Assignment 5

Due: Wednesday, December 7, 2022 11:59 pm (Atlantic Standard Time)

Thanks for your submission!

Your assignment has been received and is waiting to be graded.

Review your submission

Q1 (16 points)

3 pages submitted

Consider the function: $f(x) = e^x/(x-2)$.

a. Find the exact value of the derivative $f'(0.66)$

b. Estimate $f'(0.66)$ using the central difference formula with $h = 0.1, 0.05, 0.025$, giving the absolute error for each case. Show that the order of the approximation is $O(h^2)$.

c. Estimate $f'(0.66)$ using Richardson extrapolation with $h_1 = 0.1$ and $h_2 = 0.05$. What is the absolute error?

①

CS 3113 A5

1) $f(x) = e^x / (x-2)$

1a) $f'(0.66)$

$e^x/(x-2) = e^x (x-2)^{-1}$

$f'(x) = f'S + fS' = e^x (x-2)^{-1} + e^x (x-2)^{-2} (-1)$

$$= \frac{e^x}{(x-2)} - \frac{e^x}{(x-2)^2}$$

$$f'(0.66) = \frac{e^{(0.66)}}{(0.66-2)} - \frac{e^{(0.66)}}{(0.66-2)^2} = -2.521393441$$

1b) $f'(0.66)$ CDF $h = \underline{0.1}, \underline{0.05}, \underline{0.025}$ abs err $O(h^2)$

$h = 0.1$

$$f'(0.66) = \frac{f(0.66+0.1) - f(0.66-0.1)}{2(0.1)}$$

$$= \left(\frac{e^{0.76}}{-1.24} - \frac{e^{0.56}}{-1.44}\right) \Big/ 2(0.1)$$

$$= -2.543357575$$

abs err $= |(-2.543357575) - (-2.521393441)|$

$= 0.021964134$

(2)

$h = 0.05$

$$f'(0.66) = \frac{f(0.66+0.05) - f(0.66-0.05)}{2(0.05)}$$

$$= \left(\frac{e^{0.71}}{-1.29} - \frac{e^{0.61}}{-1.39}\right) / 2(0.05)$$

$$= -2.526860438$$

$$\text{abs err} = |(-2.526860438) - (2.521393441)$$
$$= 0.005466997$$

$h = 0.025$

$$f'(0.66) = \frac{f(0.66+0.025) - f(0.66-0.025)}{2(0.025)}$$

$$= \left(\frac{e^{0.685}}{-1.315} - \frac{e^{0.635}}{-1.365}\right) / 2(0.025)$$

$$= -2.522758696$$

$$\text{abs err} = |(-2.522758696) - (-2.521393441)|$$
$$= 0.001365255$$

$$h \to 2^0(0.1) = 2^1(0.05) = 2^2(0.025)$$
$$\text{abs err} \to 0.21964134 \simeq 4^1(0.005466997) \simeq 4^2(0.001365255)$$

$$\simeq 0.21964134 \simeq 0.021867988 \simeq 0.02184408$$

$$\therefore O(h^2)$$

3

1)c) $f'(0.66)$ RE $h_1 = 0.1$ $h_2 = 0.05$ abs err

$F = f'(x)$

$$F_2(0.05) = \frac{f(0.66+0.05) - f(0.66-0.05)}{2(0.05)}$$

$$= \left(\frac{e^{0.71}}{(0.71-2)} - \frac{e^{0.61}}{(0.61-2)}\right) / 2(0.05)$$

$$= -2.526860438$$

$$F_2(0.1) = \frac{f(0.66+0.1) - f(0.66-0.1)}{2(0.1)}$$

$$= \left(\frac{e^{0.76}}{(0.76-2)} - \frac{e^{0.56}}{(0.56-2)}\right) / 2(0.1)$$

$$= -2.543357575$$

$$F_4(h) = \frac{2^2(-2.526860438) - (-2.543357575)}{2^2 - 1}$$

$$= -2.521288667$$

$$\text{abs err} = |(-2.521288667) - (-2.521393441)|$$
$$= 0.000104774$$

Q2 (16 points)

3 pages submitted

For the integral $\int_1^2 \frac{1}{x} dx$, verify that:

a. the composite trapezoid rule has a rate of convergence $O(h^2)$ using h = 0.5, 0.25, 0.125

b. the composite Simpson's rule has a rate of convergence $O(h^4)$ using h = 0.5, 0.25, 0.125

4

2) $\int_1^2 \frac{1}{x}\,dx$

2)a) $O(h^2)$ $h = 0.5, 0.25, 0.125$ trapezoid

1 — 1.5 — 2 $h = 0.5$ $m = 2$

$\int_1^2 \frac{1}{x}\,dx = 0.25\left(\frac{1}{1} + \frac{2}{1.5} + \frac{1}{2}\right) = 0.708333333$

1 — 1.25 — 1.5 — 1.75 — 2 $m = 4$ $h = 0.25$

$\int_1^2 \frac{1}{x}\,dx = 0.125\left(\frac{1}{1} + 2\left(\frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75}\right) + \frac{1}{2}\right)$

$= 0.6970238095$

1 — 1.125 — 1.250 — 1.375 — 1.500 — 1.625 — 1.750 — 1.875 — 2 $m = 8$ $h = 0.125$

$\int_1^2 \frac{1}{x}\,dx = 0.0625\left(\frac{1}{1} + 2\left(\frac{1}{1.125} + \frac{1}{1.250} + \frac{1}{1.375} + \frac{1}{1.5} + \frac{1}{1.625} + \frac{1}{1.75} + \frac{1}{1.875}\right) + \frac{1}{2}\right)$

$= 0.6941218504$

integral ($\int$) of $\frac{1}{x} = \ln(x)$

$\ln(2) - \ln(1) = 0.69314718$

$h \rightarrow 0.5 = 2^1(0.25) = 2^2(0.125)$

(5)

abs err of $h = 0.5$: $|0.7083333333 - 0.6931471811|$
$= 0.0151861523$

abs err of $h = 0.25$: $|0.6970238095 - 0.6931471811|$
$= 0.0038766295$

abs err of $h = 0.125$: $|0.6941218505 - 0.6931471811|$
$= 0.0009746695$

abs err $\rightarrow 0.0151861523 \approx 4(0.0038766295) \simeq 16(0.0009746695)$

$\rightarrow 0.0151861523 \simeq 0.015506518 \simeq 0.015594712$

$O(h^2)$ [because of abs err & h]

2)b) Simpson $O(h^4)$ $h = 0.5, 0.25, 0.125$

$h = 0.5$ $1 — 1.5 — 2$ $m = 1$

$$\int_1^2 \frac{1}{x}\,dx = (0.5/3)\left(1 + 4\left(\frac{1}{1.5}\right) + 0.5\right) = \frac{25}{36}$$
$$= 0.6944444444$$

$h = 0.25$ $1 — 1.25 — 1.5 — 1.75 — 2$
$m = 2$

$$\int_1^2 \frac{1}{x}\,dx = (0.25/3)\left(1 + 4\left(\frac{1}{1.25} + \frac{1}{1.75}\right) + 2\left(\frac{1}{1.5}\right) + \frac{1}{2}\right)$$
$$= \frac{1747}{2520} = 0.6932539683$$

⑥

$h = 0.125$

$\quad 4 \qquad 2 \qquad 4 \qquad 2 \qquad 4 \qquad 2$

$1 — 1.125 — 1.25 — 1.375 — 1.5 — 1.625 — 1.75 — 1.875 — 2$

$m = 4$

$$\int_1^2 \frac{1}{x}\,dx$$

$$= (0.125/3)\left(1 + 4\left(\frac{1}{1.125} + \frac{1}{1.375} + \frac{1}{1.625} + \frac{1}{1.875}\right) + 2\left(\frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75}\right) + 0.5\right)$$

$$= 0.6931545307$$

$h \to 0.5 = 2^1(0.25) + 2^2(0.125)$

abs err of $h = 0.5$: $|0.6944444444 - 0.6931471811| = 0.0012972634$

abs err of $h = 0.25$: $|0.6932539683 - 0.6931471811| = 0.0001067873$

abs err of $h = 0.125$: $|0.6931545307 - 0.6931471811| = 0.0000073497$

abs err $\to 0.0012972634 \simeq 16(0.0001067873) \simeq 256(0.0000073497)$

$\to 0.0012972634 \simeq 0.0017085968 \simeq 0.0018815232$

$O(h^4)$

Q3 (12 points)

1 page submitted

Extrapolate to the limit (using Romberg integration), to get the integral of the following data between 0 and 1.0 (*hint*: extrapolate all the way to $R_{3,3}$):

x	**0.0**	**0.25**	**0.50**	**0.75**	**1.00**
$f(x)$	0.3989	0.3867	0.3521	0.3011	0.2420

⑦

3) $R_{1,1}$

$R_{2,1}$ $R_{2,2}$ $R_{3,3}$

$R_{3,1}$ $R_{3,2}$

$R_{1,1}$: $h = \frac{1-0}{1} = 1$; $m=1$

$R_{1,1} = 0.5(0.3989 + 0.2420) = 0.32045$

$R_{2,1}$: $h = \frac{1-0}{2} = \frac{1}{2}$; $m = 2$

$R_{2,1} = \frac{0.5}{2}(0.3989 + 2(0.3521) + 0.2420) = 0.336275$

$R_{3,1}$: $\frac{1-0}{4} = 0.25$; $m = 4$

$R_{3,1} = \frac{0.25}{2}(0.39898 + 2(0.3867 + 0.3521 + 0.3011) + 0.2420)$

$= 0.3400975$

$R_{2,2} = \frac{2^2(R_{2,1}) - (R_{1,1})}{2^2 - 1} = \frac{4(0.336275) - (0.32045)}{3}$

$= 0.34155$

$n=2$ $R_{3,2} = \frac{2^2(R_{3,1}) - (R_{2,1})}{2^2-1} = \frac{4(0.3400975) - 0.336275}{3}$

$= 0.3413716667$

$n = 4$

$R_{3,3} = \frac{2^4(R_{3,2}) - (R_{2,2})}{2^4 - 1} = \frac{(16(0.3413716667) - (0.34155))}{15}$

$= 0.3413597778$

Q4 (12 points)

2 pages submitted

a. Manually perform three steps of Euler's method to solve the initial value problem at $t = 2.75$, 3.0, and 3.25

$$\frac{dy}{dt} = 2y^2(3-t), \qquad y(2.5) = 3$$

with step size $h = 0.25$.

b. Repeat question 4a with the Explicit Trapezoid (Heun's) method.

⑧

4)a) Euler's $t = 2.75,\ 3.0,\ 3.25$ $h = 0.25$

$\frac{dy}{dt} = 2y^2(3-t)$, $y(2.5) = 3 \leftrightarrow y_0 = 3$

$t_0 = 2.5$

$\tilde{y}_1 = y_0 + h\left(\frac{dy}{dt}\right)$; $y_1 = 3 + 0.25\left(2(3)^2(3-2.5)\right)$

$= 21/4 = 5.25$

$t_1 = 2.75$

$\tilde{y}_2 = 5.25 + 0.25(2(5.25)^2(3-2.75) = 1113/128$

$= 8.6953125$

$t_2 = 3$

$\tilde{y}_3 = 8.6953125 + 0.25\left(2(8.6953125)^2(0)\right) =$

8.6953125

$t_3 = 3.25$

4)b) $t_0 = 2.5$ $y_0 = 3$

$a = f(2.5, 3) = 2(3)^2(3-2.5) = 9$

$b = 3 + 0.25(a) = 5.25$

$c = f(2.75, 5.25) = 2(5.25)^2(3-2.75) = 13.78125$

$\tilde{y}_1 = 3 + \frac{0.25}{2}(a + c) = 3 + \frac{0.25}{2}(9 + 13.78125)$

$= 5.84765625$ $t_1 = 2.75$

$a = f(2.75, 5.84765625) = 2(5.84765625)^2 (3-2.75)$

$= 17.09754181$

$b = 5.84765625 + 0.25(a) = 10.1220417$

$c = f(3, 10.1220417) = 2(10.1220417)^2(3-3) = 0$

$\tilde{y}_2 = 5.84765625 + \frac{0.25}{2}(a + c)$

(9)

$\tilde{y}_2 = 5.84765625 + \frac{0.25}{2}(17.09754181)$

$= 7.984848976 \qquad t_2 = 3$

$a = f(3, 7.984848976) = 2(7.984848976)^2(3-3)$
$= 0$

$b = 7.984848976 + 0.25(0) = 7.984848976$

$c = f(3.25, 7.984848976) = 2(7.984848976)^2 (3-3.25)$

$= -31.87890658$

$\tilde{y}_3 = 7.984848976 + \frac{0.25}{2}(a + c)$

$= 7.984848976 + \frac{0.25}{2}(-31.87890658)$

$= 3.999985654 \qquad t_3 = 3.25$

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