# Welcome to Advanced Data Analysis (PHYS 605)

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We continue with the main recommended references for this course:

Datalogy: Measurement, Estimation, Transformation, Reduction, Analysis, Classification, and Modeling by Dr. Brian Jackel (UCalgary)

D. G. Martinson, Quantitative Methods of Data Analysis for the Physical Sciences and Engineering, Cambridge University Press, 2018.

Fourier Methods in Science and Engineering by Wen Li, Weiming Sun, 2022 (available at the UCalgary Library)

Digital Signal Processing: A Practical Guide for Engineers and Scientists by Smith, Steven, 2013 (available at the UCalgary Library)

https://pythonnumericalmethods.berkeley.edu/notebooks/Index.html
https://www.3blue1brown.com/lessons/fourier-transforms

and some other materials cited throughout…

Often, we have to deal with time series data.

Fourier or harmonic analysis involves the decomposition of a time series into a sum of sinusoidal components (sines and cosines).

**Fourier series** (periodic functions): an expansion of a periodic function in terms of an infinite sum of sines and cosines. Fourier series makes use of the orthogonality relationships of the sine and cosine functions (Wolfram MathWorld). The computation and study of Fourier series is known as **harmonic analysis** and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms to get insight into the underlying structure and composition of the data series being analyzed.

The Fourier method has many applications in engineering and science, such as signal processing, partial differential equations, image processing, etc.

(Data analysis context) Fourier series involves a set of particular sines and cosines that combine to interpolate a data set or time series.

## Fourier Series

**NOTE:** despite the fact that we may give certain emphasis to time series and time as independent variable in this context of Fourier analysis, be aware that the method is not restricted to time series, it can be applied to spatial or any other independent variable of interest.

Fourier series is also known as the trigonometric series expansion of a periodic function. A function $f(t)$ is referred to as a periodic function if

$$f(t+T) = f(t)$$

where $T$ is the period of the function $f(t)$. The Fourier series of $f(t)$ is its expansion into an infinite series involving sines and cosines as

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi f n\, t) + \sum_{n=1}^{\infty} b_n \sin(2\pi f n\, t)$$

in which $a_n$ and $b_n$ are the **Fourier coefficients** that hold the amplitudes of the cosine and sine waves, respectively, and $f$ is the **fundamental frequency** of the signal such as $f = 1/T$ and we can also obtain the angular fundamental frequency $\omega = 2\pi f$.

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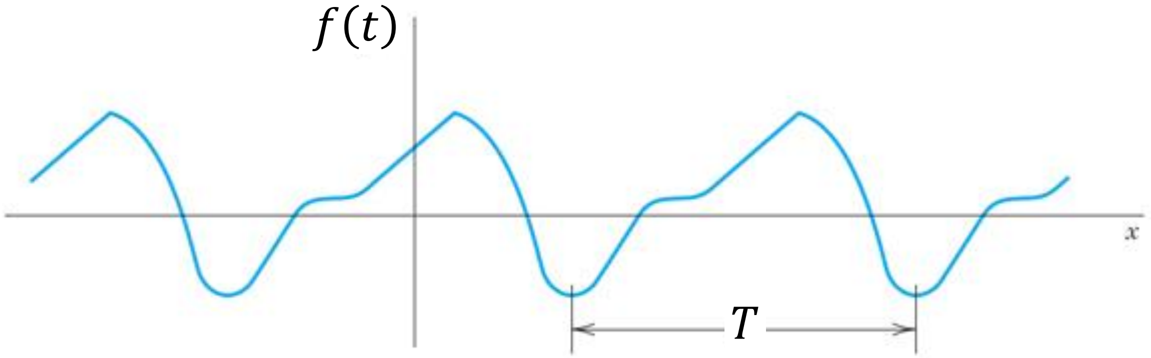
We can also define multiples $\omega_n = n\omega$, known as the **harmonic frequencies** $(\omega, 2\omega, 3\omega, \ldots)$ in which the integer $n$ can label the **harmonic mode**.




$$f(t+T) = f(t)$$

If $f(t)$ has period $T$, it also has the period $2T, 3T, \ldots$ Thus, for any integer $n = 1,2,3, \ldots$

$$f(t+nT) = f(t)$$

Advanced Engineering Mathematics, 10th Edition
Erwin Kreyszig, Herbert Kreyszig, Edward J. Norminton

**Fourier Series**

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n\, t}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n\, t}{T}\right)dt$$

Note that since the time domain signal is periodic, the expressions only need to be calculated over a single period which can be $\left[-\frac{T}{2},\frac{T}{2}\right]$, or $[0,T]$, or $[-T,0]$, etc.

The coefficients $a_n$ and $b_n$ give the "amount" of cosines and sines present in the signal $f(t)$. Their expressions can be derived using the orthogonality properties of sines and cosines:

$$\frac{2}{T}\int_{-T/2}^{T/2} dt\ \sin(n\omega t)\sin(k\omega t) = \delta_{nk} = \begin{cases} 0 & n \neq k \\ 1 & n = k \end{cases}$$

$$\frac{2}{T}\int_{-T/2}^{T/2} dt\ \cos(n\omega t)\sin(k\omega t) = 0$$

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EXAMPLE: Periodic Rectangular Wave

Find the Fourier coefficients of the periodic function

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases}$$

and $f(x+2\pi) = f(x)$ and $k$ is a constant. The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$. Functions of this kind can represent external forces acting on mechanical systems, electromotive forces in electric circuits, etc.

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$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \left\{ \int_{-\pi}^{0} (-k)\, dx + \int_{0}^{\pi} k\, dx \right\} = 0$$

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$f(x)$ is an odd function, so this integral is zero for all $a_n$ coefficients!

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To simplify things, let’s set $k = 1$ (square wave):

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,dx = \frac{2}{n\pi}[1 - (-1)^n] = \frac{4}{n\pi}\begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases}$$

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This results into the Fourier series

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(nx)}{n}$$

# EXAMPLE: Periodic Rectangular Wave

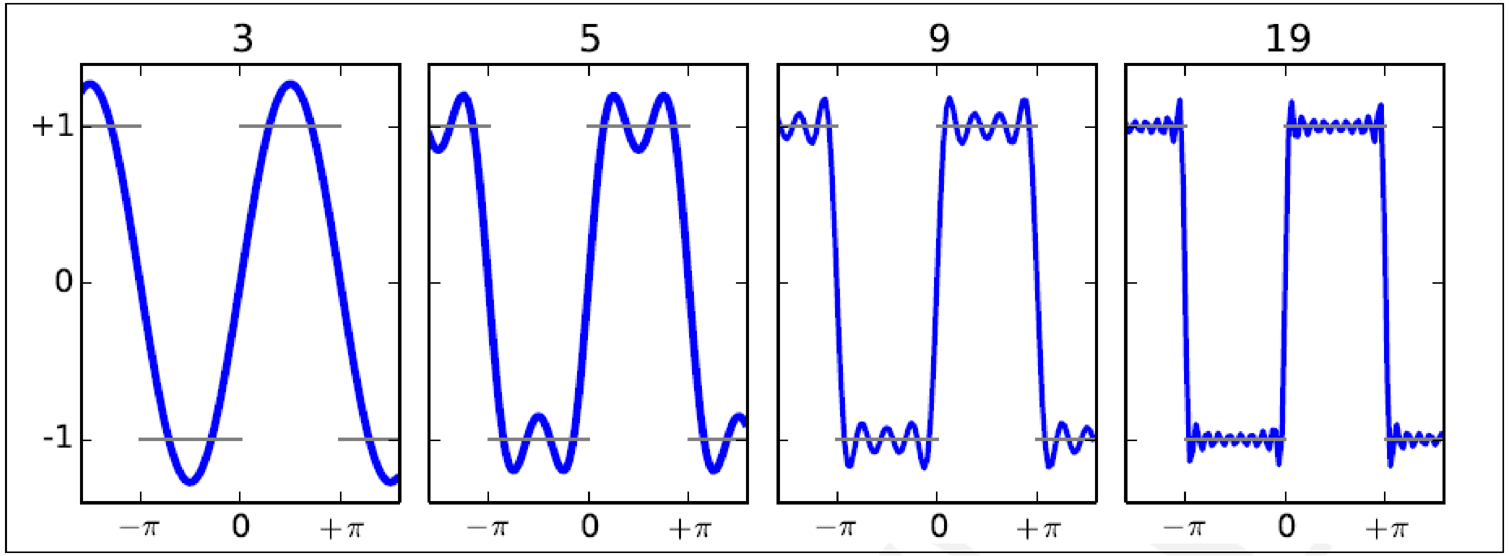



**Figure 15.3:** Reconstruction of a square-wave using sinusoids up to degree 3, 5, 9, and 19.

The figure shows a reasonable approximation of the square wave function (grey horizontal lines) when plotting against its Fourier series, particularly when summing over 10 terms in the series. We do however note the "ringing" at x = 0 and other discontinuities; this feature is known as the **Gibbs phenomenon**. Gibbs phenomenon occurs near a jump discontinuity in the signal. It says that no matter how many terms we include in the Fourier series, there will always be an error in the form of an overshoot near the discontinuity. The overshoot can be estimated to be about 9% of the size of the jump.

**Dirac Delta**

$$\delta(x) = \begin{cases} 0 & if \quad x \neq 0 \\ \infty & if \quad x = 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

$$\delta(\alpha x) = \frac{\delta(x)}{\alpha}$$

And one of many useful properties of the Dirac Delta function is “sifting”

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a) dx = f(a)$$

to evaluate a function at a selected value of $x = a$.

# Dirac Delta

Chapter 15 of Dr. Jackel's notes (will be in the D2L shell)

The Fourier series coefficients of a delta function $\delta(t - t_0)$

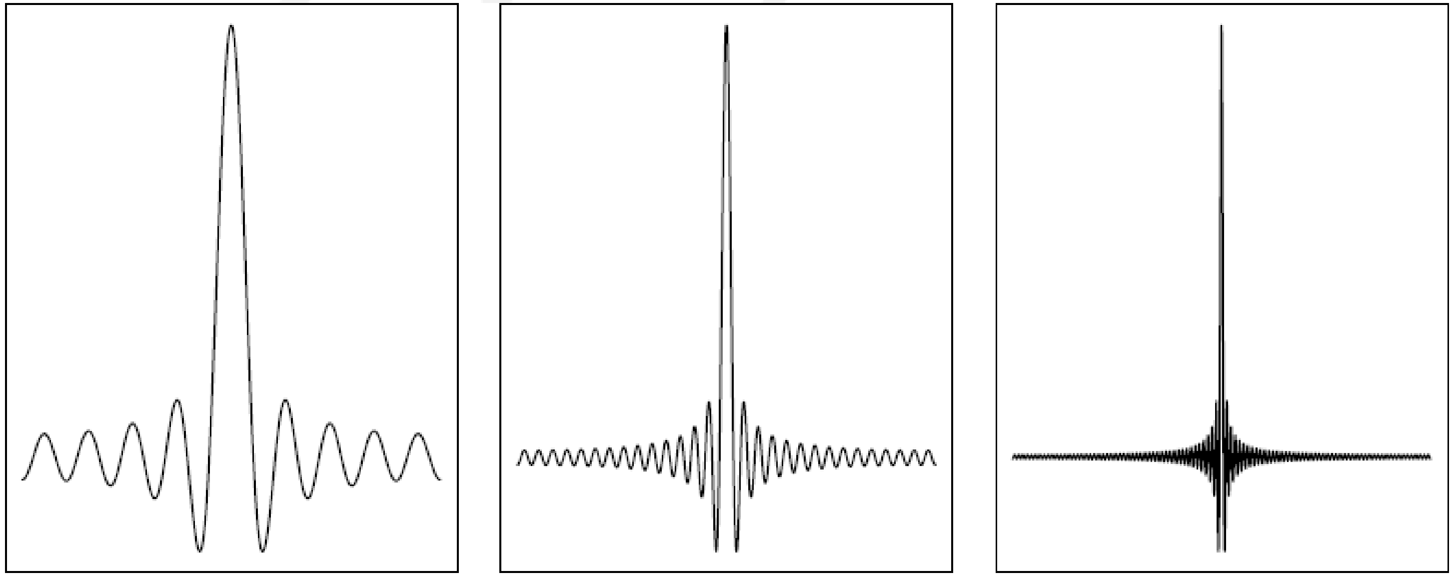
$$a_n = \frac{1}{T} \int_{-T/2}^{+T/2} \cos\left(\frac{2\pi n t}{T}\right) \delta(t - t_0) dt = \cos\left(\frac{2\pi n t_0}{T}\right) \tag{15.28}$$

$$b_n = \frac{1}{T} \int_{-T/2}^{+T/2} \sin\left(\frac{2\pi n t}{T}\right) \delta(t - t_0) dt = sin\left(\frac{2\pi n t_0}{T}\right) \tag{15.29}$$

leads to the Fourier series

$$\begin{aligned} \delta(t - t_0) &= \frac{1}{T} + \sum_{n=1}^{\infty} \left[\cos\left(\tfrac{2\pi n t}{T}\right) \cos\left(\tfrac{2\pi n t_0}{T}\right) + \sin\left(\tfrac{2\pi n t}{T}\right) \sin\left(\tfrac{2\pi n t_0}{T}\right)\right] \qquad (15.30) \\ &= \frac{1}{T} + \sum_{n=1}^{\infty} \cos\left(\frac{2\pi n (t - t_0)}{T}\right) \end{aligned}$$

## Dirac Delta



Fourier series of Dirac Delta function using 9, 29, and 99 terms.

Chapter 15 of Dr. Jackel's notes (will be in the D2L shell)

## Fourier Series: Moments

The average of the function $f(t)$

$$\bar{f}(t) = \langle f(t) \rangle = \frac{1}{T}\int_{-T/2}^{+T/2} f(t)dt \tag{15.20}$$

can be expressed in terms of the harmonic coefficients

$$\frac{1}{T}\int_{-T/2}^{+T/2} dt \left[a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)\right] = a_0 \tag{15.21}$$

so the "DC" term $a_0$ contains all information about the first moment.

The variance

$$\sigma_f^2 = \langle f(t) - \langle f(t) \rangle \rangle^2 = \int_{-T/2}^{+T/2} \left[f(t) - \bar{f}(t)\right]^2 dt \tag{15.22}$$

can also be expressed in terms of the coefficients

$$\frac{1}{T}\int_{-T/2}^{+T/2} dt \left[\sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)\right]^2 = \frac{1}{2}\sum_{n=1}^{\infty} \left[a_n^2 + b_n^2\right] = \frac{1}{2}\sum_{n=1}^{\infty} c_n^2 \tag{15.23}$$

where $c_n^2 = a_n^2 + b_n^2$.

Chapter 15 of Dr. Jackel's notes (will be in the D2L shell)

For an odd function, the integral over a symmetric interval equals zero, because half the area is negative!

Once we determine the Fourier coefficients $a_n$ and $b_n$, we can plot them as a function of $n$ to analyze their pattern. We can also combine them in an amplitude defined as

$$c_n = \sqrt{a_n^2 + b_n^2}$$

and plot as a function of $n$ to create an **amplitude spectrum**. We can also obtain a phase

$$\phi_n = \tan^{-1}(a_n/b_n)$$

that we can also visualize as a function of $n$ (**phase spectrum**).

This amplitude and phase can be derived from the Fourier series using the trigonometry identity

$$a\cos\theta + b\sin\theta = c\sin(\theta - \phi) \quad \text{with} \quad \phi = \tan^{-1}(a/b)$$

that reduces the Fourier series with a single term (in a series) with an amplitude and a phase

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \phi_n)$$

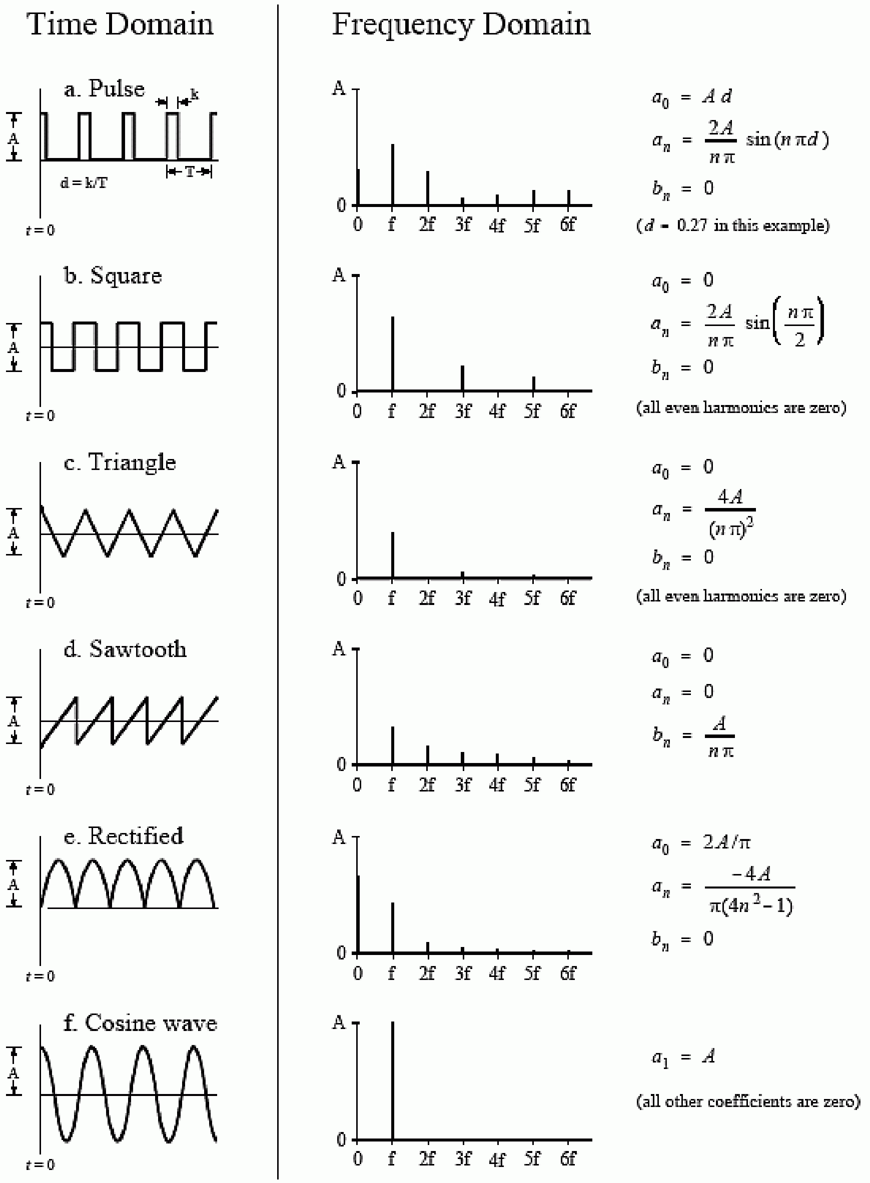



FIGURE 13-10
Examples of the Fourier series. Six common time domain waveforms are shown, along with the equations to calculate their "$a$" and "$b$" coefficients.

The Scientist and Engineer's Guide to
Digital Signal Processing
By Steven W. Smith