Q1

10 / 10

In exercises 1a and 1b, verify that the given function has a zero on the indicated interval. Next, perform the first five iterations of the bisection method (iterations 0 through 4) and verify that each approximation satisfies the theoretical error bound of the bisection method, but that the errors do not steadily increase. The exact location of the zero is indicated by the value of r.

(a) $f(x) = x^3 + x^2 - 3x - 3$, $(1, 2)$, $r = \sqrt{3}$

(b) $f(x) = 1 - \ln x$, $(2, 3)$, $r = e$

√

①

CS 3113 A2

1) a) $x^3 + x^2 - 3x - 3$ $[1,2]$, $r = \sqrt{3}$

$f(1) = (1)^3 + (1)^2 - 3(1) - 3 = -4$
$f(2) = (2)^3 + (2)^2 - 3(2) - 3 = 3$
Assuming f is a continuous function on $[1,2]$ & y is between -4 & 3, then there exists a number c that is $1 \leq c \leq 2$ such that $f(c) = y$

iter	bracket	c	sign (f)	error bound
0	[1, 2]	1.5	<0	0.5
1	[1.5, 2]	1.75	>0	0.25
2	[1.5, 1.75]	1.625	<0	0.125
3	[1.625, 1.75]	1.6875	<0	0.0625
4	[1.6875, 1.75]	1.71875	<0	0.03125

iter	absolute error
0	0.23205
1	0.1794
2	0.10705
3	0.04455
4	0.01330

b) $f(x) = 1 - \ln x$ $[2,3]$, $r = e$
$f(2) = 1 - \ln(2) = 0.30685$
$f(3) = 1 - \ln(3) = -0.09861$
Assuming f is a continuous function on $[2,3]$ & y is between 0.30685 & -0.09861, then there exists a number c that is $2 \leq c \leq 3$ such that $f(c) = y$

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②

iter	bracket	c	sign(f)	error bound
0	[2,3]	2.5	>0	0.5
1	[2.5,3]	2.75	<0	0.25
2	[2.5,2.75]	2.625	>0	0.125
3	[2.625,2.75]	2.6875	>0	0.0625
4	[2.6875,2.75]	2.71875	<0	0.03125

iter	absolute error
0	0.2183
1	0.0317
2	0.0933
3	0.0308
4	0.00047

good **10**
job!

Q2

18 / 18

Fixed point iteration: The function $f(x) = x^3 - x^2 - 10x + 7$ has a unique zero on the interval (0, 1). For this function, the following $g(x)$ were produced

- $g_1(x) = \frac{x^3 - x^2 + 7}{10}$
- $g_2(x) = \sqrt{x^3 - 10x + 7}$
- $g_3(x) = \frac{7}{-x^2 + x + 10}$

(a) Comment on the convergence properties of these functions.

(b) Verify the conclusion reached in part (a) using by computing the root for each $g(x)$ for 10 iterations starting at $x_0 = 0$. Use double precision (i.e. Matlab, Python, etc...).

10

2) a) $g_1(x) = \frac{x^3 - x^2 + 7}{10}$ $[0, 1]$ $x_0 = 0.5$

$g_1'(x) = \frac{1}{10}(3x^2 - 2x)$; $g_1'(0.5) = \frac{1}{10}(3(0.5)^2 - 2(0.5))$

$= -0.025$

it converges really fast as it is close to zero and $-1 <$ and < 1

$g_2(x) = \sqrt{x^3 - 10x + 7}$; $g_2'(x) = \frac{3x^2 - 10}{2\sqrt{x^3 - 10x + 7}}$

$g_2'(0.5) = \frac{3(0.5)^2 - 10}{2\sqrt{(0.5)^3 - 10(0.5) + 7}} = -3.17272$

doesn't converge, it is < -1

$g_3(x) = \frac{7}{-x^2 + x + 10}$; $g_3'(x) = \frac{-7(-2x+1)}{(-x^2 + x + 10)^2}$

$g_3'(0.5) = \frac{-7(-2(0.5)+1)}{((-(0.5)^2) + 0.5 + 10)^2} = 0$

This has the fastest converging among all 3.

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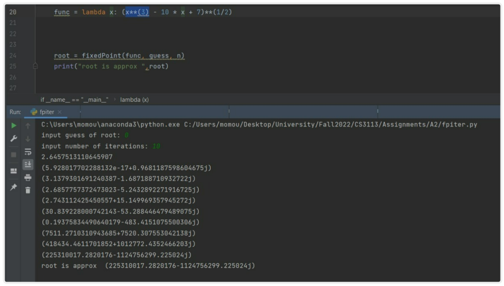
```
    func = lambda x: (x**3 - x**2 +7)/10



    root = fixedPoint(func, guess, n)
    print("root is approx ",root)
```

```
if __name__ == "__main__"  › lambda (x)
```

fpiter

```
C:\Users\momou\anaconda3\python.exe C:/Users/mo
input guess of root: 0
input number of iterations: 10
0.7
0.6853
0.6852205522477
0.6852202485635336
0.6852202474052763
0.6852202474008589
0.685220247400842
0.6852202474008419
0.6852202474008419
0.6852202474008419
root is approx  0.6852202474008419
```

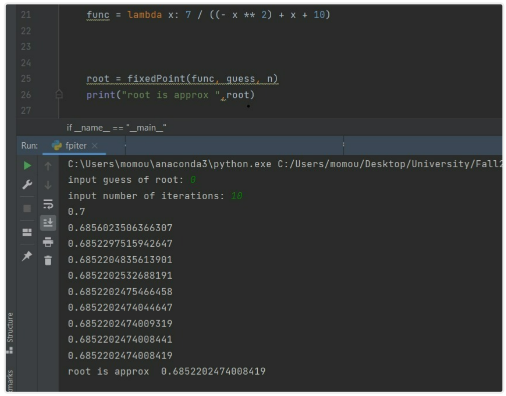


```
    func = lambda x: (x**(3) - 10 * x + 7)**(1/2)



    root = fixedPoint(func, guess, n)
    print("root is approx ",root)

```

if __name__ == "__main__" › lambda (x)

Run: fpiter

```
C:\Users\momou\anaconda3\python.exe C:/Users/momou/Desktop/University/Fall2022/CS3113/Assignments/A2/fpiter.py
input guess of root: 0
input number of iterations: 10
2.6457513110645907
(5.928017702288132e-17+0.9681187598604675j)
(3.1379301691240387-1.687188710932722j)
(2.6857757372473023-5.2432892271916725j)
(2.743112425450557+15.149969357945272j)
(30.839228000742143-53.288446479489075j)
(0.19375834490640179-483.4151075500306j)
(7511.271031094368+7520.307553042138j)
(418434.4611701852+1012772.4352466203j)
(225310017.2820176-1124756299.225024j)
root is approx  (225310017.2820176-1124756299.225024j)
```



```
    func = lambda x: 7 / ((- x ** 2) + x + 10)



    root = fixedPoint(func, guess, n)
    print("root is approx ",root)
```
if __name__ == "__main__"
Run: fpiter
```
C:\Users\momou\anaconda3\python.exe C:/Users/momou/Desktop/University/Fall2
input guess of root: 0
input number of iterations: 10
0.7
0.6856023506366307
0.6852297515942647
0.6852204835613901
0.6852202532688191
0.6852202475466458
0.6852202474044647
0.6852202474009319
0.6852202474008441
0.6852202474008419
root is approx  0.6852202474008419
```
Structure

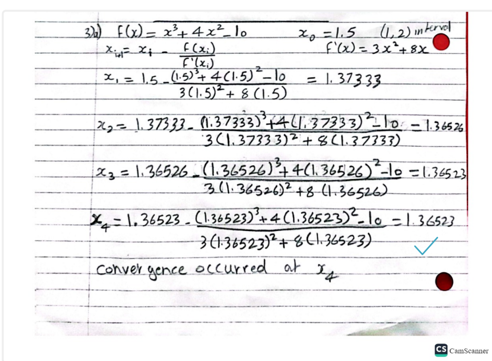
Q3

8 / 10

Newton's method: For a function $f(x) = x^3 + 4x^2 - 10$

(a) Find the root of function $f(x)$ in the interval (1, 2) starting with $x_0 = 1.5$.

(b) Compute the ratio $|x_n - r|/|x_{n-1} - r|^2$, for iterations 1, 2, 3 given $r = 1.3652300134141000$. Show that this ratio's value approaches $|f''(x)/2f'(x)|$ (i.e., the iteration converges quadratically). In error computation, keep as many digits as you can.



3)a) $f(x) = x^3 + 4x^2 - 10$ $x_0 = 1.5$ (1, 2) interval

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ $f'(x) = 3x^2 + 8x$

$x_1 = 1.5 - \frac{(1.5)^3 + 4(1.5)^2 - 10}{3(1.5)^2 + 8(1.5)} = 1.37333$

$x_2 = 1.37333 - \frac{(1.37333)^3 + 4(1.37333)^2 - 10}{3(1.37333)^2 + 8(1.37333)} = 1.36526$

$x_3 = 1.36526 - \frac{(1.36526)^3 + 4(1.36526)^2 - 10}{3(1.36526)^2 + 8(1.36526)} = 1.36523$

$x_4 = 1.36523 - \frac{(1.36523)^3 + 4(1.36523)^2 - 10}{3(1.36523)^2 + 8(1.36523)} = 1.36523$

convergence occurred at x_4

3)b) $|x_n - r| / |x_{n-1} - r|^2$ $f''(x) = 6x+8$

Show that this ratio's value approaches |f''(x)/2f'(x)| f'(x) = 3x^2 + 8x **-2**
f'(r) = 16.5134
f''(x)...
f''(r)...
f''(r)/2f'(r) = ...

Q4

8 / 9

Matlab's `fzero` function finds the root of the given function using a combination of bisection, secant, and inverse quadratic interpolation. This allows rapid convergence where possible while avoiding a diverging solution. For this question you will take a similar approach manually, by combining the bisection and Newton methods, to find the root of $f(x) = 5x^3 - 39x^2 + 22x - 60$ on the interval (7, 8). Begin by applying Newton's method, starting from $x_0 = 7$. If the solution leaves the starting interval then perform one iteration of bisection (using the current bracket of the root) and then return to Newton's method. Repeat this procedure for the starting interval (5, 10) and initial guess of $x_0 = 5$.

4) $f(x) = 5x^3 - 39x^2 + 22x - 60$ $[7,8]$ $x_0 = 7$

$f'(x) = 15x^2 - 78x + 22$

$$x_1 = 7 - \frac{5(7)^3 - 39(7)^2 + 22(7) - 60}{15(7)^2 - 78(7) + 22} = 7.48341$$

$$x_2 = 7.48341 - \frac{5(7.48341)^3 - 39(7.48341)^2 + 22(7.48341) - 60}{15(7.48341)^2 - 78(7.48341) + 22}$$

$$x_3 = 7.42596 - \frac{5(7.42596)^3 - 39(7.42596)^2 + 22(7.42596) - 60}{15(7.42596)^2 - 78(7.42596) + 22}$$

$= 7.42507$

$$x_4 = 7.42507 - \frac{5(7.42507)^3 - 39(7.42507)^2 + 22(7.42507) - 60}{15(7.42507)^2 - 78(7.42507) + 22}$$

$= 7.42507$

Converged

$[5, 10]$

$x_0 = 5$

$$x_1 = 5 - \frac{5(5)^3 - 39(5)^2 + 22(5) - 60}{15(5)^2 - 78(5) + 22} = 47.8571$$

out of interval

Newton $\rightarrow$ bisection

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$c = \frac{5+10}{2} = 7.5$

$F(5) = 5(5)^3 - 39(5)^2 + 22(5) - 60 = -300$

$F(5) < 0$

$F(10) = 5(10)^3 - 39(10)^2 + 22(10) - 60 = 1260$

$F(7.5) = 5(7.5)^3 - 39(7.5)^2 + 22(7.5) - 60 = 20.625$

$F(7.5) > 0$

bracket $[5, 7.5]$

if we use $x_0 = 5$ for Newton's method again the result will be 47.85714

therefore, bisection

$c = 6.25$

$F(6.25) = 5(6.25)^3 - 39(6.25)^2 + 22(6.25) - 60$

$= -225.23$

bracket: $[6.25, 7.5]$

Newton's method

$$x_1 = 6.25 - \frac{5(6.25)^3 - 39(6.25)^2 + 22(6.25) - 60}{15(6.25)^2 - 78(6.25) - 22}$$

$= 8.12013$

$8.12013 > 7.5$ out of interval

Bisection

$\frac{6.25 + 7.5}{2} = 6.875$

$F(7.5) = 5(7.5)^3 - 39(7.5)^2 + 22(7.5) + 22(7.5) - 60$

$= 20.625$

$F(6.875) = 5(6.875)^3 - 39(6.875)^2 + 22(6.875) - 60$

$= -127.35351$

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one iteration of bisection is enough. Interval left. -1

[6.875, 7.5]

Newton's

$$x_1 = 6.875 - \frac{5(6.875)^3 - 39(6.875)^2 + 22(6.875) - 60}{15(6.875)^2 - 78(6.875) + 22}$$

$= 7.52898 \Rightarrow$ out of interval

bisection

$c = 7.1875$

$f(7.1875) = 5(7.1875)^3 - 39(7.1875)^2 + 22(7.1875) - 60$

$= -60.08422 < 0$ -ve

[7.1875, 7.5]

Newton's

$$x_1 = 7.1875 - \frac{f(7.1875)}{f'(7.1875)} = 7.44179$$

$$x_2 = 7.44179 - \frac{f(7.44179)}{f'(7.44179)} = 7.42514$$

$$x_3 = 7.42514 - \frac{f(7.42514)}{f'(7.42514)} = 7.42507$$

$$x_4 = 7.42507 - \frac{f(7.42507)}{f'(7.42507)} = 7.42507$$

converged

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Q5

12 / 13

Use the Secant method to find the root of function $f(x) = x^3 + 4x^2 - 10$ given in problem 3. Comment on the convergence of the Secant method for the following starting points (Use tolerance level $\epsilon = 0.00001$, that is, stop when $|x_i - x_{i-1}| < \epsilon$):

(a) $x_0 = 0$, $x_1 = 1$, compare the speed of convergence with Newton's method.

(b) $x_0 = 0$, $x_1 = 3$

(c) $x_0 = -2$, $x_1 = 4$

5) a) $f(x) = x^3+4x^2-10 \qquad x_0 = 0 \qquad x_1 = 1$

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$x_2 = 1 - \frac{((1)^3+4(1)^2-10)(1-0)}{((1)^3+4(1)^2-10)-((0)^3+4(0)^2-10)} = 2$$

$$x_3 = 2 - \frac{((2)^3+4(2)^2-10)(2-1)}{((2)^3+4(2)^2-10)-((1)^3+4(1)^2-10)} = 1.26315$$

$$x_4 = 1.26315 - \frac{((1.26315)^3+4(1.26315)^2-10)(1.26315-2)}{((1.26315)^3+4(1.26315)^2-10)-((2)^3+4(2)^2-10)}$$

$= 1.33882$

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$$x_5 = 1.33882 - \frac{((1.33882)^3 + 4(1.33882)^2 - 10)(1.33882 - 1.26315)}{((1.33882)^3 + 4(1.33882)^2 - 10) - ((1.26315)^3 + 4(1.26315)^2 - 10)}$$

$$= 1.36661$$

$$x_6 = 1.36661 - \frac{f(1.36661)(1.36661 - 1.33882)}{f(1.36661) - f(1.33882)}$$

$$= 1.36521$$

$$x_7 = 1.36521 - \frac{f(1.36521)(1.36521 - 1.36661)}{f(1.36521) - f(1.36661)}$$

$$= 1.36522$$

$$x_8 = 1.36522 - \frac{f(1.36522)(1.36522 - 1.36521)}{f(1.36522) - f(1.36521)}$$

$$= 1.36523$$

$$x_9 = 1.36523 - \frac{f(1.36523)(1.36523 - 1.36522)}{f(1.36523) - f(1.36522)}$$

$$= 1.36523$$

For this question, Newton's method was faster with 3 iteration vs 8 iteration for the secant method

tolerance; x_0 & x_1: $|1 - 0| < \epsilon$ X
x_2 & x_1: $|2-1| < \epsilon$ X
x_3 & x_2: $|1.26315 - 2| < \epsilon$ X
x_4 & x_3: $|1.33882 - 1.26315| < \epsilon$ X
x_5 & x_4: $|1.36661 - 1.33882| < \epsilon$ X
x_6 & x_5: $|1.36521 - 1.36661| < \epsilon$ X

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5

x_7 & x_6: $|1.36522 - 1.36521| < \epsilon$ X
x_8 & x_7: $|1.36523 - 1.36522| < \epsilon$ X
x_9 & x_8: $|1.36523 - 1.36523| < \epsilon$ ✓

b) $x_0 = 0 \quad x_1 = 3$

$$x_2 = 3 - \frac{(f(3))(3-0)}{(f(3)) - f(0)} = 0.47619$$

tolerance: x_0 & x_1: $|3-0| < \epsilon$ X
x_2 & x_1: $|0.47619 - 3| < \epsilon$ X

$$x_3 = 0.47619 - \frac{f(0.47619)(0.47619-3)}{f(0.47619) - f(3)}$$
$$= 0.84202$$

tolerance: $|0.84202 - 0.47619| < \epsilon$ X

$$x_4 = 0.84202 - \frac{f(0.84202)(0.84202 - 0.47619)}{f(0.84202) - f(0.47619)}$$
$$= 1.83558$$

tolerance: $|1.83558 - 0.84202| < \epsilon$ X

$$x_5 = 1.83558 - \frac{f(1.83558)(1.83558 - 0.84202)}{f(1.83558) - f(0.84202)}$$
$$= 1.24405$$

tolerance: $|1.24405 - 1.83558| < \epsilon$ X

$$x_6 = 1.24405 - \frac{f(1.24405)(1.24405 - 1.83558)}{f(1.24405) - f(1.83558)}$$
$$= 1.34057$$

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tolerance: $|1.34057 - 1.24405| < \epsilon$ X

$$x_7 = 1.34057 - \frac{f(1.34057)(1.34057 - 1.24405)}{f(1.34057) - f(1.24405)}$$
$= 1.36677$

tolerance: $|1.36677 - 1.34057| < \epsilon$ X

$$x_8 = 1.36677 - \frac{f(1.36677)(1.36677 - 1.34057)}{f(1.36677) - f(1.34057)}$$
$= 1.36521$

tolerance: $|1.36521 - 1.36677| < \epsilon$ X

$$x_9 = 1.36521 - \frac{f(1.36521)(1.36521 - 1.36677)}{f(1.36521) - f(1.36677)}$$
$= 1.36522$

tolerance: $|1.36522 - 1.36521| < \epsilon$ X

$$x_{10} = 1.36522 - \frac{f(1.36522)(1.36522 - 1.36521)}{f(1.36522) - f(1.36521)}$$
$= 1.36523$

tolerance: $|1.36523 - 1.36522| < \epsilon$ X

$$x_{11} = 1.36523 - \frac{f(1.36523)(1.36523 - 1.36522)}{f(1.36523) - f(1.36522)}$$
$= 1.36523$

tolerance: $|1.36523 - 1.36523| < \epsilon$ ✓

②

c) $x_0 = -2 \quad x_1 = 4$

$$x_2 = 4 - \frac{f(4)\,(4 - -2)}{f(4) - f(-2)} = -1.9$$

tolerance: $x_0 \,\&\, x_1 = |4 - -2| < \epsilon$ X

$x_2 \,\&\, x_1 = |-1.9 - 4| < \epsilon$ X

$$x_3 = -1.9 - \frac{f(-1.9)\,(-1.9 - 4)}{f(-1.9) - f(4)} = -1.78147$$

tolerance $|-1.78147 - -1.9| < \epsilon$ X

$$x_4 = -1.78147 - \frac{f(-1.78147)(-1.78147 - -1.9)}{f(-1.78147) - f(-1.9)}$$

$= -2.43077$

tolerance: $|-2.43077 - -1.78147| < \epsilon$ X

$$x_5 = -2.43077 - \frac{f(-2.43077)(-2.43077 - -1.78147)}{f(-2.43077) - f(-1.78147)}$$

$= -2.64261$

tolerance: $|-2.64261 - -2.43077| < \epsilon$ X

$$x_6 = -2.64261 - \frac{f(-2.64261)(-2.64261 - -2.43077)}{f(-2.64261) - f(-2.43077)}$$

$= -3.17519$

$$x_7 = -3.17519 - \frac{f(-3.17519)(-3.17519 - -2.64261)}{f(-3.17519) - f(-2.64261)}$$

$= -2.40422$

tolerance: $|-2.40422 - -3.17519| < \epsilon$ X

8

$$x_8 = -2.40422 - \frac{f(-2.40422)(-2.40422 - -3.17519)}{f(-2.40422) - f(-3.17519)}$$

$= -1.74569$

tolerance: $|-1.74569 - -2.40422| < \epsilon$ X

$$x_9 = -1.74569 - \frac{f(-1.74569)(-1.74569 - -2.40422)}{f(-1.74569) - f(-2.40422)}$$

$= -2.62127$

tolerance: $|-2.62127 - -1.74569| < \epsilon$ X

$$x_{10} = -2.62127 - \frac{f(-2.62127)(-2.62127 - -1.74569)}{f(-2.62127) - f(-1.74569)}$$

$= -2.79839$

$$x_{11} = -2.79839 - \frac{f(-2.79839)(-2.79839 - -2.62127)}{f(-2.79839) - f(-2.62127)}$$

$= -1.15315$

tol: $|-1.15315 - -2.79839| < \epsilon$ X

$$x_{12} = -1.15315 - \frac{f(-1.15315)(-1.15315 - -2.79839)}{f(-1.15315) - f(-2.79839)}$$

$= -2.97104$

tol: $|-2.97104 - -1.15315| < \epsilon$ X

$$x_{13} = -2.97104 - \frac{f(-2.97104)(-2.97104 - -1.15315)}{f(-2.97104) - f(-1.15315)}$$

$= -3.28584$

tol: $|-3.28584 - -2.97104| < \epsilon$ X

9

$$x_{14} = -3.28584 - \frac{f(-3.28584)(-3.28584 - -2.97104)}{f(-3.28584) - f(-2.97104)}$$

$$= -2.76058$$

$\text{tol}: |-2.76058 - -3.28584| < \epsilon$ X

using the secant program, it starts to show convergence signs around 148^{th} iteration. It converges at a very slow rate

comments on 5a, b

a → takes a long time to converge
b → it takes longer than a because the starting bracket's range is bigger than a's

in c it does not converge