5. Infinite Impulse Response Filters

5.1 Introduction

Basically a digital filter is a linear time-invariant discrete time system. The terms Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) are used to distinguish filter types. The FIR filters are of non-recursive type, whereby the present output sample depends on the present input sample and previous input samples, whereas the IIR filters are of recursive type, whereby the present output sample depends on the present input, past input samples and output samples. The properties and the design of FIR filters are discussed in detail in chapter 6. In this chapter the design of IIR filters that are realizable and stable are discussed in detail.

The impulse response $h(n)$ for a realizable filter is

$$h(n) = 0 \quad \text{for} \quad n \leq 0 \tag{5.1a}$$

and for stability it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty. \tag{5.1b}$$

IIR digital filters have the transfer function of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{5.2}$$

The design of an IIR filter for the given specifications is to find filter coefficients a_ks and b_ks of Eq.(5.2).

5.2 Frequency Selective Filters

A filter is one, which rejects unwanted frequencies from the input signal and allow the desired frequencies. The range of frequencies of signal that are passed through the filter is called passband and those frequencies that are blocked is called stopband.

The filters are of different types.
1. Lowpass filter, 2. Highpass filter, 3.Bandpas filter, 4. Bandreject filter.

1. *Lowpass filter*

The magnitude response of an ideal lowpass filter allows [illegible]encies in the passband $0 < \Omega < \Omega_c$ to pass, whereas the higher frequencies in the stopband $\Omega > \Omega_c$ are blocked. The frequency Ω_c between the two bands is the cutoff frequency, where the magnitude $|H(j\Omega)| = 1/\sqrt{2}$.

In practice it is impossible to obtain the ideal response. The practical response of a lowpass filter is shown in solid line in Fig. 5.1a.

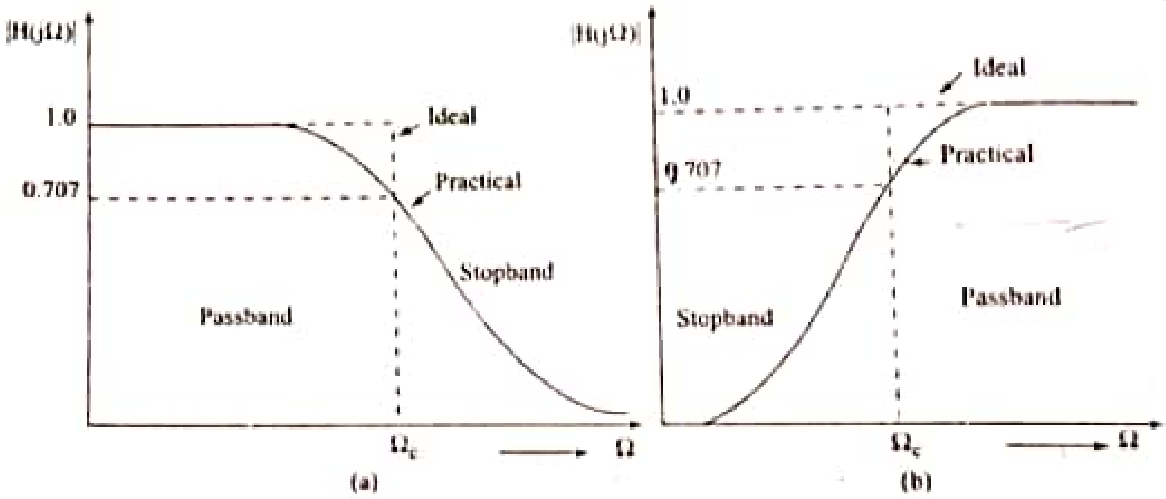

Fig. 5.1 Magnitude response of filters (a) Lowpass (b) Highpass

2. *Highpass filter*

The highpass filter allows high frequencies above $\Omega > \Omega_c$ and rejects the frequencies between $\Omega = 0$ and $\Omega = \Omega_c$. The magnitude response of an ideal and practical highpass filter is shown in Fig. 5.1b.

3. *Bandpass filter*

It allows only a band of frequencies Ω_1 to Ω_2 to pass and stops all other frequencies. The ideal and practical response of bandpass filter are shown in Fig. 5.2.

4. *Bandreject filter*

It rejects all the frequencies between Ω_1 and Ω_2 and allows remaining frequencies. The magnitude response of an ideal and practical filters is shown in Fig. 5.3.

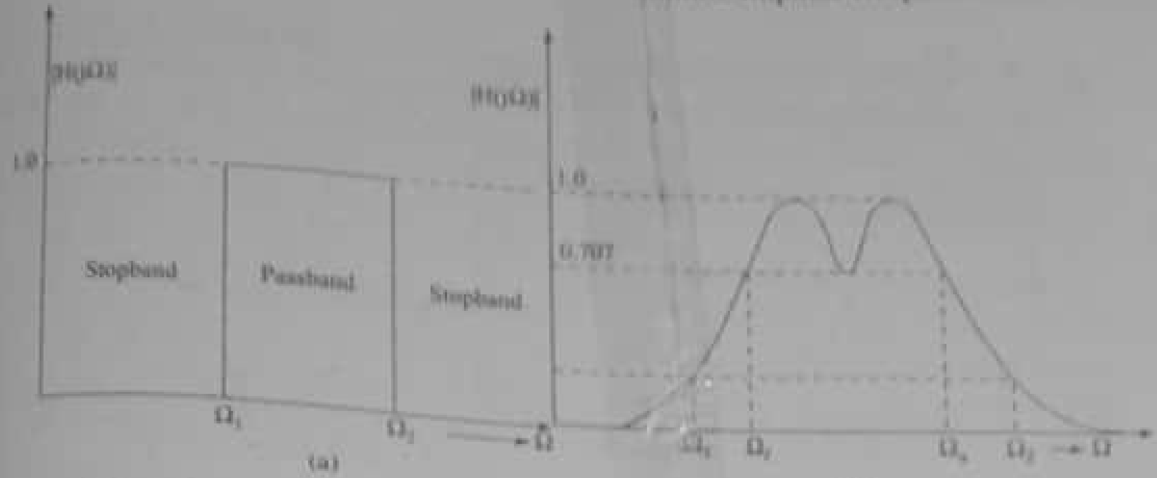

Fig. 5.2 Magnitude response of Bandpass filter (a)Ideal (b) Practical

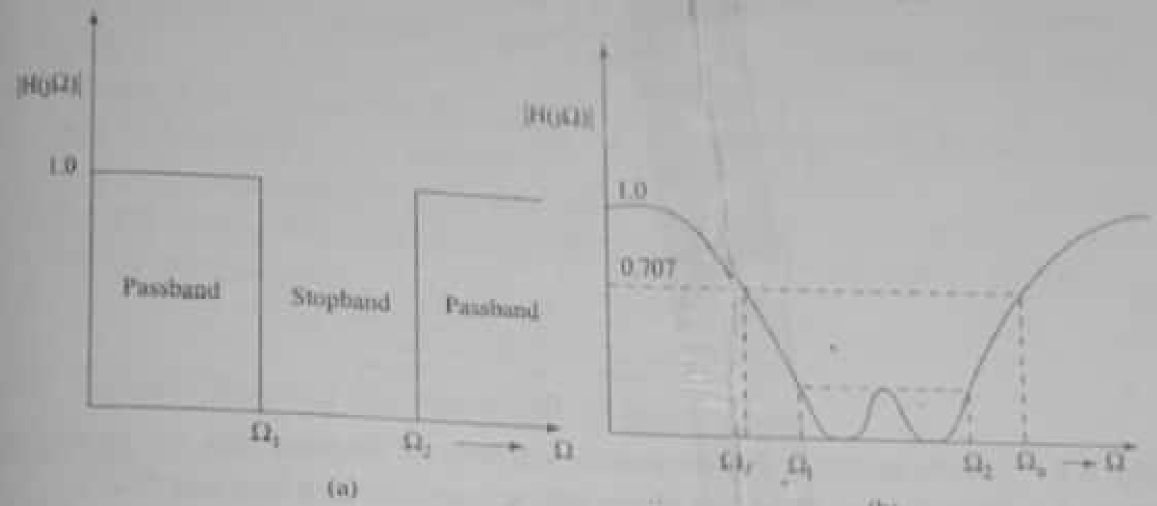

Fig. 5.3 Magnitude response of Bandreject filter (a) Ideal (b) Practical

5.3 Design of Digital filters from Analog filters

The most common technique used for designing IIR digital filters known as indirect method, involves first designing an analog prototype filter and then transforming the prototype to a digital filter. For the given specifications of a digital filter, the derivation of the digital filter transfer function requires three steps.

1. Map the desired digital filter specifications into those for an equivalent analog filter.
2. Derive the analog transfer function for the analog prototype.
3. Transform the transfer function of the analog prototype into an equivalent digital filter transfer function.

Fig. 5.4b shows the magnitude response of a digital lowpass filter. The various parameters in the figure are

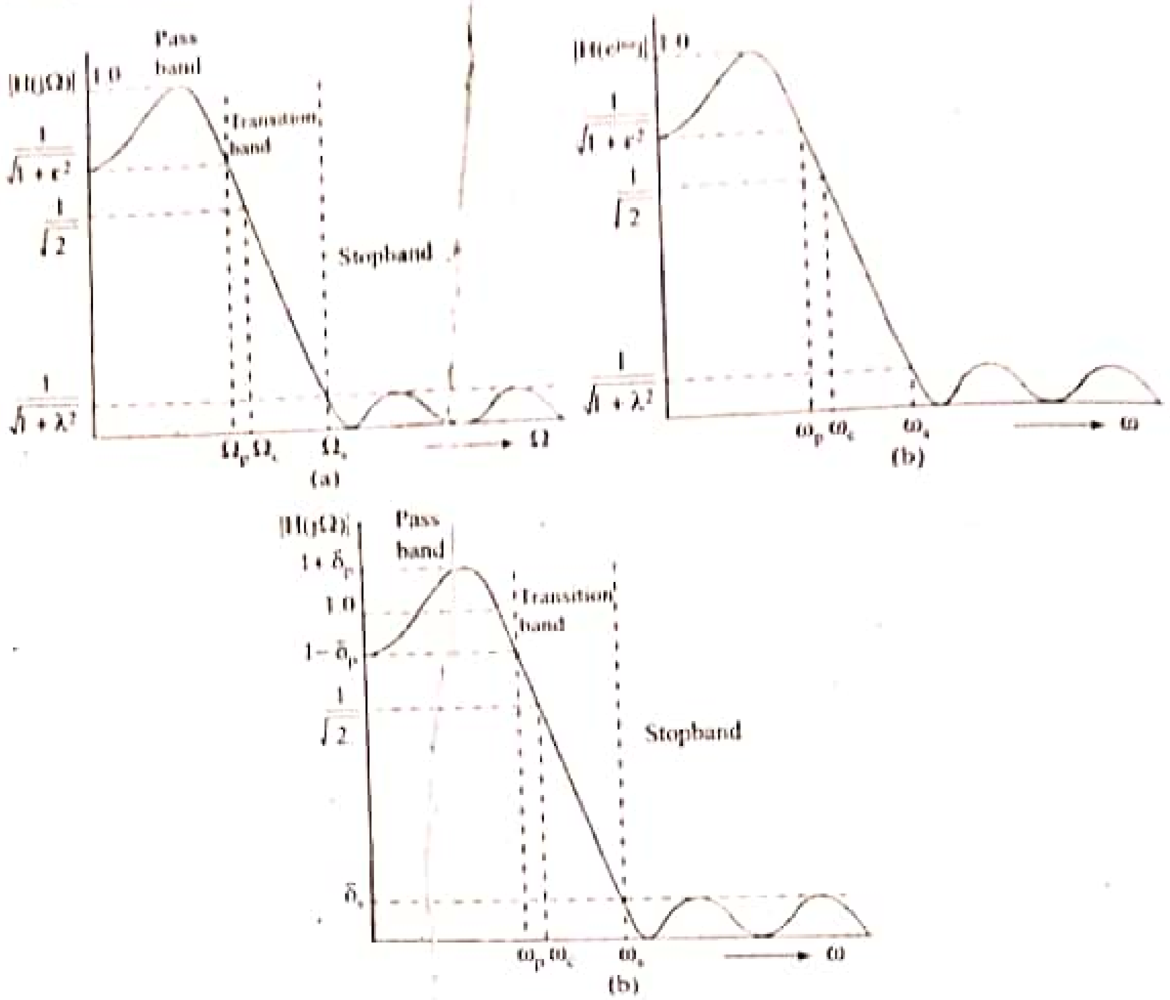

Fig. 5.4 Specifications for the magnitude response of lowpass filter (a) analog, (b) digital, (c) Alternate specifications of magnitude response of a lowpass filter

$\omega_p =$ Passband frequency in radians
$\omega_s =$ Stopband frequency in radians
$\omega_c =$ 3-dB cutoff frequency in radians
$\epsilon =$ Parameter specifying allowable passband
$\lambda =$ Parameter specifying allowable stopband

The Fig. 5.4b can be modified to apply to analog lowpass filter as shown in Fig. 5.4a. Here the digital frequencies $\omega_p, \omega s$ and ω_c are replaced by the analog frequencies Ω_p, Ω_s and Ω_c whose units are in radians/sec.

Often, a different set of parameters is used for specifying the magnitude response of a digital lowpass filter. These parameters are shown in Fig. 5.4c, where δ_p represents the passband error tolerance and δ_s represents the maximum allowable magnitude in the stopband. The relation between the parameters shown in Fig.5.4b and Fig. 5.4c are given by

$$\epsilon = 2\frac{\sqrt{\delta_p}}{1 - \delta p} \quad \text{and}$$

$$\lambda = \frac{\sqrt{(1 + \delta_p)^2 - \delta_s^2}}{\delta_s}$$

Using the analog filter specifications the transfer function of analog lowpass filter is designed and it is transformed to digital filter using suitable transformation methods. Among the transformation methods available, the bilinear transformation is usually the method of choice for the design of filters like lowpass, highpass, bandpass and bandreject filters.

5.3.1 *Digital Versus Analog Filters*

Analog Filter	Digital Filter
1. Analog filter processes analog inputs and generates analog outputs. 2. Analog filters are constructed from active or passive electronic components. 3. Analog filter is described by a differential equation. 4. The frequency response of an analog filter can be modified by changing the components.	1. A digital filter processes and generates digital data. 2. A digital filter consists of elements like adder, multiplier and delay unit. 3. Digital filter is described by a difference equation. 4. The frequency response can be changed by changing the filter coefficients.

5.3.2 *Advantages and disadvantages of digital filters*

Advantages

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature and power supply variations.
2. A digital filter is highly immune to noise and possesses considerable parameter stability.
3. Digital filters afford a wide variety of shapes for the amplitude and phase responses.
4. There are no problems of input or output impedance matching with digital filters.
5. Digital filters can be operated over a wide range of frequencies.
6. The coefficients of digital filter can be programmed and altered any time to obtain the desired characteristics.
7. Multiple filtering is possible only in digital filter.

Disadvantage

1. The quantization error arises due to finite word length in the representation of signals and parameters.

For $N = 4$, the poles can be found from Eq. (5.12) as

$$s_1 = e^{j5\pi/8} = -0.3827 + j0.9239$$
$$s_2 = e^{j7\pi/8} = -0.9239 + j0.3827$$
$$s_3 = e^{j9\pi/8} = -0.9239 - j0.3827$$
$$s_4 = e^{j11\pi/8} = -0.3827 - j0.9239$$

Now the denominator of transfer function $H(s)$ is

$$\{(s + 0.3827)^2 + (0.9239)^2\}\{(s + 0.9239)^2 + (0.3827)^2\}$$
$$= (s^2 + 1.84776s + 1)(s^2 + 0.76536s + 1)$$

For fourth order Butterworth filter the transfer function for $\Omega_c = 1$ rad/sec is given by

$$H(s) = \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)} \tag{5.13}$$

The following table 5.1 gives Butterworth polynomials for various values of N for $\Omega_c = 1$ rad/sec.

Table 5.1 List of Butterworth Polynomials

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764s + 1)$
7	$(s + 1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

The Eq. (5.12) gives us the pole locations of Butterworth filter for $\Omega_c = 1$ rad/sec and are known as normalized poles. In general, the unnormalized poles are given by

$$s_k' = \Omega_c s_k \tag{5.14}$$

The transfer function of such type of Butterworth filter can be obtained by substituting $s \rightarrow s/\Omega_c$ in the transfer function of Butterworth filter.

We can now proceed to determine the order equation given the filter specifications. In Eq. (5.5) the filter was restricted to -3 dB attenuation at Ω_c. Now let the maximum passband attenuation in positive dB is $\alpha_p (< 3\,\text{dB})$ at passband frequency Ω_p, and α_s is the minimum stopband attenuation in positive dB at the stopband frequency Ω_s. Now the magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{[1 + \varepsilon^2(\Omega/\Omega_p)^{2N}]^{1/2}} \tag{5.15a}$$

$$\Rightarrow |H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2(\Omega/\Omega_p)^{2N}} \tag{5.15b}$$

Taking logarithm on both sides we have

$$20 \log |H(j\Omega)| = 10 \log 1 - 10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}\right] \tag{5.16}$$

From Fig. 5.7 we can find that at $\Omega = \Omega_p$ the attenuation is equal to α_p.

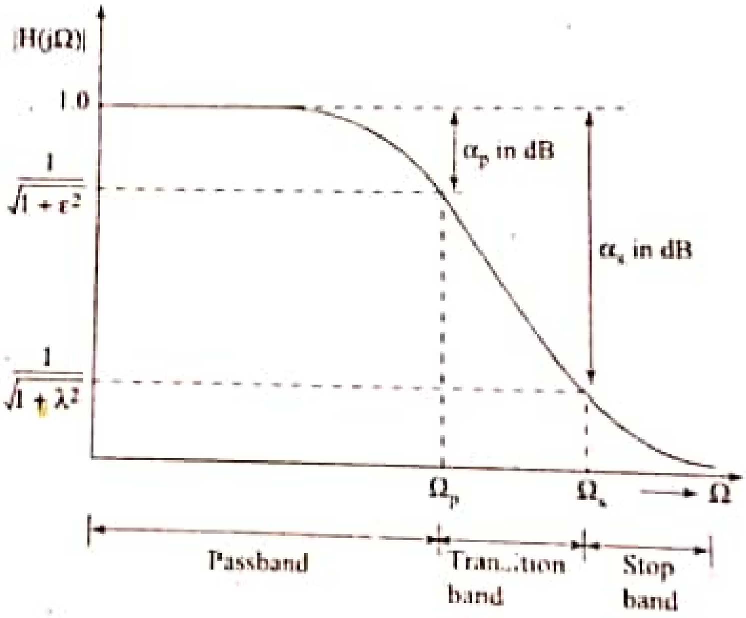

Fig. 5.7 Butterworth approximation of magnitude response.

Therefore, Eq. (5.16) can be written as

$$20 \log |H(j\Omega_p)| = -\alpha_p = -10 \log(1 + \varepsilon^2)$$

which gives us

$$\alpha_p = 10 \log(1 + \varepsilon^2)$$

$$0.1\alpha_p = \log(1 + \varepsilon^2)$$

Taking antilog on both sides

$$1 + \varepsilon^2 = 10^{0.1\alpha_p}$$

$$\Rightarrow \epsilon = (10^{0.1\alpha_p} - 1)^{1/2} \tag{5.17}$$

Reference to Fig. 5.7 shows that at $\Omega = \Omega_s$ the minimum stopband attenuation is equal to α_s. Substituting these values in Eq. (5.16)

$$20\log|H(j\Omega_s)| = 10\log 1 - 10\log\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$-\alpha_s = -10\log\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$0.1\alpha_s = \log\left[1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

After simplification, we get

$$\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s} - 1 \tag{5.18}$$

Substituting Eq. (5.17) in Eq. (5.18), we get

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1} \tag{5.19}$$

Since we are interested in finding expression for N, taking logarithm for Eq. (5.19) the following expression is obtained for the order of the filter.

$$N = \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} \tag{5.20}$$

Since this expression normally does not result in an integer value, we therefore, roundoff N to the next higher integer.

$$\text{i.e.,}\quad N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} \tag{5.21}$$

$$\geq \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\dfrac{\Omega_s}{\Omega_p}} \tag{5.22}$$

where

$$\varepsilon = (10^{0.1\alpha_p}-1)^{0.5} \tag{5.23}$$

$$\lambda = (10^{0.1\alpha_s}-1)^{0.5} \tag{5.24}$$

For simplicity of notation, we now define the parameters A and k as follows

$$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)^{0.5} \tag{5.25}$$

and

$$k = \frac{\Omega_p}{\Omega_s} \tag{5.26}$$

where k is known as transition ratio.

Finally the order equation for the lowpass Butterworth analog filter is given by

$$N \geq \frac{\log A}{\log(1/k)} \tag{5.27}$$

Example 5.1 Given the specification $\alpha_p = 1\,\text{dB}$; $\alpha_s = 30\,\text{dB}$; $\Omega_p = 200\,\text{rad/sec}$; $\Omega_s = 600\,\text{rad/sec}$. Determine the order of the filter.

Solution

From Eq. (5.25)

$$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)^{0.5}$$

$$= \left(\frac{10^3 - 1}{10^{0.1} - 1}\right)^{0.5} = 62.115$$

From Eq. (5.26)

$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

From Eq. (5.27)

$$N \geq \frac{\log A}{\log 1/k}$$

$$\geq \frac{\log 62.115}{\log 3} = 3.758$$

Rounding off N to the next higher integer we get $N = 4$.

Example 5.2 Determine the order and the poles of lowpass Butterworth filter that has a 3 dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

Solution

Given data $\alpha_p = 3\,\text{dB}$; $\alpha_s = 40\,\text{dB}$; $\Omega_p = 2 \times \pi \times 500 = 1000\pi$ rad/sec.
$\Omega_s = 2 \times \pi \times 1000 = 2000\pi$ rad/sec.

The order of the filter

$$N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log\sqrt{\dfrac{10^4-1}{10^{0.3}-1}}}{\log\dfrac{2000\pi}{1000\pi}} = 6.6$$

Rounding 'N' to nearest higher value we get $N = 7$.

The poles of Butterworth filter are given by

$$s_k = \Omega_c e^{j\phi_k} = 1000\pi e^{j\phi_k} \quad k = 1, 2, \ldots 7$$

where $\phi_k = \dfrac{\pi}{2} + \dfrac{(2k-1)\pi}{2N} \quad k = 1, 2, \ldots 7.$

Example 5.3 Prove that $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \dfrac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$

Solution

The magnitude square function of Butterworth analog lowpass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \tag{5.28}$$

From Eq. (5.15b) we know

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

Comparing Eq. (5.15b) and Eq. (5.28) we get

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = 1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N} = \left(\frac{\Omega}{\Omega_c}\right)^{2N} \tag{5.29}$$

Simplifying above Eq. (5.29) by substituting Eq. (5.17) we obtain

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_p}-1 \tag{5.30}$$