Strassen's Algorithm for Matrix Multiplication

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The lecture notes are mostly based on Section 4.2 of Cormen, Leiserson, Rivest, and Stein. Introduction to Algorithms. 3rd Ed. 2009. MIT Press. Cambridge, Massachusetts.

Contents

1 Running time of matrix multiplication

Goal: To multiply two $n \times n$ matrices A and B

$$C = AB$$

using $o(n^3)$ time (not big-O).

$$C = \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = A\,B$$

$$\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}$$

Implementing the above by divide-and-conquer, we have

$$T(n) = 8T(n/2) + \Theta(n^2)$$

which is $\Theta(n^3)$, not $o(n^3)$.

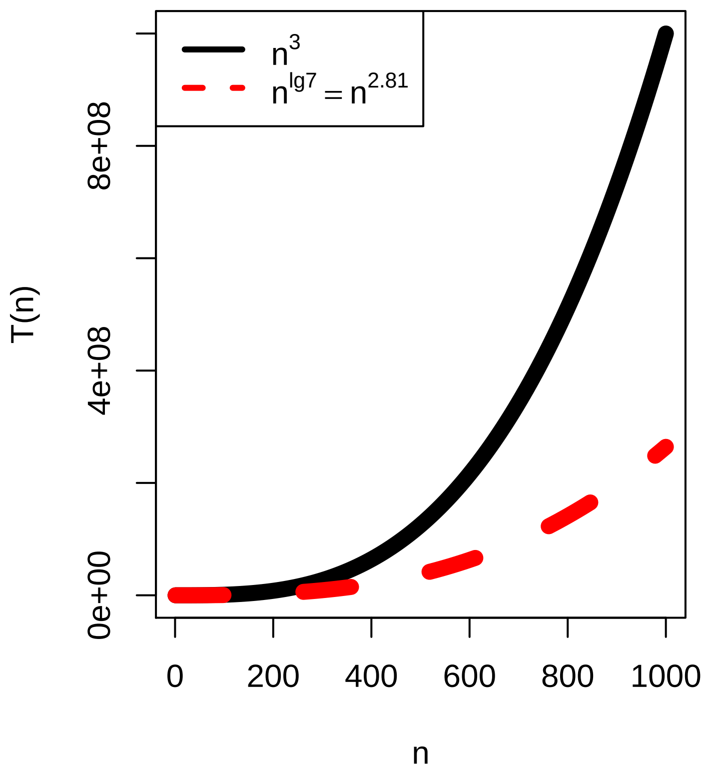
2 Description of Strassen's algorithm

Strassen method uses only

- 7 recursive multiplications of two $n/2 \times n/2$ matrices, and
- $\Theta(n^2)$ scalar additions and subtractions

yielding the recurrence

$$T(n) = 7T(n/2) + \Theta(n^2) = \Theta(n^{\lg 7}) \approx \Theta(n^{2.81})$$



Four steps:

1. Divide the input $n \times n$ matrix to four $n/2 \times n/2$ sub-matrices
2. Compute 14 $n/2 \times n/2$ intermediate sub-matrices $A_1, \ldots, A_7$ and $B_1, \ldots, B_7$ by 10 sub-matrix additions or subtractions.
3. Compute **seven** $n/2 \times n/2$ matrix products $P_1 = A_1 B_1, \ldots, P_7 = A_7 B_7$
4. Compute sub-matrices r, s, t, u in C by 8 sub-matrix additions or subtractions among P_i.

$$P_i = A_i B_i = (\alpha_{i1} a + \alpha_{i2} b + \alpha_{i3} c + \alpha_{i4} d) \cdot (\beta_{i1} e + \beta_{i2} f + \beta_{i3} g + \beta_{i4} h)$$

$$\begin{aligned} r &= ae + bg \\ &= (a\ b\ c\ d) \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \\ &= \begin{array}{ccccc} & e & f & g & h \\ a & + & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & + & \cdot \\ c & \cdot & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot \end{array} \end{aligned}$$

"+": for +1

"−": for −1

"·": for 0

$$s = af + bh = \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & + & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & + \\ c & \cdot & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$t = ce + dg = \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot \\ c & + & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & + & \cdot \end{array}$$

$$u = cf + dh = \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & + & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & + \end{array}$$

$$\begin{aligned} P_1 &= A_1 B_1 \\ &= a \cdot (f - h) \\ &= af - ah \\ &= \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & + & \cdot & - \\ b & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot \end{array} \end{aligned}$$

$$
\begin{aligned}
P_2 &= A_2 B_2 \\
&= (a+b)\cdot h \\
&= ah + bh \\
&= \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & \cdot & \cdot & + \\ b & \cdot & \cdot & \cdot & + \\ c & \cdot & \cdot & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot \end{array}
\end{aligned}
$$

$\implies s = P_1 + P_2$

$$
\begin{aligned}
P_3 &= A_3 B_3 \\
&= (c+d)\cdot e \\
&= ce + de \\
&= \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot \\ c & + & \cdot & \cdot & \cdot \\ d & + & \cdot & \cdot & \cdot \end{array}
\end{aligned}
$$

$$
\begin{aligned}
P_4 &= A_4 B_4 \\
&= d \cdot (g - e) \\
&= dg - de \\
&= \begin{array}{ccccc}
 & e & f & g & h \\
a & \cdot & \cdot & \cdot & \cdot \\
b & \cdot & \cdot & \cdot & \cdot \\
c & \cdot & \cdot & \cdot & \cdot \\
d & - & \cdot & + & \cdot
\end{array}
\end{aligned}
$$

$\Longrightarrow\ t = P_3 + P_4$

$$
\begin{aligned}
P_5 &= A_5 B_5 \\
&= (a + d) \cdot (e + h) \\
&= ae + ah + de + dh \\
&= \begin{array}{ccccc}
 & e & f & g & h \\
a & + & \cdot & \cdot & + \\
b & \cdot & \cdot & \cdot & \cdot \\
c & \cdot & \cdot & \cdot & \cdot \\
d & + & \cdot & \cdot & +
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
P_5 + P_4 - P_2 &= ae + dh + dg - bh \\
&= \begin{array}{ccccc}
 & e & f & g & h \\
a & + & \cdot & \cdot & + \\
b & \cdot & \cdot & \cdot & - \\
c & \cdot & \cdot & \cdot & \cdot \\
d & \cdot & \cdot & + & +
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
P_6 &= A_6 B_6 \\
&= (b-d)\cdot(g+h) \\
&= bg + bh - dg - dh \\
&= \begin{array}{c|cccc}
 & e & f & g & h \\
a & \cdot & \cdot & \cdot & \cdot \\
b & \cdot & \cdot & + & + \\
c & \cdot & \cdot & \cdot & \cdot \\
d & \cdot & \cdot & - & -
\end{array}
\end{aligned}
$$

$$
\implies r = P_5 + P_4 - P_2 + P_6 = ae + bg = \begin{array}{c|cccc}
 & e & f & g & h \\
a & + & \cdot & \cdot & \cdot \\
b & \cdot & \cdot & + & \cdot \\
c & \cdot & \cdot & \cdot & \cdot \\
d & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

$$
\begin{aligned}
P_5 + P_1 - P_3 &= ae + af - ce + dh \\
&= \begin{array}{c|cccc}
 & e & f & g & h \\
a & + & + & \cdot & \cdot \\
b & \cdot & \cdot & \cdot & - \\
c & - & \cdot & \cdot & \cdot \\
d & \cdot & \cdot & \cdot & +
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
P_7 &= A_7 B_7 \\
&= (a-c)\cdot(e+f) \\
&= ae + af - ce - cf \\
&= \begin{array}{ccccc} & e & f & g & h \\ a & + & + & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot \\ c & - & - & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & \cdot \end{array}
\end{aligned}
$$

$$
\Longrightarrow u = P_5 + P_1 - P_3 - P_7 = cf + dh = \begin{array}{ccccc} & e & f & g & h \\ a & \cdot & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & \cdot \\ c & \cdot & + & \cdot & \cdot \\ d & \cdot & \cdot & \cdot & + \end{array}
$$

Thus we use seven matrix products to compute $P_1, P_2, \ldots, P_7$. All remaining operations are additions.

3 Summary

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f) \\
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

There are:

- Seven (7) multiplications of two $n/2 \times n/2$ submatrices
- Eighteen (18) additions or subtractions of two $n/2 \times n/2$ submatrices, leading to a total number of

  $$18 \times (n/2) \times (n/2) = \frac{9n^2}{2}$$

  operations

Thus

$$T(n) = 7T(n/2) + \frac{9n^2}{2}$$

giving rise to

$$T(n) = \Theta(n^{\lg 7})$$