1 Perfect Square

(a) : Prove that if n^2 is odd, then n must also be odd.

Proof. We will proceed with a direct proof. Assume n^2 is odd, i.e.$n^2 = 2k+1$ ($k \in \mathbb{N}$) then $n^2 - 1 = 2k$ (n-1)(n+1)=2k if n is even then (n-1),(n+1) are even which is qualified. If n is even, then (n-1),(n+1) are odd, (n-1)(n+1) is odd, which is not qualified. So n must be odd. □

(b) : Prove that if n^2 is odd, then n^2 can be written in the form 8k+1 for some integer k.

Proof. We will proceed with a direct proof. Assume n^2 is odd. Then according to (a), we know that n is odd, i.e. n can be written in the form of 2t+1 for some integer t. Then $n^2 = 4t^2 + 4t + 1$, which can be also written as $n^2 = 4t(t+1) + 1$. It's obvious that t(t+1) can be divided by 2. Accordingly, 4t(t+1) can be written in the form of 8k, which means n^2 can be written in the form of 8k+1 for some integer k.□

2 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

Proof. We will proceed with contradiction. Assume that there is no column that is all red, i.e. for every column there exists at least one blue pebble. Then we can choose the pebble in the way that we just pick the blue pebble from each column(because every column has at least one blue pebble to choose), where the pebbles we pick are all blue. That's contradictory to the fact that for every way of choosing one pebble from each column there exists a red pebble among the chosen ones.□

3 Numbers of Friends

Prove that if there are n $\geqslant$ 2 people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

Proof. We will proceed with cases.

When $n > 2$:

Case 1 : Everyone has at least 1 friend, i.e. there is no one who don't have friends. According to the assumption and the fact that there are $n \geqslant 2$

people at the party, the number of friends ranges from 1 to n-1 for each person. From the Pigeonhole Principle, we know because there are n people and n-1 number there must exist two people have the same number of friends.

Case 2 : There are at least 2 people who don't have any friends. This is clearly what we need to prove.

Case 3 : There is only 1 person who doesn't have any friends, i.e. for the rest of people at party they all have at least 1 friend. For the rest n-1 people, it's just the same of the case 1.

When n=2 :

Either they are friends or they aren't friends is qualified for the same number of friends.□