University of New Brunswick
Faculty of Computer Science
CS2333: Computability and Formal Languages
Homework Assignment 4, ***Due Time, Date*** *5:00 PM, February 18, 2022*

Student Name: ____________________________ Matriculation Number: ________________

Instructor: Rongxing Lu
The marking scheme is shown in the left margin and [100] constitutes full marks.

[**100**] 1. Use the pumping lemma to show that the following languages are not regular.

[20] (a) $A_0 = \{a^n b^n | n \geq 0\}$.

[40] (b) $A_1 = \{0^n 1^n 2^n | n \geq 0\}$.

[40] (c) $A_2 = \{ww | w \in \{a, b\}^*\}$.

Solutions.

1. Use the pumping lemma to show that the following languages are not regular.

   (a) $A_0 = \{a^n b^n | n \geq 0\}$.
   <u>This is a sample solution, you still need to check and write it down in your own assignment.</u>
   <u>You can also follow what I discussed in the class and tutorial to finish the assignment 4.</u>

   Proof by Contradiction. (We only need to find one counter example.)
   Assume that A_0 is regular.
   Let p be the pumping length given by the pumping lemma. Choose s to be the string $a^p b^p$. Because s is a member of A_0 and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^i z$ is in A_0.

   i. The string y consists only of as, or only of bs. In both cases, the string $xyyz$ will not have equal number of as, and bs, leading to a contradiction.
   ii. The string y consists of two kind of symbol. In this case $xyyz$ will have as, bs out of order. Hence it is not a member of A_0, which a contradiction.

   Therefore, A_0 is not regular.
   For clarity, consider $p = 7$ as an example, then

   $$s = a^p b^p = aaaaaaabbbbbbb$$

   According to the pumping lemma, three conditions should be satisfied, i) $xy^i z \in A_0$ for every $i \geq 0$; ii) $|y| > 0$; and iii) $xy \leq p$. Therefore, based on ii) and iii), we first consider cases of y in the splitting of $s = xyz$. Clear, we will have three cases for $|y| = 4$:

   - case 1: The y is in the a part

     $$s = \underbrace{aa}_{x}\underbrace{aaaa}_{y}\underbrace{abbbbbbb}_{z}$$

     Consider $xy^i z$ as $xy^2 z$ when $i = 2$, we have

     $$xy^2 z = \underbrace{aa}_{x}\underbrace{aaaaaaaa}_{y^2}\underbrace{abbbbbbb}_{z}$$

     Clearly, $11 \neq 7$, and thus $xy^2 z \notin A_0$, the condition i) is not satisfied.
   - case 2: The y is in the b part

     $$s = \underbrace{aaaaaaabb}_{x}\underbrace{bbbb}_{y}\underbrace{b}_{z}$$

     Consider $xy^i z$ as $xy^2 z$ when $i = 2$, we have

     $$xy^2 z = \underbrace{aaaaaaabb}_{x}\underbrace{bbbbbbbb}_{y^2}\underbrace{b}_{z}$$

     Clearly, $7 \neq 11$, and thus $xy^2 z \notin A_0$, the condition i) is not satisfied.

- case 3: The y is in the a and b parts

$$s = \underbrace{aaaaa}_{x}\underbrace{aabb}_{y}\underbrace{bbbbb}_{z}$$

Consider xy^iz as xy^2z when $i = 2$, we have

$$xy^2z = \underbrace{aaaaa}_{x}\underbrace{aabbaabb}_{y}\underbrace{bbbbb}_{z}$$

Clearly, as, bs are out of order in xy^2z, and thus $xy^2z \notin A_0$, the condition i) is not satisfied.

(b) $A_1 = \{0^n 1^n 2^n | n \geq 0\}$.

Proof by Contradiction. (We only need to find one counter example.)
Assume that A_1 is regular.
Let p be the pumping length given by the pumping lemma. Choose s to be the string $0^p 1^p 2^p$. Because s is a member of A_1 and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^i z$ is in A_1.

i. The string y consists only of 0s, only of 1s or only of 2s. In these cases, the string $xyyz$ will not have equal number of 0s, 1s and 2s, leading to a contradiction.
ii. The string y consists of more than one kind of symbol. In this case $xyyz$ will have 0s, 1s and 2s out of order. Hence it is not a member of A_1, which a contradiction.

Therefore, A_1 is not regular.
For clarity, consider $p = 3$ as an example, then

$$s = 0^p 1^p 2^p = 000111222$$

According to the pumping lemma, three conditions should be satisfied, i) $xy^i z \in A_1$ for every $i \geq 0$; ii) $|y| > 0$; and iii) $xy \leq p$. Therefore, based on ii) and iii), we first consider cases of y in the splitting of $s = xyz$. Clear, we will have five cases for $|y| = 2$:

- case 1: The y is in the 0 part

$$s = \underbrace{0}_{x}\underbrace{00}_{y}\underbrace{111222}_{z}$$

Consider $xy^i z$ as $xy^2 z$ when $i = 2$, we have

$$xy^2 z = \underbrace{0}_{x}\underbrace{0000}_{y^2}\underbrace{111222}_{z}$$

Clearly, $5 \neq 3 = 3$, and thus $xy^2 z \notin A_1$, the condition i) is not satisfied.

- case 2: The y is in the 1 part

$$s = \underbrace{0001}_{x}\underbrace{11}_{y}\underbrace{222}_{z}$$

Consider $xy^i z$ as $xy^2 z$ when $i = 2$, we have

$$xy^2 z = \underbrace{0001}_{x}\underbrace{1111}_{y^2}\underbrace{222}_{z}$$

Clearly, $3 \neq 5 \neq 3$, and thus $xy^2 z \notin A_1$, the condition i) is not satisfied.

- case 3: The y is in the 2 part

$$s = \underbrace{000111}_{x}\underbrace{22}_{y}\underbrace{2}_{z}$$

Consider xy^iz as xy^2z when $i = 2$, we have

$$xy^2z = \underbrace{000111}_{x}\underbrace{2222}_{y^2}\underbrace{2}_{z}$$

Clearly, $3 = 3 \neq 5$, and thus $xy^2z \notin A_1$, the condition i) is not satisfied.

- case 4: The y is in the 0 and 1 parts

$$s = \underbrace{00}_{x}\underbrace{01}_{y}\underbrace{11222}_{z}$$

Consider xy^iz as xy^2z when $i = 2$, we have

$$xy^2z = \underbrace{00}_{x}\underbrace{0101}_{y^2}\underbrace{11222}_{z}$$

Clearly, 0s, 1s and 2s are out of order in xy^2z, and thus $xy^2z \notin A_1$, the condition i) is not satisfied.

- case 5: The y is in the 1 and 2 parts

$$s = \underbrace{00011}_{x}\underbrace{12}_{y}\underbrace{22}_{z}$$

Consider xy^iz as xy^2z when $i = 2$, we have

$$xy^2z = \underbrace{00011}_{x}\underbrace{1212}_{y^2}\underbrace{22}_{z}$$

Clearly, 0s, 1s and 2s are out of order in xy^2z, and thus $xy^2z \notin A_1$, the condition i) is not satisfied.

(c) $A_2 = \{ww | w \in \{a, b\}^*\}$.

Proof by Contradiction. (We only need to find one counter example.)

Assume that A_2 is regular.

Let p be the pumping length given by the pumping lemma. Choose s to be the string $a^p b a^p b$. Because s is a member of A_2 and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string xy^iz is in A_2.

i. The string y consists of as only. In both cases that y is left or right of the middle b, $xyyz$ would not be in the form of ww. Hence it is not a member of A_2, which is a contradiction.

ii. The string y is $a^p b$. The condition $|xy| \leq p$ would be violated.

iii. The string y contains the center b and $y \neq a^p b$. Therefore, $xyyz$ would not be in the form of ww. Hence it is not a member of A_2, which is a contradiction.

Therefore, A_2 is not regular.

For clarity, consider $p = 3$ as an example, then

$$s = a^p b a^p b = aaabaaab$$

According to the pumping lemma, three conditions should be satisfied, i) $xy^iz \in A_2$ for every $i \geq 0$; ii) $|y| > 0$; and iii) $xy \leq p$. Therefore, based on ii) and iii), we first consider cases of y in the splitting of $s = xyz$. Clear, we will have five cases for $|y| = 2$:

- case 1: The y is in the a part

  $$s = \underbrace{a}_{x}\underbrace{aa}_{y}\underbrace{baaab}_{z} \qquad \text{or} \qquad s = \underbrace{aaab}_{x}\underbrace{aa}_{y}\underbrace{ab}_{z}$$

  Consider xy^iz as xy^2z when $i = 2$, we have

  $$xy^2z = \underbrace{a}_{x}\underbrace{aaaa}_{y^2}\underbrace{baaab}_{z} \qquad \text{or} \qquad xy^2z = \underbrace{aaab}_{x}\underbrace{aaaa}_{y^2}\underbrace{ab}_{z}$$

  Clearly, both cases are not in the form of ww, and thus $xy^2z \notin A_2$, the condition i) is not satisfied.

- case 2: The y contains the middle b

  $$s = \underbrace{aa}_{x}\underbrace{ab}_{y}\underbrace{aaab}_{z} \qquad \text{or} \qquad s = \underbrace{aaa}_{x}\underbrace{ba}_{y}\underbrace{aab}_{z}$$

  Consider xy^iz as xy^2z when $i = 2$, we have

  $$xy^2z = \underbrace{aa}_{x}\underbrace{abab}_{y^2}\underbrace{aaab}_{z} \qquad \text{or} \qquad xy^2z = \underbrace{aaa}_{x}\underbrace{baba}_{y^2}\underbrace{aab}_{z}$$

  Clearly, both cases are not in the form of ww, and thus $xy^2z \notin A_2$, the condition i) is not satisfied.