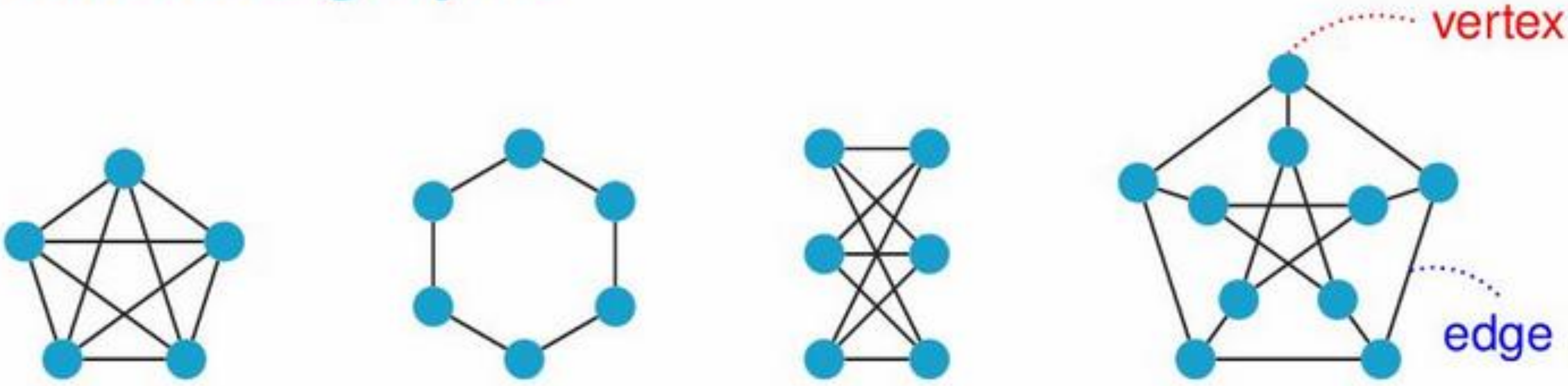
Graph Theory

Lecture 2

What is a graph?



A graph is a pair of **vertices** and **edges**, usually written as

$$G = (\underbrace{\boxed{V}}_{\text{vertex set}}, \underbrace{\boxed{E}}_{\text{edge set}})$$

In this talk, we only consider **finite** graphs, i.e., $|V|, |E| < +\infty$.

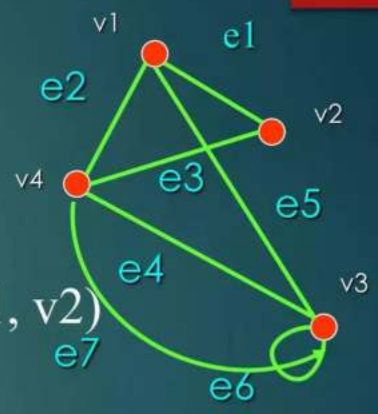
Introduction to Graphs

In Fig. G has graph 4 vertices namely

v1, v2, v3, v4& 7 edges

Namely e1, e2, e3, e4, e5, e6, e7 Then e1=(v1, v2)

Similarly for other edges.



Graph G

In short, we can represent G=(V,E) where V=(v1, v2, v3, v4) &

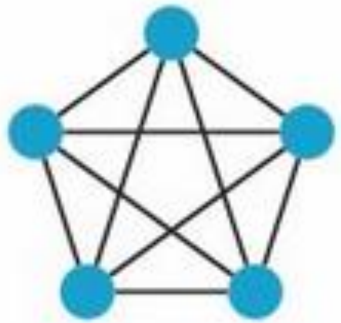
E=(e1, e2, e3, e4, e5, e6,e7)

Loops, parellel edges, simple graphs

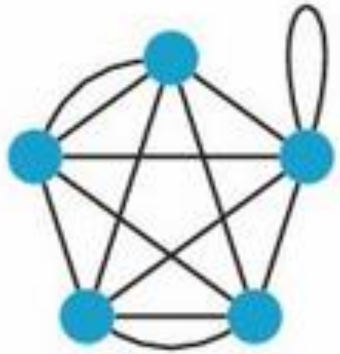
Loop

Parallel edges

Graphs without loops and parallel edges are called **simple** graphs.



Simple



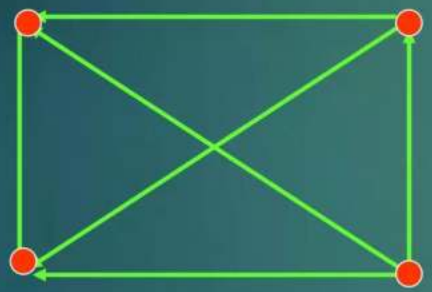
Not simple

In this talk, we only consider simple graphs.

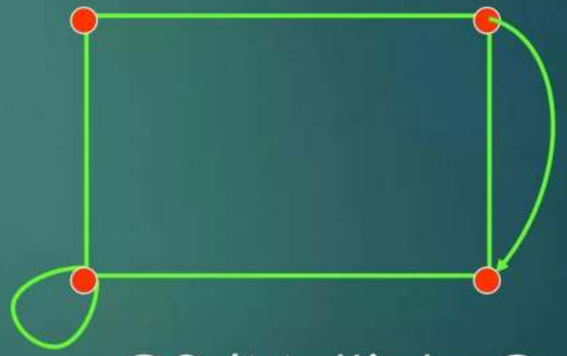
Simple & Multiple Graphs

Definition: A graph that has neither self loops or parallel edge is called as **Simple Graph** otherwise it is called as **Multiple Graph.**

For Example,



G1 (Simple Graph)



G2 (Multiple Graph)

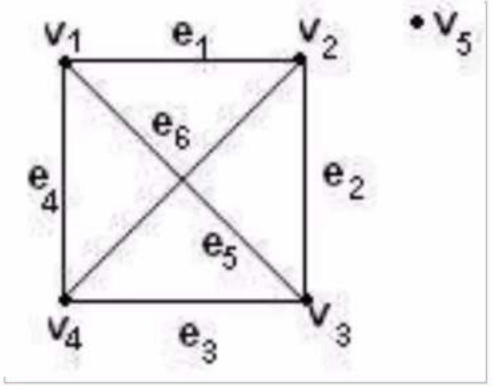
Order and Size of a Graph

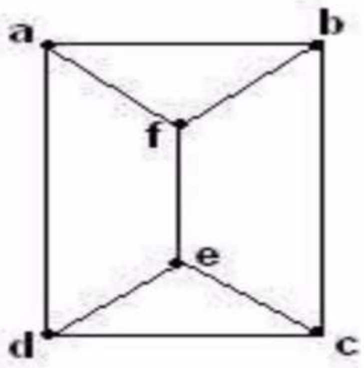
Order of a graph is the number of vertices in the graph.

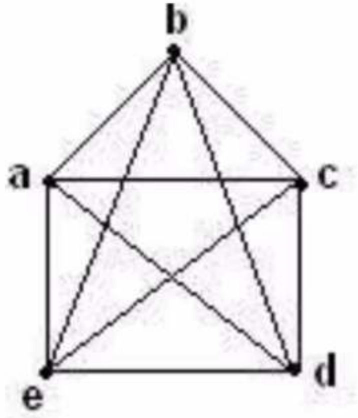
Size of a graph is the number of edges in the graph.

Order and Size of a Graph

- The magnitude of graph G is characterized by **number of vertices** $|V|$ (called the **order of G**) and **number of edges** $|E|$ (**size** of G)
- The running time of algorithms are measured in terms of the order and size

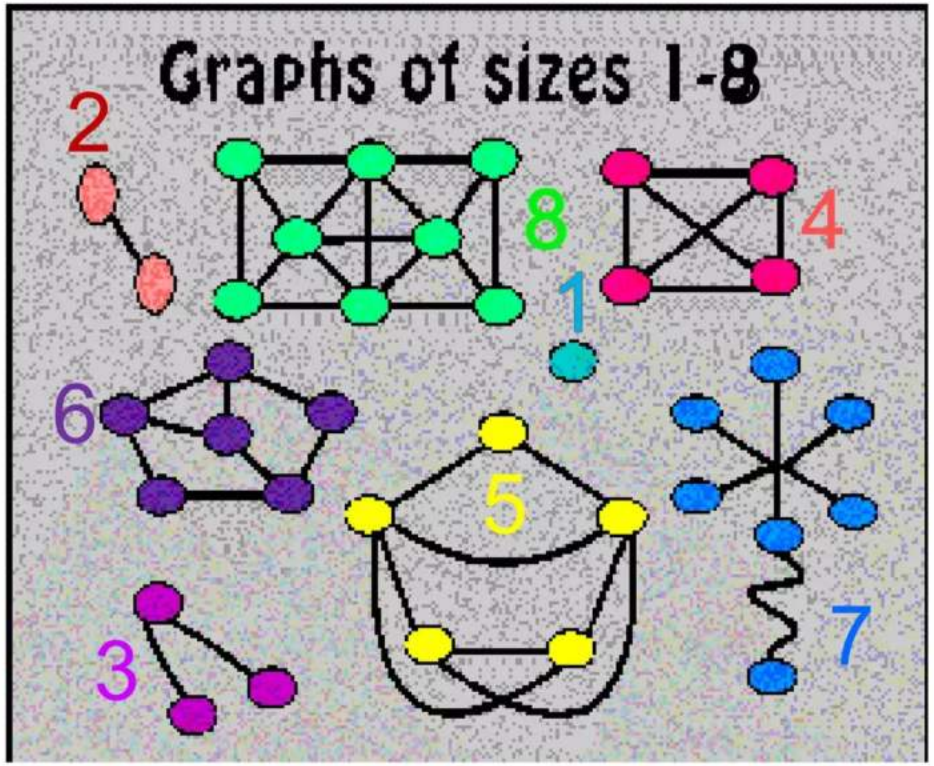






$|V|=5$ and $|E|=6$ *$|V|=6$ and $|E|=9$* *$|V|=5$ and $|E|=10$*

Size – of a graph is the number of vertices that the graph has



• **Degree of a Node:** No. of edges connected to a node.

(a) Indegree: No. of edges ending at a node.

(b) Outdegree: No. of edges beginning at a node.

Example:

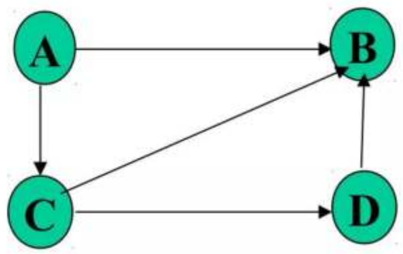


Figure 3: Directed Graph G

In graph G	
Indeg(A)= 0	Outdeg(A)=2
Indeg(B)= 3	Outdeg(B)= 0
Indeg(C)= 1	Outdeg(C)= 2
Indeg(D)= 1	Outdeg(D)= 1

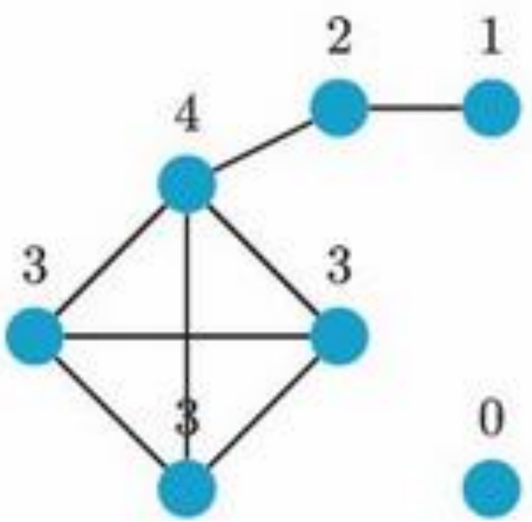
Note:

- A node u is called **source** if it has a positive outdegree and 0 indegree (A).
- A node u is called **sink** if it has a positive indegree and 0 outdegree (B).
- For a directed graph, a loop adds one to the indegree and one to the outdegree.
- For undirected graph, a loop adds two to the degree.

Degree

$G = (V, E)$: undirected graph

The **degree** of a vertex v is the number of edges incident to v. Denoted by $d(v)$.



Theorem ("First theorem of graph theory")

$$\sum_{v \in V} d(v) = 2|E|$$

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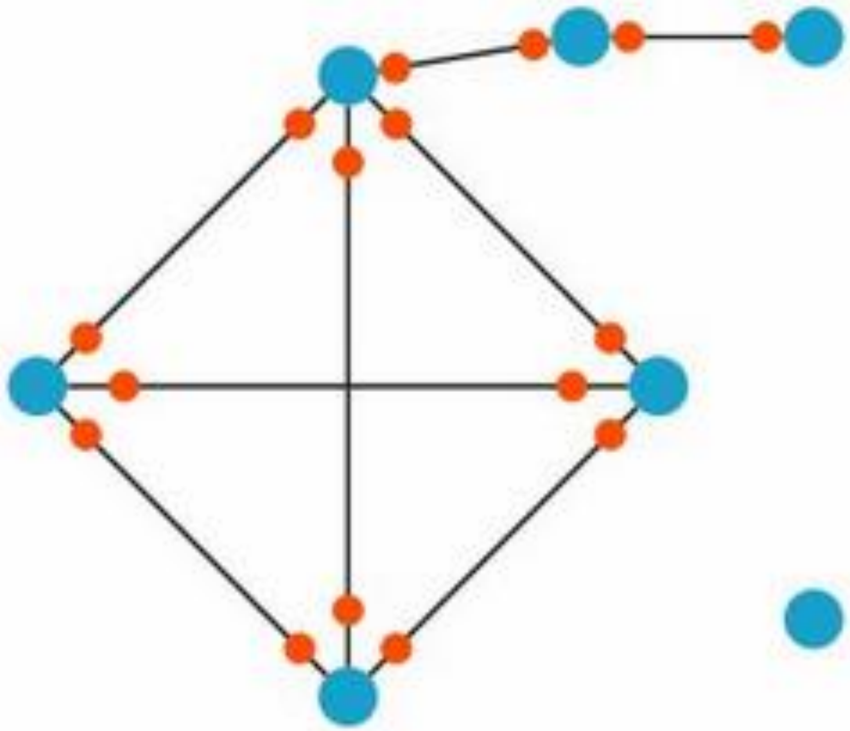
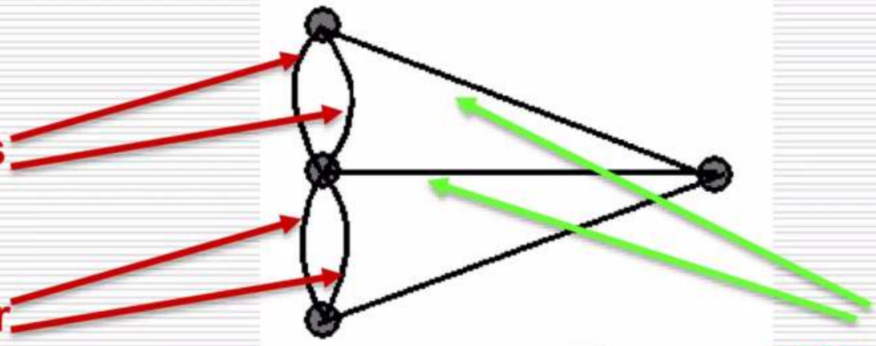


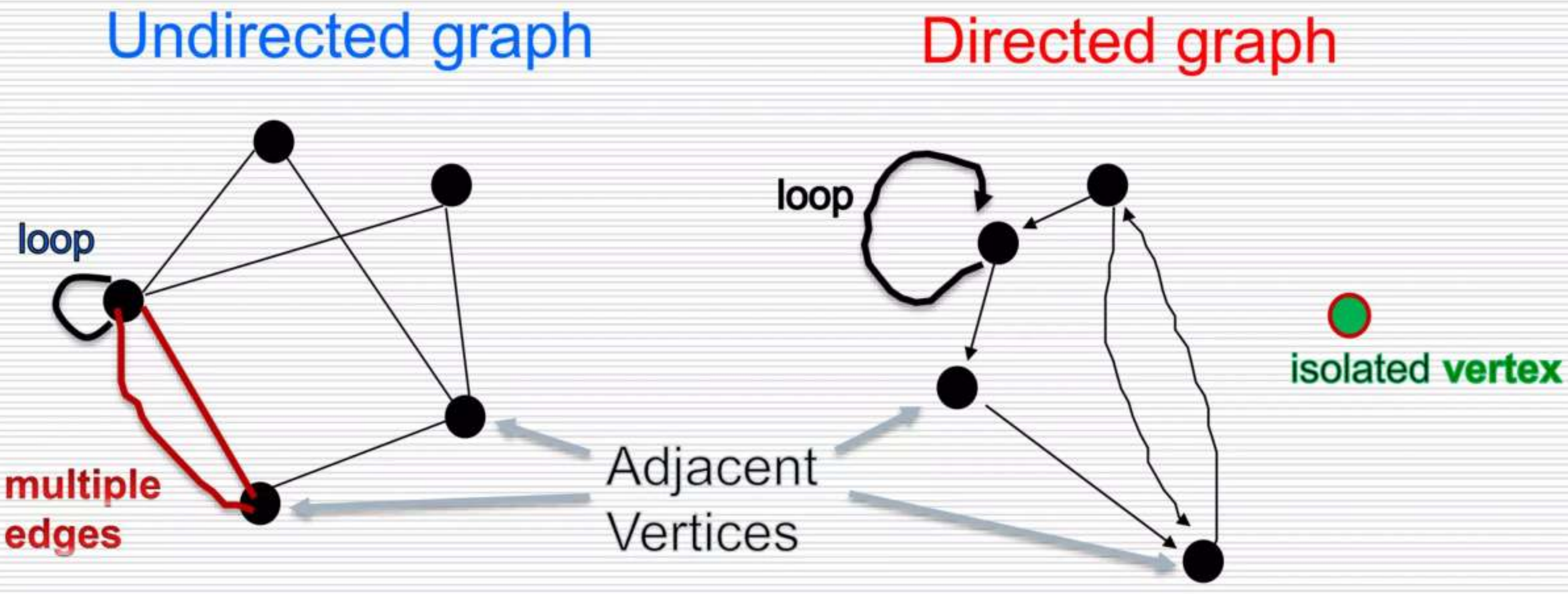
Loops and Multiple Edges

An edge where the two end vertices are the same is called a **loop**, or a **self-loop**

Multiple Edges (or) Parallel Edges: Two or more edges joining the same pair of vertices.



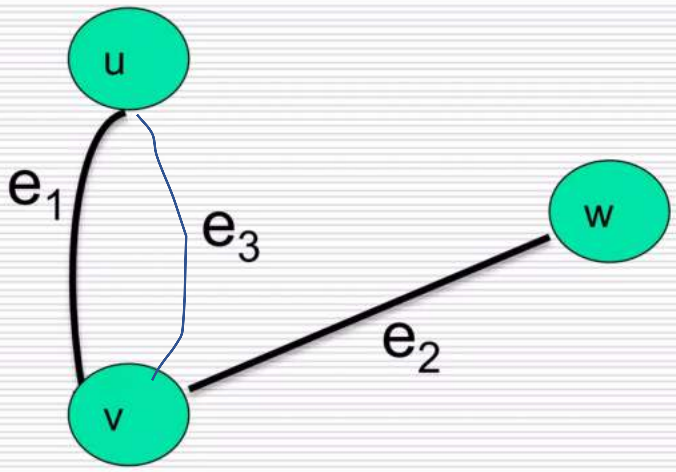
$G=(V,E)$



isolated **vertex** ?

Multigraph: A graph that contains multiple edges but no loops is called a ***multigraph***

The edges $\mathbf{e_1}$ and $\mathbf{e_3}$ are called multiple or parallel edges.

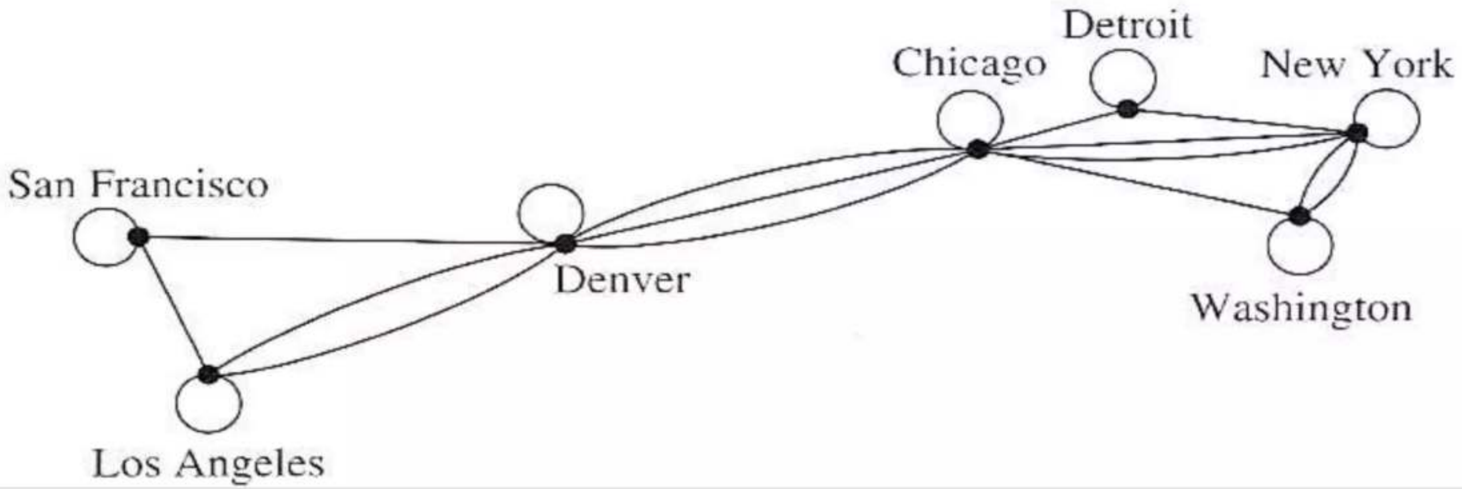


Pseudograph

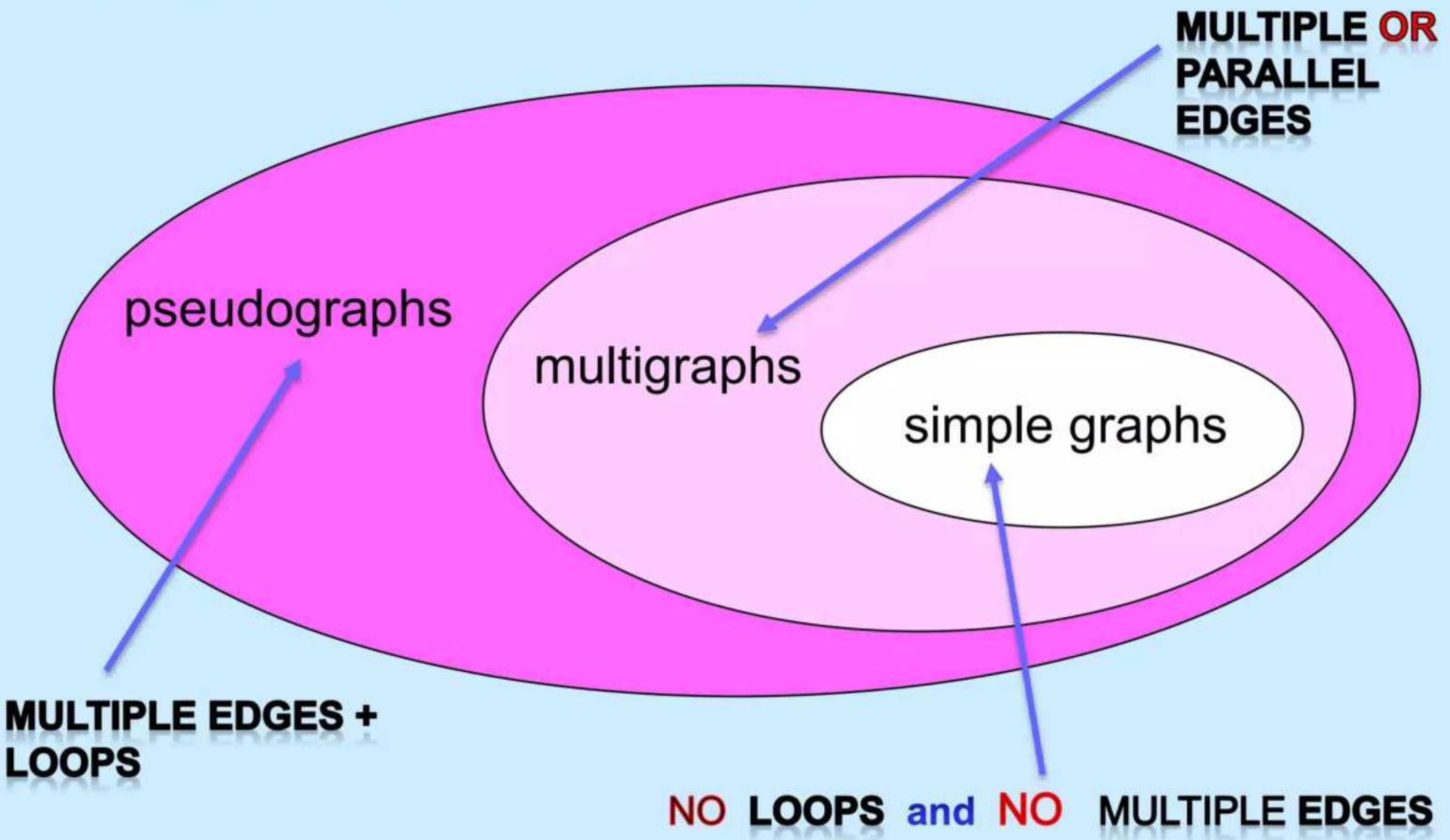
A graph that may contain multiple edges and **loops** is called a ***pseudograph.***

Example of a Pseudograph

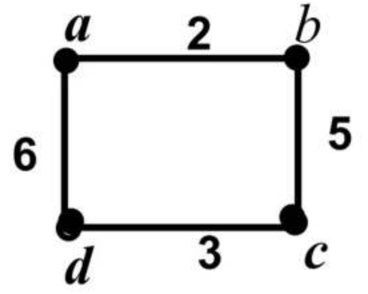
- A computer network may contain vertices with loops, which are edges from a vertex to itself.



Undirected Graphs

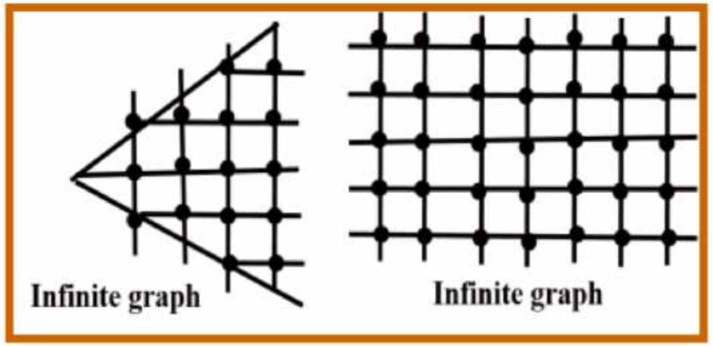


Labeled graphs: Labels are just the names we give vertices and edges so we can tell them apart.



Finite graphs: A graph with finite number of vertices and edges is called a finite graph.

Infinite graphs: A graph with infinite number of vertices and edges is called an infinite graph.



Finite & Infinite Graph

Definition: A graph is Finite no. of vertices as well as finite no. of edges called as **Finite Graph** otherwise it is **Infinite Graph.**

For Example, The graph G1 & G2 is Finite Graph.

Definition: A graph G=(V,E) is called as **Labeled Graph** if its edges are labeled with some names or data.

For Example, Graph G is labeled graph.

Basic Terminology

- ✓ *Incident :* An edge is said to be incident with the vertices it joins.
- ✓ *Adjacent :* Two vertices are said to be adjacent if they are joined by an edge.
- ✓ Two edges are said to be adjacent if they are joined by common vertices.
- ✓ *Degree of Vertices:* No. of edges incident on a particular vertex are called degree of that vertex.
- ✓ *Indegree & Outdegree:* Number of edges incident on to a vertex & number of vertex incident out of a vertex.

Basic.....

- ✓ *Loop:* If the initial vertex v_i and the terminal vertex v_j are same for an edge e_{ij}, then e_{ij} are called self loop or simply loop.

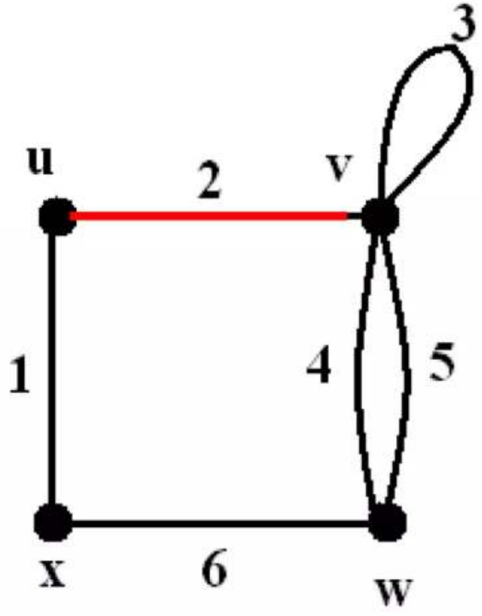
- ✓ *Parallel edges:* If there are more than one edges associated with a given pair of vertices then those edges are called parallel edges or *multiple edges*.

- ✓ *Isolated Vertex:* A vertex is said to be isolated vertex if no edge is incident on it.

- ✓ *Pendant vertex:* A vertex with degree 1 is called a Pendant vertex.

Which of the following statements hold for this graph?

(a) nodes ***v*** and ***u*** are adjacent;
(b) nodes ***v*** and ***x*** are adjacent;
(c) node ***u*** is incident with edge **2**;
(d)Edge **5** is incident with node ***x***.

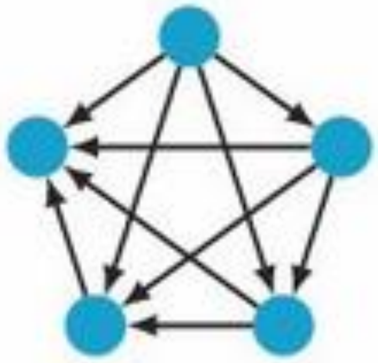


Ans: (a) Yes , connected by the edge 2
(b) **No**, No edge joins the vertices ***v*** and ***x***
(c) Yes, node ***v*** is also incident with edge **2**
(d) **No**, edge 5 is incident with nodes ***v*** and ***w***

- Types of graphs

Directed graphs (digraphs)

A **directed graph** or **digraph** is a graph where edges have directions (**directed edges** or **arcs**).

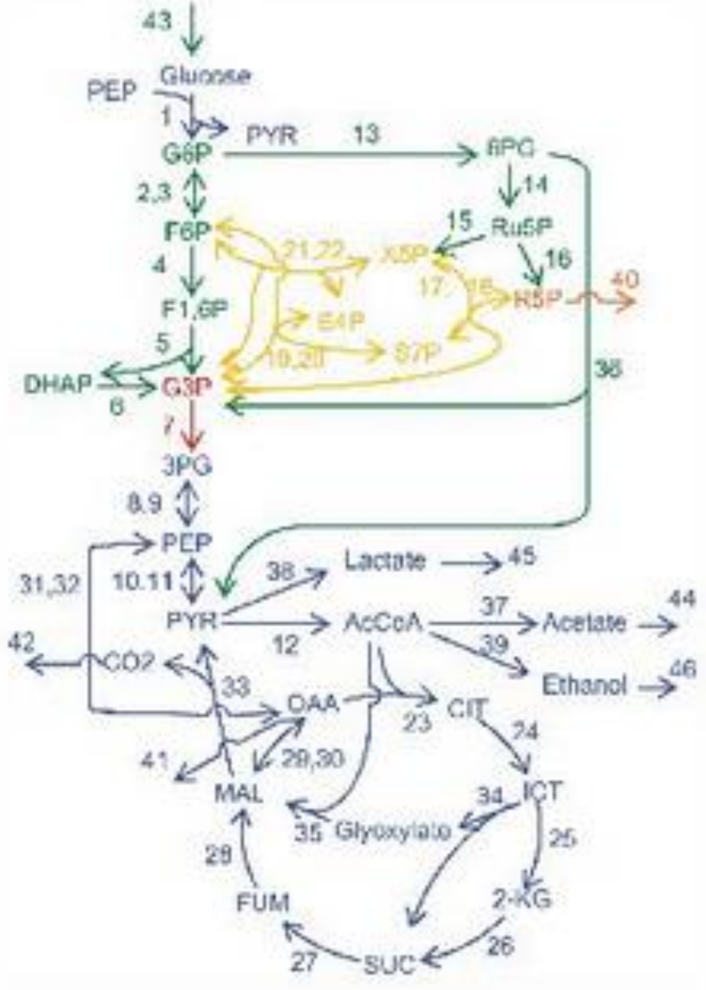


Example: Railway network

Vertex=station, Edge=railway

Example: Chemical reaction networks

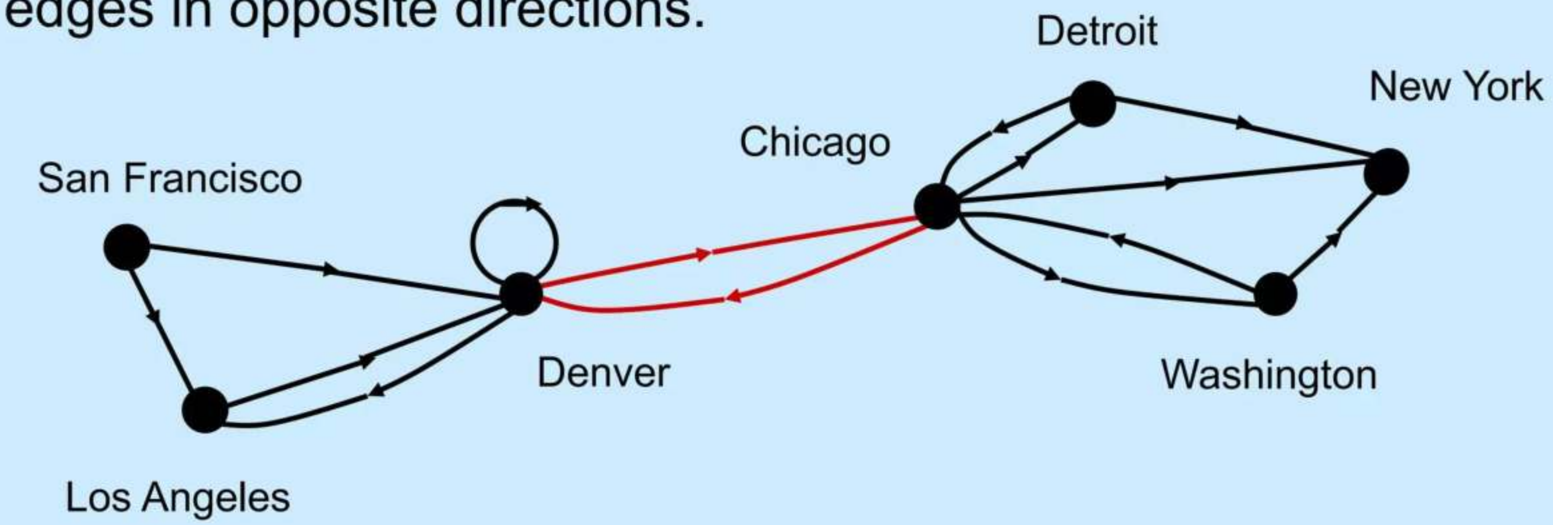
Vertex=chemicals, Edge=reactions



A Directed Graph

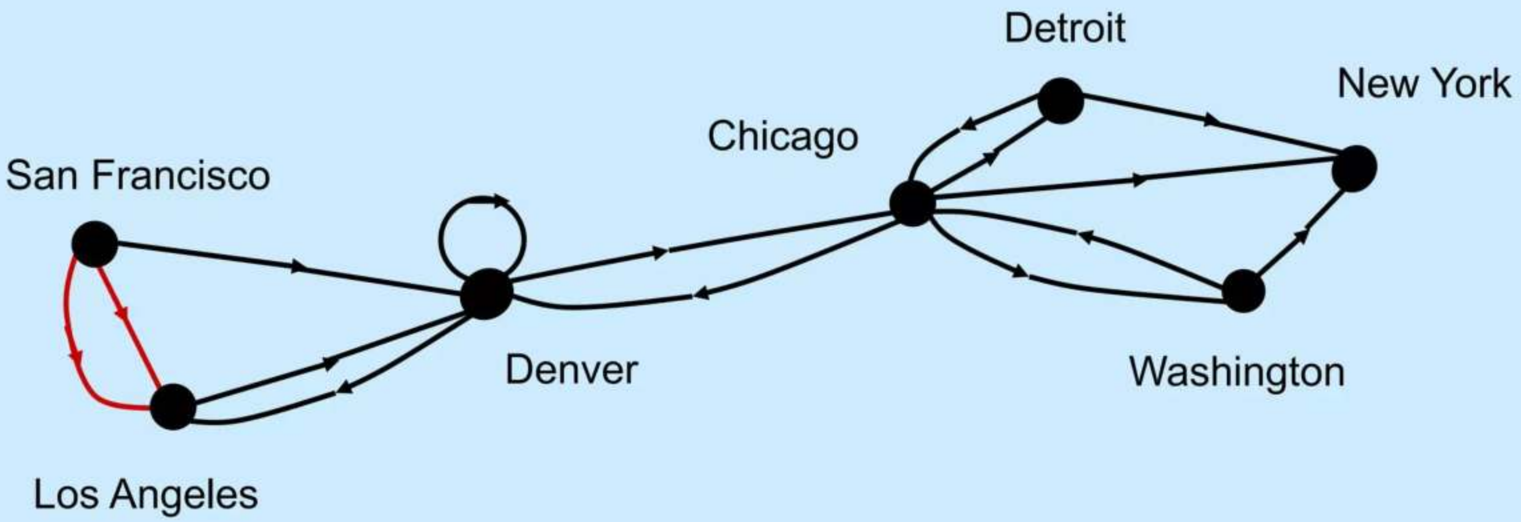
SOME TELEPHONE LINES IN THE NETWORK MAY OPERATE IN ONLY ONE DIRECTION .

Those that operate in two directions are represented by pairs of edges in opposite directions.



A Directed Multigraph

THERE MAY BE SEVERAL ONE-WAY LINES IN THE SAME DIRECTION FROM ONE COMPUTER TO ANOTHER IN THE NETWORK.



Types of Graphs

TYPE	EDGES	MULTIPLE EDGES ALLOWED?	LOOPS ALLOWED?
Simple graph	Undirected	NO	NO
Multigraph	Undirected	YES	NO
Pseudograph	Undirected	YES	YES
Directed graph	Directed	NO	YES
Directed multigraph	Directed	YES	YES

The ***maximum number of edges*** possible in a simple graph with n vertices is ${}^{n}C_{2}$

where ${}^{n}C_{2} = \frac{n(n-1)}{2}$

The maximum number of edges with n=3 vertices is

$${}^{n}C_{2} = \frac{n(n-1)}{2} = \frac{3(3-1)}{2} = \frac{3(2)}{2} = 3 \text{ edges.}$$

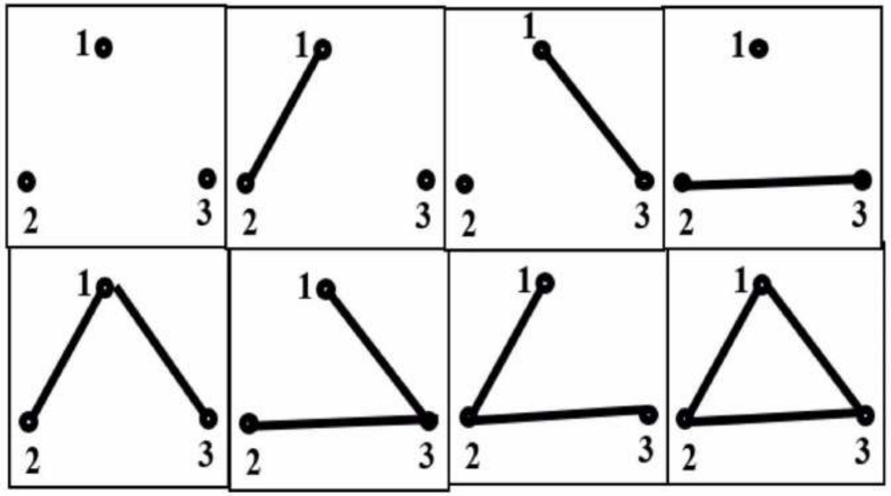
The ***number of simple graphs*** ***possible with n***

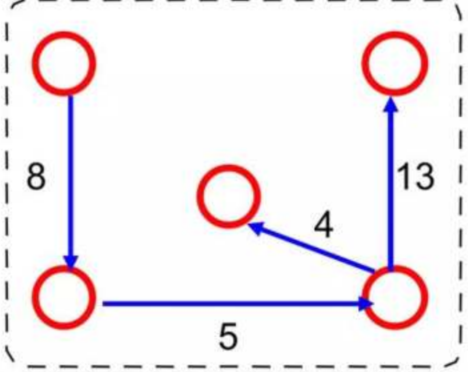
vertices = $2^{^nC_2} = 2^{\frac{n(n-1)}{2}}$

The maximum number of simple graphs with n=3 vertices is

$$2^{^nC_2} = 2^{\frac{n(n-1)}{2}} = 2^{\frac{3(3-1)}{2}} = 2^{\frac{3(2)}{2}} = 2^3 = 8$$

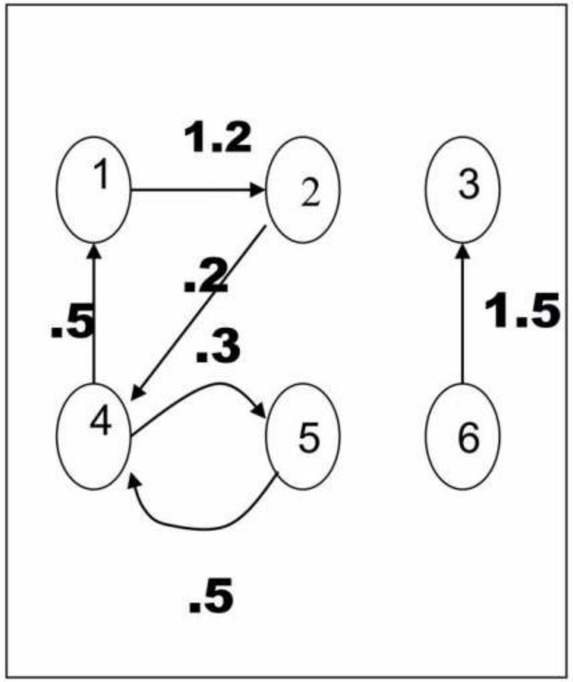
These 8 graphs are as shown in the alongside figure.

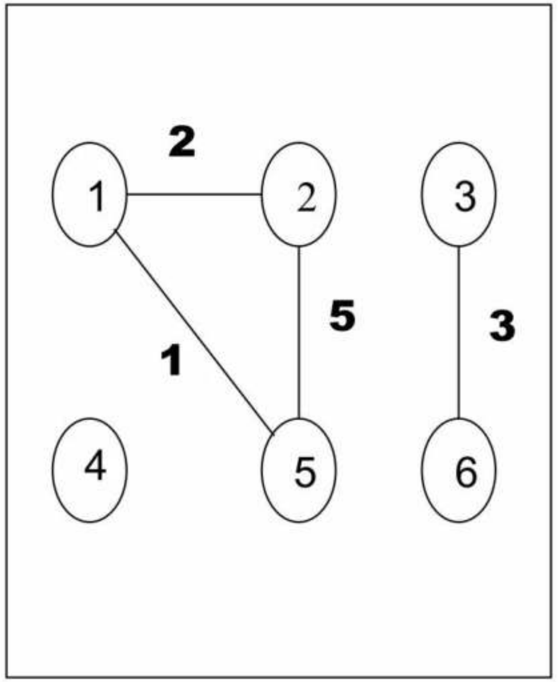


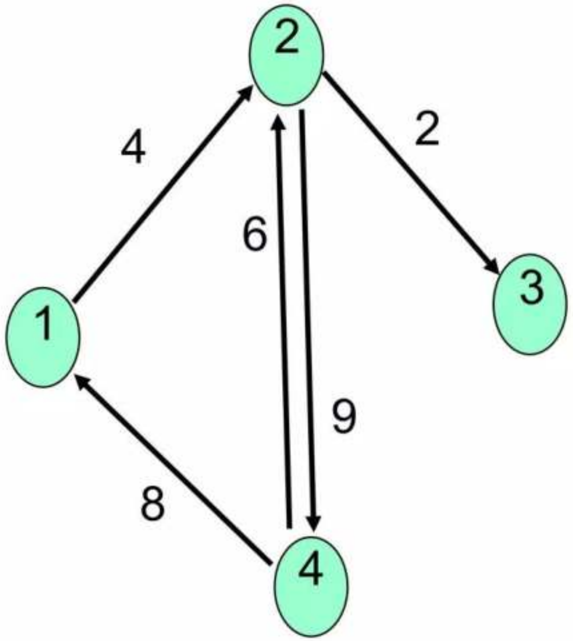


Weighted graphs

- is a graph for which each edge has an associated ***weight***, usually given by a ***weight function*** $w: E \rightarrow \mathbf{R}$.



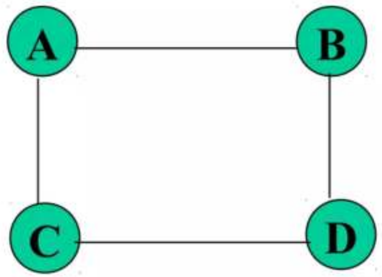




Types of Graph

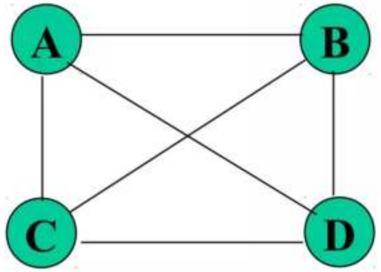
- **Connected Graph:** A graph is called connected if there is a simple path between any two of its nodes.

Example:



- **Complete Graph:** A graph G is complete if every node in graph G is adjacent to every other nodes. A complete graph with n nodes will have **n(n-1)/2** edges.

Example:



In Graph G

- Vertices, n = 4
- Edges: **n(n-1)/2 = 6**

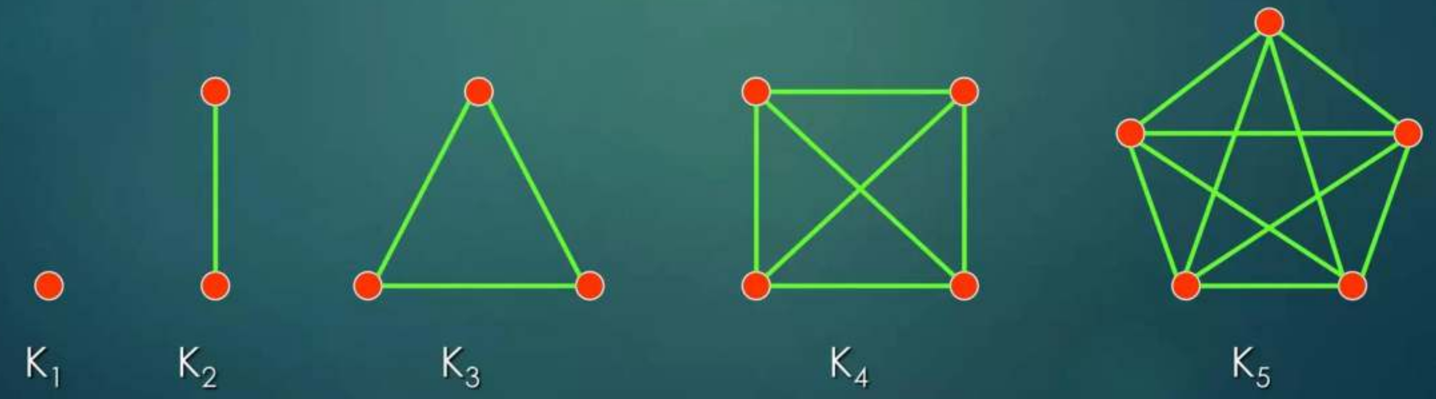
Complete Graph

Definition: Let G be simple graph on n vertices. If the degree of each vertex is (n-1) then the graph is called as **complete graph.**

Complete graph on n vertices, it is denoted by $\mathbf{K_n}$.

In complete graph K_n, the number of edges are

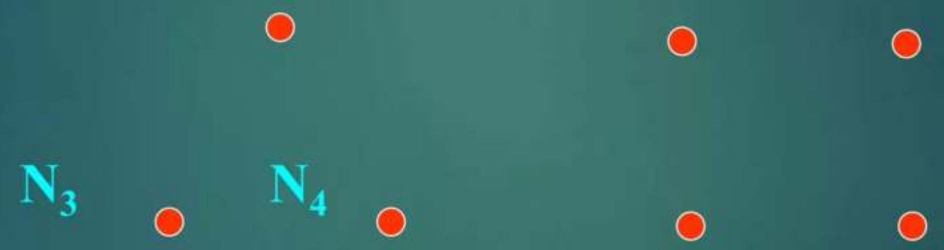
n(n-1)/2,For example,



Null Graph

Definition: If the edge set of any graph with n vertices is an empty set, then the graph is known as **null graph.**

It is denoted by N_n **For Example,**

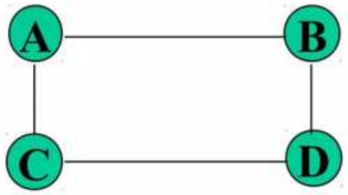


- **<u>Tree Graph:</u>** A connected graph with no cycle. If a tree graph has m nodes, then there are m-1 edges.

Example:

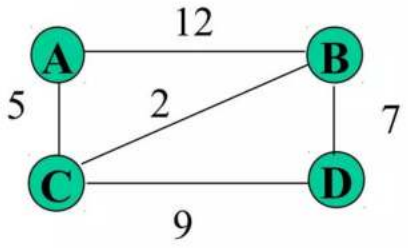
- **<u>Unweighted Graph:</u>** A graph G is said to be un weighted if its edges are not assigned any value.

Example:

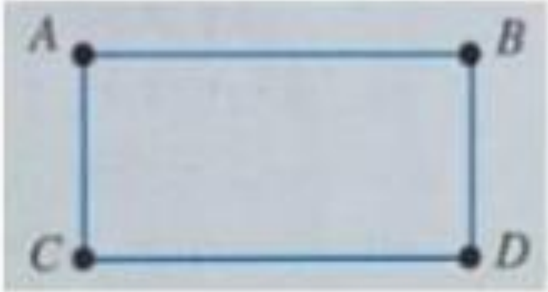


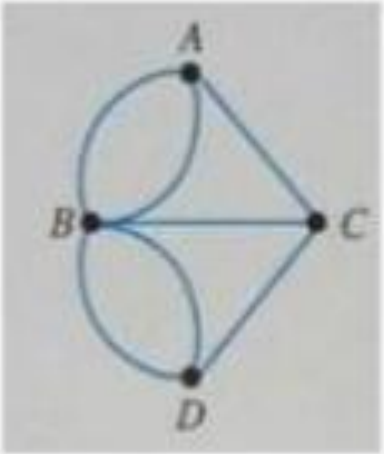
- **<u>Weighted Graph:</u>** A labeled graph where each edge is assigned a numerical value w(e).

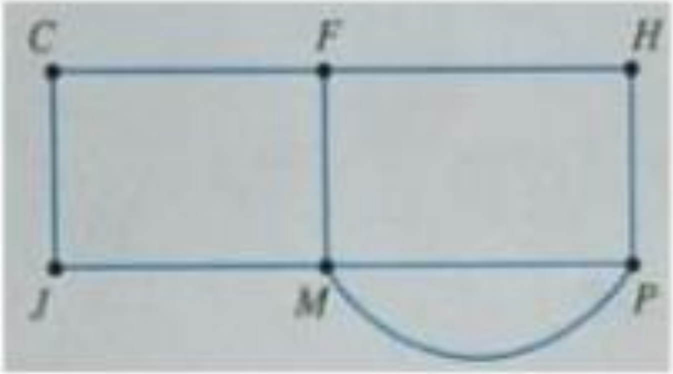
Example:



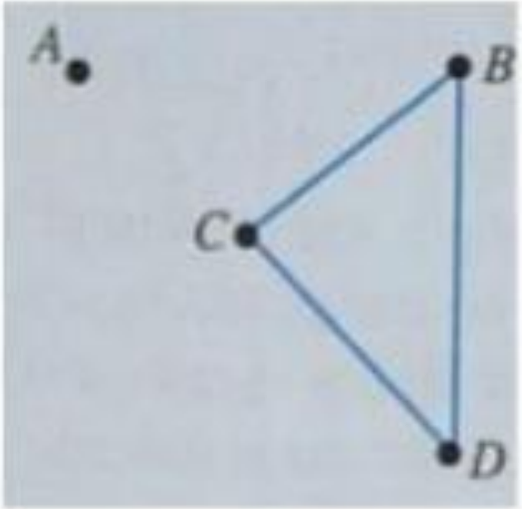
On a *connected graph*, you can draw a path from one vertex to any other vertex.

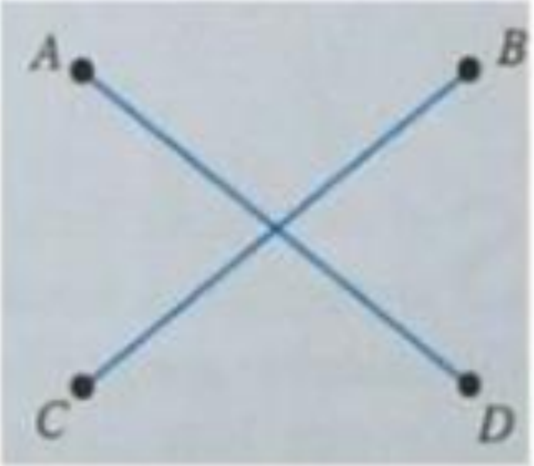


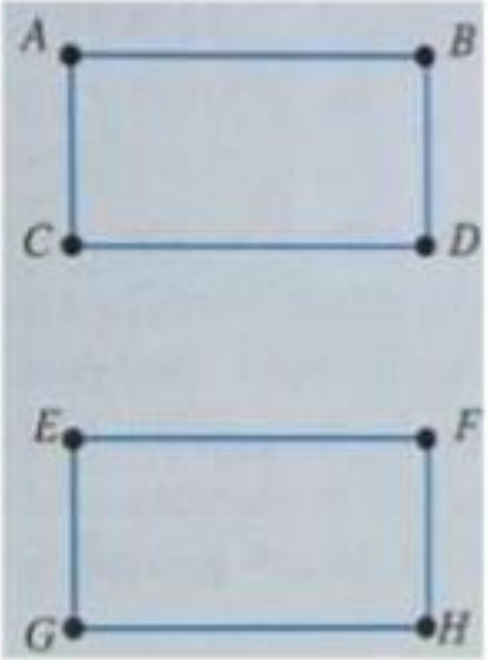




If a graph is not connected, it is ***disconnected***.



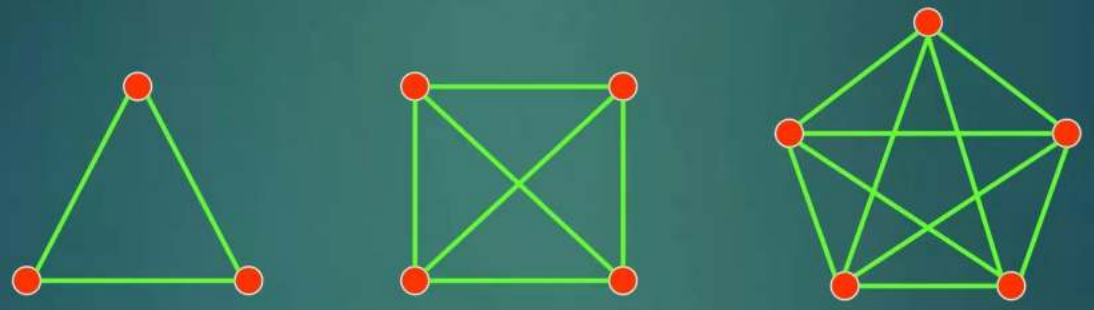




Regular Graph

Definition: If the degree of each vertex is same say 'r' in any graph G then the graph is said to be a **regular graph** of degree r.

For example,



K_3 K_4 K_5