Q1

3 / 6

Determine whether each of the following systems of equations has a solution.

(a)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 - x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

(b)

$$\begin{aligned} 5x_1 - 4x_2 + 2x_3 &= 11 \\ -2x_1 + 3x_2 + x_3 &= 1 \\ 16x_1 - 7x_2 + 4x_3 &= 19 \end{aligned}$$

CS3113 A3

1)a) $x_1 + 2x_2 - x_3 = 5$
$2x_1 - x_2 + 3x_3 = 0$
$x_1 + 3x_2 - 2x_3 = 7$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 1 & 3 & -2 & 7 \end{array}\right] \quad R_2 = R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 1 & 3 & -2 & 7 \end{array}\right] \quad R_3 = R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 1 & -1 & 2 \end{array}\right] \quad R_3 = 5R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system of equations has infinitely many solutions, according to R_3.

$0x_1 + 0x_2 + 0x_3 = 0$

1) b) $\left[\begin{array}{ccc|c} 5 & -4 & 2 & 11 \\ -2 & 3 & 1 & 1 \\ 16 & -7 & 4 & 19 \end{array}\right]$ $R_2 = R_2 + \frac{2}{5} R_1$

$\left[\begin{array}{ccc|c} 5 & -4 & 2 & 11 \\ 0 & 7/5 & 9/5 & 27/5 \\ 16 & -7 & 4 & 19 \end{array}\right]$ $R_3 = R_3 - \frac{16}{5} R_1$

$\left[\begin{array}{ccc|c} 5 & -4 & 2 & 11 \\ 0 & 7/5 & 9/5 & 27/5 \\ 0 & 29/5 & -12/5 & -16.2 \end{array}\right]$ $R_3 = R_3 - \left(\frac{29}{5} \div \frac{7}{5} R_2\right)$

$\left[\begin{array}{ccc|c} 5 & -4 & 2 & 11 \\ 0 & 7/5 & 9/5 & 27/5 \\ 0 & 0 & -69/7 & -270/7 \end{array}\right]$

according to R_3, $0x_1 + 0x_2 - \frac{69}{7} x_3 = \frac{-270}{7}$

$x_3 = 3.91$

$R_2 \rightarrow \frac{7}{5} x_2 + \frac{9}{5}(3.91) = 27/5$

$x_2 = -1.17$

$R_1 \rightarrow 5x_1 - 4(-1.17) + 2(3.91) = 11$

$x_1 = -0.3$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.3 \\ -1.17 \\ 3.91 \end{bmatrix}$

You have to calculate the determinant to see if it has a unique solution.

The first determinant is zero, so the first system does not have a unique solution. The second determinant is nonzero, so the second system will have a unique solution

Q2

16 / 17

Consider the following system $Ax = b$:

$$\begin{aligned} 8x_1 + 8x_2 + 7x_3 &= 23 \\ 8x_1 + 7x_2 + 6x_3 &= 21 \\ 7x_1 + 6x_2 + 5x_3 &= 18 \end{aligned}$$

(a) Solve this system using Gaussian elimination with partial pivoting, and **using floating point arithmetic with 3 decimal digits and rounding**.

(b) Compute the relative residual ($||r||/||b||$) of your solution in 2(a), and comment on its magnitude.

(c) Compute the exact solution x of this system, and compute the relative error of your approximate solution in 2(a).

(d) Compute the condition number of A, and use it to explain the magnitude of the error in 2(c)

2)a) $\left[\begin{array}{ccc|c} 8 & 8 & 7 & 23 \\ 8 & 7 & 6 & 21 \\ 7 & 6 & 5 & 18 \end{array}\right]$ $R_2 = R_2 - R_1$

$R_3 = R_3 - \frac{7}{8} R_1 \rightarrow 0.875$

$0.875 \times 8 = 7$

$0.875 \times 7 = 6.125 \approx 6.13$

$0.875 \times 23 = 20.125 \approx 20.1$

$6 - 7 = -1$

$5 - 6.13 = -1.13$

$18 - 20.1 = -2.1$

$\left[\begin{array}{ccc|c} 8 & 8 & 7 & 23 \\ 0 & -1 & -1 & -2 \\ 0 & -1 & -1.13 & -2.1 \end{array}\right]$ $R_3 = R_3 - R_2$

$\left[\begin{array}{ccc|c} 8 & 8 & 7 & 23 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -0.13 & -0.1 \end{array}\right]$

$-0.13 x_3 = -0.1$ $x_3 = \frac{\cancel{-}0.1}{\cancel{-}0.13} = 0.769$

$-x_2 - 1(0.769) = -2$ $x_2 = 1.23$

$8x_1 + 8(1.23) + 7(0.769) = 23$

$x_1 = \frac{23 - 7(0.769) - 8(1.23)}{8}$

$x_1 = 0.972$

2) b) $r = b - Ax_a$

$$\begin{bmatrix} 23 \\ 21 \\ 18 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 7 \\ 8 & 7 & 6 \\ 7 & 6 & 5 \end{bmatrix} \begin{bmatrix} 0.972 \\ 1.23 \\ 0.769 \end{bmatrix}$$

$$\begin{bmatrix} 23 \\ 21 \\ 18 \end{bmatrix} - \begin{bmatrix} 22.999 \\ 21 \\ 18.029 \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0 \\ -0.29 \end{bmatrix}$$

$\|r\| = 0.291$

$\|b\| = 23 + 21 + 18 = 62$

$\|r\| / \|b\| = \frac{0.291}{62} = 0.004694 \approx 4.7 \times 10^{-3}$

residual is almost 0, therefore x_a is very close to the exact answer x.

---

2) c)
$$\left[\begin{array}{ccc|c} 8 & 8 & 7 & 23 \\ 8 & 7 & 6 & 21 \\ 7 & 6 & 5 & 18 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 - 0.875R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 8 & 8 & 7 & 23 \\ 0 & -1 & -1 & -2 \\ 0 & -1 & -1.125 & -2.125 \end{array}\right] \quad R_3 = R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 8 & 8 & 7 & 23 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -0.125 & -0.125 \end{array}\right]$$

$-0.125x_3 = -0.125$

$x_3 = 1$

$-x_2 - 1) = -2$

$x_2 = 1$

$8x_1 + 8 + 7 = 23$

$8x_1 = 8 \qquad x_1 = 1$

$\|x\| = 1+1+1 = 3$

$\|x_a\| = 0.972 + 1.23 + 0.769 = 2.971$

$$\frac{\|x\| - \|x_a\|}{\|x\|} = \frac{3 - 2.971}{3} = 0.0096666$$

$$\simeq 9.7 \times 10^{-3}$$

2)d) $A = \begin{bmatrix} 8 & 8 & 7 \\ 8 & 7 & 6 \\ 7 & 6 & 5 \end{bmatrix}$ to get A^{-1}:

$$\left[\begin{array}{ccc|ccc} 8 & 8 & 7 & 1 & 0 & 0 \\ 8 & 7 & 6 & 0 & 1 & 0 \\ 7 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_2 = R_2 - R_1$$

$$\left[\begin{array}{ccc|ccc} 8 & 8 & 7 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 7 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_2 = -1 \times R_2$$

$$\left[\begin{array}{ccc|ccc} 8 & 8 & 7 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 7 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_1 = R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 7 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_1 = R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 7 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 = R_3 - 5R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 7 & 1 & 0 & -5 & 5 & 1 \end{array}\right] \quad R_3 = R_3 - 7R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -9 & 8 \end{array}\right] \quad R_2 = R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 8 & -8 \\ 0 & 1 & 0 & 2 & -9 & 8 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 & -9 & 8 \\ 0 & 0 & 1 & -1 & 8 & -8 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -9 & 8 \\ -1 & 8 & -8 \end{bmatrix}$$

$$\text{cond } A = \|A\| \times \|A^{-1}\| = 23 \times 19$$
$$= 437$$
$$= 437.0 = 4.37 \times 10^2$$

According to the rule of thumb, log_10 437 ~= 3 digits of precision are lost, so there are none left. **-1**

Q3

14 / 16

(textbook, exercise 4 from section 2.4) Solve the system by finding the $PA = LU$ factorization and then carrying out the two-step back-substitution

(a)

$$\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 17 \\ 3 \end{bmatrix}$$

3)a) $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 = R_2 - R_1$
$R_3 = R_3 - \frac{1}{2}R_1$ $\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

$R_3 = R_3 - \frac{1}{2}R_2$ $\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$PA = LU$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$Lc = Pb$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

$c_1 = 2$; $1(2) + 1(c_2) = 4$; $0.5(2) + 0.5(2) + c_3 = 6$
$c_2 = 2$; $c_3 = 4$

$c = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$

$Ux = c$

$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ $x_1 = 1$; $x_2 = -1$; $x_3 = 2$

$x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

3)b) $\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 17 \\ 3 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$ $\quad R_2 = R_2 + \frac{1}{2} R_1$ $\quad R_3 = R_3 + \frac{1}{2} R_1$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 1.5 \\ 0 & 2.5 & 0.5 \end{bmatrix}$ $\quad R_2 \longleftrightarrow R_3$

$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0.5 & 1.5 \end{bmatrix}$ $\quad R_3 = R_3 - 0.2 R_2$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 1.4 \end{bmatrix}$

$PA = LU$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0.2 & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 1.4 \end{bmatrix}$

$Lc = Pb$

$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 17 \\ 3 \end{bmatrix}$

$c_1 = 17; \quad -1/2(17) + c_2 = 3$

$c_2 = 11.5$ ✓

Where is the rest? **-2**

Q4

21 / 21

(textbook, exercise 2 from section 2.5) Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of the Jacobi and Gauss-Seidel methods from starting vector $[0, \ldots, 0]$.

(a)

$$\begin{cases} x_1 + 3x_2 & = -1 \\ 5x_1 + 4x_2 & = 6 \end{cases}$$

(b)

$$\begin{cases} x_1 - 8x_2 - 2x_3 & = 1 \\ x_1 + x_2 + 5x_3 & = 4 \\ 3x_1 - x_2 + x_3 & = -2 \end{cases}$$

(c)

$$\begin{cases} x_1 + 4x_2 & = 5 \\ x_2 + 2x_3 & = 2 \\ 4x_1 + 3x_3 & = 0 \end{cases}$$

4)a) $x_1 + 3x_2 = -1$
$5x_1 + 4x_2 = 6$

$$\begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$$

Jacobi

$$x_1^{(k+1)} = 0.2(6-4x_2^{(k)}) \qquad \vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2^{(k+1)} = 1/3(-1-x_1^{(k)})$$

$$x_1^{(1)} = 0.2(6-4(0)) = 1.2$$

$$x_2^{(1)} = 1/3(-1-(0)) = -1/3$$

$$x_1^{(2)} = 0.2(6-4(-1/3)) = 22/15$$

$$x_2^{(2)} = 1/3(-1-(1.2)) = -11/15$$

Guass - Seidell

$$x_1^{(k+1)} = 0.2(6-4x_2^{(k)})$$

$$x_2^{(k+1)} = 1/3(-1-x_1^{(k+1)})$$

$$x_1^{(1)} = 0.2(6-4(0)) = 1.2$$

$$x_2^{(1)} = 1/3(-1-(1.2)) = -11/15$$

$$x_1^{(2)} = 0.2(6-4(-11/15)) = 134/75$$

$x_2^{(2)} = 1/3(-1-(134/75)) = -209/225$

---

b)

$$\begin{aligned} x_1 - 8x_2 - 2x_3 &= 1 \\ x_1 + x_2 + 5x_3 &= 4 \\ 3x_1 - x_2 + x_3 &= -2 \end{aligned}$$

$$\begin{bmatrix} 1 & -8 & -2 \\ 1 & 1 & 5 \\ 3 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix}$$

Jacobi

$$x_1^{(k+1)} = 1/3(-2 + x_2^{(k)} - x_3^{(k)})$$

$$x_2^{(k+1)} = -1/8(1 - x_1^{(k)} + 2x_3^{(k)})$$

$$x_3^{(k+1)} = 1/5(4 - x_1^{(k)} - x_2^{(k)})$$

$$x_1^1 = 1/3(-2+(0)-(0)) = -2/3$$

$$x_2^1 = -1/8(1-(0)+2(0)) = -1/8$$

$$x_3^1 = 1/5(4-(0)-(0)) = 4/5$$

$$x_1^2 = 1/3(-2+(-1/8)-(4/5)) = -39/40$$

$$x_2^2 = -1/8(1-(-2/3)+2(4/5)) = -49/120$$

$$x_3^2 = 1/5(4-(-2/3)-(-1/8)) = 23/24$$

Gauss - Seidel

$x_1^{(k+1)} = 1/3(-2 + x_2^{(k)} - x_3^{(k)})$

$x_2^{(k+1)} = -1/8(1 - x_1^{(k+1)} + 2x_3^{(k)})$

$x_3^{(k+1)} = 1/5(4 - x_1^{(k+1)} - x_2^{(k+1)})$

$x_1^1 = 1/3(-2 + (0) - (0)) = -2/3$

$x_2^1 = -1/8(1 - (-2/3) + 2(0)) = -5/24$

$x_3^1 = 1/5(4 - (-2/3) - (-5/24)) = 39/40$

$x_1^2 = 1/3(-2 + (-5/24) - (39/40)) = -191/180$

$x_2^2 = -1/8(1 - (-191/180) + 2(39/40)) = -361/720$

$x_3^3 = 1/5(4 - (-191/180) - (-361/720)) = 89/80$

4) c)

$$x_1 + 4x_2 = 5$$
$$x_2 + 2x_3 = 2$$
$$4x_1 + 3x_3 = 0$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 2 \\ 4 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Jacobi

$$x_1^{(k+1)} = 1/4\,(0 - 3x_3^{(k)})$$
$$x_2^{(k+1)} = 1/4\,(5 - x_1^{(k)})$$
$$x_3^{(k+1)} = 1/2\,(2 - x_2^{(k)})$$

$$\vec{x}^{(0)} = [0 \quad 0 \quad 0]$$

$$x_1^1 = 1/4\,(0 - 3\,(0)) = 0$$
$$x_2^1 = 1/4\,(5 - (0)) = 5/4$$
$$x_3^1 = 1/2\,(2 - (0)) = = 1$$
$$x_1^2 = 1/4\,(0 - 3(1) = -3/4$$
$$x_2^2 = 1/4\,(5 - (0)) = 5/4$$
$$x_3^2 = 1/2\,(2 - (5/4)) = 3/8$$

Guass - Seidell

$x_1^{(k+1)} = 1/4(0 - 3x_3^{(k)})$

$x_2^{(k+1)} = 1/4(5 - x_1^{(k+1)})$

$x_3^{(k+1)} = 1/2(2 - x_2^{(k+1)})$

$x_1^1 = 1/4(0 - 3(0)) = 0$

$x_2^1 = 1/4(5 - (0)) = 5/4$

$x_3^1 = 1/2(2 - (5/4)) = 3/8$

$x_1^2 = 1/4(0 - 3(3/8)) = -9/32$

$x_2^2 = 1/4(5 - (-9/32)) = 169/128$

$x_3^2 = 1/2(2 - (169/128)) = 87/256$