This Brightness Constancy Assumption states that the intensities of a pixel in the first and the second frame are the same due to a short duration between the frames. It can be written using the following mathematical expression:

$$f(x, y, t) = f(x + dx, y + dy, t + dt)$$

Taylor

$$f(x, y, t) = f(x, y, t) + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial t} dt + \cdots$$

After neglecting the higher order terms and simplifying the above expansion further, we can write the new expression as:

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial t} dt = 0$$

Next, we will perform certain notation changes such as

$$\frac{\partial f}{\partial x} = f_x, \qquad \frac{\partial f}{\partial y} = f_y, \qquad \frac{\partial f}{\partial t} = f_t$$

We will further divide both sides by dt to arrive at the following equation:

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_t = 0$$

Here, we can say that every pixel with coordinates (x, y) in the first image, moves along the motion vector (u, v) where u is the pixel's velocity in the x direction and v is the pixel's velocity in the y direction, respectively.

Using the equation above, we can say that

$$\frac{dx}{dt} - u$$

is the pixel's velocity in the x direction, and

$$\frac{dy}{dt} - v$$

is the pixel's velocity in the y direction.

Furthermore, since we have sampled a video sequence for our example, we can say that we are working at discrete times and not continuous time. Thus, we can set $dt = 1$ and arrive at the equation below.

$$f_x u + f_y v + f_t = 0$$

The Smoothness Constraint Assumption

$$E = \iint_{x,y} \left[\left(f_x u + f_y v + f_t \right)^2 + \lambda \left(u_x^2 + u_y^2 + v_x^2 + v_y^2 \right) \right] dx dy$$

Moving Object Masks

In the digital domain, we often use moving object masks and convolve them with the images. Consider the images below.

Frame t

$$f_{x1} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad f_{y1} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad f_{t1} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Frame $t + dt$

$$f_{x2} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad f_{y2} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad f_{t2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then

$$f_x = \frac{1}{2} [(\text{Frame } t) \times f_{x1} + (\text{Frame } t + dt) \times f_{x2}]$$

$$f_y = \frac{1}{2} \left[(\text{Frame } t) \times f_{y1} + (\text{Frame } t + dt) \times f_{y2} \right]$$

$$f_t = [(\text{Frame } t) \times f_{t1} + (\text{Frame } t + dt) \times f_{t2}]$$

Now that we have calculated f_x, f_y and f_t we shall see how moving object masks can be used to reduce our method's sensitivity to noise. This is done by computing the Laplacian operator $\Delta^2 u$ by using a spatial filter or a Laplacian mask which is usually applied after the smoothing process of the image.

Consider the computation below.

$$\begin{bmatrix} \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{6} & -1 & \frac{1}{6} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{bmatrix}$$

As you can see in the above matrices, the diagonals are given lesser weight as compared to the four neighbor which have higher weightage. We can represent the Laplacian mask mathematically as written below.

$$\Delta^2 u = u_{avg} - u$$

$$\Delta^2 v = v_{avg} - v$$

In the equations above, u_{avg} and v_{avg} represent the average of neighborhood pixels, respectively. After we have found the average, we can write the middle pixel as an average of its neighboring pixels as $u_{avg} - u$. We can perform the same averaging operation for v as well.

If you look at it intuitively, we are simply converting an analog equation into a discrete equation. The only difference is that instead of using the term $\Delta^2 u$, we are representing the same thing with the term $u_{avg} - u$ and $v_{avg} - v$,respectively.

Great! Looks like we have finally solved the unconstrained system problem that we were having earlier where we had a single equation but two variables to be calculated. Now, we have two equations and two unknowns. Let's get to solving these two equations.

$$(f_x u + f_y v + f_t) f_x - \lambda (u_{avg} - u) = 0$$

$$f_x^2 u + f_y f_x v + f_t f_x - \lambda u_{avg} + \lambda u = 0$$

$$(f_x^2 + \lambda) u + f_y f_x v = -f_t f_x + \lambda u_{avg}$$

$$(f_x u + f_y v + f_t) f_y - \lambda (v_{avg} - v) = 0$$

$$f_x f_y u + f_y^2 v + f_t f_y - \lambda v_{avg} + \lambda v = 0$$

$$f_x f_y u + (f_y^2 + \lambda) v = -f_t f_y + \lambda v_{avg}$$

Here, we can use the Cramer's rule by creating three determinants, represented by D, D_u and D_v respectively. This is how we apply the Cramer's rule to define these three determinants:

$$D = \begin{vmatrix} f_x^2 + \lambda & f_x f_y \\ f_x f_y & f_y^2 + \lambda \end{vmatrix}, \qquad D_u = \begin{vmatrix} -f_t f_x + \lambda u_{avg} & f_x f_y \\ -f_t f_y + \lambda v_{avg} & f_y^2 + \lambda \end{vmatrix}, \qquad D_v = \begin{vmatrix} f_x^2 + \lambda & -f_t f_x + \lambda u_{avg} \\ f_x f_y & -f_t f_y + \lambda v_{avg} \end{vmatrix}$$

Now, we solve these determinants further as follows.

$$D = (f_x^2 + \lambda)(f_y^2 + \lambda) - (f_x f_y)^2 = f_x^2 f_y^2 + \lambda f_y^2 + \lambda f_x^2 + \lambda^2 - (f_x f_y)^2 = \lambda (f_y^2 + f_x^2 + \lambda)$$

$$\begin{aligned} D_u &= (-f_t f_x + \lambda u_{avg})(f_y^2 + \lambda) - (-f_t f_y + \lambda v_{avg}) f_x f_y \\ &= -f_t f_x f_y^2 - \lambda f_t f_x + \lambda f_y^2 u_{avg} + \lambda^2 u_{avg} + f_t f_x f_y^2 - \lambda v_{avg} f_x f_y \end{aligned}$$

$$\begin{aligned} D_v &= (f_x^2 + \lambda)(-f_t f_y + \lambda v_{avg}) - f_x f_y (-f_t f_x + \lambda u_{avg}) \\ &= -f_t f_x^2 f_y - \lambda f_t f_y + \lambda f_x^2 v_{avg} + \lambda^2 v_{avg} + f_t f_x^2 f_y - \lambda u_{avg} f_x f_y \end{aligned}$$

$$\begin{aligned} u = \frac{D_u}{D} &= \frac{-f_t f_x f_y^2 - \lambda f_t f_x + \lambda f_y^2 u_{avg} + \lambda^2 u_{avg} + f_t f_x f_y^2 - \lambda v_{avg} f_x f_y}{\lambda (f_y^2 + f_x^2 + \lambda)} \\ &= \frac{-f_t f_x + f_y^2 u_{avg} + \lambda u_{avg} - v_{avg} f_x f_y}{f_y^2 + f_x^2 + \lambda} \end{aligned}$$

$$v = \frac{D_v}{D} = \frac{-f_t f_x^2 f_y - \lambda f_t f_y + \lambda f_x^2 v_{avg} + \lambda^2 v_{avg} + f_t f_x^2 f_y - \lambda u_{avg} f_x f_y}{\lambda\left(f_y^2 + f_x^2 + \lambda\right)}$$

$$= \frac{-f_t f_y + f_x^2 v_{avg} + \lambda v_{avg} - u_{avg} f_x f_y}{f_y^2 + f_x^2 + \lambda}$$

$$u = \frac{u_{avg}\left(f_y^2 + f_x^2 + \lambda\right) - f_x\left(f_x u_{avg} + f_y v_{avg} + f_t\right)}{f_y^2 + f_x^2 + \lambda} = u_{avg} - f_x \frac{f_x u_{avg} + f_y v_{avg} + f_t}{f_y^2 + f_x^2 + \lambda}$$

$$v = \frac{v_{avg}\left(f_y^2 + f_x^2 + \lambda\right) - f_y\left(f_x u_{avg} + f_y v_{avg} + f_t\right)}{f_y^2 + f_x^2 + \lambda} = v_{avg} - f_y \frac{f_x u_{avg} + f_y v_{avg} + f_t}{f_y^2 + f_x^2 + \lambda}$$

Then

$$u = u_{avg} - f_x \frac{P}{D}, \qquad v = v_{avg} - f_y \frac{P}{D}$$

Here,

$$P = f_x u_{avg} + f_y v_{avg} + f_t, \qquad D = f_y^2 + f_x^2 + \lambda$$