Distributed leader-follower Consensus for Linear Multi-Agent Systems with Directed Graphs

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***Abstract*—This report represents an implementation of fully distributed leader-follower consensus for multi-agent systems with linear dynamics and directed communication graphs. Previous consensus procedures are designed by utilizing the smallest real component of the nonzero eigenvalues of the communication graph's Laplacian matrix. However, this is global information, which means that previous consensus protocols are not fully distributed. The studied and implemented distributed consensus protocol achieves leader-follower consensus for any communication graph containing a directed spanning tree with the leader as the root node with the assumption that the leader does not receive any input and each agent has controllable states. The proposed adaptive protocol is entirely distributed and independent of any global information in the communication graph since each agent's input is based only on agent dynamics and the relative states of its neighboring agents. The consensus protocol is implemented on three examples with different communication graphs and system dynamics to show convergence to the leader.**

***Index Terms*—distributed consensus, linear multi-agent system, leader-follower, directed graph**

I. Introduction

In recent years, multi-agent system consensus has been a hot issue in the systems and control community. Multi-agent system consensus protocol is agents' agreement on a quantity of interest or, synchronization of their state trajectory. The consensus control problem has been investigated by numerous researchers due to its potential applications such as sensor network, spacecraft formation flying, and cooperative surveillance . Existing consensus algorithms can be divided into two categories: consensus without a leader and consensus with a leader (also known as leader-follower consensus or distributed tracking). Generally, in multi-agent system one agent does not have access to the output of all other agents. As a result, the input of that agent should rely only on the output of the agent and its neighbors instead of assuming the knowledge of the whole communication graph. Consensus protocols that are based on the output of the agent's neighbors are called distributed consensus protocols.

This report studies and implements the distributed leader-follower consensus protocol in [1]. As stated in [2], previous leader-follower consensus protocols require knowledge of the communication graph's Laplacian matrix. More specifically, those protocols are based on the minimum real part of the nonzero eigenvalues of the Laplacian matrix. However, knowledge of the Laplacian matrix means that the whole communication graph is known. As a result, the leader-follower consensus procedures described in papers before [1] are not fully distributed since agents need the knowledge of the whole graph. It is important to mention that some distributed leader-follower consensus protocols were proposed before [1], which do not require the Laplacian matrix, but could be applied only to systems with undirected communication graphs.

This report first presents the proposed fully distributed leader-follower consensus protocol in [1]. This protocol is for linear multi-agent systems with the assumption that the communication graph contains a directed spanning tree with the leader as the root node, the leader does not receive any input, and each agent has controllable states. Then, the protocol is implemented for three different linear multi-agent systems with directed communication graphs to show that states of each agent do converge to the states of the leader.

II. Problem Formulation

Consider a linear multi-agent system with $N+1$ agents (N followers and a leader). Assume the dynamics of the i-th agent is as follows

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 0, ..., N \tag{1}$$

where $x_i \in R^n$ is the state, $u_i \in R^m$ is the control input, A and B are constant matrices. Assume

- agent indexed by 0 is the leader and its control input (u_0) is zero.
- (A,B) is controllable.

The communication graph also must satisfy the following assumptions

- The graph contains a directed spanning tree.
- The leader is the root node and does not get any information from other agents.

The goal of this report is to implement a distributed consensus protocol that controls the states of the N followers to converge to the states of the leader. Mathematically,

$$\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0, \quad \forall i = 1, ..., N \tag{2}$$

Many studies have proposed controllers to achieve (2). For example, the input control in (3) is proposed in [3] and [4].

$$u_i = c\tilde{K} \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad \forall i = 1, ..., N \tag{3}$$

where $c > 0$ is the common coupling weight among neighboring agents, $\tilde{K} \in R^{m*n}$ is the feedback gain matrix, and a_{ij} is the weight of edge (v_j, v_i). From article [3] and [4], one can obtain $\tilde{K}$ and c as

$$\tilde{K} = -B^T P^{-1} \tag{4}$$

$$c \geq \frac{1}{\min_{i=1,\dots,N}\left(Re(\lambda_i)\right)} \tag{5}$$

where λ_i is the i-th nonzero eigenvalue of the Laplacian matrix of the communication graph, and $P > 0$ can be obtained from the following linear matrix inequality (LMI):

$$AP + PA^T - 2BB^T < 0 \tag{6}$$

One can obtain one of the solutions of the LMI in (6) by solving the following Ricatti equation:

$$A^T Q + QA + I - QBB^T Q = 0 \tag{7}$$

where $Q = P^{-1}$.

The limitation of protocol (3) is that one must obtain eigenvalues of the Laplacian matrix to calculate c. This is not a fully distributed protocol since the whole communication graph should be known to obtain eigenvalues of the Laplacian matrix. To overcome this limitation, article [1] proposes the following completely distributed consensus protocol.

$$\begin{aligned} u_i &= c_i(1 + \xi_i^T P^{-1} \xi_i)^3 K \xi_i, \\ \dot{c}_i &= \xi_i^T \Gamma \xi_i, \quad i = 1, \dots, N \end{aligned} \tag{8}$$

where $\xi_i \triangleq \sum_{j=0}^{N} a_{ij}(x_i - x_j)$, $P > 0$ can be obtained from solving the LMI in (6), c_i is the coupling weight of the i-th follower with the initial condition $c_i(0) \geq 1$, $K = -B^T P^{-1}$, and $\Gamma = P^{-1} BB^T P^{-1}$. It is proved in [1] that the distributed protocol in (8) achieves $\lim_{t \to \infty} \|x_i(t) - x_0(t)\|$.

III. Simulation Examples

In this report, the proposed distributed leader-follower consensus protocol is implemented for three examples.

- The first example is chosen from [1] to validate if our implementation gives the same result.
- In the second example, the dynamics of each agent is modeled as a double integrator and a different communication graph is considered.
- In the third example, the A and B matrices are chosen randomly to validate if the proposed protocol works for all linear multi-agents.

A. *Example from Paper [1]*

Consider a multi-agent system, which the i-th agent has the following dynamics

$$\dot{x}_i = Ax_i + Bu_i \ , \ i = 0, \dots, 6 \tag{9}$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $u_0 = 0$

and the associated communication graph is given in Fig. 1.

One creates matrix $\mathcal{A} = [a_{ij}]$ in order to obtain $\xi_i = \sum_{j=0}^{N} a_{ij}(x_i - x_j)$. Matrix $\mathcal{A}$ is given in (10).

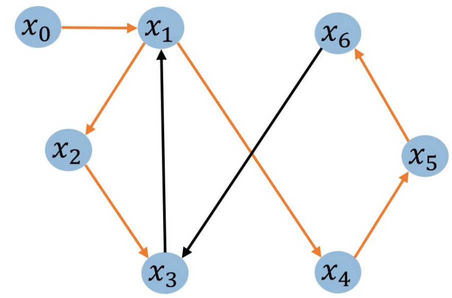

Fig. 1: Communication graph for the first example

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{10}$$

Solving the Ricatti equation (7) gives a solution

$$P = \begin{pmatrix} 0.8536 & -0.5 & 0.1464 \\ -0.5 & 0.7071 & -0.5 \\ 0.1464 & -0.5 & 0.8536 \end{pmatrix} \tag{11}$$

As a result, one can obtain feedback gains Γ and K as

$$K = \begin{pmatrix} -1 & -2.41 & -2.41 \end{pmatrix} \tag{12}$$

$$\Gamma = \begin{pmatrix} 1 & 2.41 & 2.41 \\ 2.41 & 5.82 & 5.82 \\ 2.41 & 5.82 & 5.82 \end{pmatrix} \tag{13}$$

With the given K, Γ, P one can implement the protocol in (8). The Simulink block diagram of the consensus control system is given in Fig 2.

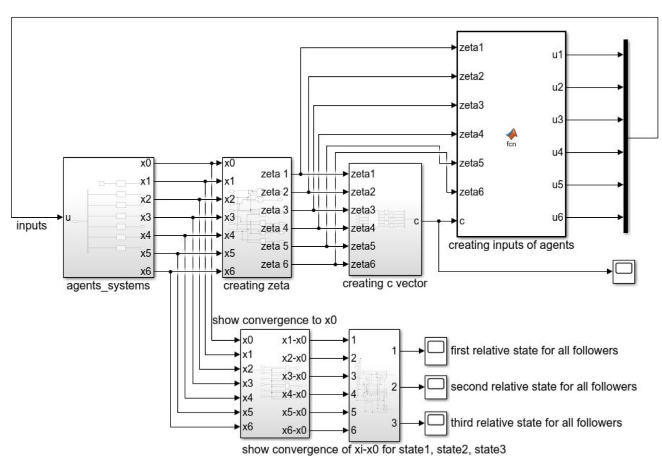

Fig. 2: Simulink block diagram of the first example's control system

The relative state $(x_i - x_0)$ over time for the first states, the second states, and the third states of agents are given in Fig 3, Fig 4, Fig 5 respectively.

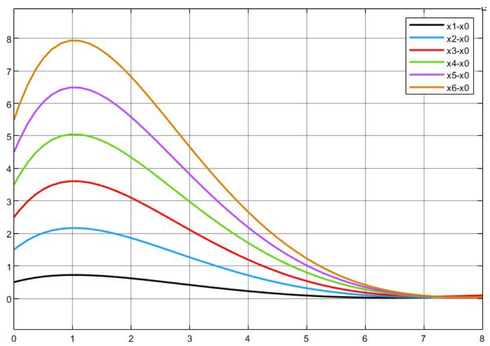

Fig. 3: The relative state $(x_i - x_0)$ over time for the first states in example-1

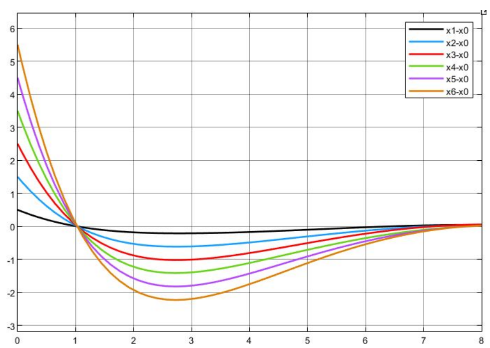

Fig. 4: The relative state $(x_i - x_0)$ over time for the second states in example-1

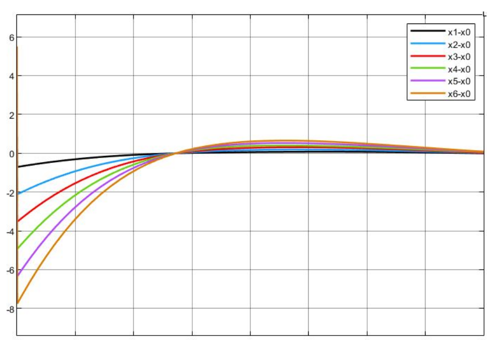

Fig. 5: The relative state $(x_i - x_0)$ over time for the third states in example-1

B. Double Integrator Example

Consider a multi-agent system, which the i-th agent is modeled as a double integrator. As a result, the i-th agent dynamcis is

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 0, ..., 7$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u_0 = 0$

The associated communication graph is given in Fig. 6 and matrix $\mathcal{A}$ is given in (14).

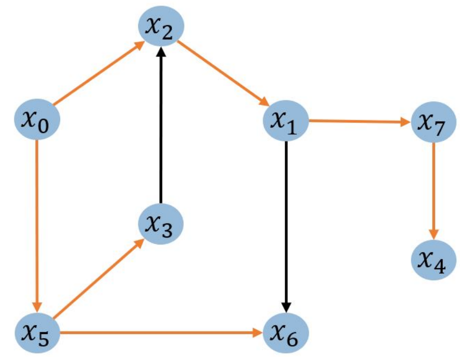

Fig. 6: Communication graph for the second example

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Solving the Ricatti equation (7) gives a solution

$$P = \begin{pmatrix} 0.86 & -0.5 \\ -0.5 & 0.86 \end{pmatrix} \tag{15}$$

As a result, one can obtain feedback gains Γ and K as

$$K = \begin{pmatrix} -1 & -1.73 \end{pmatrix} \tag{16}$$

$$\Gamma = \begin{pmatrix} 1 & 1.7 \\ 1.7 & 3 \end{pmatrix} \tag{17}$$

With the given K, Γ, P one can implement the protocol in (8). The relative state $(x_i - x_0)$ over time for the first states, and the second states of agents are given in Fig 7, Fig 8 respectively.

C. Random Linear Multi-agent System and Weighted Communication Graph

Consider a multi-agent system, which the i-th agent is modeled as

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 0, ..., 7$$

where $A = \begin{pmatrix} 0.04 & 0.69 & 0.03 \\ 0.09 & 0.31 & 0.43 \\ 0.82 & 0.95 & 0.38 \end{pmatrix}$, $B = \begin{pmatrix} 0.76 \\ 0.79 \\ 0.18 \end{pmatrix}$ are random matrices. The associated weighted communication graph is given in Fig. 9, and matrix $\mathcal{A}$ is given in (18).

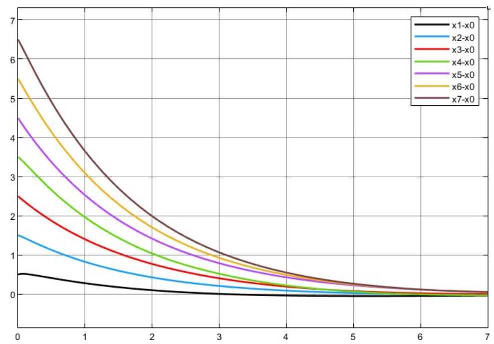

Fig. 7: The relative state $(x_i - x_0)$ over time for the first states in example-2

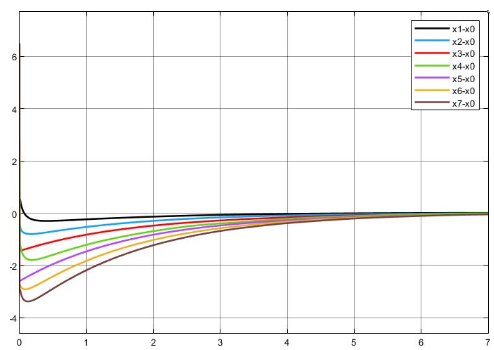

Fig. 8: The relative state $(x_i - x_0)$ over time for the second states in example-2

$$
\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.9 \\ 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 2.9 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}
$$

Solving the Ricatti equation (7) gives a solution

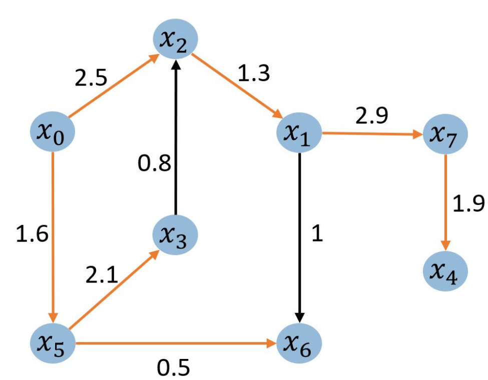

Fig. 9: Weighted communication graph for the third example

$$
P = \begin{pmatrix} 0.6411 & 0.0763 & -0.1053 \\ 0.0763 & 0.6330 & -0.3934 \\ -0.1053 & -0.3934 & 0.5857 \end{pmatrix} \tag{19}
$$

As a result, one can obtain feedback gains Γ and K as

$$
K = \begin{pmatrix} -1.2619 & -2.4780 & -2.2104 \end{pmatrix} \tag{20}
$$

$$
\Gamma = \begin{pmatrix} 1.5924 & 3.1270 & 2.7893 \\ 3.1270 & 6.1406 & 5.4774 \\ 2.7893 & 5.4774 & 4.8858 \end{pmatrix} \tag{21}
$$

With the given K, Γ, P one can implement the protocol in (8). The relative state $(x_i - x_0)$ over time for the first states, the second states, and the third states of agents are given in Fig 10, Fig 11, Fig 12 respectively.

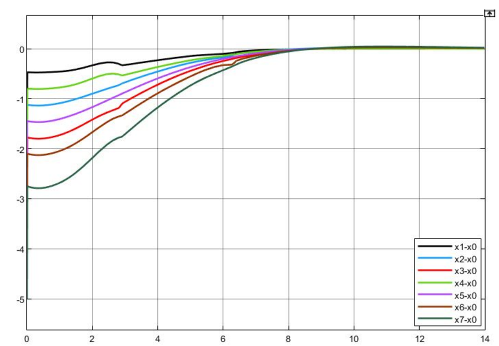

Fig. 10: The relative state $(x_i - x_0)$ over time for the first states in example-3

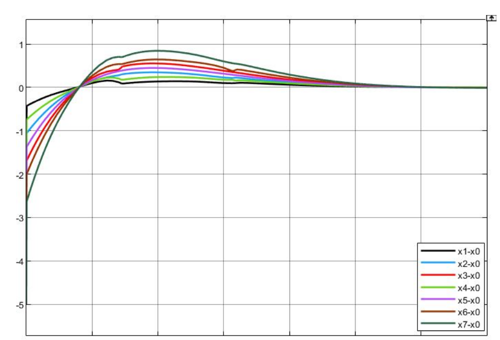

Fig. 11: The relative state $(x_i - x_0)$ over time for the second states in example-3

IV. Conclusion

In this report, the proposed method by paper [1] to address the distributed leader-follower consensus problem for a multi-agent system with linear dynamics and a directed communication graph is studied. Then, the consensus protocol is implemented for the example in [1] and two other examples, double integrator agents and random dynamics for agents with arbitrary weighted graphs. The convergence of the followers' states to the leader's states is achieved only with knowledge from neighbors of each agent.

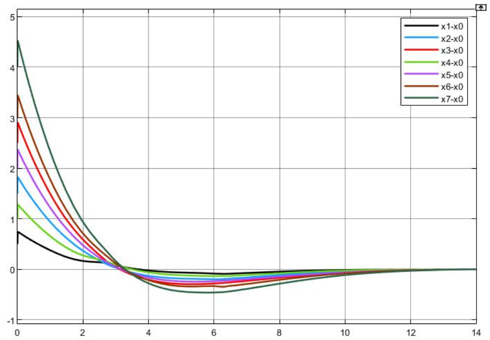

Fig. 12: The relative state $(x_i - x_0)$ over time for the third states in example-3

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