1 Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1 + x)^n \geqslant 1 + nx$.

Proof. We proceed by induction on n.

Base case($n = 0$): LHS=1⩾1=RHS.

Induction Hypothesis: For some arbitrary n=$k \geqslant 0$, assume $(1 + x)^k \geqslant 1 + kx$.

Inductive Step: We prove for n=k+1. According to induction hypothesis,

$$(1+x)^{k+1} = (1+x)^k(1+x) \geqslant (1+kx)(1+x) = 1+(k+1)x+kx^2 \geqslant 1+(k+1)x$$

We conclude that $(1 + x)^{k+1} \geqslant 1 + (k + 1)x$. Thus, by the principle of induction, $\forall n \in \mathbb{N}$ and x>0, $(1 + x)^n \geqslant 1 + nx$. □

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \cdots$ is defined by $a_1 = 1$ and $a_{n+1} = 3{a_n}^2$ for $n \geqslant 1$. We want to prove that

$$a_n \leqslant 3^{2^n}$$

for every positive integer n.

(a) **Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leqslant 3^{2^n}$? Attempt an induction proof with this hypothesis to show why this does not work.**

Failed.

(b) **Try to instead prove the statement $a_n \leqslant 3^{2^n-1}$ using induction.**

Proof. We proceed by induction on n.

Base case($n = 1$): $a_1 = 1 \leqslant 3^{2-1} \leqslant 3^2$.

Induction Hypothesis: For some arbitrary n=k, we assume that $a_k \leqslant 3^{2^k-1}$.

Inductive Step: We prove for n=k+1. According to the recursive relationship $a_{n+1} = 3{a_n}^2$ and the hypothesis,

$$a_{k+1} = 3{a_k}^2 \leqslant 3 \cdot (3^{2^k-1})^2 = 3^{2^{k+1}-1},$$

And we conclude that $a_k \leqslant 3^{2^k-1}$. Thus, by the principle of induction, $\forall n \in \mathbb{N}$,$a_n \leqslant 3^{2^n-1}$. □

(c) **Why does the hypothesis in part (b) imply the overall claim?**

The induction hypothesis is weak.

3 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$\boldsymbol{n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots c_1 \cdot 2^1 + c_0 \cdot 2^0,}$$

where $\boldsymbol{k \in \mathbb{N}}$ and $\boldsymbol{c_i \in \{0, 1\}}$ for all $\boldsymbol{i \leqslant k}$.

Proof. We proceed by induction on n.

Base case($n = 1$): $n = 1 = 1 \cdot 2^0$, i.e.$c_0 = 1$.

Induction Hypothesis: For some arbitrary n=$k \geqslant 1$, assume k can be written in binary, i.e.

$$k = c_j \cdot 2^j + c_{j-1} \cdot 2^{j-1} + \cdots c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $j \in \mathbb{N}$ and $c_t \in 0, 1$ for all $t \leqslant j$.

Inductive Step: We prove for n=k+1. According to induction hypothesis,

$$n = k + 1 = c_j \cdot 2^j + c_{j-1} \cdot 2^{j-1} + \cdots c_1 \cdot 2^1 + c_0 \cdot 2^0 + 1 =$$

$$c_j \cdot 2^j + c_{j-1} \cdot 2^{j-1} + \cdots c_1 \cdot 2^1 + c_0 \cdot 2^0 + 1 \cdot 2^0 = c_j \cdot 2^j + c_{j-1} \cdot 2^{j-1} + \cdots c_1 \cdot 2^1 + (c_0 + 1) \cdot 2^0,$$

If $c_0 = 0$, then for n=k+1, it's obvious that it can be written in binary. If $c_0 = 1$, then for n=k+1, c_0 becomes 0, and c_1 becomes $c_1 + 1$, which is similar with the case of c_0. One by one continue the process until the digit doesn't change. And we conclude $\forall c_s (s \leqslant j + 1)$, $c_s \in \{0, 1\}$. Thus the claim is true for k+1. Thus by the principle of induction, the claim is true for $\forall n \in \mathbb{N}$. $\square$