Further simplifying Eq. (5.30) we get

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} \tag{5.31}$$

$$= \frac{\Omega_p}{\epsilon^{1/N}} \tag{5.32}$$

From Eq. (5.19) we have

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$\Omega_s = \Omega_p \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{1/2N}$$

$$= \Omega_c(10^{0.1\alpha_p} - 1)^{1/2N} \cdot \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{1/2N}$$

$$\Rightarrow \quad \Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}} \tag{5.32a}$$

Therefore

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

5.6 Steps to design an analog Butterworth lowpass filter

1. From the given specifications find the order of the filter N.
2. Round off it to the next higher integer.
3. Find the transfer function $H(s)$ for $\Omega_c = 1$ rad/sec for the value of N.
4. Calculate the value of cutoff frequency Ω_c.
5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s \rightarrow \frac{s}{\Omega_c}$ in $H(s)$.

Example 5.4 Design an analog Butterworth filter that has a -2 dB passband attenuation at a frequency of 20 rad/sec and atleast -10 dB stopband attenuation at 30 rad/sec.

Solution

Given $\alpha_p = 2\,\text{dB}; \quad \Omega_p = 20\,\text{rad/sec}$

$\alpha_s = 10\,\text{dB}; \quad \Omega_s = 30\,\text{rad/sec}$

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\dfrac{10-1}{10^{0.2}-1}}}{\log \dfrac{30}{20}}$$

$$\geq 3.37$$

Rounding off N to the next highest integer we get

$$N = 4$$

The normalized lowpass Butterworth filter for $N = 4$ can be found from table 5.1 as

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

From Eq. (5.31) we have

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{20}{(10^{0.2} - 1)^{1/8}} = 21.3868$$

> **Note:**
> To find the cutoff frequency Ω_c either (5.31) or (5.32a) can be used. The Eq. (5.31) satisfies passband specification at Ω_p, while the stopband specification at Ω_s is exceeded. The Eq. (5.32a) satisfies the stopband specification Ω_s, while the passband specification at Ω_p is exceeded. All the examples in this chapter are solved using Eq. (5.31). Students are advised to solve the exercise problems using Eq. (5.32a).

The transfer function for $\Omega_c = 21.3868$ can be obtained by substituting

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

i.e.,
$$H(s) = \frac{1}{\left(\dfrac{s}{21.3868}\right)^2 + 0.76537\left(\dfrac{s}{21.3868}\right) + 1} \times \frac{1}{\left(\dfrac{s}{21.3868}\right)^2 + 1.8477\left(\dfrac{s}{21.3868}\right) + 1}$$

$$= \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

Example 5.5 For the given specifications design an analog Butterworth filter. $0.9 \leq |H(j\Omega)| \leq 1$ for $0 \leq \Omega \leq 0.2\pi$. $|H(j\Omega)| \leq 0.2$ for $0.4\pi \leq \Omega \leq \pi$.

Solution

From the data we find $\Omega_p = 0.2\pi$; $\Omega_s = 0.4\pi$; $\frac{1}{\sqrt{1+\epsilon^2}} = 0.9$ and $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$ from which we obtain

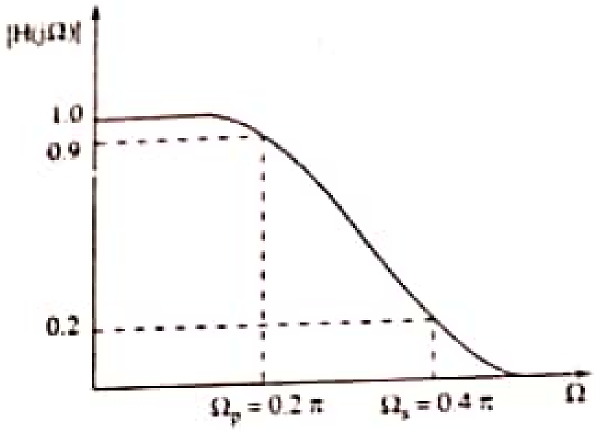

Fig. 5.8 Magnitude response of example 5.5

$$\varepsilon = 0.484 \text{ and } \lambda = 4.898$$

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\frac{\Omega_s}{\Omega_p}} = \frac{\log\frac{4.898}{0.484}}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34$$

i.e., $N = 4$

From the table 5.1, for $N = 4$, the transfer function of normalised Butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

we know $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \dfrac{\Omega_p}{\varepsilon^{1/N}} = \dfrac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi.$

$H(s)$ for $\Omega_c = 0.24\pi$ can be obtained by substituting $s \to \dfrac{s}{0.24\pi}$ in $H(s)$ i.e.,

$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right) + 1\right\}} \times \frac{1}{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right) + 1}$$

$$= \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$

Practice Problem 5.1 For the given specifications find the order of the Butterworth filter

$$\alpha_p = 3\,\text{dB}; \quad \alpha_s = 18\,\text{dB}; \quad f_p = 1\,\text{kHz}; \quad f_s = 2\,\text{kHz}.$$

Practice Problem 5.2 Design an analog Butterworth filter that has

$$\alpha_p = 0.5\,\text{dB}; \quad \alpha_s = 22\,\text{dB}; \quad f_p = 10\,\text{kHz}; \quad f_s = 25\,\text{kHz}.$$

---

6. The numerator of the transfer function depends on the value of N.

   (a) For N odd substitute $s = 0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer function.
   ($\because$ For N odd the magnitude response $|H(j\Omega)|$ starts at 1.)

   (b) For N even substitute $s = 0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to the numerator.

Example 5.6 Given the specifications $\alpha_p = 3\,\text{dB}$; $\alpha_s = 16\,\text{dB}$; $f_p = 1\,\text{KHz}$ and $f_s = 2\,\text{KHz}$. Determine the order of the filter using Chebyshev approximation. Find $H(s)$.

Solution

From the given data we can find

$$\Omega_p = 2\pi \times 1000\,\text{Hz} = 2000\,\pi\ \text{rad/sec}$$
$$\Omega_s = 2\pi \times 2000\,\text{Hz} = 4000\,\pi\ \text{rad/sec}$$

and $\alpha_p = 3\,\text{dB}$; $\alpha_s = 16\,\text{dB}$.

Step 1:

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \cosh^{-1}\frac{\sqrt{\dfrac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\dfrac{4000\pi}{2000\pi}}$$
$$= 1.91$$

Step 2: Rounding N to next higher value we get $N = 2$.

For N even, the oscillatory curve starts from $\dfrac{1}{\sqrt{1+\varepsilon^2}}$.

Step 3: The values of minor axis and major axis can be found as below

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = (10^{0.3} - 1)^{0.5} = 1$$
$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$
$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 2000\pi \frac{[(2.414)^{1/2} - (2.414)^{-1/2}]}{2} = 910\pi$$
$$b = \Omega_p \frac{[\mu^{1/N} + \mu^{-1/N}]}{2} = 2000\pi \frac{[(2.414)^{1/2} + (2.414)^{-1/2}]}{2} = 2197\pi$$

Step 4: The poles are given by

$$s_k = a\cos\phi_k + jb\sin\phi_k, \quad k = 1,2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1,2$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225^\circ$$

$$s_1 = a\cos\phi_1 + j\,b\sin\phi_1 = -643.46\pi + j1554\pi$$

$$s_2 = a\cos\phi_2 + j\,b\sin\phi_2 = -643.46\pi - j1554\pi$$

Step 5: The denominator of $H(s) = (s + 643.46\pi)^2 + (1554\pi)^2$

Step 6: The numerator of $H(s) = \dfrac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}} = (1414.38)^2\pi^2$

The transfer function $H(s) = \dfrac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$.

Example 5.7 Obtain an analog Chebyshev filter transfer function that satisfies the constraints $\dfrac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1; \quad 0 \le \Omega \le 2$

$$|H(j\Omega)| < 0.1; \quad \Omega \ge 4$$

Solution

Step 1: From the given data we can find that

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{1+\lambda^2}} = 0.1,$$

$\Omega_p = 2$ and $\Omega_s = 4$, from which we can obtain $\varepsilon = 1$ and $\lambda = 9.95$.

We know

$$N \ge \frac{\cosh^{-1}\dfrac{\lambda}{\varepsilon}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1} 9.95}{\cosh^{-1} 2} = 2.269$$

Step 2: Rounding N to next higher value we get $N = 3$. For N odd, the oscillatory curve starts from unity as shown in Fig. 5.12.

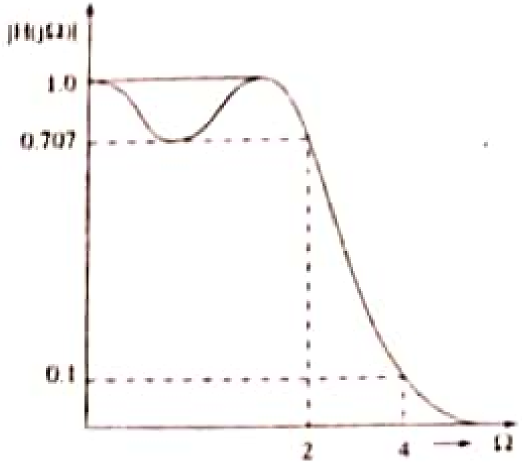

Fig. 5.12 Magnitude response of example 5.7

Step 3: Finding the values of a and b

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 2.414$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right] = 2 \left[\frac{(2.414)^{1/3} - (2.414)^{-1/3}}{2} \right]$$
$$= 0.596$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right] = 2 \left[\frac{(2.414)^{1/3} + (2.414)^{-1/3}}{2} \right]$$
$$= 2.087$$

Step 4: To calculate the poles of Chebyshev filter

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, 3$$
$$\phi_1 = 120^\circ,\ \phi_2 = 180^\circ,\ \phi_3 = 240^\circ$$

We know $s_k = a \cos \phi_k + jb \sin \phi_k \quad k = 1, 2, 3$ from which we get

$$s_1 = a \cos \phi_1 + jb \sin \phi_1 = 0.596 \cos 120^\circ + j2.087 \sin 120^\circ = -0.298 + j1.807$$
$$s_2 = a \cos \phi_2 + jb \sin \phi_2 = 0.596 \cos 180^\circ + j2.087 \sin 180^\circ = -0.596$$
$$s = a \cos \phi_3 + jb \sin \phi_3 = 0.596 \cos 240^\circ + j2.087 \sin 240^\circ = -0.298 - j1.807$$

Step 5: The denominator polynomial is given by

$$(s + 0.596)\{(s + 0.298) - j1.807\}\{(s + 0.298) + j1.807\}$$
$$= (s + 0.596)[(s + 0.298)^2 + (1.807)^2]$$
$$= (s + 0.596)(s^2 + 0.596s + 3.354)$$

Step 6: The numerator of $H(s)$ can be obtained by substituting $s = 0$ (for N odd) in the denominator.

Therefore the numerator of $H(s) = 2$

The transfer function of Chebyshev filter for the given specifications is given by

$$H(s) = \frac{2}{(s + 0.596)(s^2 + 0.596s + 3.354)}$$

Example 5.8 Determine the order and the poles of a type I lowpass Chebyshev filter that has a 1 dB ripple in the passband and passband frequency $\Omega_p = 1000\pi$, a stopband frequency of 2000π and an attenuation of 40 dB or more.

Solution

Given data $\alpha_p = 1$ dB; $\Omega_p = 1000\pi$ rad/sec; $\alpha_s = 40$ dB
$\Omega_s = 2000\pi$ rad/sec

$$N \geq \frac{\cos h^{-1}\sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cos h^{-1}\dfrac{\Omega_s}{\Omega_p}} \geq \frac{\cos h^{-1}\sqrt{\dfrac{10^4 - 1}{10^{0.1} - 1}}}{\cos h^{-1}\dfrac{2000\pi}{1000\pi}} = 4.536$$

i.e., $N = 5$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 0.508; \quad \mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 4.17$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 289.5\pi; \quad b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 1041\pi$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \ldots 5$$

$$\phi_1 = 180°; \ \phi_2 = 144°; \ \phi_3 = 180°; \ \phi_4 = 216°; \ \phi_5 = 252°$$

$$s_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2, \ldots 5$$

$$s_1 = -89.5\pi + j989\pi; \quad s_2 = -234.2\pi + j612\pi; \quad s_3 = -289.5\pi$$

$$s_4 = -234.2\pi - j612\pi; \quad s_5 = -89.5\pi - j989\pi$$

Example 5.9 Design a Chebyshev filter with a maximum passband attenuation of 2.5 dB; at $\Omega_p = 20$ rad/sec and the stopband attenuation of 30 dB at $\Omega_s = 50$ rad/sec.

(AU ECE May'07)

Solution

Given

$$\Omega_p = 20 \ \text{rad/sec}; \quad \alpha_p = 2.5\ \text{dB};$$
$$\Omega_s = 50 \ \text{rad/sec}; \quad \alpha_s = 30\ \text{dB};$$

We know

$$N = \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1} 1/k}$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = 31.607$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 0.882$$

$$k = \frac{\Omega_p}{\Omega_s} = 0.4$$

Now

$$N \geq \frac{\cosh^{-1} \frac{31.607}{0.882}}{\cosh^{-1} \frac{1}{0.4}} = 2.726$$

i.e., $N = 3$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 2.65$$

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 6.6$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 21.06$$

$$s_k = a\cos\phi_k + jb\sin\phi_k; \quad k = 1, 2, 3$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi; \quad k = 1, 2, 3$$

$$\phi_1 = 120^\circ, \phi_2 = 180^\circ, \phi_3 = 240^\circ$$

$$s_1 = -3.3 + j18.23$$

$$s_2 = -6.6$$

$$s_3 = -3.3 - j18.23$$

Denominator of $H(s) = (s + 6.6)(s^2 + 6.6s + 343.2)$

Numerator of $H(s) = (6.6)(343.2) = 2265.27$

Transfer function $H(s) = \dfrac{2265.27}{(s + 6.6)(s^2 + 6.6s + 343.2)}$

Practice Problem 5.4 For the given specifications find the order of the Chebyshev-I filter

$$\alpha_p = 1.5\,\text{dB}; \quad \alpha_s = 10\,\text{dB}; \quad \Omega_p = 2\,\text{rad/sec}; \quad \Omega_s = 30\,\text{rad/sec}$$

Practice Problem 5.5 Find the pole locations of a normalized Chebyshev filter of order 5.

These poles map to z-plane poles z_1 and z_2, via impulse invariant mapping.

$$z_1 = e^{s_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T} \tag{5.76a}$$

$$z_2 = e^{s_2 T} = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} \cdot e^{j\Omega T + j2\pi}$$

$$= e^{\sigma T} \cdot e^{j\Omega T} \quad (\because e^{j2\pi} = 1) \tag{5.76b}$$

From Eq.(5.76a) and Eq.(5.76b) we find that these poles map to the same location in the z-plane. There are an infinite number of s-plane poles that map to the same location in the z-plane. They must have the same real parts and imaginary parts that differ by some integer multiple of $\frac{2\pi}{T}$. This is the main disadvantage of impulse invariant mapping. The s-plane poles having imaginary parts greater than $\frac{\pi}{T}$ or less than $-\frac{\pi}{T}$ cause aliasing, when sampling analog signals. The analog poles will not be aliased by the impulse invariant mapping if they are confined to the s-plane's "Primary strip" (within π/T of the real axis).

Let $H_a(s)$ is the system function of an analog filter. This can be expressed in partial fraction form as

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \tag{5.77}$$

where $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients in the partial fraction expansion.

The inverse Laplace transform of Eq.(5.77) is

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t} \quad t \geq 0 \tag{5.78}$$

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} c_k e^{p_k nT} \tag{5.79}$$

We know

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{5.80}$$

Substituting Eq.(5.79) in Eq.(5.80) we obtain

$$H(z) = \sum_{n=0}^{\infty}\sum_{k=1}^{N} c_k e^{p_k n T} z^{-n}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n$$

$$= \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}} \tag{5.81a}$$

That is if $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$ then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$.

For high sampling rates (for small T), the digital filter gain is high. Therefore, instead of Eq. (5.81a) we can use

$$H(z) = \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}} \tag{5.81b}$$

Due to the presence of aliasing, the impulse invariant method is appropriate for the design of lowpass and bandpass filters only. The impulse invariance method is unsuccessful for implementing digital filters such as a highpass filter.

Steps to design a digital filter using Impulse Invariance method

1. For the given specifications, find $H_a(s)$, the transfer function of an analog filter.
2. Select the sampling rate of the digital filter, T seconds per sample.
3. Express the analog filter transfer function as the sum of single-pole filters.

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

4. Compute the z-transform of the digital filter by using the formula

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}.$$

For high sampling rates use

$$H(z) = \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}}$$