Asymptotic Orders of Recurrence Functions

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The lecture notes are mostly based on Chapter 4.3 of Cormen, Leiserson, Rivest, and Stein. Introduction to Algorithms. 3rd Ed. 2009. MIT Press. Cambridge, Massachusetts.

Contents

Introduction

Definition 1 (Recurrence)**.** *A recurrence is an equation or inequality that defines the value of a function by the values of the same function on smaller input sizes.*

Boundary condition: $T(n) = \Theta(1)$ for small n.

Motivation: What are the growth rates of the following two functions?

$$f_1(n) = f_1(n-1) + 2$$

and

$$f_2(n) = f_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n$$

Do they have the same growth rate?

They are both linear $\Theta(n)$, though not obviously.

Computer programs can be written recursively leading to these recurrence equations.

That is why we need to study how to evaluate the growth rate of a recurrence function.

1 Substitution method

Two steps:

1. Guess a solution – a closed asymptotic form;
2. Use mathematical induction to prove the solution works by the definition of asymptotic bounds.

Example 1.

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

Guess:

$$T(n) = O(n \lg n)$$

Mathematical induction:

Must prove the proposition

$$T(n) \leq cn \lg n$$

but not

$$T(n) = O(n \lg n)$$

in the induction step! Because using the asymptotic order within an induction proof is logically inconsistent.

Proof. (by induction)

Hypothesis:

$$T(\lfloor n/2 \rfloor) \leq c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$

Induction:

$$\begin{aligned} T(n) &\leq 2c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor) + n \\ &\leq cn \lg(n/2) + n \quad (\because \lfloor n/2 \rfloor \leq n/2) \\ &= cn \lg n - cn \lg 2 + n \\ &= cn \lg n - cn + n \\ &\leq cn \lg n \quad (\text{if } c \geq 1) \end{aligned}$$

Base case(s):

- When $n = 1$, $T(1) = 1 \nleq cn \lg n = 0$ for any $c > 0$!

  We assume $T(1)$ to be a constant 1 for ease of argument but without loss of generality.

- When $n = 2$, $T(2) = 4$. To make $T(2) = 4 \leq c2 \lg 2$, we must have $c \geq 2$;

- We also want to consider $n = 3$ in the base case, because it will be reduced to input of size 1 by recurrence.

$T(3) = 5$. To make $T(3) = 5 \le c3\lg 3$, we must have $c \ge 5/(3\lg 3) = 1.52$.

Thus $c = 2$ will make the base cases work for $n > 1$ as well as the induction.

□

Example 2.

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

Guess:

$$T(n) = \Omega(n \lg n) \tag{1}$$

Mathematical induction attempt 1:

$$T(n) \ge cn \lg n$$

Proof. (by induction)

Hypothesis:

$$T(\lfloor n/2 \rfloor) \ge c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$

Induction:

$$\begin{aligned} T(n) &\ge 2c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor) + n \\ &\ngeq cn \lg(n/2) + n \quad (\because \lfloor n/2 \rfloor \ngeq n/2) \end{aligned}$$

The induction does not necessarily follow from here. This attempt is thus unsuccessful.

□

Mathematical induction attempt 2:

$$T(n) \ge c(n+1)\lg(n+1) \ (\ge cn \lg n)$$

Proof. (by induction)

Hypothesis:

$$T(\lfloor n/2 \rfloor) \geq c(\lfloor n/2 \rfloor + 1)\lg(\lfloor n/2 \rfloor + 1)$$

Induction:

$$\begin{aligned}
T(n) \geq & 2c(\lfloor n/2 \rfloor + 1)\lg(\lfloor n/2 \rfloor + 1) + n & (2)\\
\geq & 2c[(n+1)/2]\lg[(n+1)/2] + n \quad (\because \lfloor n/2 \rfloor + 1 \geq (n+1)/2) & (3)\\
= & c(n+1)\lg(n+1) - c(n+1)\lg 2 + n & (4)\\
= & c(n+1)\lg(n+1) - c(n+1) + n & (5)\\
\geq & c(n+1)\lg(n+1) \quad (\text{if } -c(n+1) + n \geq 0) & (6)
\end{aligned}$$

Proof of $\lfloor n/2 \rfloor + 1 \geq (n+1)/2$:

When even $n = 2k$ $(k \geq 1)$:

$$\lfloor n/2 \rfloor + 1 = k + 1 > k + \frac{1}{2} = (n+1)/2$$

When odd $n = 2k - 1$ $(k \geq 1)$:

$$\lfloor n/2 \rfloor + 1 = \lfloor (2k-1)/2 \rfloor + 1 = k - 1 + 1 = k = (n+1)/2$$

Now let's determine some $c > 0$

$$\begin{aligned}
& -c(n+1) + n \geq 0\\
\Leftarrow & n \geq \frac{c}{1-c}, 1 - c > 0\\
\Leftarrow & n \geq 1, c = 0.5
\end{aligned}$$

Base case(s): When $n = 1$, $T(1) = 1 \geq c(1+1)\lg(1+1) = 2c$. Thus, $c \leq 0.5$ will make the base case work.

Therefore, $c = 0.5$ will make both the base case and the induction work for $n \geq 1$.

$\square$

Example 3.

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

Guess: $T(n) = O(n)$.

Proposition: $T(n) \leq cn$.

Mathematical Induction:

Proof.

Hypothesis: $T(\lfloor n/2 \rfloor) \leq c\lfloor n/2 \rfloor$ and $T(\lceil n/2 \rceil) \leq c\lceil n/2 \rceil$.

Induction:

$$T(n) \leq c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 = cn + 1 \quad (\because n \in Z)$$

which does not imply $T(n) \leq cn$ for any choice of c.

STOP! Make another guess for $T(n)$!

Base cases:

New proposition: $T(n) \leq cn - b$. Then the induction becomes

$$\begin{aligned} T(n) &\leq (c\lfloor n/2 \rfloor - b) + (c\lceil n/2 \rceil - b) + 1 \\ &= cn - 2b + 1 \\ &= cn - b - (b - 1) \\ &\leq cn - b \quad (\text{if } b \geq 1) \end{aligned}$$

$\square$

Example 4. *Why we cannot use O or Ω in induction?*

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

Proposition: $T(n) = O(n)$

By hypothesis $T(\lfloor n/2 \rfloor) \le c\lfloor n/2 \rfloor$.

Thus,

$$\begin{aligned} T(n) \le& 2(c\lfloor n/2 \rfloor) + n \\ \le& cn + n \\ =& O(n) \quad \leftarrow \textit{WRONG!} \end{aligned}$$

Remark: *This obviously contradicts* $T(n) = \Omega(n \lg n)$ *proven in Eq. (1). We MUST prove the exact algebraic form of the inductive hypothesis, instead of the asymptotic form.*

Example 5.

$$T(n) = 2T(\sqrt{n}) + \lg n$$

Renaming $m = \lg n$ *yields*

$$T(2^m) = 2T(2^{m/2}) + m$$

Let

$$S(m) = T(2^m)$$

We get

$$S(m) = 2S(m/2) + m$$

From Example 1, $S(m) = O(m \lg m)$. *Thus*

$$T(n) = S(\lg n) = O(\lg n \lg \lg n) = O(\lg n \lg^{(1)} n)$$

2 Recursion tree method

Roles of a recursion tree method:

- Estimate the order of a recurrence function.
- Approximation/simplification allowed.
- The estimate can be used as a guess required by the substitution method.

The sum of geometric series:

$$a^0 + a^1 + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a} \quad a \neq 1$$

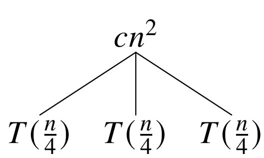
Example 6.

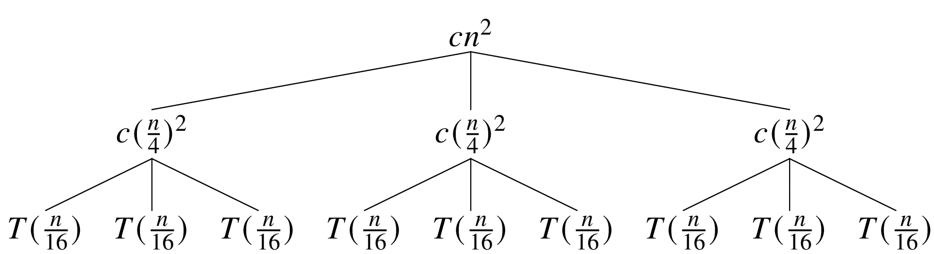
$$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$$

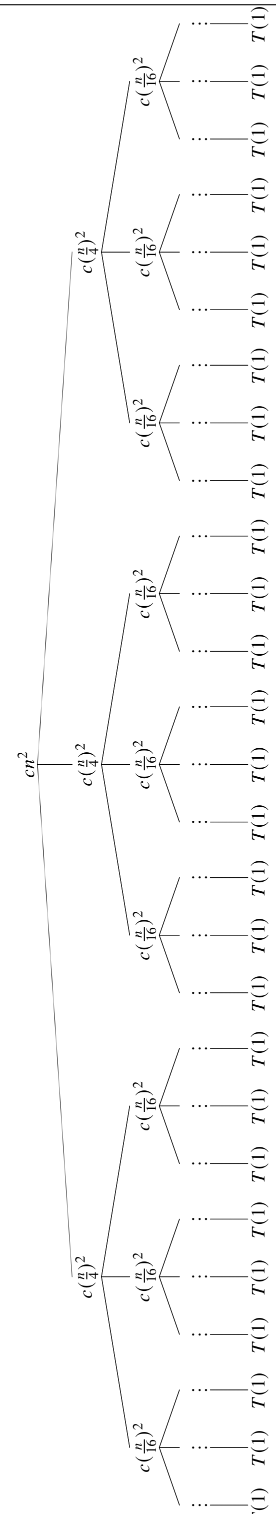
Simplification (that would not change the order of $T(n)$ in this case):

- *We assume n is exact power of 4 to approximate $\lfloor n/4 \rfloor$ by $n/4$*

$T(n)$






cn^2
$c(\frac{n}{4})^2$
$c(\frac{n}{4})^2$
$c(\frac{n}{4})^2$
$c(\frac{n}{16})^2$
$c(\frac{n}{16})^2$
$c(\frac{n}{16})^2$
$c(\frac{n}{16})^2$
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$c(\frac{n}{16})^2$
$c(\frac{n}{16})^2$
$c(\frac{n}{16})^2$
$c(\frac{n}{16})^2$
$T(1)$

The depth of the root node is 0;

The depth of a leaf node is $\log_4 n$*;*

For a node at depth i, its value is approximated by $c(n/4^i)^2$.

There are 3^i *nodes at depth i.*

The sum of all nodes at depth i is

$$c(n/4^i)^2 3^i = cn^2(3/16)^i$$

There are $3^{\log_4 n} = n^{\log_4 3}$ *leaf nodes, each with value* $T(1)$.

From depth 0 *to* $\log_4 n$*, the total approximates* $T(n)$ *by*

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n - 1} cn^2(3/16)^i + \Theta(n^{\log_4 3}) \\
&= cn^2 \frac{1-(3/16)^{\log_4 n}}{1-(3/16)} + \Theta(n^{\log_4 3}) \\
&\leq \sum_{i=0}^{\infty} cn^2(3/16)^i + \Theta(n^{\log_4 3}) \\
&= cn^2 \frac{1}{1-(3/16)} + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2)
\end{aligned}$$

Remarks:

- *The root node dominates the total;*
- *As* $O(n^2)$ *is the upper bound and* $T(n)$ *is evidently* $\Omega(n^2)$ *by its definition,* $T(n)$ *must be* $\Theta(n^2)$.

Proposition: $T(n)$ is $O(n^2)$.

Proof. We want to show

$$T(n) \le dn^2 \quad \text{for some } d > 0$$

$$\begin{aligned}
T(n) &\le 3T(\lfloor n/4 \rfloor) + cn^2 \\
&\le 3d\lfloor n/4 \rfloor^2 + cn^2 \\
&\le 3d(n/4)^2 + cn^2 \\
&= \frac{3}{16}dn^2 + cn^2 \\
&\le dn^2
\end{aligned}$$

The last inequality holds as long as $d \ge \frac{16}{13}c$. $\square$

Example 7.

$$T(n) = T(n/3) + T(2n/3) + O(n)$$

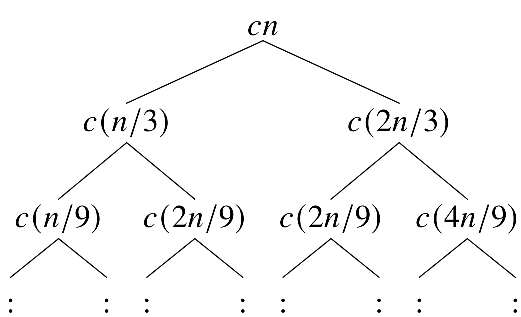

Figure 1: Recursion tree for $T(n) = T(n/3) + T(2n/3) + O(n)$.

There are at most $1 + \log_{3/2} n$ levels by looking at the rightmost branch of the tree.

Each depth has a total of cn.

Therefore, we guess $T(n) = cn(1 + \log_{3/2} n) = O(n \lg n)$.

We will show that $T(n) = O(n \lg n)$ *by applying substitution method on* $T(n) \leq dn \lg n$ *for some constant* d.

$$
\begin{aligned}
T(n) &\leq T(n/3) + T(2n/3) + cn \\
&\leq d(n/3)\lg(n/3) + d(2n/3)\lg(2n/3) + cn \\
&= [d(n/3)\lg n - d(n/3)\lg 3] + [d(2n/3)\lg 2n - d(2n/3)\lg 3] + cn \\
&= dn\lg n - d[(n/3)\lg 3 + (2n/3)\lg(3/2)] + cn \\
&= dn\lg n - dn(\lg 3 - 2/3) + cn \\
&\leq dn\lg n,
\end{aligned}
$$

as long as $d \geq c/(\lg 3 - (2/3))$.

Remarks: Were the tree complete, the number of leaves would be $2^{\log_{3/2} n} = n^{\log_{3/2} 2} = n^{1.7} = \omega(n \lg n)$. *However, as the tree is generally incomplete and by our analysis above,* $T(n) \neq \omega(n \lg n)$.

3 The Master method

The recurrence

$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1, f(n) > 0 \text{ (for } n > n_0 > 0)$$

can be thought of as a model for divide-and-conquer algorithms:

Divide: The problem is divided into a subproblems;

Conquer: Each subproblem has an input size of n/b and is solved in time $T(n/b)$;

Combine: $f(n)$ is the total cost for dividing the problem and combining results of the subproblems.

Example 8 (Merge-Sort)**.**

$$a = 2; \quad b = 2; \quad f(n) = \Theta(n)$$

Theorem 1 (Master Theorem)**.** *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be an asymptotically positive function, and let $T(n)$ be defined on the nonnegative integers by the recurrence*

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.

1. *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*

   Note: $\log_b a - \epsilon = (\log_b a) - \epsilon$

2. *If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \lg n)$.*

3. *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if*

   $$af(n/b) \leq cf(n)$$

   for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

   Note: $\log_b a + \epsilon = (\log_b a) + \epsilon$

Remarks 1. *The order of $T(n)$ is determined by $n^{\log_b a}$ and the order of $f(n)$.*

1. *If $n^{\log_b a}$ is "polynomially" larger than $f(n)$, $T(n) = \Theta(n^{\log_b a})$;*

2. *If $n^{\log_b a}$ and $f(n)$ are the same order, then*

   $$T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(f(n) \lg n);$$

3. *If $f(n)$ is "polynomially" larger than $n^{\log_b a}$, $T(n) = \Theta(f(n))$;*

Limitations of the Master Theorem: *It does not cover all possible asymptotically positive $f(n)$.*

Example 9. *An example that the Master Theorem does not cover.*

$T(n) = T(n/2) + n(2 - \cos n\pi)$. *$T(n)$ does satisfy the asymptotic condition for case 3*

$$n(2 - \cos n\pi) \geq n = \Omega(n^{\log_2 1+0.5}) \quad (\epsilon = 0.5)$$

but it violates the regularity condition. It is shown as follows.

$$af(n/b) = n - \frac{n}{2}\cos\frac{n\pi}{2}$$

The maxima of $af(n/b)$ is $\frac{3}{2}n$, occurring at $n = 2, 6, 10, 14, \ldots$

The minima of $cf(n) = cn(2 - \cos n\pi)$ is cn, occurring at $n = 0, 2, 4, 6, 8, 10, 12, 14, \ldots$ Thus we cannot find $0 < c < 1$ to make

$$af(n/b) < cf(n)$$

for $n = 2, 6, 10, \ldots$. Therefore, function $f(n)$ violates the regularity condition in case 3.

The correct solution (not from the textbook): Let

$$T_1(n) = T_1(n/2) + n \text{ and } T_1(1) = T(1)$$

and

$$T_2(n) = T_2(n/2) + 3n \text{ and } T_2(1) = T(1)$$

By the Master theorem, both $T_1(n)$ and $T_2(n)$ can be solved by case 3 to satisfy

$$T_1(n) = T_2(n) = \Theta(n)$$

It also follows by definitions of $T_1(n)$ and $T_2(n)$ that

$$T_1(n) \le T(n) \le T_2(n)$$

Therefor, we have

$$T(n) = \Theta(n)$$

Example 10.

$$T(n) = 9T(n/3) + n$$

As

$$n = O(n) = O(n^{\log_3 9 - 1}) \quad (\epsilon = 1),$$

we have case 1:

$$T(n) = \Theta(n^2)$$

Example 11.

$$T(n) = T(2n/3) + 1$$

As

$$1 = \Theta(n^0) = \Theta(n^{\log_{3/2} 1})$$

we have case 2:

$$T(n) = \Theta(\lg n)$$

Example 12.

$$T(n) = 3T(n/4) + n \lg n$$

Case 3.

$$n \lg n = \Omega(n) = \Omega(n^{\log_4 3 + \log_4 (4/3)})$$

Thus we can choose $\epsilon = \log_4(4/3) > 0$.

Regularity condition:

$$af(n/b) = 3(n/4)\lg(n/4)$$

and

$$cf(n) = cn \lg n$$

To have

$$\begin{aligned}
&3(n/4)\lg(n/4) < cn \lg n \\
\Leftarrow &3/4 \lg(n/4) < c \lg n \quad (n > 0) \\
\Leftarrow &3/4 \lg n \leq c \lg n
\end{aligned}$$

Thus $c = 3/4 \in (0, 1)$ *will satisfy the regularity condition.*

$$T(n) = \Theta(n \lg n)$$

Example 13.

$$T(n) = 2T(n/2) + n \lg n$$

If we apply case 3, we get

$$T(n) = \Theta(n \lg n) \longleftarrow \textit{WRONG!}$$

because $T(n) = n \lg n$ *does not satisfy*

- *case 1.* $T(n) = n \lg n \neq O(n^{\log_2 2 - \epsilon}) = O(n^{1-\epsilon})$
- *case 2.* $T(n) = n \lg n \neq \Theta(n^{\log_2 2}) = \Theta(n)$
- *case 3.* $T(n) = n \lg n \neq \Omega(n^{\log_2 2 + \epsilon}) = \Omega(n^{1+\epsilon})$

This example does not belong to any case of Master Theorem.

Quiz 4 (10 points)

Use the Master method to determine the growth rate of $T(n)$ defined as

$$A: \quad T(n) = 2T(n/2) + n2^n$$
$$B: \quad T(n) = 2T(n/2) + n3^n$$

You must determine a $\epsilon > 0$ for case 1 or 3; If it is case 3, you must show the regularity condition holds.

Correct answer 5 points. Correct proof 5 points.

Quiz 4 (10 points)

Use the Master method to determine the growth rate of $T(n)$ defined as

$$A: \quad T(n) = 2T(n/2) + n2^n$$
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You must determine a $\epsilon > 0$ for case 1 or 3; If it is case 3, you must show the regularity condition holds.

Correct answer 5 points. Correct proof 5 points.

Solution:

A:

Case 3. $T(n) = \Theta(n2^n)$.

As 2^n is $\omega(n)$, we have

$$n2^n = \Omega(n^{\log_2 2+1})$$

where $\epsilon = 1$.

Regularity condition:

$$\begin{aligned}
& af(n/b) = 2(n/2)(2^{n/2}) < cn2^n = cf(n) \\
\Leftrightarrow & n(\sqrt{2})^n < cn2^n \\
\Leftrightarrow & \frac{\sqrt{2}}{2} n(\sqrt{2})^{n-1} < cn2^{n-1} \\
\Leftarrow & \frac{\sqrt{2}}{2} n2^{n-1} < cn2^{n-1} \\
\Leftarrow & c > \frac{\sqrt{2}}{2} \quad (n > 0)
\end{aligned}$$

where we can choose $c = \frac{\sqrt{3}}{2} \in (0, 1)$ to satisfy the regularity condition.

B:

Case 3. $T(n) = \Theta(n3^n)$.

As 3^n is $\omega(n)$, we have

$$n3^n = \Omega(n^{\log_2 2+1})$$

where $\epsilon = 1$.

Regularity condition:

$$\begin{aligned}
&af(n/b) = 2(n/2)(3^{n/2}) < cn3^n = cf(n) \\
\Leftrightarrow & n(\sqrt{3})^n < cn3^n \\
\Leftrightarrow & \frac{\sqrt{3}}{3} n(\sqrt{3})^{n-1} < cn3^{n-1} \\
\Leftarrow & \frac{\sqrt{3}}{3} n3^{n-1} < cn3^{n-1} \\
\Leftarrow & c > \frac{\sqrt{3}}{3} \quad (n > 0)
\end{aligned}$$

where we can choose $c = \frac{\sqrt{3}}{3} \in (0, 1)$ to satisfy the regularity condition.