Question 1: Solution to Moments of a Triangular Distribution

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Given a triangular probability density function (PDF) $p_T(x|\alpha,\beta)$ defined as:

$$p_T(x|\alpha,\beta) = \begin{cases} \frac{2(x-\alpha)}{(\beta-\alpha)^2} & \text{for } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

we need to calculate the first, second, third, and fourth moments.

Definition of Moments

The n-th moment about the origin for a probability density function $p(x)$ is given by:

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n \, p(x) \, dx$$

In our case, since $p_T(x)$ is only non-zero for $x \in [\alpha,\beta]$, the integral simplifies to:

$$\mathbb{E}[X^n] = \int_{\alpha}^{\beta} x^n \, p_T(x|\alpha,\beta) \, dx$$

We will calculate each of the first four moments by evaluating this integral for $n = 1, 2, 3$, and 4.

First Moment

The first moment (mean) is:

$$\mathbb{E}[X] = \int_{\alpha}^{\beta} x \, p_T(x|\alpha,\beta) \, dx \tag{1}$$

Substitute $p_T(x|\alpha,\beta) = \frac{2(x-\alpha)}{(\beta-\alpha)^2}$:

$$\mathbb{E}[X] = \int_{\alpha}^{\beta} x \cdot \frac{2(x-\alpha)}{(\beta-\alpha)^2} \, dx = \frac{2}{(\beta-\alpha)^2} \int_{\alpha}^{\beta} x(x-\alpha) \, dx \tag{2}$$

Expanding $x(x-\alpha) = x^2 - \alpha x$:

$$\mathbb{E}[X] = \frac{2}{(\beta-\alpha)^2} \int_{\alpha}^{\beta} (x^2 - \alpha x) \, dx \tag{3}$$

Now, integrate each term separately: first part will be:

$$\int_{\alpha}^{\beta} x^2 \, dx = \left[\frac{x^3}{3}\right]_{\alpha}^{\beta} = \frac{\beta^3}{3} - \frac{\alpha^3}{3}, \tag{4}$$

and the second part is:

$$\int_\alpha^\beta \alpha x\,dx = \alpha\left[\frac{x^2}{2}\right]_\alpha^\beta = \alpha\left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right) \tag{5}$$

Combining these results:

$$\mathbb{E}[X] = \frac{2}{(\beta-\alpha)^2}\left(\frac{\beta^3}{3} - \frac{\alpha^3}{3} - \alpha\frac{\beta^2}{2} + \alpha\frac{\alpha^2}{2}\right)$$

Simplify this to get the mean. and the answer of the simplification is:

$$\mathbb{E}[X] = \frac{1}{3}(2\beta + \alpha)$$

Second Moment

The second moment is calculated as:

$$\mathbb{E}[X^2] = \int_\alpha^\beta x^2\, p_T(x|\alpha,\beta)\,dx = \frac{2}{(\beta-\alpha)^2}\int_\alpha^\beta x^2(x-\alpha)\,dx \tag{6}$$

Expand $x^2(x-\alpha) = x^3 - \alpha x^2$ and integrate each term separately:

$$\int_\alpha^\beta x^3\,dx = \left[\frac{x^4}{4}\right]_\alpha^\beta = \frac{\beta^4}{4} - \frac{\alpha^4}{4}$$
$$\int_\alpha^\beta \alpha x^2\,dx = \alpha\left[\frac{x^3}{3}\right]_\alpha^\beta = \alpha\left(\frac{\beta^3}{3} - \frac{\alpha^3}{3}\right)$$

Thus,

$$\mathbb{E}[X^2] = \frac{2}{(\beta-\alpha)^2}\left(\frac{\beta^4}{4} - \frac{\alpha^4}{4} - \alpha\frac{\beta^3}{3} + \alpha\frac{\alpha^3}{3}\right) = \frac{1}{6}(3\beta^2 + 2\alpha\beta + \alpha^2)$$

Third Moment

The third moment is:

$$\mathbb{E}[X^3] = \int_\alpha^\beta x^3\, p_T(x|\alpha,\beta)\,dx = \frac{2}{(\beta-\alpha)^2}\int_\alpha^\beta x^3(x-\alpha)\,dx \tag{7}$$

Expanding $x^3(x-\alpha) = x^4 - \alpha x^3$, we integrate each term:

$$\int_\alpha^\beta x^4\,dx = \left[\frac{x^5}{5}\right]_\alpha^\beta = \frac{\beta^5}{5} - \frac{\alpha^5}{5}$$
$$\int_\alpha^\beta \alpha x^3\,dx = \alpha\left[\frac{x^4}{4}\right]_\alpha^\beta = \alpha\left(\frac{\beta^4}{4} - \frac{\alpha^4}{4}\right)$$

Thus,

$$\mathbb{E}[X^3] = \frac{2}{(\beta-\alpha)^2}\left(\frac{\beta^5}{5} - \frac{\alpha^5}{5} - \alpha\frac{\beta^4}{4} + \alpha\frac{\alpha^4}{4}\right) = \frac{1}{10}(4\beta^3 + 3\alpha\beta^2 + 2\alpha^2\beta + \alpha^3)$$

Forth Moment

The fourth moment is:

$$\mathbb{E}[X^4] = \int_\alpha^\beta x^4\,p_T(x|\alpha,\beta)\,dx = \frac{2}{(\beta-\alpha)^2}\int_\alpha^\beta x^4(x-\alpha)\,dx \tag{8}$$

Expanding $x^4(x-\alpha) = x^5 - \alpha x^4$, we integrate each term:

$$\int_\alpha^\beta x^5\,dx = \left[\frac{x^6}{6}\right]_\alpha^\beta = \frac{\beta^6}{6} - \frac{\alpha^6}{6}$$
$$\int_\alpha^\beta \alpha x^4\,dx = \alpha\left[\frac{x^5}{5}\right]_\alpha^\beta = \alpha\left(\frac{\beta^5}{5} - \frac{\alpha^5}{5}\right)$$

Thus,

$$\mathbb{E}[X^4] = \frac{2}{(\beta-\alpha)^2}\left(\frac{\beta^6}{6} - \frac{\alpha^6}{6} - \alpha\frac{\beta^5}{5} + \alpha\frac{\alpha^5}{5}\right) = \frac{1}{15}(5\beta^4 + 4\alpha\beta^3 + 3\alpha^2\beta^2 + 2\alpha^3\beta + \alpha^4)$$

Substitute the limits and simplify each expression to find the final values for the moments. Now I want to calculate the median and mode of this distribution.

Mode of the Triangular Distribution

The mode of a distribution is the point where the probability density function (PDF) reaches its maximum value.

1. For the triangular distribution given here, the PDF $p_T(x|\alpha,\beta)$ is monotonically increasing within the range $[\alpha,\beta]$. 2. The PDF reaches its peak value at the right endpoint $x = \beta$.

Thus, for this triangular distribution, the mode is simply:

$$\text{Mode} = \beta$$

Median of the Triangular Distribution

The median is the point at which the cumulative distribution function (CDF) equals 0.5. In other words, it is the value x_m such that:

$$\int_{\alpha}^{x_m} p_T(x|\alpha, \beta)\, dx = 0.5$$

Using $p_T(x|\alpha, \beta) = \frac{2(x-\alpha)}{(\beta-\alpha)^2}$, we need to solve for x_m such that:

$$\int_{\alpha}^{x_m} \frac{2(x-\alpha)}{(\beta-\alpha)^2}\, dx = 0.5 \tag{9}$$

To find x_m, we first evaluate the integral:

$$\int_{\alpha}^{x_m} \frac{2(x-\alpha)}{(\beta-\alpha)^2}\, dx$$

Let $u = x - \alpha$, then $du = dx$, and the integral limits transform as follows: - When $x = \alpha$, $u = 0$. - When $x = x_m$, $u = x_m - \alpha$.

The integral becomes:

$$\frac{2}{(\beta-\alpha)^2} \int_{0}^{x_m-\alpha} u\, du$$

Now integrate with respect to u:

$$= \frac{2}{(\beta-\alpha)^2} \left[\frac{u^2}{2}\right]_0^{x_m-\alpha} = \frac{2}{(\beta-\alpha)^2} \cdot \frac{(x_m-\alpha)^2}{2} = \frac{(x_m-\alpha)^2}{(\beta-\alpha)^2}$$

Now, set the result equal to 0.5 to find x_m:

$$\frac{(x_m-\alpha)^2}{(\beta-\alpha)^2} = 0.5$$

Multiply both sides by $(\beta-\alpha)^2$:

$$(x_m-\alpha)^2 = 0.5 \cdot (\beta-\alpha)^2$$

Take the square root of both sides:

$$x_m - \alpha = \sqrt{0.5} \cdot (\beta - \alpha)$$

So,

$$x_m = \alpha + \sqrt{0.5} \cdot (\beta - \alpha)$$

Since $\sqrt{0.5} = \frac{\sqrt{2}}{2}$, we can rewrite the median as:

$$\text{Median} = x_m = \alpha + \frac{\sqrt{2}}{2}(\beta - \alpha)$$

In summary, the median and mode of the triangular distribution are:

$$\text{Mode} = \beta$$

$$\text{Median} = \alpha + \frac{\sqrt{2}}{2}(\beta - \alpha)$$

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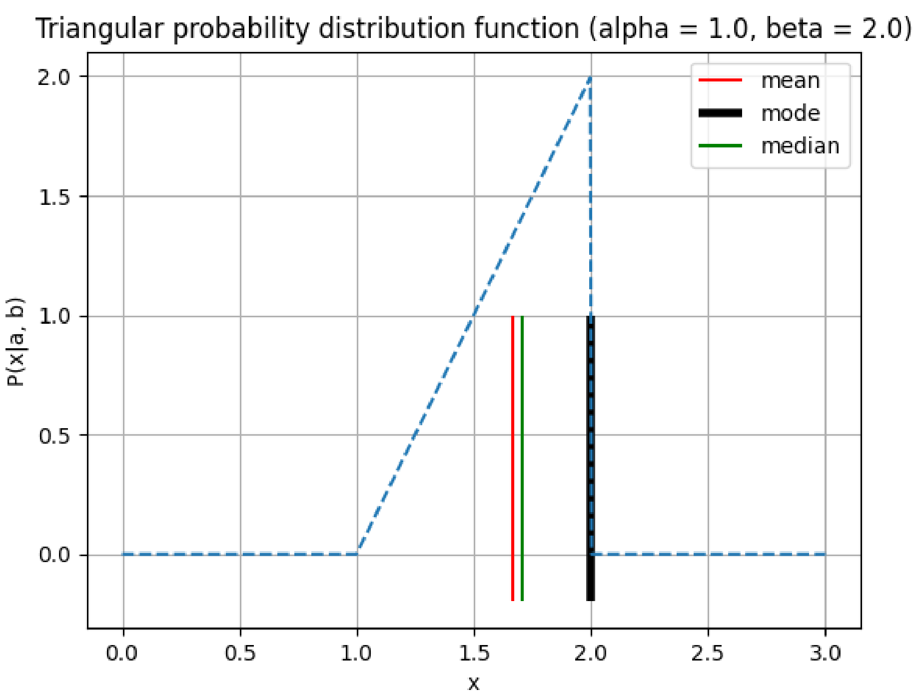

Figure 1: Plot of the triangular distribution $p_T(x \mid \alpha = 1, \beta = 2)$ showing the mean = 1.66, median = 1.70, and mode = 2.

The triangular distribution in Figure 1 illustrates the probability density function $p_T(x|\alpha = 1, \beta = 2)$, with vertical lines marking the mean, median, and mode.