The maximum-subarray problem

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The lecture notes are mostly based on Chapter 4.1 of Cormen, Leiserson, Rivest, and Stein. Introduction to Algorithms. 3rd Ed. 2009. MIT Press. Cambridge, Massachusetts.

Contents

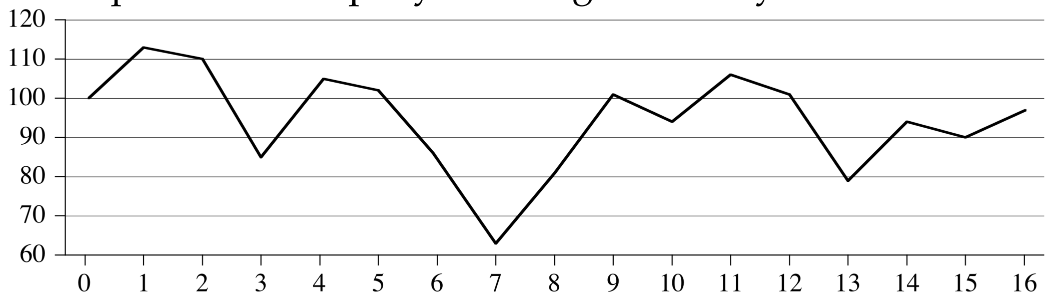
1 An example of trading stocks

Problem: If we magically know the price of a stock in advance, can we write a computer program to buy and sell stocks to maximize the profit?

Solutions:

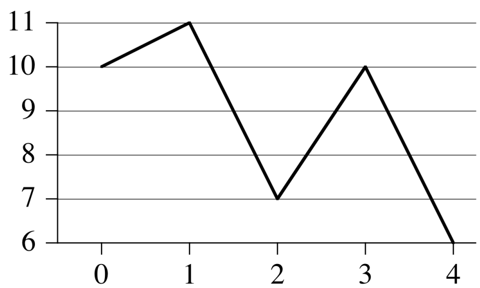
1. *buy low and sell high*. $O(n)$ (Will this always work?) No. The highest may come before the lowest. Counter example below:

   Stock price of a company in a range of 17 days:



Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	−3	−25	20	−3	−16	−23	18	20	−7	12	−5	−22	15	−4	7

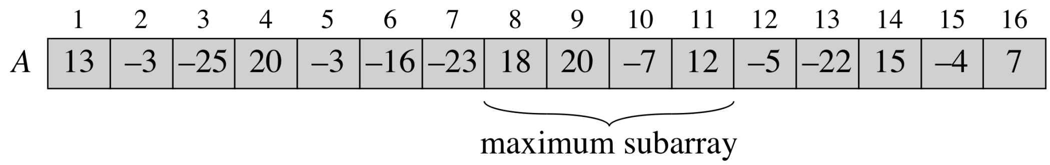
2. *buy low or sell high*. This will not always work. Counter example below:

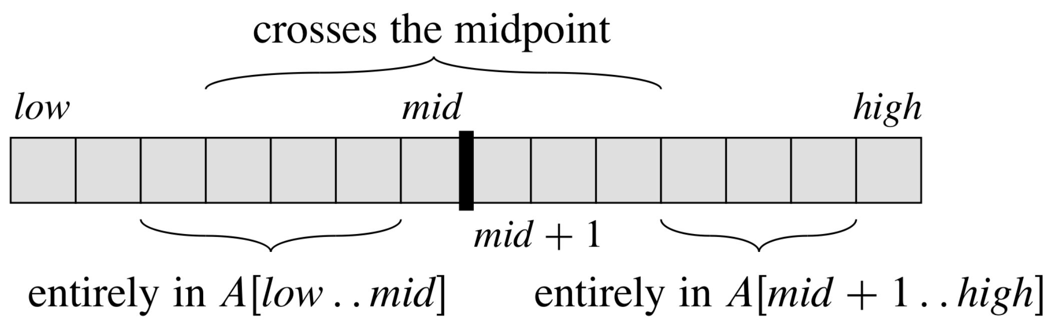


Day	0	1	2	3	4
Price	10	11	7	10	6
Change		1	−4	3	−4

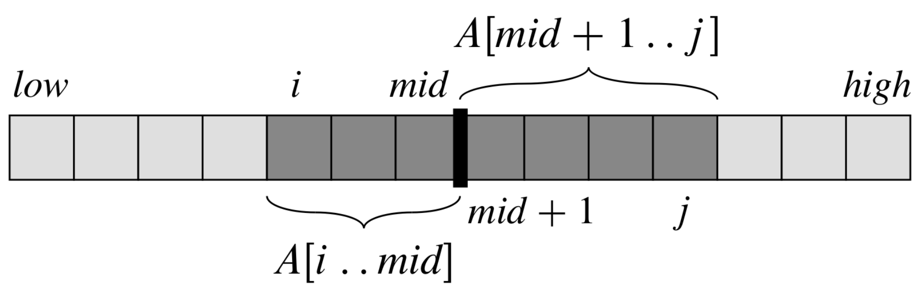
3. Try all possible (buy, sell) days and find the maximum profit. $O(n^2)$. Slow!

4. The maximum subarray concept:





(a)



(b)

2 The algorithm

Input: array A

Output: a maximum subarray in A, whose sum is the largest.

```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
// Find a maximum subarray of the form A[i..mid]
left.sum = -∞
sum = 0
for i = mid downto low
    sum = sum + A[i]
    if sum > left.sum
        left.sum = sum
        max.left = i
// Find a maximum subarray of the form A[mid + 1..j]
right.sum = -∞
sum = 0
for j = mid + 1 to high
    sum = sum + A[j]
    if sum > right.sum
        right.sum = sum
        max.right = j
// Return the indices and the sum of the two subarrays
return (max.left, max.right, left.sum + right.sum)
```

```
FIND-MAXIMUM-SUBARRAY(A, low, high)
if high == low
    return (low, high, A[low]) // base case: only one element
else mid = ⌊(low + high)/2⌋
    (left.low, left.high, left.sum)
        = FIND-MAXIMUM-SUBARRAY(A, low, mid)
    (right.low, right.high, right.sum)
        = FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
    (cross.low, cross.high, cross.sum)
        = FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
    if left.sum ≥ right.sum and left.sum ≥ cross.sum
        return (left.low, left.high, left.sum)
    elseif right.sum ≥ left.sum and right.sum ≥ cross.sum
        return (right.low, right.high, right.sum)
    else return (cross.low, cross.high, cross.sum)
```

3 Running time

Let $T(n)$ be the runtime of the algorithm with an input array of size n. As

$$T(n) = 2T(n/2) + n$$

by the master theorem, we have

$$T(n) = \Theta(n \lg n)$$