

## 5.40 Digital Signal Processing

**Example 5.11** For the analog transfer function  $H(s) = \frac{2}{(s+1)(s+2)}$  determine  $H(z)$  using impulse invariance method. Assume  $T = 1$  sec.

**Solution**

Given  $H(s) = \frac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$\begin{aligned} H(s) &= \frac{A}{s+1} + \frac{B}{s+2} \\ H(s) &= \frac{2}{s+1} - \frac{2}{s+2} \\ &= \frac{2}{s - (-1)} - \frac{2}{s - (-2)} \end{aligned}$$

$$\begin{aligned} A &= (s+1) \frac{2}{(s+1)(s+2)} \Big|_{s=-1} \\ &= 2 \\ B &= (s+2) \frac{2}{(s+1)(s+2)} \Big|_{s=-2} \\ &= -2 \end{aligned}$$

Using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^N \frac{c_k}{s - p_k} \quad \text{then} \quad H(z) = \sum_{k=1}^N \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

i.e.,  $(s - p_k)$  is transformed to  $1 - e^{p_k T} z^{-1}$ .

There are two poles  $p_1 = -1$  and  $p_2 = -2$ . So

$$H(z) = \frac{2}{1 - e^{-T} z^{-1}} - \frac{2}{1 - e^{-2T} z^{-1}}$$

For  $T = 1$  sec

$$\begin{aligned} H(z) &= \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}} \\ &= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}} \\ H(z) &= \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.04976 z^{-2}} \end{aligned}$$

**Example 5.12** Using impulse invariance with  $T = 1$  sec determine  $H(z)$  if  $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$

**Solution**

$$\text{Given } H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$\begin{aligned} h(t) &= L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2 + \sqrt{2}s + 1}\right] \\ &= L^{-1}\left[\frac{1}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right] \\ &= L^{-1}\left[\sqrt{2} \cdot \frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right] \\ &= \sqrt{2}L^{-1}\left[\frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right] = \sqrt{2}e^{-t/\sqrt{2}} \sin(t/\sqrt{2}) \end{aligned}$$

Let  $t = nT$

$$h(nT) = \sqrt{2} e^{-nT/\sqrt{2}} \sin \frac{nT}{\sqrt{2}}$$

If  $T = 1$  sec

$$h(n) = \sqrt{2} e^{-n/\sqrt{2}} \sin \frac{n}{\sqrt{2}}$$

$$\begin{aligned} H(z) &= Z[h(n)] = \sqrt{2} \left[ \frac{e^{-1/\sqrt{2}} z^{-1} \sin 1/\sqrt{2}}{1 - 2e^{-1/\sqrt{2}} z^{-1} \cos \frac{1}{\sqrt{2}} + e^{-\sqrt{2}} z^{-2}} \right] \\ &= \frac{0.453z^{-1}}{1 - 0.7497z^{-1} + 0.2432z^{-2}} \end{aligned}$$

**Example 5.13** Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period  $T = 1$  sec.

**Solution**

From the table 5.1, for  $N = 3$ , the transfer function of a normalised Butterworth filter is given by

$$\begin{aligned} H(s) &= \frac{1}{(s+1)(s^2+s+1)} \\ &= \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866} \end{aligned}$$

### 5.42 Digital Signal Processing

$$A = (s+1) \frac{1}{(s+1)(s^2+s+1)} \Big|_{s=-1} = \frac{1}{(-1)^2 - 1 + 1} = 1$$

$$B = (s+0.5+j0.866) \frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)} \Big|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-j0.866-j0.866)}$$

$$= \frac{1}{-j1.732(0.5-j0.866)} = \frac{1}{-j0.866-1.5}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

Hence

$$\begin{aligned} H(s) &= \frac{1}{s+1} + \frac{-0.5+j0.288j}{s+0.5+j0.866} + \frac{-0.5-j0.288j}{s+0.5-j0.866} \\ &= \frac{1}{s-(-1)} + \frac{-0.5+j0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-j0.288j}{s-(-0.5+j0.866)} \end{aligned}$$

In impulse invariant technique

$$\text{if } H(s) = \sum_{k=1}^N \frac{c_k}{s-p_k}, \text{ then } H(z) = \sum_{k=1}^N \frac{c_k}{1-e^{p_k T} z^{-1}}$$

Therefore,

$$\begin{aligned} H(z) &= \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5+j0.866}z^{-1}} \\ &= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}} \end{aligned}$$

**Example 5.14** Apply impulse invariant method and find  $H(z)$  for  $H(s) = \frac{s+a}{(s+a)^2+b^2}$ .

**Solution** The inverse Laplace transform of given function is

$$h(t) = \begin{cases} e^{-at} \cos(bt) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sampling the function produces

$$\begin{aligned}
 h(nT) &= \begin{cases} e^{-anT} \cos(bnT) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
 H(z) &= \sum_{n=0}^{\infty} e^{-anT} \cos(bnT) z^{-n} \\
 &= \sum_{n=0}^{\infty} \left[ e^{-anT} z^{-n} \left( \frac{e^{jbnT} + e^{-jbnT}}{2} \right) \right] \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} \left[ (e^{-(a-jb)T} z^{-1})^n + (e^{-(a+jb)T} z^{-1})^n \right] \\
 &= \frac{1}{2} \left[ \frac{1}{1 - e^{-(a-jb)T} z^{-1}} + \frac{1}{1 - e^{-(a+jb)T} z^{-1}} \right] \\
 &= \frac{1 - e^{-aT} \cos(bT) z^{-1}}{1 - 2e^{-aT} \cos(bT) z^{-1} + e^{-2aT} z^{-2}}
 \end{aligned}$$

**Example 5.15** An analog filter has a transfer function  $H(s) = \frac{10}{s^2 + 7s + 10}$ . Design a digital filter equivalent to this using impulse invariant method for  $T = 0.2$  sec.

**Solution**

Given

$$\begin{aligned}
 H(s) &= \frac{10}{s^2 + 7s + 10} \\
 &= \frac{-3.33}{s + 5} + \frac{3.33}{s + 2} = \frac{-3.33}{s - (-5)} + \frac{3.33}{s - (-2)}
 \end{aligned}$$

Using Eq. (5.81b) we have

$$\begin{aligned}
 H(z) &= T \left[ \frac{-3.33}{1 - e^{-5T} z^{-1}} + \frac{3.33}{1 - e^{-2T} z^{-1}} \right] = 0.2 \left[ \frac{-3.33}{1 - e^{-1} z^{-1}} + \frac{3.33}{1 - e^{-0.4} z^{-1}} \right] \\
 &= \left[ \frac{-0.666}{1 - 0.3678 z^{-1}} + \frac{0.666}{1 - 0.67 z^{-1}} \right] \\
 &= \frac{0.2012 z^{-1}}{1 - 1.0378 z^{-1} + 0.247 z^{-2}}
 \end{aligned}$$

**Practice Problem 5.7** An analog filter has a transfer function

$$H(s) = \frac{5}{s^3 + 6s^2 + 11s + 6}$$

Design a digital filter equivalent to this using impulse invariant method for  $T = 1$  sec.

**Practice Problem 5.8** An analog filter has a transfer function

$$H(s) = \frac{s + 3}{s^2 + 6s + 25}$$

Design a digital filter equivalent to this using impulse invariant method for  $T = 1$  sec.

### 5.12.3 Design of IIR Filter Using Bilinear Transformation

The bilinear transformation is a conformal mapping that transforms the  $j\Omega$  axis into the unit circle in the  $z$ -plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LHP of ' $s$ ' are mapped inside the unit circle in the  $z$ -plane and all points in the RHP of ' $s$ ' are mapped into corresponding points outside the unit circle in the  $z$ -plane.

Let us consider an analog linear filter with system function

$$H(s) = \frac{b}{s + a} \quad (5.82)$$

which can be written as

$$\begin{aligned} \frac{Y(s)}{X(s)} &= \frac{b}{s + a} \\ \text{so } sY(s) + aY(s) &= bX(s) \end{aligned} \quad (5.83)$$

This can be characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad (5.84)$$

$y(t)$  can be approximated by the trapezoidal formula.

Thus

$$y(t) = \int_{t_0}^t y'(\tau) d\tau + y(t_0) \quad (5.85)$$

where  $y'(t)$  denotes the derivative of  $y(t)$ .

The approximation of the integral in Eq.(5.85) by the trapezoidal formula at  $t = nT$  and  $t_0 = nT - T$  yields

$$y(nT) = \frac{T}{2} [y'(nT) + y'(nT - T)] + y(nT - T) \quad (5.86)$$

From the differential Eq.(5.84) we obtain

$$y'(nT) = -ay(nT) + bx(nT) \quad (5.87)$$

Substituting Eq.(5.87) in Eq.(5.86) we get

$$y(nT) = \frac{T}{2} [-ay(nT) + bx(nT) - ay(nT - T) + bx(nT - T)] + y(nT - T)$$



which implies

$$y(nT) + \frac{aT}{2} y(nT) = \left(1 - \frac{aT}{2}\right) y(nT - T) + \frac{bT}{2} [x(nT) + x(nT - T)] \quad (5.88)$$

With  $y(n) = y(nT)$  and  $x(n) = x(nT)$  we obtain the result

$$\left(1 + \frac{aT}{2}\right) y(n) = \left(1 - \frac{aT}{2}\right) y(n-1) + \frac{bT}{2} [x(n) + x(n-1)]$$

The  $z$ -transform of this difference equation is

$$\left(1 + \frac{aT}{2}\right) Y(z) = \left(1 - \frac{aT}{2}\right) z^{-1} Y(z) + \frac{bT}{2} [1 + z^{-1}] X(z)$$

The system function of the digital filter is

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{\frac{bT}{2} (1 + z^{-1})}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}} \\ &= \frac{\frac{bT}{2} (1 + z^{-1})}{(1 - z^{-1}) + \frac{aT}{2} (1 + z^{-1})} \end{aligned}$$

Dividing numerator and denominator by  $\frac{T}{2} (1 + z^{-1})$  we get

$$H(z) = \frac{b}{\frac{2}{T} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) + a} \quad (5.89)$$

Comparing Eq.(5.82) and Eq.(5.89), the mapping from  $s$ -plane to the  $z$ -plane can be obtained as

$$s = \frac{2}{T} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad (5.90)$$

This relationship between  $s$  and  $z$  is known as bilinear transformation.

Let  $z = re^{j\omega}$  and

$$s = \sigma + j\Omega \quad (5.90a)$$

then Eq.(5.90) can be expressed as

$$\begin{aligned} s &= \frac{2(z-1)}{T(z+1)} \\ &= \frac{2}{T} \left[ \frac{re^{j\omega} - 1}{re^{j\omega} + 1} \right] = \frac{2}{T} \left[ \frac{r \cos \omega - 1 + jr \sin \omega}{r \cos \omega + 1 + jr \sin \omega} \right] \\ &= \frac{2}{T} \left[ \frac{r \cos \omega - 1 + jr \sin \omega}{r \cos \omega + 1 + jr \sin \omega} \right] \left[ \frac{r \cos \omega + 1 - jr \sin \omega}{r \cos \omega + 1 - jr \sin \omega} \right] \\ &= \frac{2}{T} \left[ \frac{r^2 \cos^2 \omega - 1 + r^2 \sin^2 \omega + j2r \sin \omega}{(r \cos \omega + 1)^2 + r^2 \sin^2 \omega} \right] \\ &= \frac{2}{T} \left[ \frac{r^2 \cos^2 \omega - 1 + r^2 \sin^2 \omega + j2r \sin \omega}{1 + r^2 \cos^2 \omega + 2r \cos \omega + r^2 \sin^2 \omega} \right] \end{aligned}$$

Separating imaginary and real parts, we have

$$s = \frac{2}{T} \left[ \frac{r^2 - 1}{1 + r^2 + 2r \cos \omega} + j \frac{2r \sin \omega}{1 + r^2 + 2r \cos \omega} \right] \quad (5.90b)$$

Comparing Eq. (5.90a) and Eq. (5.90b), we have

$$\sigma = \frac{2}{T} \left[ \frac{r^2 - 1}{1 + r^2 + 2r \cos \omega} \right]; \Omega = \frac{2}{T} \left[ \frac{2r \sin \omega}{1 + r^2 + 2r \cos \omega} \right] \quad (5.91)$$

From Eq. (5.91), we find that if  $r \leq 1$ , then  $\sigma < 0$  and if  $r > 1$ , then  $\sigma > 0$ . Consequently the LHP in 's' maps into the inside of the unit circle in the z-plane and the RHP in the 's' maps into the outside of the unit circle. When  $r = 1$ , then  $\sigma = 0$  and

$$\begin{aligned} \Omega &= \frac{2}{T} \frac{\sin \omega}{1 + \cos \omega} = \frac{2}{T} \frac{2 \sin \frac{\omega}{2} \cos \frac{\omega}{2}}{2 \cos^2 \frac{\omega}{2}} \\ &= \frac{2}{T} \tan \frac{\omega}{2} \end{aligned} \quad (5.92)$$

or

$$\omega = 2 \tan^{-1} \frac{\Omega T}{2} \quad (5.93)$$

### ✓ The warping effect

Let  $\Omega$  and  $\omega$  represent the frequency variables in the analog filter and the derived digital filter respectively. From Eq. (5.92) we have

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

For small value of  $\omega$

$$\Omega = \frac{2}{T} \cdot \frac{\omega}{2} = \frac{\omega}{T}$$

$$\omega = \Omega T$$

$\therefore \text{for small value of } \theta$   
 $\tan \theta = \theta$

(5.94)

For low frequencies the relationship between  $\Omega$  and  $\omega$  are linear, as a result, the digital filter have the same amplitude response as the analog filter. For high frequencies, however, the relationship between  $\omega$  and  $\Omega$  becomes non-linear (see Fig. 5.24) and distortion is introduced in the frequency scale of the digital filter to that of the analog filter. This is known as the warping effect.

The influence of the warping effect on the amplitude response is shown in Fig. 5.25 by considering an analog filter with a number of passbands centered at regular intervals. The derived digital filter will have same number of passbands. But the center frequencies and bandwidth of higher frequency passband will tend to reduce disproportionately.

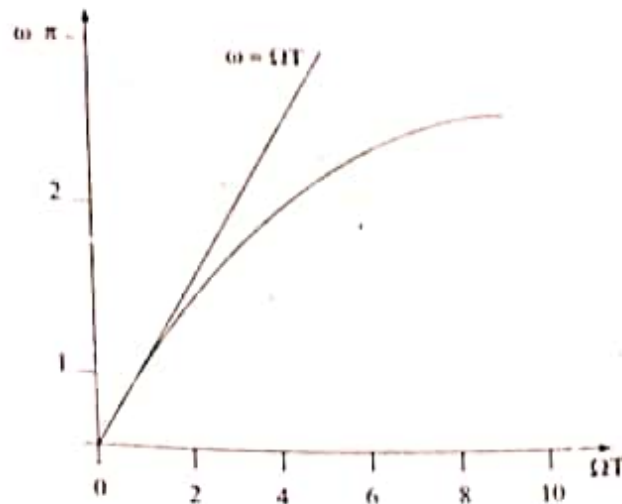
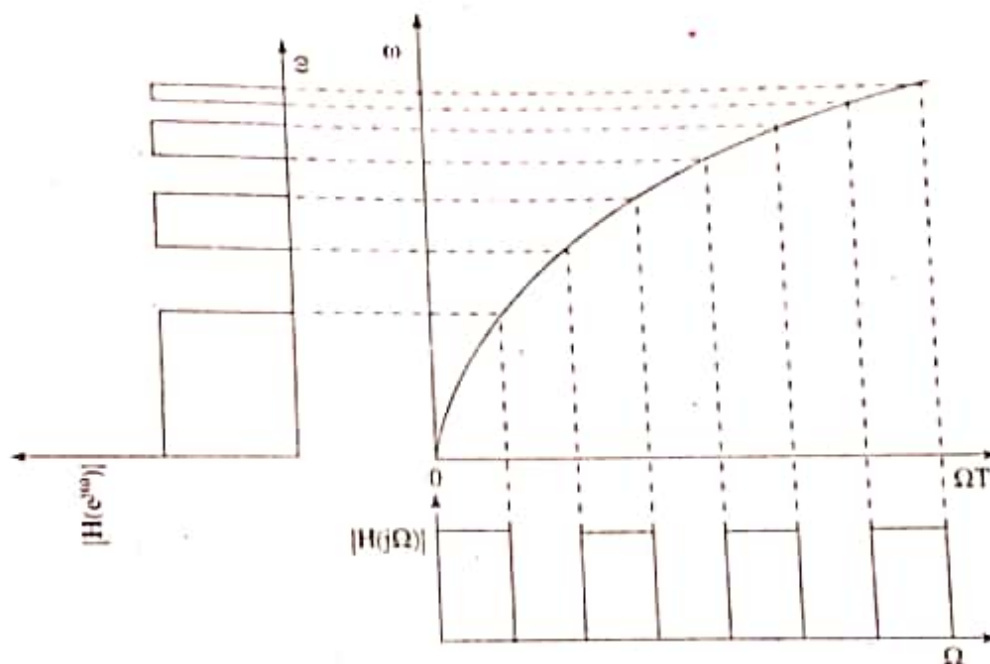

 Fig. 5.24 Relationship between  $\Omega$  and  $\omega$ .


Fig. 5.25 The effect on magnitude response due to warping effect.

The influence of the warping effect on the phase response is shown in Fig. 5.26. Considering an analog filter with linear phase response, the phase response of the derived digital filter will be non-linear.

### Prewarping

The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2} \quad (5.95)$$



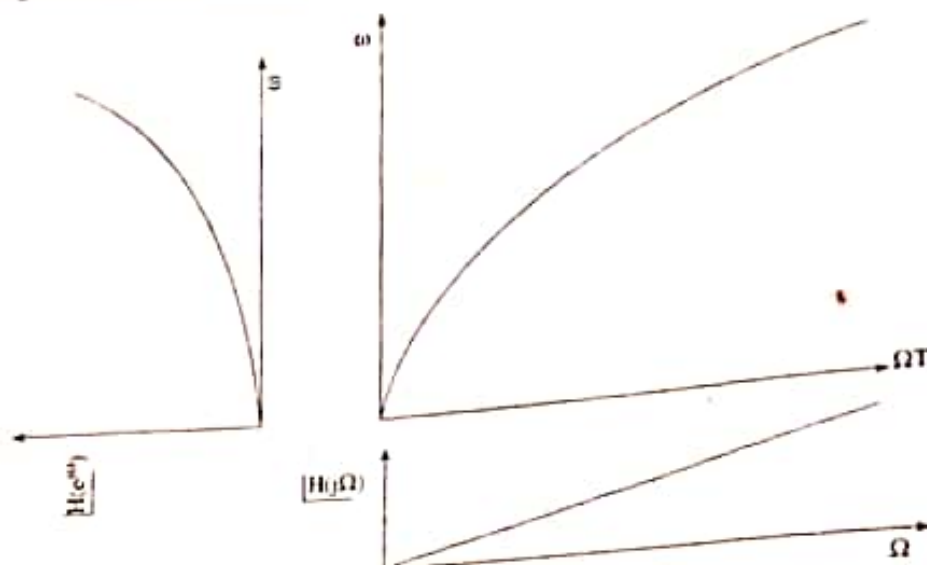


Fig. 5.26 The effect on phase response due to warping effect

Therefore, we have

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} \quad (5.96a)$$

and

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} \quad (5.96b)$$

### Steps to design digital filter using bilinear transform technique.

1. From the given specifications, find prewarping analog frequencies using formula  $\Omega = \frac{2}{T} \tan \frac{\omega}{2}$ .
2. Using the analog frequencies find  $H(s)$  of the analog filter.
3. Select the sampling rate of the digital filter, call it  $T$  seconds per sample.
4. Substitute  $s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$  into the transfer function found in step 2.

#### 5.12.4 The matched $z$ -transform

Another method for converting an analog filter into an equivalent digital filter is to map the poles and zeros of  $H(s)$  directly into poles and zeros in the  $z$ -plane. If

$$H(s) = \frac{\prod_{k=1}^M (s - z_k)}{\prod_{k=1}^N (s - p_k)}$$

where  $\{z_k\}$  are the zeros and  $\{p_k\}$  are the poles of the filter, then the system function of the digital filter is

$$H(z) = \frac{\prod_{k=1}^M (1 - e^{z_k T} z^{-1})}{\prod_{k=1}^N (1 - e^{p_k T} z^{-1})} \quad (5.97)$$

where  $T$  is the sampling interval. Thus each factor of the form  $(s - a)$  in  $H(s)$  is mapped into the factor  $(1 - e^{aT} z^{-1})$ . This mapping is called the matched  $z$ -transform.

**Example 5.16** Apply bilinear transformation to  $H(s) = \frac{2}{(s+1)(s+2)}$  with  $T = 1$  sec and find  $H(z)$ .

**Solution**

$$\text{Given } H(s) = \frac{2}{(s+1)(s+2)}$$

$$\text{Substitute } s = \frac{2}{T} \left[ \frac{1-z^{-1}}{1+z^{-1}} \right] \text{ in } H(s) \text{ to get } H(z)$$

$$\begin{aligned} H(z) &= H(s) \Big|_{s=\frac{2}{T} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)} \\ &= \frac{2}{(s+1)(s+2)} \Big|_{s=\frac{2}{T} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)} \end{aligned}$$

Given  $T = 1$  sec

$$\begin{aligned} H(z) &= \frac{2}{\left\{ 2 \left( \frac{1-z^{-1}}{1+z^{-1}} \right) + 1 \right\} \left\{ 2 \left( \frac{1-z^{-1}}{1+z^{-1}} \right) + 2 \right\}} \\ &= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)} \\ &= \frac{(1+z^{-1})^2}{6-2z^{-1}} \\ &= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})} \end{aligned}$$

**Example 5.17** Using the bilinear transform, design a highpass filter, monotonic in passband with cutoff frequency of 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.