Q1

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Consider the function $f(x) = \ln x$.

(a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(1, \ln 1)$, $(2, \ln 2)$, and $(3, \ln 3)$.

(b) Use the polynomial obtained in part (a) to estimate both $\ln(1.5)$ and $\ln(2.4)$. What is the absolute error in each approximation?

(c) Establish the theoretical error bound for using the polynomial found in part (a) to approximate $\ln(1.5)$. Compare the theoretical error bound to the error found in part (b)

①

CS 3113 A4

1) a) Lagrange $\overset{x_1\ \ y_1}{(1, \ln 1)}$ $\overset{x_2\ \ y_2}{(2, \ln 2)}$ $\overset{x_3\ \ y_3}{(3, \ln 3)}$

$$P_2(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$L_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2}$$

$$L_2 = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$= \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{(x-1)\quad(x-3)}{-1}$$

$$L_3 = \frac{(x-x_2)(x-x_1)}{(x_3-x_2)(x_3-x_1)}$$

$$= \frac{(x-2)(x-1)}{(3-2)(3-1)} = \frac{(x-2)(x-1)}{2}$$

②

$$P_2(x) = (\ln 1)\left(\frac{(x-2)(x-3)}{2}\right) + (\ln 2)(-(x-1)(x-3))$$

$$+ (\ln 3)\left(\frac{(x-2)(x-1)}{2}\right)$$

b) $P_2(1.5) = \ln(1)\left(\frac{(1.5-2)(1.5-3)}{2}\right) + \ln(2)(-(1.5-1)$

$(1.5-3)) + \ln(3)\left(\frac{(1.5-2)(1.5-1)}{2}\right)$

$= 0.38253$

$P_2(2.4) = \ln(1)\left(\frac{(2.4-2)(2.4-3)}{2}\right) + \ln(2)$

$(-(2.4-1)(2.4-3)) + \ln(3)$

$\left(\frac{(2.4-2)\ (2.4-1)}{2}\right)$

$= 0.88985$

$P_2(1.5)$ absolute error $= |f(1.5) - P_2(1.5)|$

$= |0.40546 - 0.38253|$

$= 0.02293$

③

p(2.4) absolute error $= |f(2.4) - P_2(2.4)|$
$= 0.01438$

c) Theoretical error bound

$$\frac{(x-x_1)(x-x_2)(x-x_1)}{n!} \; f^{(n)}(c)$$

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$ $\qquad f'''(x) = \frac{2}{x^3}$

$f''(x) = \frac{-1}{x^2}$

let c=1: $\frac{(1.5-1)(1.5-2)(1.5-3)}{6} \times \frac{2}{1^3} = 0.125$

let c=3: $\frac{(1.5-1)(1.5-2)(1.5-3)}{6} \times \frac{2}{3^3} = 0.00462$

$1 < c < 3$

$0.00462 <$ p(1.5) error < 0.125

Q2

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(a) Construct the divided difference table for the points $(1, \ln 1)$, $(2, \ln 2)$, $(3, \ln 3)$, and $(4, \ln 4)$

(b) Construct the quadratic Newton polynomial passing through the first three points.

(c) Show that the polynomial in (b) is equivalent to the Lagrange polynomial in 1(a).

(d) Estimate the error in the polynomial in (b) at $x = 1.5$ using the point $(4, \ln 4)$.

2) a)

x	y	First	2nd
1	$\ln 1$		
		$F[1\ 2] = \frac{\ln(2) - \ln(1)}{2-1} = 0.69314$	
2	$\ln 2$		$F[1\ 2\ 3]$ $\frac{0.40546 - 0.69314}{3-1} = -0.14384$
		$F[2\ 3] = \frac{\ln(3) - \ln(2)}{3-2} = 0.40546$	
3	$\ln 3$		$F[2\ 3\ 4]$ $\frac{0.28768 - 0.40546}{4-2} = -0.05889$
		$F[3\ 4] = \frac{\ln(4) - \ln(3)}{4-3} = 0.28768$	
4	$\ln 4$		

3rd

$$F[1\ 2\ 3\ 4] = \frac{-0.05889 - (-0.14384)}{4-1} = 0.02831$$

5

2)b) $P_2(x) = F[x_1] + F[x_1, x_2](x-x_1) + F[x_1, x_2, x_3](x-x_1)(x-x_2)$

$= \ln(1) + 0.69314(x-1) - 0.14384(x-1)(x-2)$

2c) Polynomial from 1a

$(\ln(1))\left(\frac{(x-2)(x-3)}{2}\right) + (\ln(2))(-(x-1)(x-3))$

$+(\ln(3))\left(\frac{(x-2)(x-1)}{2}\right)$

$= 0 + 0.69314(-(x-1)(x-3)) + 1.09861 \left(\frac{(x-2)(x-1)}{2}\right)$

$= -0.69314x^2 + 2.77256x - 2.07942 + 0.549305(x-2)(x-1)$

$= -0.69314x^2 + 2.77256x - 2.07942 + 0.549305x^2 - 1.647915x + 1.09861$

$= -0.143835x^2 + 1.124645x - 0.98081$

2b polynomial: $0 + 0.69314x - 0.69314 - 0.14384x^2 + 0.43152x - 0.28768$

$= -0.14384x^2 + 1.12466x - 0.98082$

As we can see the polynomials of 2b and 1a are approximately equal [would've been equal if I didn't round] $\longrightarrow$ simplified

⑥

2) d) $P_3(x) = P_2(x) + F[1\ 2\ 3\ 4](x-x_1)(x-x_2)(x-x_3)$

$= \ln(1) + 0.69314(x-1) - 0.14384(x-1)(x-2) + 0.02831(x-1)(x-2)(x-3)$

$P_2(1.5) = \ln(1) + 0.69314(1.5-1) - 0.14384(1.5-1)(1.5-2)$

$= 0.38253$

$P_3(1.5) = 0.38253 + 0.02831(1.5-1)(1.5-2)(1.5-3)$

$= 0.39314625$

error $= |P_3(1.5) - P_2(1.5)| = |0.3931425 - 0.38253|$

$= 0.0106125$

Good! 12

Q3

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The following data was taken from a polynomial of degree at most five. Find the polynomial.

x	-2	-1	0	1	2	3
y	39	3	-1	-3	-9	-1

3)

x	y	1st	2nd	3rd	4th	5th
-2	39	$F[-2\ -1] = \frac{3-39}{-1+2} = -36$	$F[-2\ -1\ 0] = \frac{-4+36}{0+2} = 16$	-5	1	0
-1	3	$F[-1\ 0] = \frac{-1-3}{0+1} = -4$	$F[-1\ 0\ 1] = \frac{-2+4}{1+1} = 1$	-1	1	
0	-1	$F[0\ 1] = \frac{-3+1}{1} = -2$	$F[0\ 1\ 2] = -2$	3		
1	-3	$F[1\ 2] = \frac{-9+3}{2-1} = -6$	$F[1\ 2\ 3] = 7$			
2	-9	$F[2\ 3] = 8$				
3	-1					

$P_5(x) = 39 - F[-2\ -1](x+2) - F[-2\ -1\ 0](x+2)(x+1) + F[-2\ -1\ 0\ 1](x+2)(x+1)(x+0) + F[-2\ -1\ 0\ 1\ 2](x+2)(x+1)(x-0)(x-1) + F[-2\ -1\ 0\ 1\ 2\ 3](x+2)(x+1)(x-0)(x-1)(x-2)$

$= 39 - 36(x+2) + 16(x+2)(x+1) - 5(x+2)(x+1)(x-0) + 1(x+2)(x+1)(x-0)(x-1) + 0$

$= x^4 - 3x^3 - 1$

good work! **10**

Q4

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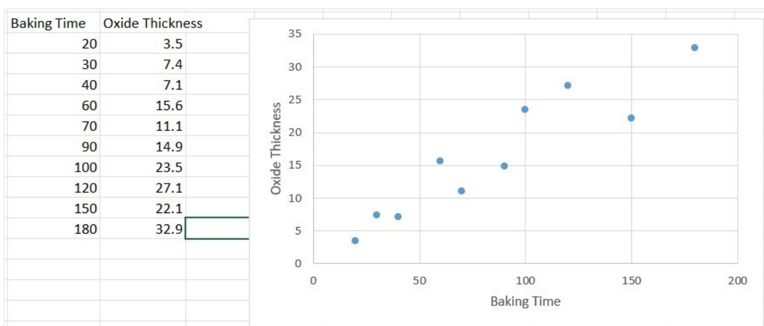
Experimental data relating the oxide thickness, measured in Angstroms, of a thin film to the baking time of the film, in minutes, is given in the table below.

Baking time	20	30	40	60	70	90	100	120	150	18
Oxide thickness	3.5	7.4	7.1	15.6	11.1	14.9	23.5	27.1	22.1	32.

(a) Construct a scatter plot of this data. What functional form is most appropriate for fitting this data? (*Hint*: try linear, exponential, and power laws.)

(b) Fit the data to the function you determined in part (a).

(c) Predict the oxide thickness for a film which is baked for 45 minutes.



Baking Time	Oxide Thickness
20	3.5
30	7.4
40	7.1
60	15.6
70	11.1
90	14.9
100	23.5
120	27.1
150	22.1
180	32.9

4)a) Linear, as the points on the scatterplot look like they follow a linear pattern.

power law is the best one.
-0.5

b) $Ax = b \Longrightarrow A^TA\bar{x} = A^Tb$

$$A^TA = \begin{bmatrix} 20 & 30 & 40 & 60 & 70 & 90 & 100 & 120 & 150 & 180 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 20 & 1 \\ 30 & 1 \\ 40 & 1 \\ 60 & 1 \\ 70 & 1 \\ 90 & 1 \\ 100 & 1 \\ 120 & 1 \\ 150 & 1 \\ 180 & 1 \end{bmatrix}$$

$$A^TAx = \begin{bmatrix} 98800 & 860 \\ 860 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^Tb = A^T \begin{bmatrix} 3.5 \\ 7.4 \\ 7.1 \\ 15.6 \\ 11.1 \\ 14.9 \\ 23.5 \\ 27.1 \\ 22.1 \\ 32.9 \end{bmatrix} = \begin{bmatrix} 18469 \\ 165.2 \end{bmatrix}$$

$$A^TA\bar{x} = A^Tb$$

$$\begin{bmatrix} 98800 & 860 \\ 860 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18469 \\ 165.2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 98800 & 860 & 18469 \\ 860 & 10 & 165.2 \end{array}\right] \quad R_2 = R_2 - \frac{860}{98800} R_1$$

$$\left[\begin{array}{cc|c} 98800 & 860 & 18469 \\ 0 & 2.51417 & 4.4375 \end{array}\right]$$

$$2.51417\, x_2 = 4.4375$$

$$x_2 = 1.76499$$

$$98800\, x_1 + 860(1.76499) = 18469$$

$$x_1 = 0.17156$$

$$f(x) = 0.17156\, x + 1.76499$$

4) c) $f(45) = 0.17156(45) + 1.76499$

$$= 9.48519$$

Q5

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Given the following data:

x	**0.0**	**1.0**	**2.0**	**3.0**	**4.0**	**5.0**	**6.0**	**7.0**	**8.0**
y	12.0	10.5	10.0	8.0	7.0	8.0	7.5	8.5	9.0

Determine the coefficients in a least squares model, $y = ax^2 + bx + c$, for the data.

5) $Ax = b \Rightarrow A^T A \bar{x} = A^T b$

$y = ax^2 + bx + c$

$$A^T A = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \\ 36 & 6 & 1 \\ 49 & 7 & 1 \\ 64 & 8 & 1 \end{bmatrix}$$

$$A^T A \bar{x} = \begin{bmatrix} 8772 & 1296 & 204 \\ 1296 & 204 & 36 \\ 204 & 36 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A^T b = A^T \begin{bmatrix} 12 \\ 10.5 \\ 10 \\ 8 \\ 7 \\ 8 \\ 7.5 \\ 8.5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1697 \\ 299 \\ 80.5 \end{bmatrix}$$

$A^T A \bar{x} = A^T b$

$$\begin{bmatrix} 8772 & 1296 & 204 \\ 1296 & 204 & 36 \\ 204 & 36 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1697 \\ 299 \\ 80.5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 8772 & 1296 & 204 & 1697 \\ 1296 & 204 & 36 & 299 \\ 204 & 36 & 9 & 80.5 \end{array}\right]$$

For easier elimination $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 9 & 36 & 204 & 80.5 \\ 36 & 204 & 1296 & 299 \\ 204 & 1296 & 8772 & 1697 \end{array}\right] \quad \begin{array}{l} R_2 = R_2 - \frac{36}{9} R_1 \\ R_3 = R_3 - \frac{204}{9} R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 9 & 36 & 204 & 80.5 \\ 0 & 60 & 480 & -23 \\ 0 & 480 & 4148 & -127.666 \end{array}\right] \quad R_3 = R_3 - \frac{48\cancel{0}}{6\cancel{0}} R_2$$

$$\left[\begin{array}{ccc|c} 9 & 36 & 204 & 80.5 \\ 0 & 60 & 480 & -23 \\ 0 & 0 & 308 & 56.334 \end{array}\right]$$

$308 \ x_1 = 56.334 \qquad x_1 = 0.1829$

$60x_2 + 480(0.1829) = -23$

$x_2 = -1.84653$

$9x_3 + 36(-1.84653) + 204(0.1829) = 80.5$

$x_3 = 12.18483$

$y = 0.1829x^2 - 1.84653x + 12.18483$

good work! 8