Graph Theory - Basic Properties

Graphs come with various properties which are used for characterization of graphs depending on their structures. These properties are defined in specific terms pertaining to the domain of graph theory. In this chapter, we will discuss a few basic properties that are common in all graphs.

Distance between Two Vertices

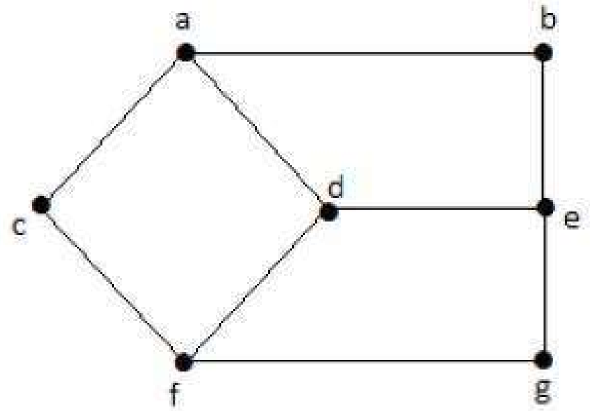
It is number of edges in a shortest path between Vertex U and Vertex V. If there are multiple paths connecting two vertices, then the shortest path is considered as the distance between the two vertices.

Notation − d(U,V)

There can be any number of paths present from one vertex to other. Among those, you need to choose only the shortest one.

Example

Take a look at the following graph −



Here, the distance from vertex 'd' to vertex 'e' or simply 'de' is 1 as there is one edge between them. There are many paths from vertex 'd' to vertex 'e' −

- da, ab, be
- df, fg, ge
- de (It is considered for distance between the vertices)
- df, fc, ca, ab, be
- da, ac, cf, fg, ge

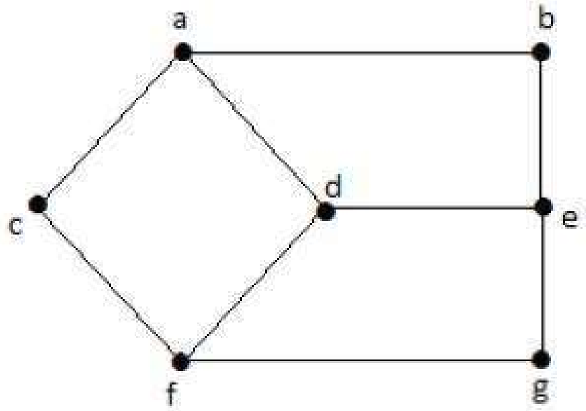
Eccentricity of a Vertex

The maximum distance between a vertex to all other vertices is considered as the eccentricity of vertex.

Notation − e(V)

The distance from a particular vertex to all other vertices in the graph is taken and among those distances, the eccentricity is the highest of distances.

Example



In the above graph, the eccentricity of 'a' is 3.

The distance from 'a' to 'b' is 1 ('ab'),

from 'a' to 'c' is 1 ('ac'),

from 'a' to 'd' is 1 ('ad'),

from 'a' to 'e' is 2 ('ab'-'be') or ('ad'-'de'),

from 'a' to 'f' is 2 ('ac'-'cf') or ('ad'-'df'),

from 'a' to 'g' is 3 ('ac'-'cf'-'fg') or ('ad'-'df'-'fg').

So the eccentricity is 3, which is a maximum from vertex 'a' from the distance between 'ag' which is maximum.

In other words,

e(b) = 3

e(c) = 3

e(d) = 2

e(e) = 3

e(f) = 3

e(g) = 3

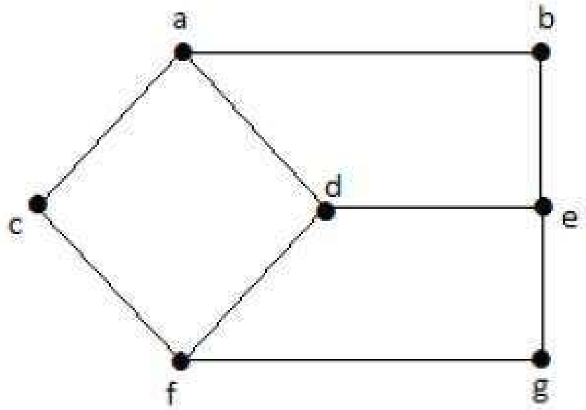
Radius of a Connected Graph

The minimum eccentricity from all the vertices is considered as the radius of the Graph G. The minimum among all the maximum distances between a vertex to all other vertices is considered as the radius of the Graph G.

Notation − r(G)

From all the eccentricities of the vertices in a graph, the radius of the connected graph is the minimum of all those eccentricities.

Example



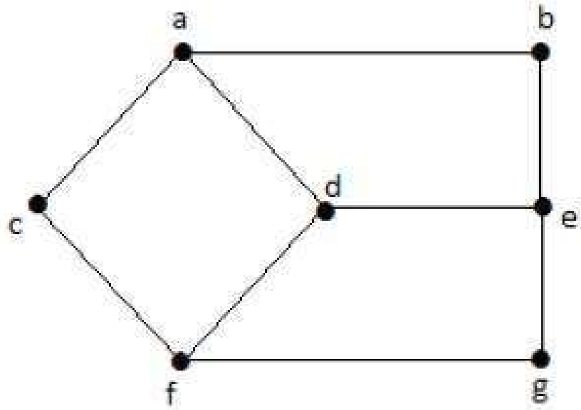
In the above graph r(G) = 2, which is the minimum eccentricity for ‘d’.

Diameter of a Graph

The maximum eccentricity from all the vertices is considered as the diameter of the Graph G. The maximum among all the distances between a vertex to all other vertices is considered as the diameter of the Graph G.

Notation − d(G) − From all the eccentricities of the vertices in a graph, the diameter of the connected graph is the maximum of all those eccentricities.

Example



In the above graph, d(G) = 3; which is the maximum eccentricity.

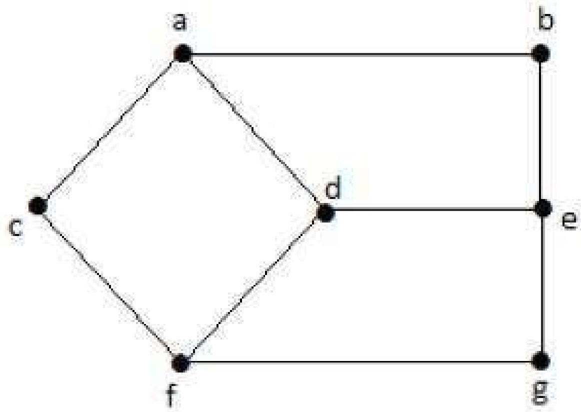
Central Point

If the eccentricity of a graph is equal to its radius, then it is known as the central point of the graph. If

e(V) = r(V),

then 'V' is the central point of the Graph 'G'.

Example



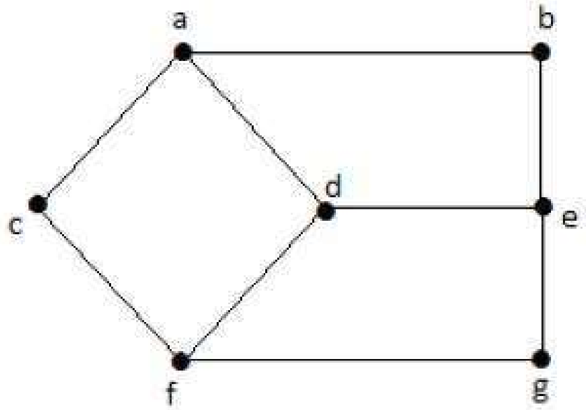
In the example graph, 'd' is the central point of the graph.

e(d) = r(d) = 2

Centre

The set of all central points of 'G' is called the centre of the Graph.

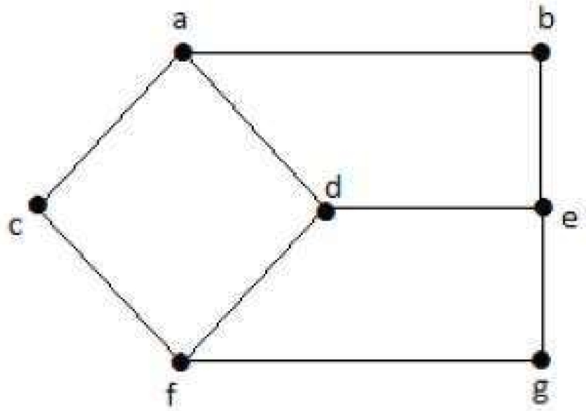
Example



In the example graph, {‘d’} is the center of the Graph.

Circumference

The **number of edges in the longest cycle of ‘G’** is called as the circumference of ‘G’.



Example

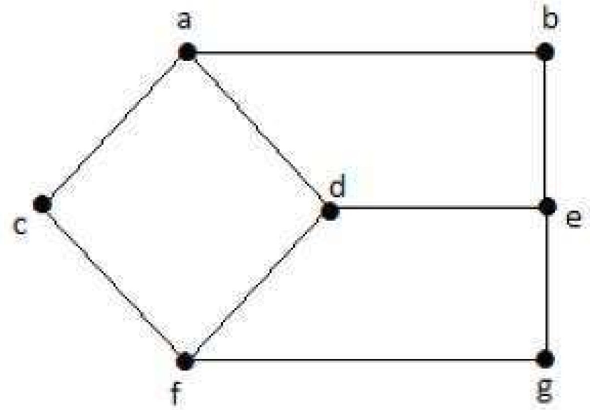
In the example graph, the circumference is 6, which we derived from the longest cycle a-c-f-g-e-b-a or a-c-f-d-e-b-a.

Girth

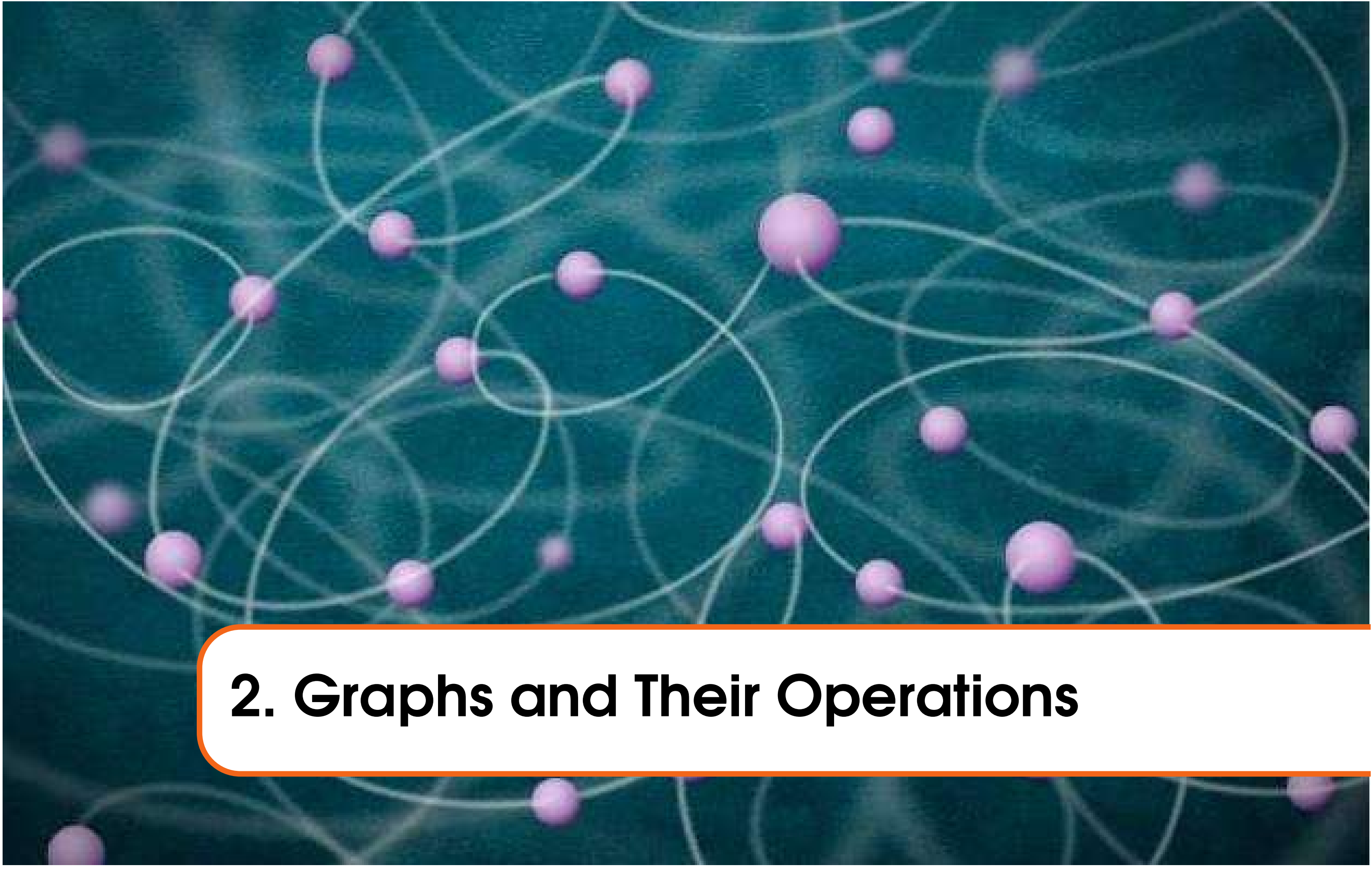
The number of edges in the shortest cycle of ‘G’ is called its Girth.

Notation: g(G).

Example –



In the example graph, the Girth of the graph is 4, which we derived from the shortest cycle a-c-f-d-a or d-f-g-e-d or a-b-e-d-a.



2. Graphs and Their Operations

We have already seen that the notion of subgraphs can be defined for any graphs as similar to the definition of subsets to sets under consideration. Similar to the definitions of basic set operations, we can define the corresponding basic operations for graphs also. In addition to these fundamental graph operations, there are some other new and useful operations are also defined on graphs. In this chapter, we discuss some basic graph operation.

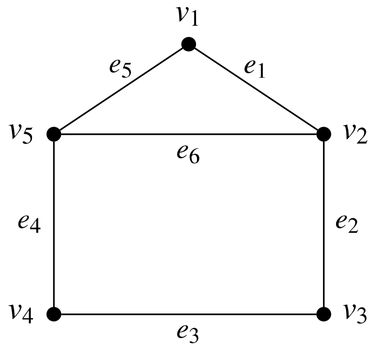
2.1 Union, Intersection and Ringsum of Graphs

Definition 2.1.1 — Union of Graphs. The *union* of two graphs G_1 and G_2 is a graph G, written by $G = G_1 \cup G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

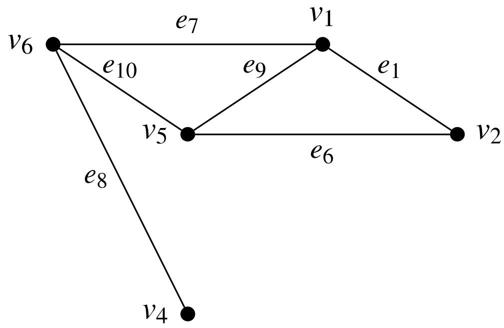
Definition 2.1.2 — Intersection of Graphs. The *intersection* of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \cap G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

Definition 2.1.3 — Ringsum of Graphs. The *ringsum* of two graphs G_1 and G_2 is another graph G, written by $G = G_1 \oplus G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \oplus E(G_2)$, where $\oplus$ is the symmetric difference (XOR Operation) of two sets.

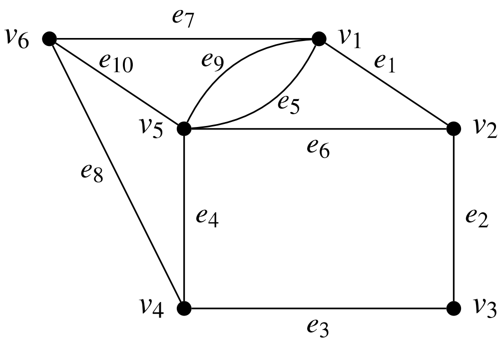
Figure 2.1 illustrates the union, intersection and ringsum of two given graphs.



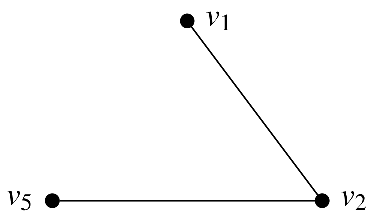
(a) G_1



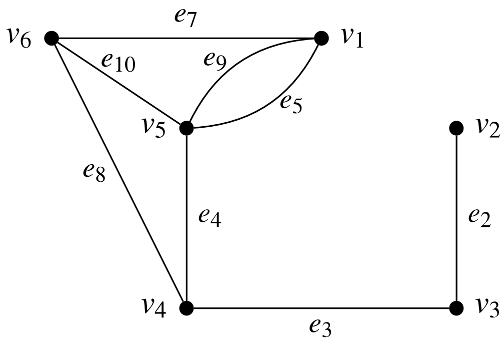
(b) G_2



(c) $G_1 \cup G_2$



(d) $G_1 \cap G_2$



(e) $G_1 \oplus G_2$

Figure 2.1: Illustrations to graph operations

R

1. The union, intersection and ringsum operations of graphs are commutative. That is, $G_1 \cup G_2 = G_2 \cup G_1, G_1 \cap G_2 = G_2 \cap G_1$ and $G_1 \oplus G_2 = G_2 \oplus G_1$.
2. If G_1 and G_2 are edge-disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$.
3. If G_1 and G_2 are vertex-disjoint, then $G_1 \oplus G_2$ is empty.
4. For any graph G, $G \cap G = G \cup G$ and $G \oplus G$ is a null graph.

Definition 2.1.4 — Decomposition of a Graph. A graph G is said to be *decomposed* into two subgraphs G_1 and G_2 , if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a null graph.

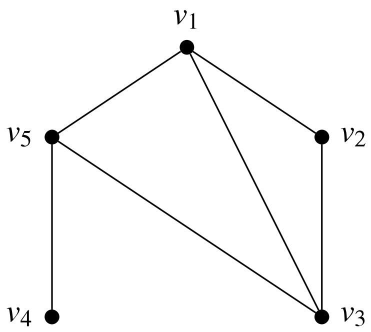
2.2 Complement of Graphs

Definition 2.2.1 — Complement of Graphs. The *complement* or *inverse* of a graph G, denoted by $\bar{G}$ is a graph with $V(G) = V(\bar{G})$ such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

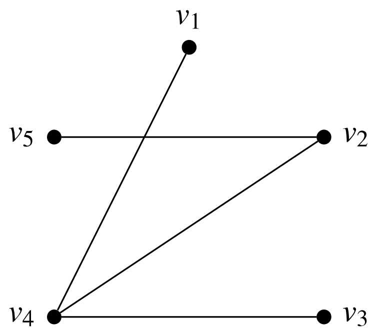
R Note that for a graph G and its complement $\bar{G}$, we have

(i) $G_1 \cup \bar{G} = K_n$;
(ii) $V(G) = V(\bar{G})$;
(iii) $E(G) \cup E(\bar{G}) = E(K_n)$;
(iv) $|E(G)| + |E(\bar{G})| = |E(K_n)| = \binom{n}{2}$.

A graph and its complement are illustrated below.



(a) G



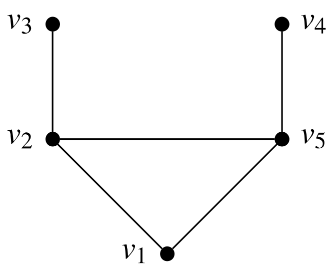
(b) $\bar{G}$

Figure 2.2: A graph and its complement

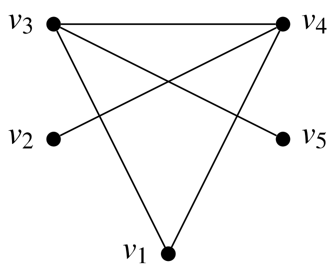
2.2.1 Self-Complementary Graphs

Definition 2.2.2 — Self-Complementary Graphs. A graph G is said to be *self-complementary* if G is isomorphic to its complement. If G is self complementary, then $|E(G)| = |E(\bar{G})| = \frac{1}{2}|E(K_n)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$.

The following are two examples of self complementary graphs.

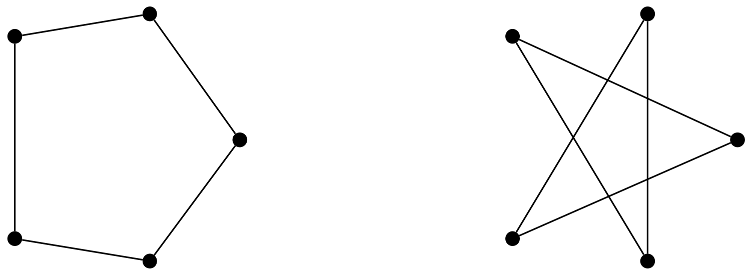


(a) Bull graph G.



(b) $\bar{G}$.

Figure 2.3: Example of self-complementary graphs



(a) G_2 (b) G_2.

Figure 2.4: Example of self-complementary graphs

Problem 2.1 For any self-complementary graph G of order n, show that $n \equiv 0, 1 (\text{mod } 4)$.
Solution: For self-complementary graphs, we have

(i) $V(G) = V(\bar{G})$;
(ii) $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$;
(iii) $|E(G)| = |E(\bar{G})|$.

Therefore, $|E(G)| = |E(\bar{G})| = \frac{n(n-1)}{4}$. This implies, 4 divides either n or $n-1$. That is, for self-complementary graphs of order n, we have $n \equiv 0, 1 (\text{mod } 4)$. ■

(Note that we say $a \equiv b (\text{mod } n)$, which is read as "$a$ is congruent to b modulo n", if $a - b$ is completely divisible by n).

2.3 Join of Graphs

Definition 2.3.1 The *join* of two graphs G and H, denoted by $G + H$ is the graph such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

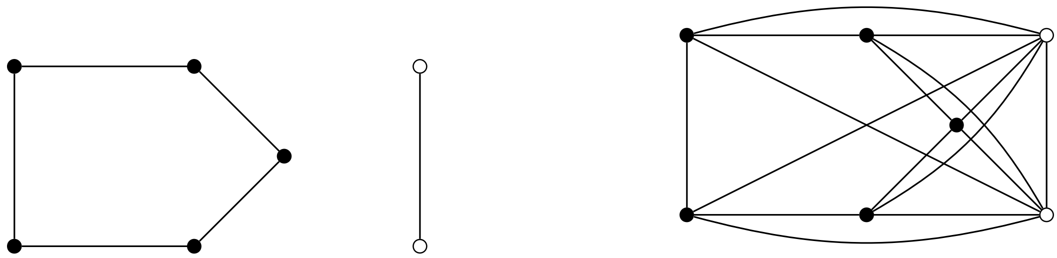
In other words, the join of two graphs G and H is defined as the graph in which every edge of the first graph is adjacent to all vertices of the second graph.

Figure 2.5 illustrates the join of two graphs P_3 and P_4 and Figure 2.6 illustrates the join of two graphs C_5 and P_2.

Figure 2.5: The join of the paths P_4 and P_3.

2.4 Deletion and Fusion

Definition 2.4.1 — Edge Deletion in Graphs. If e is an edge of G, then $G - e$ is the graph obtained by removing the edge of G. The subgraph of G thus obtained is called an *edge-deleted subgraph* of G. Clearly, $G - e$ is a spanning subgraph of G.



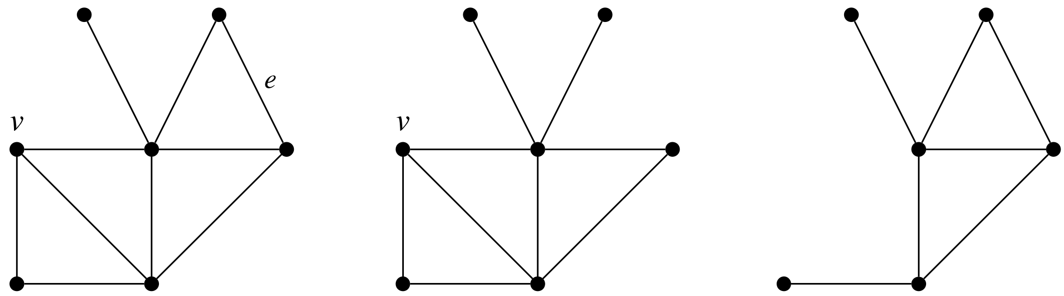
(a) C_5 (b) P_2 (c) $C_5 + P_2$.

Figure 2.6: The join of the cycle C_5 and the path P_2.

Similarly, vertex-deleted subgraph of a graph is defined as follows:

> **Definition 2.4.2 — Vertex Deletion in Graphs.** If v is a vertex of G, then $G - v$ is the graph obtained by removing the vertex v and all edges G that are incident on v. The subgraph of G thus obtained is called an *vertex-deleted subgraph* of G. Clearly, $G - v$ will not be a spanning subgraph of G.

Figure 2.7 illustrates the edge deletion and the vertex deletion of a graph G.



(a) G (b) $G - e$ (c) $G - v$

Figure 2.7: Illustrations to edge deletion and vertex deletion

> **Definition 2.4.3 — Fusion of Vertices.** A pair of vertices u and v are said to be *fused* (or *merged* or *identified*) together if the two vertices are together replaced by a single vertex w such that every edge incident with either u or v is incident with the new vertex w (see Figure 2.8).

Note that the fusion of two vertices does not alter the number of edges, but reduces the number of vertices by 1.

2.4.1 Edge Contraction

> **Definition 2.4.4 — Edge Contraction in Graphs.** An *edge contraction* of a graph G is an operation which removes an edge from a graph while simultaneously merging its two end vertices that it previously joined. Vertex fusion is a less restrictive form of this operation.

A graph obtained by contracting an edge e of a graph G is denoted by $G \circ e$.