Working with the amplitude-phase form of the Fourier series allows us to do certain things much more easily than sine-cosine. For instance, determining the effect of a shift in the independent variable $(x \longrightarrow x + \delta)$:

$$f(x+\delta) = a_0 + \sum_{n=1}^{\infty} c_n \cos(nx + n\delta + \phi_n) = a_0 + \sum_{n=1}^{\infty} c_n \cos(nx + \delta_n)$$

with $\delta_n = n\delta + \phi_n$. The expression above quickly tells us that the amplitudes are invariant, while the phase increases linearly with increasing n.

Fourier Series: Complex Notation

A more compact notation for the Fourier series requires the introduction of an indexed angular frequency

$$\omega_n = \frac{2\pi n}{T}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + \sum_{n=1}^{\infty} b_n \sin(\omega_n t)$$

The sines and cosines can be re-written as complex exponentials

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i} \right) =$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - ib_n}{2} \right) e^{i\omega_n t} + \left(\frac{a_n + ib_n}{2} \right) e^{-i\omega_n t} \right]$$

from which we can define the coefficients:

Fourier Series: Complex Notation

$$F_{+n} = \frac{a_n - ib_n}{2} \qquad\qquad F_{-n} = \frac{a_n + ib_n}{2} \qquad\qquad F_0 = a_0$$

So, the Fourier series can be rewritten as

$$f(t) = F_0 + \sum_{n=1}^{\infty} F_{+n}\, e^{i\omega_n t} + \sum_{n=1}^{\infty} F_{-n}\, e^{-i\omega_n t}$$

Since $-\omega_n = -2\pi n/T = \omega_{-n}$ and $F_n^* = F_{-n}$, we can obtain the general Fourier series in complex form

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n\, e^{+i\omega_n t}$$

with coefficients

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_n t}\, dt \qquad\qquad n = 0, \pm 1, \pm 2, \dots$$

Fourier Transform

The Fourier series expresses a function of time over a finite interval in terms of a sum of functions at discrete frequencies ω_n. This can be extended into a **Fourier transform** which allows the treatment of a non-periodic function $f(t)$. This transformation starts by expanding the function in terms of cosine and sine functions but, in this case, the expansion is a **Fourier integral** over a **continuous range of frequencies**, instead of a sum over a discrete set of frequencies.

The Fourier integral may be viewed as the limit of a Fourier series in the limit $T \to \infty$. The summation over n in the Fourier series is replaced by an integration over ω

$$f(t) = \int_0^\infty d\omega\, [a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)]$$

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\, \cos(\omega t)$$

$$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\, \sin(\omega t)$$

Fourier Transform

The Fourier Transform can be derived from the Fourier series in complex notation and transforming the sum over discrete frequencies into a continuous integration over ω

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{+i\omega_n t} \, \Delta n$$

We did not change anything here because $\Delta n = 1$. A frequency spacing can then be defined as $\Delta\omega = 2\pi \, \Delta n / T$ and included above to set the integral increment

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n \, e^{+i\omega_n t} \left(\frac{T}{2\pi}\right) \Delta\omega$$

We now let the period T over which $f(t)$ is defined goes to infinity, so that the frequencies ω_n become so close together that we can replace the discrete ω_n with the continuous variable ω, $\Delta\omega$ by $d\omega$, and the sum with an integral. The result is a function $f(t)$ expressed as

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \, e^{+i\omega t} \, d\omega = \mathcal{F}^{-1}\{F(\omega)\}}$$

Inverse Fourier transform

Fourier Transform

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \mathcal{F}\{f(t)\}}$$

Fourier transform

We have resolved $f(t)$ into its frequency components. Note that $F(\omega)$ is only a function of the angular frequency, so we have transformed a function of time into a function of angular frequency in this case.

Attention to other conventions: other sources may follow other conventions for the Fourier transform and the inverse Fourier transform. Some may normalize the integrals by $1/\sqrt{2\pi}$ and some may even have the Fourier transform written with a positive exponential and the inverse transform with a negative exponential. For now, we will follow the convention of the equations written above.

Fourier Transform

What is the Fourier transform of a Dirac delta $\delta(t)$?

Fourier Transform

What is the Fourier transform of a Dirac delta $\delta(t)$?

That is constant at all frequencies!

$$F(\omega) = \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = e^{i\omega t}\Big|_{t=0} = 1$$

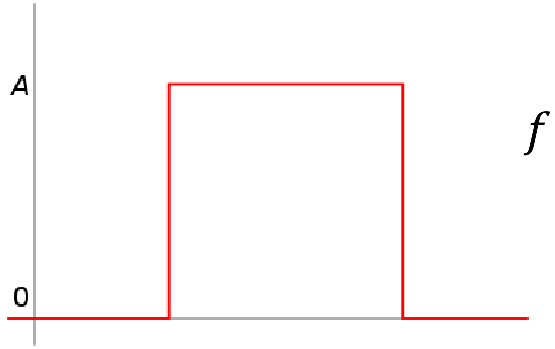
A single zero-frequency (DC) term can be represented using the Dirac delta function as $F(\omega) = \delta(\omega)$. What is its inverse Fourier transform?

$$f(t) = \mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi} e^{i\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

In both cases, a feature that is narrowly localized in one domain requires an equal contribution from all elements in the other domain.

Fourier Transform

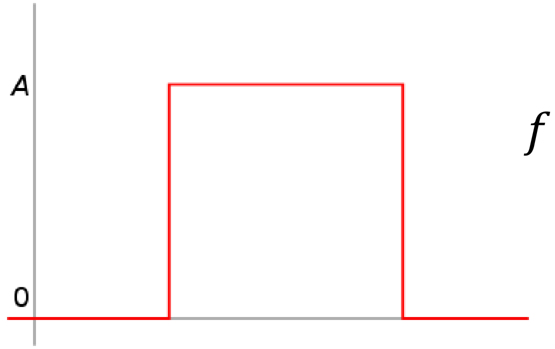
What is the Fourier transform of a “boxcar”?



$$f(t) = \begin{cases} 1, & -0.5 < t < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

Fourier Transform

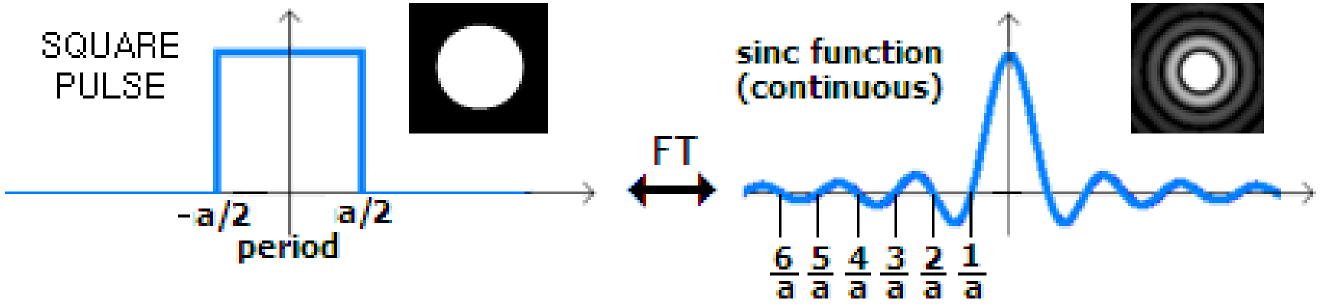
What is the Fourier transform of a “boxcar”?



$$f(t) = \begin{cases} 1, & -0.5 < t < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-0.5}^{0.5} e^{-i\omega t}\, dt = \left.\frac{e^{-i\omega t}}{i\omega}\right|_{-0.5}^{0.5} = \frac{\sin(\omega/2)}{\omega/2}$$

The Fourier transform of a “boxcar” is a sinc function.



Fourier Transform

What is the Fourier transform of a Gaussian?

Fourier Transform

What is the Fourier transform of a Gaussian?

$$\mathcal{F}\left\{e^{-t^2}\right\} = e^{-\omega^2}$$

The Fourier transform of a Gaussian is another Gaussian!

More specifically, if we have a Gaussian function given by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}$$

its Fourier transform is given by

$$F(\omega) = e^{-(\sigma\omega)^2/2}$$

Fourier Transform: Shifting

Shifting a function in one domain transforms to a linear phase term in the other domain.

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{+\infty} f(t-t_0)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{+\infty} f(t-t_0)e^{-i\omega(t-t_0)}e^{-i\omega t_0}d(t-t_0)$$

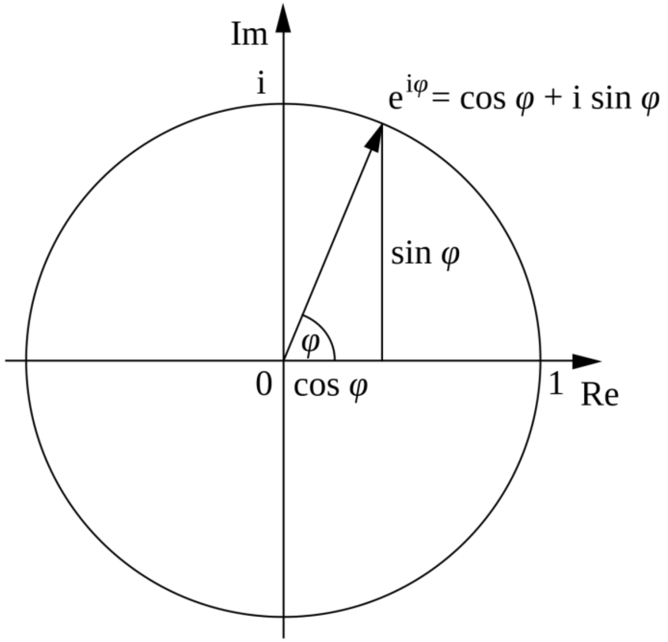
$$= e^{-i\omega t_0}\int_{-\infty}^{+\infty} f(s)e^{-i\omega s}ds = e^{-i\omega t_0}F(\omega)$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Shifting

Example: Dirac delta

$$\mathcal{F}\{\delta(t-t_0)\} = e^{-i\omega t_0}$$



Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Linearity

Adding functions with Fourier transforms $f(t) \overset{\mathcal{F}}{\rightleftharpoons} F(\omega)$ and $g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(\omega)$ produces the sum of the transforms

$$\int_{-\infty}^{+\infty} [f(t) + g(t)]\, e^{i\omega t} dt = \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt + \int_{-\infty}^{+\infty} g(t) e^{i\omega t} dt = F(\omega) + G(\omega) \qquad (16.22)$$

and a corollary is that $a\, f(t) \overset{\mathcal{F}}{\rightleftharpoons} a\, F(\omega)$.

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Linearity

16.4.1 Example: Real cosine

An even pair of real delta functions transforms to a purely real cosinusoid as shown in Figure 16.8

$$\delta(t+t_0)+\delta(t-t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{+iz}+e^{-iz} \qquad z \equiv \omega t_0 \tag{16.23}$$
$$\overset{\mathcal{F}}{\rightleftharpoons} [\cos(z)+i\sin(z)]+[\cos(-z)+i\sin(-z)]$$
$$\overset{\mathcal{F}}{\rightleftharpoons} 2\cos(z)$$

16.4.2 Example: Imaginary sinusoid

An odd pair of real delta functions transforms to an imaginary sinusoid as shown in Figure 16.9

$$\delta(t+t_0)-\delta(t-t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{+iz}-e^{-iz} \qquad z \equiv \omega t_0 \tag{16.24}$$
$$\overset{\mathcal{F}}{\rightleftharpoons} [\cos(z)+i\sin(z)]-[\cos(-z)+i\sin(-z)]$$
$$\overset{\mathcal{F}}{\rightleftharpoons} 2i\sin(z)$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

16.4.3 Example: Real sinusoid

An odd pair of imaginary delta functions transforms to an real sinusoid as shown in Figure 16.9

$$
\begin{aligned}
i\delta(t+t_0) - i\delta(t-t_0) &\overset{\mathcal{F}}{\rightleftharpoons} ie^{+iz} - ie^{-iz} \qquad\qquad z \equiv \omega t_0 \qquad (16.25)\\
&\overset{\mathcal{F}}{\rightleftharpoons} i\left[\cos(z) + i\sin(z)\right] - i\left[\cos(-z) + i\sin(-z)\right]\\
&\overset{\mathcal{F}}{\rightleftharpoons} 2\sin(z)
\end{aligned}
$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Scaling

16.5 Scaling

Compression along the time axis leads to expansion of the frequency scale

$$\int_{-\infty}^{+\infty} f(at)e^{i\omega t}dt = \frac{1}{|a|}\int_{-\infty}^{+\infty} f(at)e^{i\frac{\omega}{a}(at)}d(at) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \qquad (16.26)$$

and expansion in the time domain leads to compression in the frequency domain

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

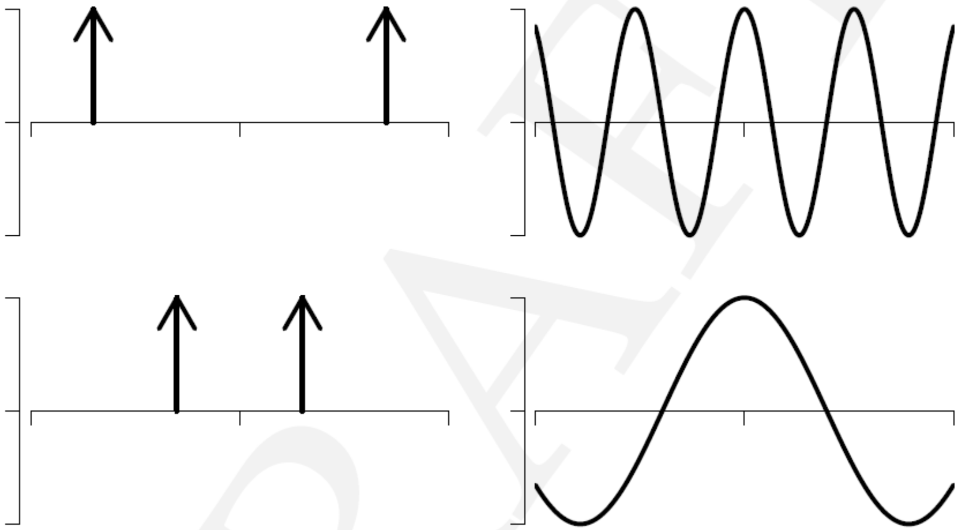
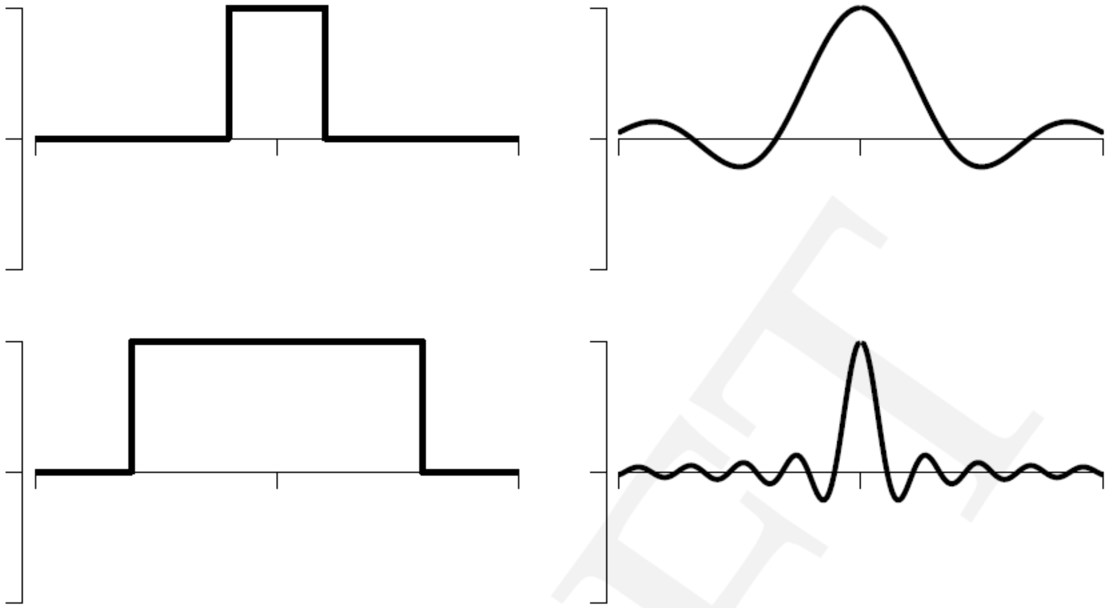


Figure 16.11: Expanding a cosinusoid by shifting delta functions



Fourier Transform: Derivatives

the first derivative of $f(t)$ with respect to t

$$f'(t) = \frac{d\,f(t)}{dt} = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, F(\omega) e^{+i\omega t} \tag{16.28}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, F(\omega)\, \frac{d}{dt} e^{+i\omega t} \tag{16.29}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \underbrace{i\omega\, F(\omega)}\, e^{+i\omega t} \tag{16.30}$$

has a Fourier transform that is the transform of $f(t)$ multiplied by a (complex) term that is linear in angular frequency

$$\boxed{f'(t) \overset{\mathcal{F}}{\rightleftharpoons} i\omega\, F(\omega)} \tag{16.31}$$

Similarly, the second derivative

$$f''(t) = \frac{1}{2\pi}\frac{d^2}{dt^2}\int_{-\infty}^{+\infty} d\omega\, F(\omega) e^{+i\omega t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \underbrace{(i\omega)^2\, F(\omega)}\, e^{+i\omega t} \tag{16.32}$$

has a transform that is weighted by (minus) frequency squared

$$\boxed{f'(t) \overset{\mathcal{F}}{\rightleftharpoons} \cancel{-i\omega^2} F(\omega)} \quad f''(t) \overset{\mathcal{F}}{\Leftrightarrow} (i\omega)^2 F(\omega) \tag{16.33}$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Derivatives

and an arbitrary derivative of degree n

$$f^{(n)}(t) = \frac{1}{2\pi}\frac{d^n}{dt^n}\int_{-\infty}^{+\infty} d\omega\, F(\omega)e^{+i\omega t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \underbrace{(i\omega)^n F(\omega)}\, e^{+i\omega t} \qquad (16.34)$$

transforms to a multiplication by the nth power of frequency.

$$\boxed{f^{(n)}(t) \overset{\mathcal{F}}{\rightleftharpoons} (i\omega)^n F(\omega)} \qquad (16.35)$$

This indicates that differentiation in the time domain could equivalently be expressed as a three step process

1. forward transformation to the frequency domain
2. easy multiplication by a polynomial term
3. inverse transformation back to the time domain

Setting aside for now the question of whether this is computationally wise, it does illustrate an important concept. It is well known that differentiation amplifies noise; the Fourier transform tells us that the transformed effect is multiplication of high frequency components.

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

16.7.1 Zeroth

The definite integral of any function $f(x)$ over the range $[-\infty, +\infty]$ is equal to the value of the transform at the origin

$$\int_{-\infty}^{+\infty} f(t)dt = \left. \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \right|_{\omega=0} = F(0) \qquad (16.36)$$

This result is true in general, but is particularly useful if $f(x)$ is positive definite, so that the total area under the curve is given by $F(0)$. For example, a probability density function normalized to unit area will have $F(0) = 1$.

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

16.7.2 First

Starting with the usual Fourier transform pair

$$\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt = F(\omega) \tag{16.37}$$

and taking the derivative of both sides with respect to frequency

$$-i\int_{-\infty}^{+\infty} t\,f(t)e^{-i\omega t}\,dt = F'(\omega) \tag{16.38}$$

then evaluating at zero frequency

$$-i\int_{-\infty}^{+\infty} t\,f(t)\,dt = F'(\omega=0) \tag{16.39}$$

gives a relationship between the first ~~(central)~~ moment of a function and the DC component of the transform first derivative

$$\boxed{\int_{-\infty}^{+\infty} t\,f(t)\,dt = i\,F'(0)} \tag{16.40}$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

Centroid

The central mean of a function normalized to unit area

$$\langle t \rangle = \frac{\int_{-\infty}^{+\infty} t\, f(t)\, dt}{\int_{-\infty}^{+\infty} f(t)\, dt} \tag{16.41}$$

is now easy to express in terms of Fourier transform derivatives

$$\boxed{\langle t \rangle = \frac{i\, F'(0)}{F(0)}} \tag{16.42}$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

16.7.3 Second

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

The second central moment

$$\int_{-\infty}^{+\infty} t^2 f(t)\, dt \tag{16.43}$$

involves a second derivative

$$\boxed{\int_{-\infty}^{+\infty} t^2 f(t)\, dt = -F''(0)} \tag{16.44}$$

Variance

So the variance

$$\sigma^2 = \langle t - \langle t \rangle \rangle^2 = \overset{\langle t^2 \rangle}{\cancel{\langle () t^2 \rangle}} - \langle t \rangle^2 \tag{16.45}$$

becomes

$$\boxed{\sigma^2 = -\frac{F''(0)}{F(0)} + \frac{[F'(0)]^2}{[F(0)]^2}} \tag{16.46}$$

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

16.7.4 Nth

In general, the nth central moment

$$\int_{-\infty}^{+\infty} t^n f(t)\, dt \tag{16.47}$$

is equal to the nth derivative of the Fourier transform evaluated at the origin

$$\boxed{\int_{-\infty}^{+\infty} t^n f(t)\, dt = \frac{F^{(n)}(0)}{(-i)^n}} \tag{16.48}$$

Chapter 16 of Dr. Jackel's notes (will be in the D2L shell)

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

Cross-correlation (sometimes referred as cross-covariance)

$$(f \star g)(\tau) = \int_{-\infty}^{\infty} f^*(t) g(t+\tau)\, dt = \int_{-\infty}^{\infty} f^*(t-\tau) g(t) dt$$

where τ is called *displacement* or *lag.* For highly correlated f and g which have a maximum cross-correlation at a particular τ means that a feature in f at time t also occurs later in g at $\mathrm{t}+\tau$.

If f is a Hermitian function, then $f \star g = f * g$ (convolution denoted by *) and if both f and g are Hermitian functions, then $f \star g = g \star f$.

And taking the Fourier transform of the cross-correlation, we get

$$\mathcal{F}\{f \star g\} = F^*(\omega) G(\omega)$$

when coupled with Fast Fourier Transform, this property can be exploited for efficient numerical computation of cross-correlations.

Cross-correlation theorem: the Fourier transform of the cross-correlation function yields the so-called **cross-spectral density function.**

Fourier Transform: Moments (expressing moments in terms of Fourier transform)

Auto-correlation function (ACF) of continuous-time signal

$$(f \star f)(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau)\, dt = \int_{-\infty}^{\infty} f^*(t-\tau) f(t) dt$$

where τ is called *displacement* or *lag.* In discrete form, a lag 1 autocorrelation (i.e., $\tau = 1$) is the correlation between values that are one time step apart. More generally, a lag τ autocorrelation is the correlation between values that are τ time steps apart. Since the autocorrelation is a specific type of cross-correlation, it maintains all the properties of cross-correlation. **The ACF is a way to measure the linear relationship between an observation at time t and the observations at previous times.**

And taking the Fourier transform of the ACF, we get

$$\mathcal{F}\{f \star f\} = |F(\omega)|^2$$

when coupled with Fast Fourier Transform, this property can be exploited for efficient numerical computation of cross-correlations.

Autocorrelation theorem: the Fourier transform of the autocorrelation function is equal to the power spectrum of $f(t)$, i.e., $|F(\omega)|^2$.