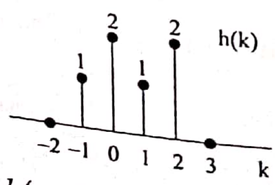
**Example 1.12** Determine the convolution sum of two sequences $x(n) = \{3, 2, 1, 2\}$; $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

**Solution**

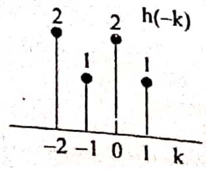
**Step 1** The sequence $x(n)$ starts at $n = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore the starting time for evaluating the output sequence $y(n)$ is $n = n_1 + n_2 = 0 + (-1) = -1$

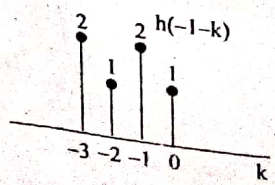
**Step 2** Express both sequences in terms of the index $k$.




**Step 3** Fold $h(k)$ about $k = 0$ to obtain $h(-k)$




As starting time to evaluate $y(n)$ is $-1$, shift $h(k)$ by one unit to left to obtain $h(-1-k)$

From Eq. (1.123)

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$

Multiply the two sequences $x(k)$ and $h(-1-k)$ element by element and sum the products

$$\Rightarrow y(-1) = 0(2) + 0(1) + 0(2) + 3(1) + 2(0) + 1(0) + 2(0)$$
$$= 3$$

Increment the index by 1, shift the sequence to right to obtain $h(-k)$ and multiply the two sequences $x(k)$ and $h(-k)$ element by element and sum the products

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

$$= 0(2) + 0(1) + 3(2) + 2(1) + 1(0) + 2(0) = 8$$

Similarly

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

$$= 0(2) + 3(1) + 2(2) + 1(1) + 2(0)$$

$$= 8$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$

$$= 3(2) + 2(1) + 1(2) + 2(1)$$

$$= 12$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= 3(0) + 2(2) + 1(1) + 2(2)$$

$$= 9$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= 3(0) + 2(0) + 1(2) + 2(1) + 0(2)$$

$$= 4$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k)$$

$$= 3(0) + 2(0) + 1(0) + 2(2) + 0(1) + 0(2) + 0(1)$$

$$= 4$$

$$y(n) = \{3, 8, 8, 12, 9, 4, 4\}$$

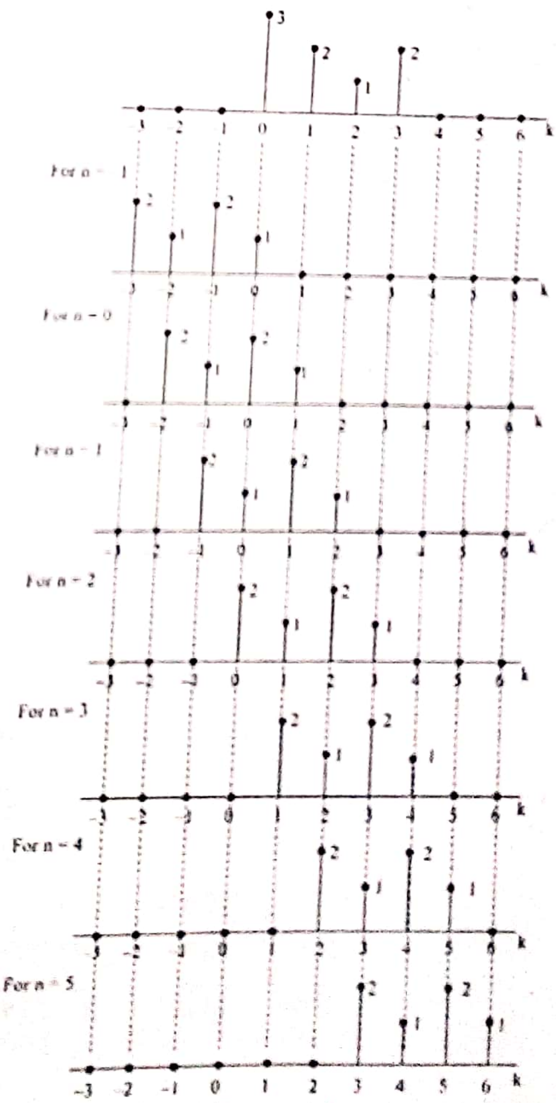


Fig. 1.58 Operation on signals $x(n)$ and $h(n)$ to compute convolution

To check the correctness of the result sum all the samples in $x(n)$ and multiply by the sum of all samples in $h(n)$. This value must be equal to sum of all samples in $y(n)$.

In the given problem $\sum_n x(n) = 8$, $\sum_n h(n) = 6$ $\sum_n y(n) = 48$

$\sum_n x(n) \cdot \sum_n h(n) = \sum_n y(n)$ proved. Therefore, the result is correct.

---

### *Method 2*

Tabulate the sequence $x(k)$ and shifted version of $h(k)$ as shown below

The starting value of $n = -1$

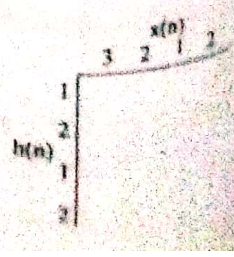
| | $k$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x(k)$ | | | | | 3 | 2 | 1 | 2 | | | | |
| $n=-1$ | $h(-1-k)$ | | 2 | 1 | 2 | 1 | | | | | | | |
| $n=0$ | $h(0-k)$ | | | 2 | 1 | 2 | 1 | | | | | | |
| $n=1$ | $h(1-k)$ | | | | 2 | 1 | 2 | 1 | | | | | |
| $n=2$ | $h(2-k)$ | | | | | 2 | 1 | 2 | 1 | | | | |
| $n=3$ | $h(3-k)$ | | | | | | 2 | 1 | 2 | 1 | | | |
| $n=4$ | $h(4-k)$ | | | | | | | 2 | 1 | 2 | 1 | | |
| $n=5$ | $h(5-k)$ | | | | | | | | 2 | 1 | 2 | 1 | |

$$y(-1) = 3(1) = 3$$
$$y(0) = 3(2) + 2(1) = 8$$
$$y(1) = 3(1) + 2(2) + 1(1) = 8$$
$$y(2) = 3(2) + 2(1) + 1(2) + 2(1) = 12$$
$$y(3) = 3(0) + 2(2) + 1(1) + 2(2) = 9$$
$$y(4) = 1(2) + 2(1) = 4$$
$$y(5) = 2(2) = 4$$
$$y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$$

### *Method 3*

Given $x(n) = \{3, 2, 1, 2\}$, $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

**Step 1:** Write down the sequence $x(n)$ and $h(n)$ as shown.



**Step 2:** Multiply each and every sample in $h(n)$ by the samples of $x(n)$ and tabulate the values.

**Step 3:** Divide the elements in the table by drawing diagonal lines as shown.

| $h(n)$ \ $x(n)$ | 3 | 2 | 1 | 2 |
|---|---|---|---|---|
| 1 | 3 | 2 | 1 | 2 |
| 2 | 6 | 4 | 2 | 4 |
| 1 | 3 | 2 | 1 | 2 |
| 2 | 6 | 4 | 2 | 4 |

**Step 4:** Starting from the left, sum all the elements in each strip and write down in the same order

$$= 3, 6+2, 3+4+1, 6+2+2+2, 4+1+4, 2+2, 4$$
$$= 3, 8, 8, 12, 9, 4, 4$$

**Step 5:** The starting value of $n = -1$, mark the symbol $\uparrow$ at time origin $(n = 0)$.

$$y(n) = \{3,\ \underset{\uparrow}{8},\ 8,\ 12,\ 9,\ 4,\ 4\}$$

---

**Example 1.13** Find the convolution of the signals

$$x(n) = 1 \quad n = -2, 0, 1$$
$$= 2 \quad n = -1$$
$$= 0 \quad \text{elsewhere}$$
$$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$$ AU IT Dec'03

**Solution** For $n = -2$

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$
$$= 1(1) = 1$$

For $n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$
$$= 1(-1) + 2(1) = 1$$

For $n = 0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$
$$= 1(1) + 2(-1) + 1(1) = 0$$

For $n = 1$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

$$= 1(-1) + 2(1) + 1(-1) + 1(1)$$

$$= 1$$

For $n = 2$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$

$$= 2(-1) + 1(1) + 1(-1)$$

$$= -2$$

For $n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= 1(-1) + 1(1)$$

$$= 0$$

For $n = 4$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= 1(-1) = -1$$

$$\therefore y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

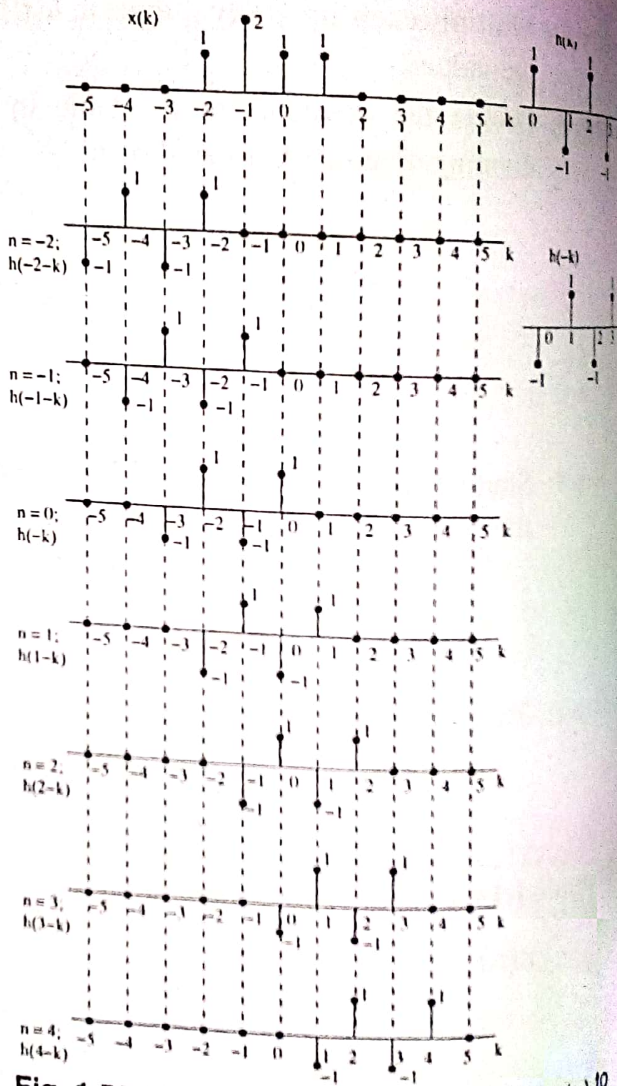
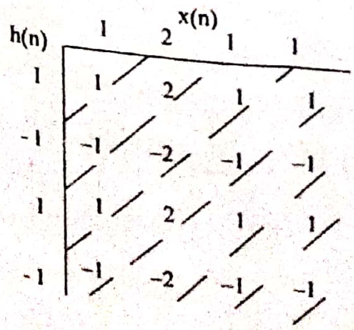



**Fig. 1.59** Operation on Signals $x(n)$ and $h(n)$ to perform convolution

**Another Method**

Given $x(n) = \{1,\ 2,\ \underset{\uparrow}{1},\ 1\}; h(n) = \{1,\ -1,\ 1, -1\}$

The starting value of $n = n_1 + n_2 = -2 + 0 = -2$




$$y(n) = 1, -1 + 2, 1 - 2 + 1, -1 + 2 - 1 + 1, -2 + 1 - 1, -1 + 1, -1$$

$$y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

From above examples we find that if the length of the sequence $x(n)$ is $N_1$, and the length of the sequence $h(n)$ is $N_2$, then the convolution of these two sequences produce a sequence $y(n)$, whose length is equal to $N_1 + N_2 - 1$.

The limits in the convolution sum can be modified according to the type of sequence and system. For a causal system the impulse response $h(n) = 0$ for $n < 0$. (1.141)

The limits in the convolution sum of a causal system is modified as

$$y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) \tag{1.142}$$

$$= \sum_{k=0}^{\infty} h(k)x(n-k) \tag{1.143}$$

If the input to the causal system is a causal sequence (i.e., $x(n) = 0$ for $n < 0$) the limits in the convolution sum is modified as

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) \tag{1.144}$$

$$= \sum_{k=0}^{n} h(k)x(n-k) \tag{1.145}$$

---

**Example 1.14** Find the convolution of two finite duration sequences

$h(n) = a^n u(n)$ for all $n$ (i) when $a \neq b$

$x(n) = b^n u(n)$ for all $n$ (ii) when $a = b$

**Solution**

The impulse response $h(n) = 0$ for $n < 0$, so the given system is causal ar $x(n) = 0$ for $n < 0$, hence the sequence is a causal sequence.

Using Eq. (1.144) we have

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} b^k a^{n-k} = a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k = a^n \left[1 + \frac{b}{a} + \frac{b^2}{a^2} \cdots (n+1) \text{ terms}\right] \tag{1.146}$$

$$= a^n \left[\frac{1-(b/a)^{n+1}}{1-b/a}\right] \quad \boxed{\text{Hint} \left(\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}\right)} \tag{1.147}$$

When $a = b$, the Eq. (1.147) reduces to indeterminate form.

Therefore, applying L'Hospital's rule we get

$$y(n) = a^n \lim_{b \to a} \left[ \frac{-(1/a)^{n+1}(n+1)b^n}{(-1/a)} \right]$$

$$= a^n \lim_{b \to a}(n+1)\left(\frac{b}{a}\right)^n = a^n(n+1) \quad \text{or}$$

When $a = b$ the Eq. (1.146) reduces to

$$y(n) = a^n \sum_{k=0}^{n} (1)^k = a^n(n+1) \qquad (1.14\ldots)$$

$\boxed{\text{Hint } (\because\ 1 + 1 + 1 \ldots n + 1 \text{ terms} = n + 1)}$

---

**Example 1.15** Find $y(n)$ if $x(n) = n + 2$ for $0 \leq n \leq 3$

$$h(n) = a^n u(n) \quad \text{for all } n$$

**Solution**

We have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Given $x(n) = n + 2$ for $0 \leq n \leq 3$

$$h(n) = a^n u(n) \quad \text{for all } n$$

$h(n) = 0$ for $n < 0$. So, the system is causal.

$x(n)$ is a causal finite sequence whose value is zero for $n > 3$.

Therefore,

$$y(n) = \sum_{k=0}^{3} x(k)h(n-k)$$

$$= \sum_{k=0}^{3} (k+2)a^{n-k}u(n-k)$$

$$= 2a^n u(n) + 3a^{n-1}u(n-1) + 4a^{n-2}u(n-2) + 5a^{n-3}u(n-3)$$

---

**Example 1.16** Determine the response of the relaxed system characterized by the impulse response $h(n) = (1/2)^n u(n)$ to the input signal $x(n) = 2^n u(n)$.

**Solution**

Given $x(n) = 2^n u(n)$;   $h(n) = (1/2)^n u(n)$

A causal signal is applied to a causal system.

Therefore,

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} 2^k \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^k \left(\frac{1}{2}\right)^{-k} = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^{2k}$$

$$= \left(\frac{1}{2}\right)^n [1 + 2^2 + 2^4 + 2^6 \ldots (n+1) \text{ terms}]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{(2^2)^{n+1} - 1}{(2)^2 - 1}\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{4 \cdot 4^n - 1}{3}\right]$$

$$= \left(\frac{1}{2}\right)^n \left(\frac{4^{n+1} - 1}{3}\right)$$

---

**Practice Problem 1.9** Compute convolution of following sequences

(i) $x(n) = h(n) = \{\underset{\uparrow}{1}, 2, -1\}$   Ans: $\{1, 4, 2, -4, 1\}$

(ii) $x(n) = u(n-1)$; $h(n) = (3)^n u(-n-1)$

Ans: $0.5(3)^n$ for $n < 0$;  $0.5$ for $n \geq 0$

(iii) $x(n) = u(n)$; $h(n) = 5\left(-\frac{1}{2}\right)^n u(n)$   Ans: $\dfrac{5\left[1 - (-\frac{1}{2})^{n+1}\right]}{1 + \frac{1}{2}}$

---

## 1.26 Interconnection of LTI Systems

### 1.26.1 (i) *Parallel Connection of Systems*

Consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in parallel as shown in Fig. 1.60(a).

From Fig. 1.60(a) the output of system 1 is

$$y_1(n) = x(n) * h_1(n) \tag{1.149}$$

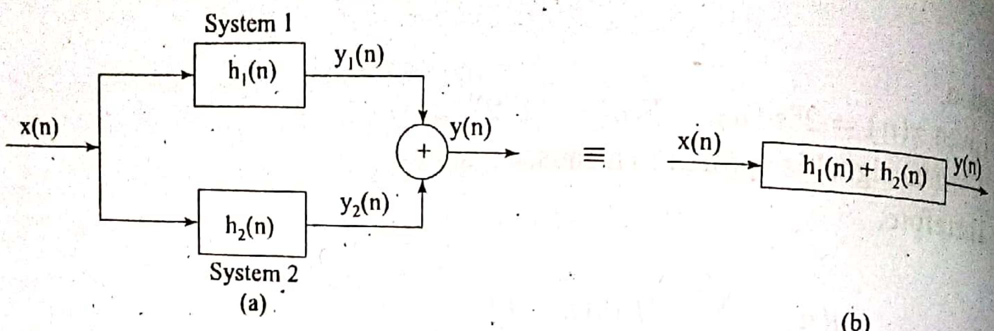



**Fig. 1.60** (a) Parallel connection of two systems; (b) Equivalent system

and the output of system 2 is

$$y_2(n) = x(n) * h_2(n). \tag{1.150}$$

The output

$$\begin{aligned} y(n) &= y_1(n) + y_2(n) \\ &= x(n) * h_1(n) + x(n) * h_2(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)h_2(n-k) \\ &= \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k) + h_2(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= x(n) * h(n) \end{aligned}$$

where $h(n) = h_1(n) + h_2(n)$. (1.151)

Thus if the two-systems are connected in parallel, then the overall impulse response is equal to sum of two impulse responses.

### 1.26.2 *(ii) Cascade Connection of Two Systems*

Consider two LTI systems with impulse-responses $h_1(n)$ and $h_2(n)$ connected in cascade. Let

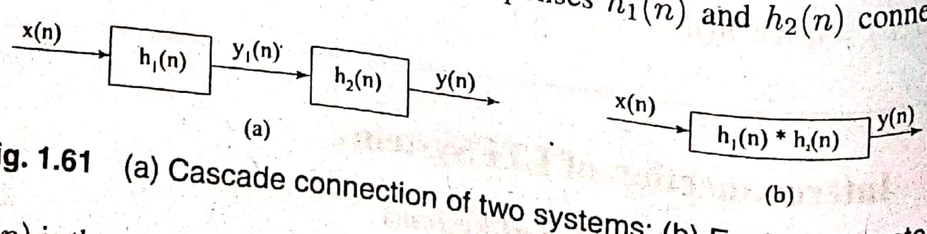



**Fig. 1.61** (a) Cascade connection of two systems; (b) Equivalent system

$y_1(n)$ is the output of the first system. Then

$$\begin{aligned} y_1(k) &= x(k) * h_1(k) \\ &= \sum_{v=-\infty}^{\infty} x(v)h_1(k-v). \end{aligned} \tag{1.152}$$

The output

$$y(n) = y_1(k) * h_2(k)$$

$$= \left[\sum_{v=-\infty}^{\infty} x(v)h_1(k-v)\right] * h_2(k) \qquad (1.153)$$

$$y(n) = \sum_{k=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} x(v)h_1(k-v)h_2(n-k)$$

Let $\qquad k - v = p$

$$y(n) = \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p)h_2(n-v-p)$$

$$= \sum_{v=-\infty}^{\infty} x(v)h(n-v)$$

$$= x(n) * h(n)$$

$$\text{where } h(n) = \sum_{k=-\infty}^{\infty} h_1(k)h_2(n-k)$$

$$= h_1(n) * h_2(n) \qquad (1.154)$$

Hence the impulse response of two LTI systems connected in cascade is the convolution of the individual impulse responses.

---

**Example 1.17** An interconnection of LTI systems is shown in Fig. 1.62. The impulse responses are $h_1(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-3)]$; $h_2(n) = \delta(n)$ and $h_3(n) = u(n-1)$. Let the impulse response of the overall system from $x(n)$ to $y(n)$ be denoted as $h(n)$. (a) Express $h(n)$ in terms of $h_1(n), h_2(n)$ and $h_3(n)$ (b) Evaluate $h(n)$.

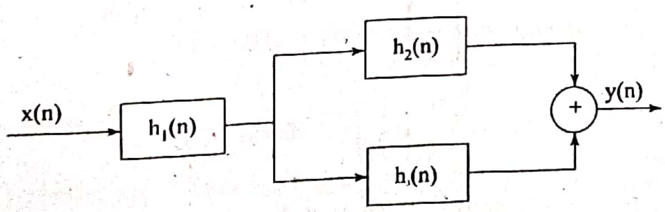



**Fig. 1.62**

**Solution**

The systems with impulse responses $h_2(n)$ and $h_3(n)$ are connected in parallel. This can be replaced by an equivalent system whose impulse response is sum of two individual impulse responses. That is,

$$h'(n) = h_2(n) + h_3(n)$$

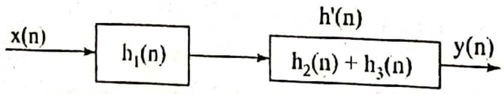



**Fig. 1.63**

Now the systems with impulse responses $h_1(n)$ and $h'(n)$ are connected in cascade. Therefore, the overall impulse response

$$h(n) = h_1(n) * h'(n)$$
$$= h_1(n) * [h_2(n) + h_3(n)]$$
$$= h_1(n) * h_2(n) + h_1(n) * h_3(n)$$

Given $h_1(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-3)]$

$$h_2(n) = \delta(n)$$
$$h_3(n) = u(n-1)$$

$h_1(n) * h_2(n)$

$$= \left[\left(\frac{1}{2}\right)^n u(n) - u(n-3)\right] * \delta(n)$$
$$= \left(\frac{1}{2}\right)^n [u(n) - u(n-3)] \qquad \boxed{\because\ x(n) * \delta(n) = x(n)}$$

$h_1(n) * h_3(n)$

$$= \left\{\left(\frac{1}{2}\right)^n [u(n) - u(n-3)]\right\} * u(n-1)$$
$$= \left(\frac{1}{2}\right)^n u(n) * u(n-1) - \left(\frac{1}{2}\right)^n u(n-3) * u(n-1)$$

Let us take

$$y_1(n) = \left(\frac{1}{2}\right)^n u(n) * u(n-1)$$
$$y_1(n) = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \quad \text{for } n \geq 1$$
$$= 0 \quad \text{for } n < 1$$
$$\Rightarrow y_1(n) = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$$
$$\Rightarrow y_1(n) = 2\left[1 - \frac{1}{2}^n\right] \quad \text{for } n \geq 1$$
$$= 0 \quad \text{for } n < 1$$

Therefore,

$$y_1(n) = 2\left[1 - \left(\frac{1}{2}\right)^n\right] u(n-1)$$

$$y_2(n) = \left(\frac{1}{2}\right)^n u(n-3) * u(n-1)$$

$$= \sum_{k=3}^{n-1} \left(\frac{1}{2}\right)^k \quad \text{for } n \geq 4$$

$$= 0 \qquad \text{for } n < 4$$

$$\Rightarrow y_2(n) = \left(\frac{1}{8}\right) \frac{\left[1 - \left(\frac{1}{2}\right)^{n-3}\right]}{1 - \frac{1}{2}} \quad \text{for} \quad n \geq 4$$

$$= \frac{1}{4}\left[1 - 8\left(\frac{1}{2}\right)^n\right] \quad \text{for} \quad n \geq 4$$

$$= \left[\frac{1}{4} - 2\left(\frac{1}{2}\right)^n\right] \quad \text{for} \quad n \geq 4$$

$$= \frac{1}{4}u(n-4) - 2\left(\frac{1}{2}\right)^n u(n-4)$$

$$\Rightarrow h(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-3)] + 2\left[1 - \left(\frac{1}{2}\right)^n\right] u(n-1)$$

$$+ \left[\frac{1}{4} - 2\left(\frac{1}{2}\right)^n\right] u(n-4)$$

**Practice Problem 1.10** An interconnection of LTI system is shown in Fig.1.62. The impulse response $h_1(n) = \left(\frac{1}{2}\right)^n u(n)$; $h_2(n) = u(n)$; $h_3(n) = u(n-4)$. Evaluate $h(n)$.

**Ans:** $h(n) = 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right]$ for $n < 4$

$= 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] + 2\left[1 - \left(\frac{1}{2}\right)^{n-3}\right]$ for $n \geq 4$

## 1.27 Correlation of Two Sequences

So far we have discussed about the convolution of two signals which is used to find the output $y(n)$ of a system, if the impulse response $h(n)$ of the system and the input signal $x(n)$ are known. In this section, we will study a mathematical operation known as correlation that closely resembles convolution. Correlation is basically used to compare two signals. It occupies a significant place in signal processing. It has applications in radar and sonar system where the location of the target is measured by comparing the transmitted and reflected signals. Other applications of correlation include image processing and control engineering etc.

**b) Determine the inverse Fourier transform of the first order recursive filter**

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

**Solution**

a) Given

$$Y(e^{j\omega}) = \frac{1}{3}\left[\frac{e^{j\omega}}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{-j\omega}} + \frac{e^{-j\omega}}{1 - ae^{-j\omega}}\right]$$

using time shifting property we have

$$y(n) = \frac{1}{3}[a^{n+1}u(n+1) + a^n u(n) + a^{n-1}u(n-1)]$$

b) Given $H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$
Taking inverse Fourier transform we have

$$h(n) = a^n u(n)$$

---

**Example 1.78** Test the causality and stability of the following systems
(PU EIE Nov. 2002)

(i) $y(n) = x(n) - x(-n-1) + x(n-1)$ (ii) $y(n) = \sin(x(n))$

**Solution**

$$y(n) = x(n) - x(-n-1) + x(n-1)$$

The output depends on future values of input. Therefore non causal. The impulse response is

$$h(n) = \delta(n+1) - \delta(-n-1) + \delta(n-1)$$

The impulse response is absolutely summable.
Hence the system is stable.

$$(ii) \quad y(n) = \sin[x(n)]$$

causal
If $|x(n)| < M$, then $|y(n)| = |\sin x(n)| < M$
Hence the system is BIBO stable.

**Solution**

From example (1.38) we can find

$$X(e^{j\omega}) = \frac{\sin\frac{\omega L}{2}}{\sin\frac{\omega}{2}} e^{-j(L-1)\omega/2}$$

The energy density spectrum is given by

$$S_{xx}(e^{j\omega}) = |X(e^{j\omega})|^2 = \frac{\sin^2\frac{\omega L}{2}}{\sin^2\frac{\omega}{2}}$$

---

**Example 1.82** A system has unit sample response $h(n)$ given by $h(n) = -\frac{1}{4}\delta(n+1) + \frac{1}{2}\delta(n) - \frac{1}{4}\delta(n-1)$. Is the system is BIBO stable? Is the filter causal? Justify your answer

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**Solution**

For the system to be BIBO stable .

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given impulse response is absolutely summable. Therefore the system is stable.

The output of the filter depends on future values. Hence the system is non-causal.

---

**Example 1.83** Let $X(e^{j\omega})$ denote the DFT of a real sequence $x(n)$. Express the inverse DTFT $y(n)$ of $Y(e^{j\omega}) = X(e^{j3\omega})$ in terms of $x(n)$.

Given

$$Y(e^{j\omega}) = X(e^{j3\omega})$$

**Solution**

The frequency spectrum $X(e^{j3\omega})$ is three fold repetition of $X(e^{j\omega})$. Therefore the period is given by $\frac{2\pi}{3}$. We have

$$\begin{aligned}
y(n) &= \frac{1}{2\pi/3}\int_{-\pi/3}^{\pi/3} X(e^{j3\omega})e^{j\omega n}d\omega \\
&= \frac{3}{2\pi}\int_{-\pi}^{\pi} X(e^{j\phi})e^{j\phi(\frac{n}{3})}\left(\frac{d\phi}{3}\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\phi})e^{j\phi(\frac{n}{3})}d\phi \\
&= x\left(\frac{n}{3}\right)
\end{aligned}$$

$$\Rightarrow \quad y(n) = x\left(\frac{n}{3}\right) \quad \text{if } n \text{ is multiple of 3}$$

$$= 0 \quad \text{otherwise}$$

**Example 1.86** Determine if the system $y(n) = x(n^2)$ is time invariant

MU oct',02

Given

$$y(n) = T[x(n)] = x(n^2)$$

The output due to the delayed input is

$$y(n,k) = T[x(n-k)] = x(n^2 - k)$$

The delayed output is

$$y(n-k) = [x(n-k)^2]$$
$$y(n,k) \neq y(n-k)$$

Therefore the system is time-variant

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**Example 1.87** Check whether the system given by $y(n) = x(-n+2)$ is linear, causal or not

MU Oct'02

$$\text{Given} \quad y(n) = T[x(n)] = x(-n+2)$$

linearity check. The output due to input $x_1(n)$ is given by $y_1(n) = x_1(-n+2)$.

The output due to input $x_2(n)$ is given by $y_2(n) = x_2(-n+2)$

The output due to weighted sum of input is given by $y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = a_1x_1(-n+2) + a_2x_2(-n+2)$

The weighted sum of output is

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(-n+2) + a_2x_2(-n+2)$$
$$y_3(n) = a_1y_1(n) + a_2y_2(n)$$

Therefore the system is linear.

Causality check: $y(n) = x(-n+2)$ The output at $n = -1$ depends on the input n=3; That is, the output depends on future input. Therefore, the system is non-causal

---

**Example 1.88** A discrete-time signal

$x(n) = -2, -1, 0, \underset{\uparrow}{1}, -1, 1$ is multiplied by $u(-n-2)$. What is the resulting signal.

**Solution**

The sequence $u(-n-2)$ is given by

$$u(-n-2) = 1 \text{ for } n \leq -2$$
$$= 0 \text{ for } n > -2$$

then the resulting signal is $-2, -1, 0, \underset{\uparrow}{0}$

Comparing Eq.(1) and Eq.(2) we g...

$$h(0) = 1$$
$$h(1) = a$$
$$h(2) = a^2$$
$$h(3) = a^3$$
$$\vdots \quad \vdots$$
$$\Rightarrow \; h(n) = a^n u(n)$$

---

**Example 1.97** Check the following systems for linearity, causality, time invariance and stability using appropriate tests.

(i) $y(n) = ne^{|x(n)|}$

(ii) $y(n) = a^{x(n)} \cos(2\pi n/N)$

**Solution**

(i) $y(n) = ne^{|x(n)|}$

$$y_1(n) = ne^{|x_1(n)|}$$
$$y_2(n) = ne^{|x_2(n)|}$$
$$a_1 y_1(n) + a_2 y_2(n) = a_1 ne^{|x_1(n)|} + a_2 ne^{|x_2(n)|}$$
$$x_3(n) = a_1 x_1(n) + a_2 x_2(n)$$
$$y_3(n) = ne^{|a_1 x_1(n) + a_2 x_2(n)|}$$
$$y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$$

Hence the system is non-linear.

The output depends on the present input. Hence the system is causal.

$$y(n-k) = ne^{|x(n-k)|}$$
$$y(n,k) = ne^{|x(n-k)|}$$
$$y(n,k) = y(n-k)$$

Hence the system is time invariant.

Even $|x(n)| < M$, as $n \to \infty, y(n) \to \infty$. Hence the system is unstable

Ans : linear, causal, time invariant and unstable.

(ii) $y(n) = a^{x(n)} \cos(2\pi n/N)$

Linear, causal, time invariant

If $|x(n)| < M$, then $y(n)$ is bounded. Hence the system is stable.

**Example 1.98** Compute discrete time Fourier transform of the following sequence.

$$x(n) = \{1, -2, 3, 4\}$$

**Solution**

$$\text{Given } x(n) = \{1, -2, 3, 4\}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega}$$

$$= 1 - 2e^{-j\omega} + 3e^{-j2\omega} + 4e^{-j3\omega}$$

---

**Example 1.99** By direct evaluation of the convolution sum, determine the step response of a linear shift invariant system whose unit sample response $y(n)$ is given by

$$h(n) = a^{-n}u(-n), 0 < a < 1$$

JNTU Apr'06

**Solution**

$$\text{Given} \quad h(n) = a^{-n}u(-n)$$

$$h(n) = u(n) = 1 \text{ for } n \geq 0$$

$$= \text{ for } \quad n < 0$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} a^{-k}u(-k)\,u(n-k)$$

$$\text{For } \quad n \leq 0 \quad y(n) = \sum_{k=-\infty}^{-n} a^{-k}$$

$$= \sum_{k=n}^{\infty} a^{k} = \frac{a^n}{1-a}$$

$$\text{For } \quad n \geq 0 \quad y(n) = \sum_{k=-\infty}^{0} a^{-k}$$

$$= \sum_{k=0}^{\infty} a^{k}$$

$$= \frac{1}{1-a}$$