Lecture 4

Graph Theory

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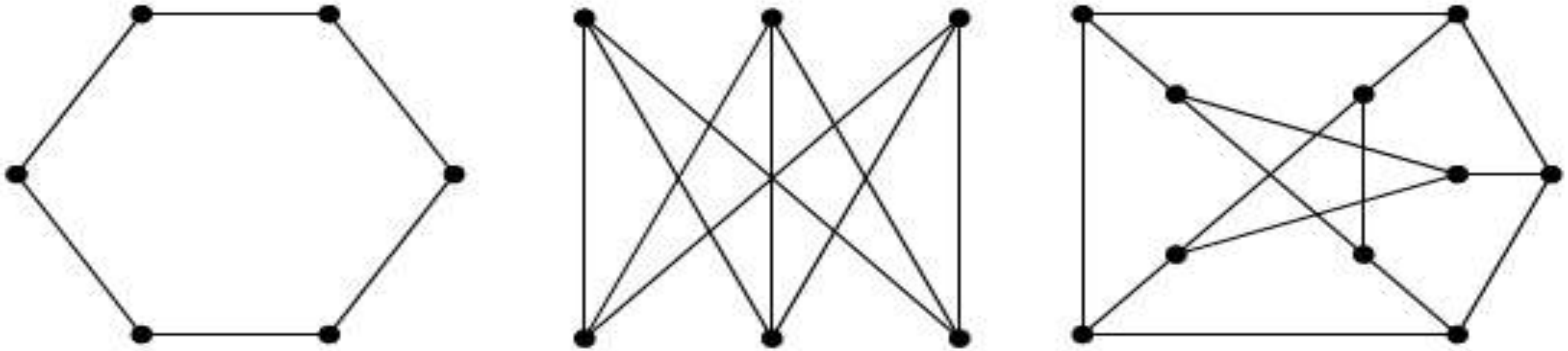
Regular Graphs

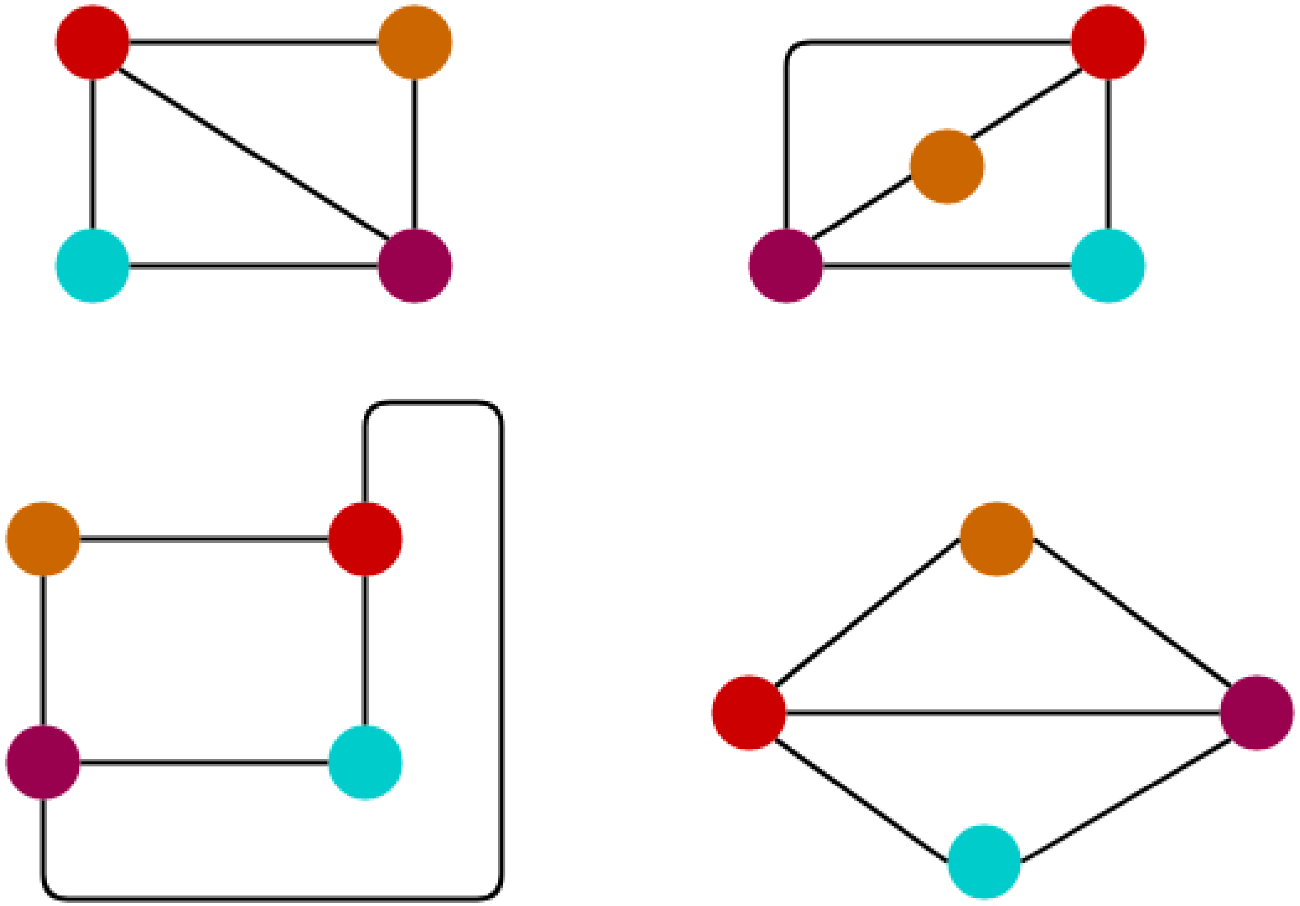
Definition 1.4.4 — Regular Graphs. A graph G is said to be a *regular graph* if all its vertices have the same degree. A graph G is said to be a *k-regular graph* if $d(v) = k \ \forall \ v \in V(G)$. Every complete graph is an $(n-1)$-regular graph.

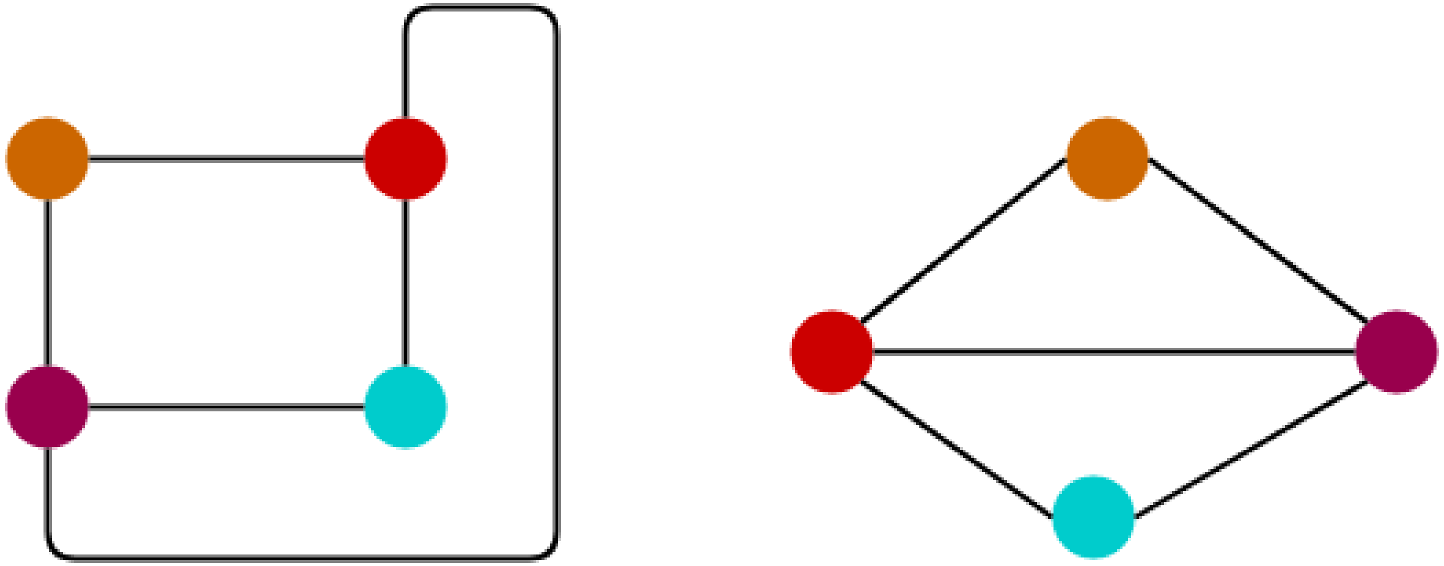
The degree of all vertices in each partition of a complete bipartite graph is the same. Hence, the complete bipartite graphs are also called *biregular graphs*. Note that, for the complete bipartite graph $K_{|X|,|Y|}$, we have $d_X(v) = |Y|$ and $d_Y(v) = |X|$.



(a) A 2-regular graph (b) A 3-regular graph (c) Petersen Graph

Graph Isomorphism Example-





Graph Isomorphism Example

Here,

- The same graph exists in multiple forms.
- Therefore, they are **Isomorphic graphs**.

Graph Isomorphism Conditions-

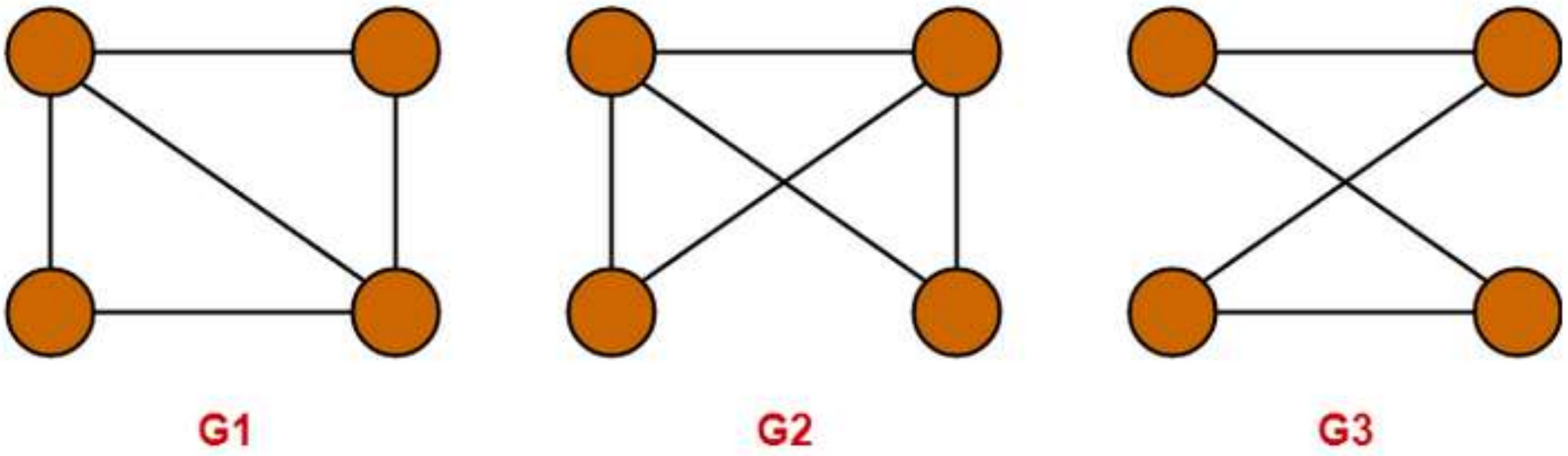
For any two graphs to be isomorphic, following 4 conditions must be satisfied-

- Number of vertices in both the graphs must be same.
- Number of edges in both the graphs must be same.
- Degree sequence of both the graphs must be same.
- If a cycle of length k is formed by the vertices { v_1 , v_2 , ….. , v_k } in one graph, then a cycle of same length k must be formed by the vertices { $f(v_1)$, $f(v_2)$, ….. , $f(v_k)$ } in the other graph as well.

Important Points-

- The above 4 conditions are just the necessary conditions for any two graphs to be isomorphic.
- They are not at all sufficient to prove that the two graphs are isomorphic.
- If all the 4 conditions satisfy, even then it can't be said that the graphs are surely isomorphic.
- However, if any condition violates, then it can be said that the graphs are surely not isomorphic.

Which of the following graphs are isomorphic?

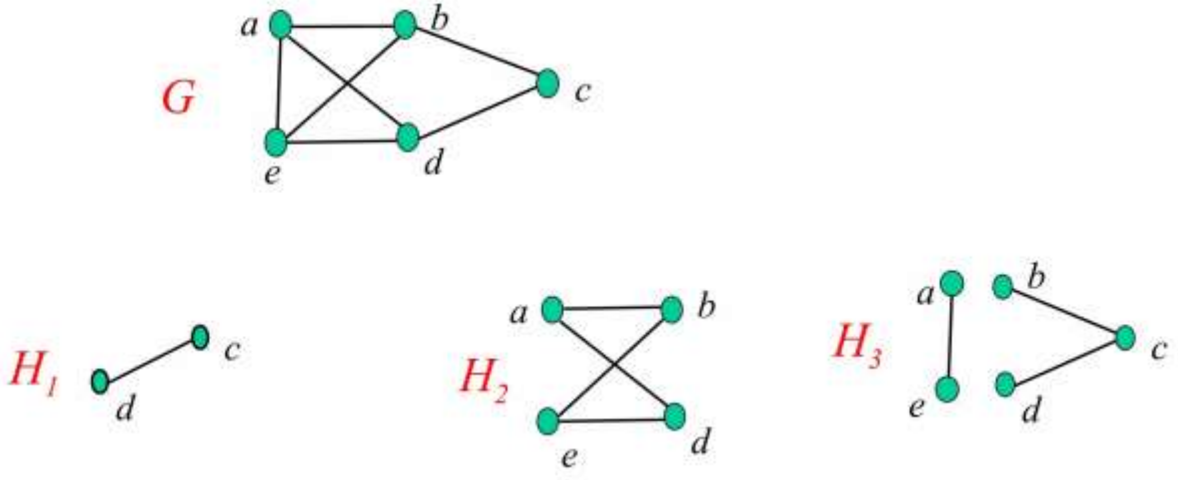


Subgraphs

- A ***subgraph*** of a graph G is a graph H such that:
  - $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and
  - The assignment of endpoints to edges in H is the same as in G.

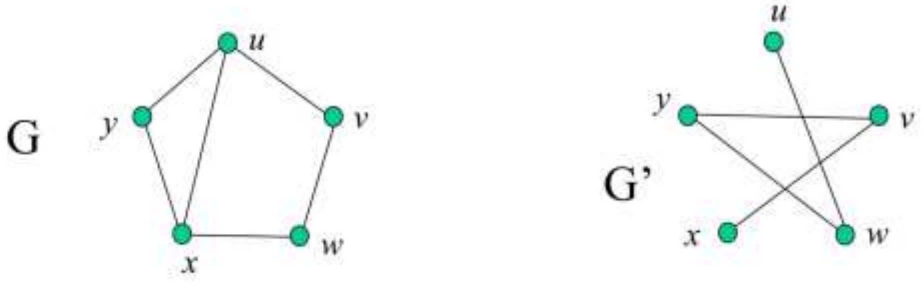
Subgraphs

- Example: H_1, H_2, and H_3 are subgraphs of G



Complement

- ***Complement of G***: The complement G' of a simple graph G :
  - A simple graph
  - $V(G') = V(G)$
  - $E(G') = \{ uv \mid uv \notin E(G) \}$



Self-Complementary Graphs

Definition 2.2.2 — Self-Complementary Graphs. A graph G is said to be *self-complementary* if G is isomorphic to its complement. If G is self complementary, then $|E(G)| = |E(\bar{G})| = \frac{1}{2}|E(K_n)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$.

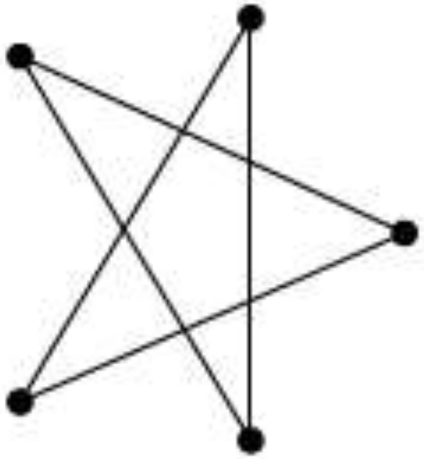
The following are two examples of self complementary graphs.

(a) Bull graph G.

(b) $\bar{G}$.

Figure 2.3: Example of self-complementary graphs

(a) G_2



(b) G_2.

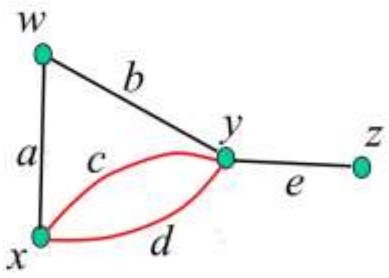
Figure 2.4: Example of self-complementary graphs

Adjacency Matrices for Graphs

- Adjacency matrix: an *n* x *n* matrix for a graph with *n* vertices where each entry is the number of edges from each vertex to all the others

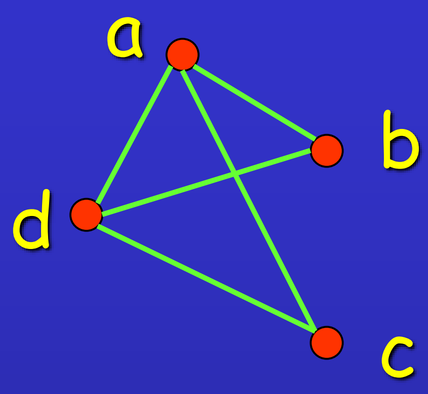
Adjacency matrix

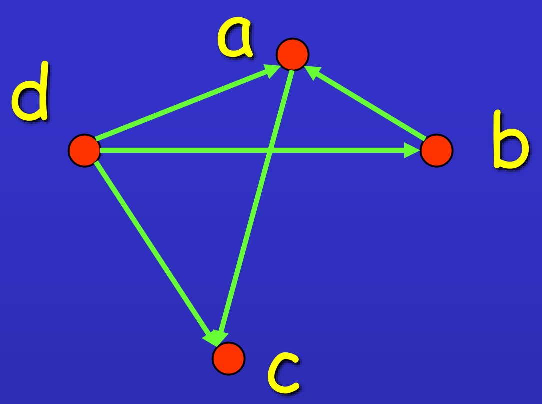
- Let $G = (V, E)$, $|V| = n$ and $|E|=m$
- The ***adjacency matrix*** of G written $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$.



$$\begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Representing Graphs





Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Initial Vertex	Terminal Vertices
a	c
b	a
c	
d	a, b, c

Representing Graphs

Definition: Let $G = (V, E)$ be a simple graph with $|V| = n$. Suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.

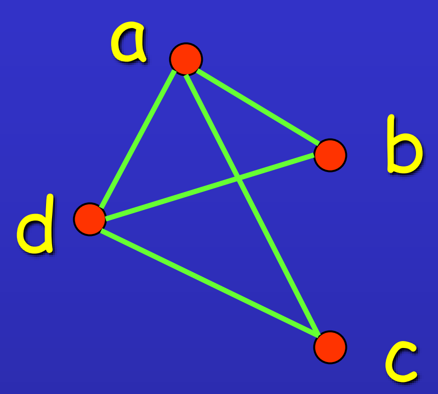
The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j) entry when v_i and v_j are adjacent, and 0 otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$,

$a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,
$a_{ij} = 0$ otherwise.

Representing Graphs

Example: What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Solution:

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note: Adjacency matrices of undirected graphs are always symmetric.

Example 1

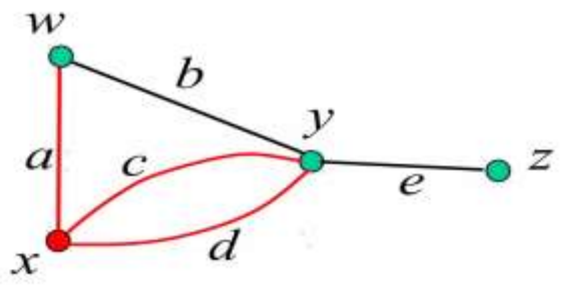
- Suppose A, B and C are three cities. Every day there are 2 nonstop flights from A to B, 3 from B to C and 1 from A to C. There are 2 nonstop flights from B to A, 2 from C to B, and 1 from C to A. Write the adjacency matrix for this information.

Example 2

- Draw a picture of a directed graph that has the following adjacency matrix: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

Incidence Matrix

- Let $G = (V, E)$, $|V| = n$ and $|E|=m$
- The ***incidence matrix*** $M(G)$ is the n-by-m matrix in which entry $m_{i,j}$ is 1 if v_i is an endpoint of e_i and otherwise is 0.



$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} w \\ x \\ y \\ z \end{array} &
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Representing Graphs

Definition: Let $G = (V, E)$ be an undirected graph with $|V| = n$. Suppose that the vertices and edges of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$, respectively.

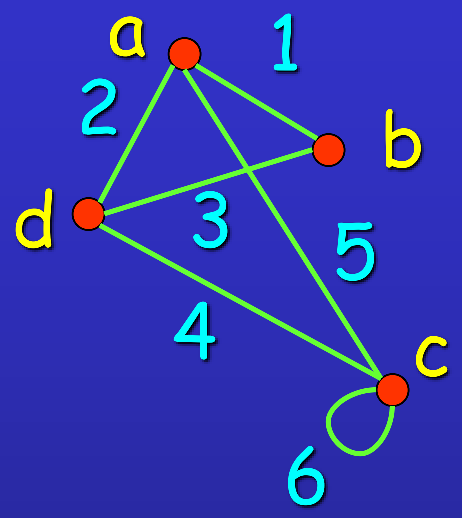
The **incidence matrix** of G with respect to this listing of the vertices and edges is the $n \times m$ zero-one matrix with 1 as its (i, j) entry when edge e_j is incident with v_i, and 0 otherwise.

In other words, for an incidence matrix $M = [m_{ij}]$,

$m_{ij} = 1$ if edge e_j is incident with v_i
$m_{ij} = 0$ otherwise.

Representing Graphs

Example: What is the incidence matrix M for the following graph G based on the order of vertices a, b, c, d and edges 1, 2, 3, 4, 5, 6?



Solution:

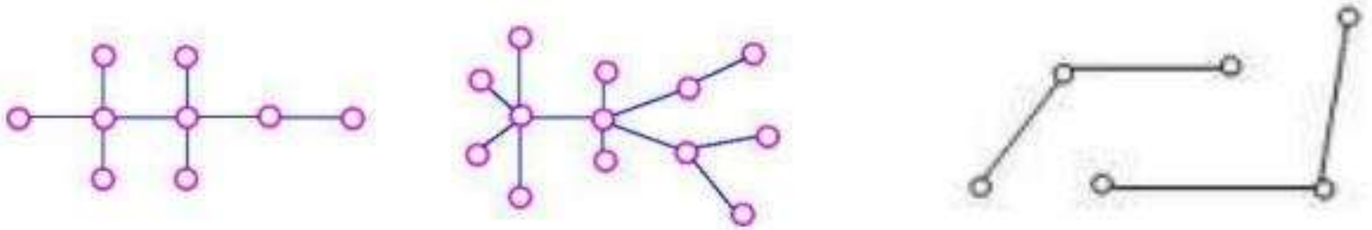
$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Note: Incidence matrices of directed graphs contain two 1s per column for edges connecting two vertices and one 1 per column for loops.

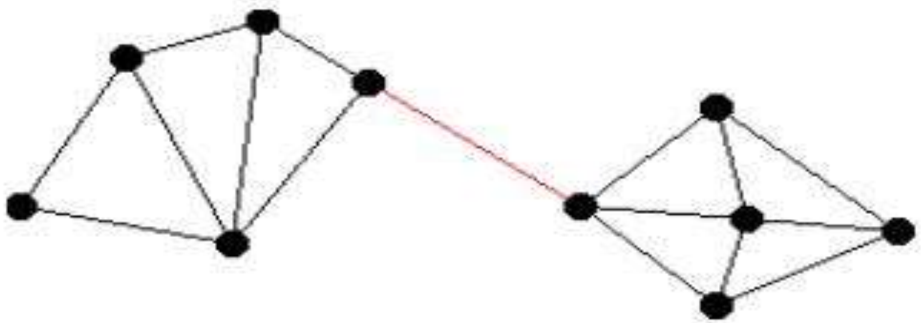
A graph is a connected graph if it is possible to travel from one vertex to any other vertex by moving along successive edges.

A graph is a disconnected graph if it is not possible to travel from one vertex to any other vertex by moving along successive edges.

Examples:



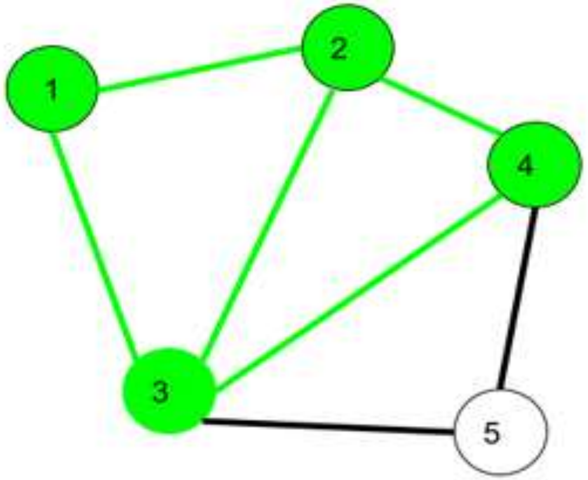
A Bridge in a connected graph is an edge such that if it were removed the graph is no longer connected.



1. Walk –

A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk.

Edge and Vertices both can be repeated.



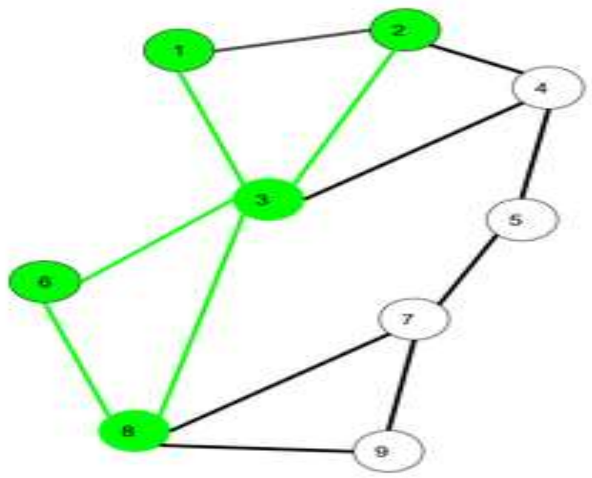
- Here, 1->2->3->4->2->1->3 is a walk.

- **Open walk-** A walk is said to be an open walk if the starting and ending vertices are different i.e. the origin vertex and terminal vertex are different.
  Closed walk- A walk is said to be a closed walk if the starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.
- In the above diagram:
  1->2->3->4->5->3 is an open walk.
  1->2->3->4->5->3->1 is a closed walk.

2. Trail –

Trail is an open walk in which no edge is repeated.

Vertex can be repeated.

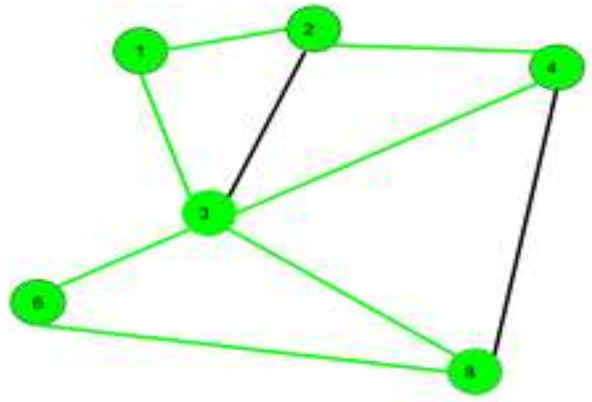


- Here 1->3->8->6->3->2 is trail
  Also 1->3->8->6->3->2->1 will be a closed trail

3. Circuit –

Traversing a graph such that not an edge is repeated but vertex can be repeated and it is closed also i.e. it is a closed trail.

Vertex can be repeated.
Edge can not be repeated.



Here 1->2->4->3->6->8->3->1 is a circuit.

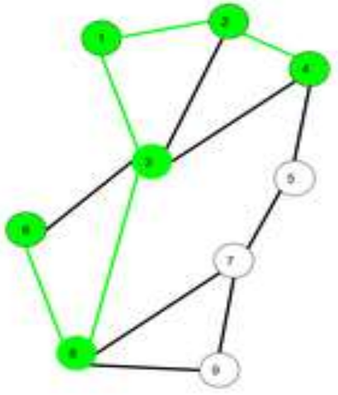
```
Circuit is a closed trail.
These can have repeated vertices only.
```

4. Path –

It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge. As path is also a trail, thus it is also an open walk.

Another definition for path is a walk with no repeated vertex. This directly implies that no edges will ever be repeated and hence is redundant to write in the definition of path.

Vertex not repeated
Edge not repeated



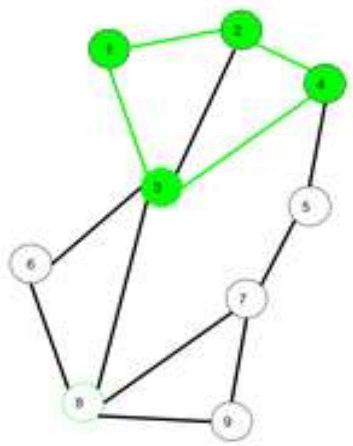
Here 6->8->3->1->2->4 is a Path

5. Cycle –

Traversing a graph such that we do not repeat a vertex nor we repeat a edge but the starting and ending vertex must be same i.e. we can repeat starting and ending vertex only then we get a cycle.

Vertex not repeated

Edge not repeated



Here 1->2->4->3->1 is a cycle.

```
Cycle is a closed path.
These can not have repeat anything (neither edges nor vertices).
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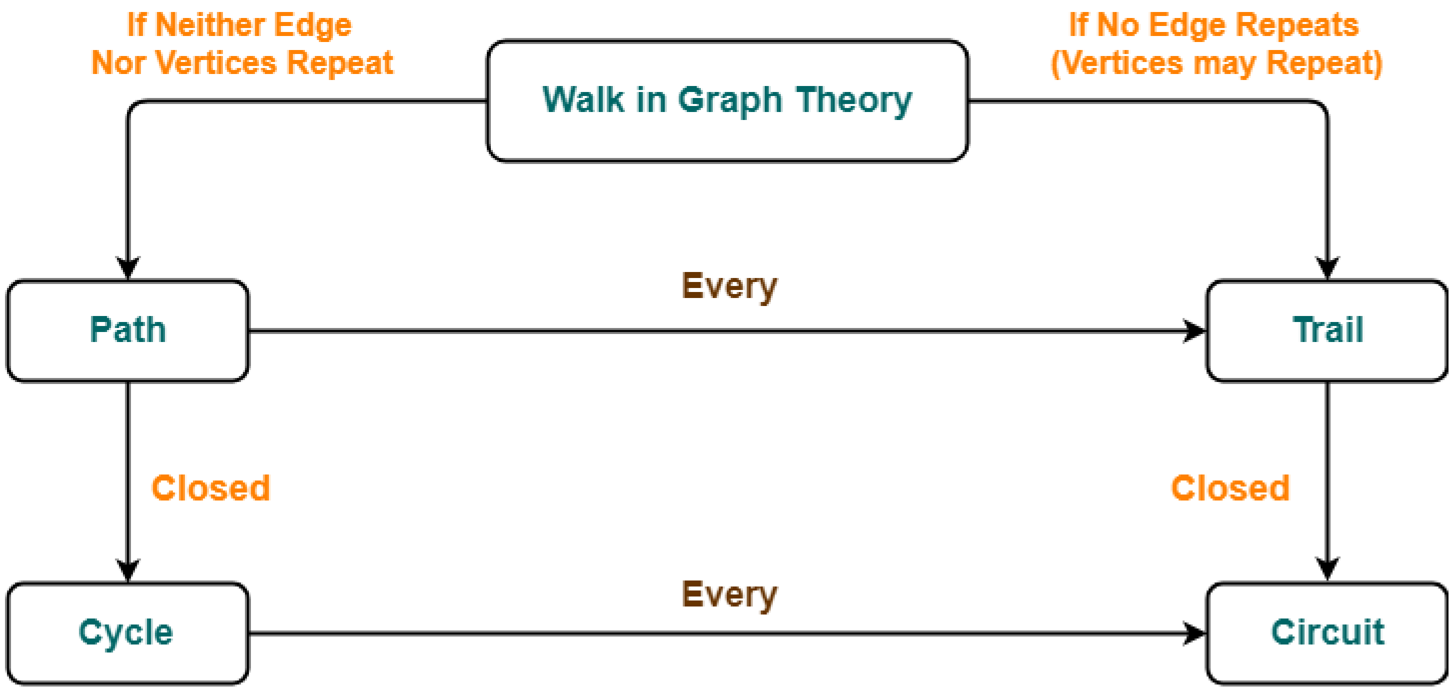
Vocabulary

A walk is a series of adjacent vertices that can go backwards as long as there is a line segment.

A path is a walk which does not repeat vertices.

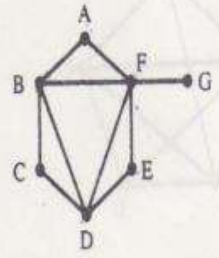
A Circuit is a path whose endpoints are the same vertex. Note: you start and finish with the same vertex without repeating any edges.



Important Chart to Remember

	Repeat Vertex	Repeat Edge	Closed	Open
Walk	✓	✓	✓	✓
Trail	✓	✗	✗	✓
Path	✗	✗	✗	✓
Cycle	✗	✗	✓	✗
Circuit	✓	✗	✓	✗

Exercises 27–29 refer to the following graph.



27. Which of the following are walks in the graph? If not, why not?

a) A → B → C
b) B → A → D
c) E → F → A → E
d) B → D → F → B → D
e) D → E
f) C → B → C → B

28. Which of the following are paths in the graph? If not, why not?

a) B → D → E → F
b) D → F → B → D
c) B → D → F → B → D
d) D → E → F → G → F → D
e) B → C → D → B → A
f) A → B → E → F → A

29. Which of the following are circuits in the graph? If not, why not?

a) A → B → C → D → E → F
b) A → B → D → E → F → A
c) C → F → E → D → C
d) G → F → D → E → F
e) F → D → F → E → D → F