Lab works : Subject 1

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Theoretical Study in the case of sine wave

We have the following signal :

$$x(t) = a\cos(2\pi f_0 t + \phi)$$

a) We already know that $\mathcal{F}_{cc}\{\cos(2\pi f_0 t)\} = \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$ and $\mathcal{F}_{cc}\{\sin(2\pi f_0 t)\} = \frac{1}{2j}(\delta(f - f_0) - \delta(f + f_0))$

$x(t) = a\cos(2\pi f_0 t + \phi) = a(\cos(2\pi f_0 t)\cos(\phi) - \sin(2\pi f_0 t)\sin(\phi)) \Rightarrow$

For the sake of notation simplicity, let's denote $\boxed{X(f) = \mathcal{F}_{cc}x(f)}$ and $\boxed{X_s(\lambda) = \mathcal{F}_{dc}x_s(\lambda)}$

$$X(f) = \frac{a}{2}\cos(\phi)(\delta(f - f_0) + \delta(f + f_0)) + \frac{a}{2j}\sin(\phi)(\delta(f - f_0) - \delta(f + f_0)) \Rightarrow \boxed{X(f) = \frac{a}{2}(\delta(f - f_0)e^{j\phi} + \delta(f + f_0)e^{-j\phi})}$$

b) The sampled signal $x_s[n] = x(nT_s) = x(\frac{n}{f_s}) = a\cos(2\pi\frac{f_0}{f_s}n + \phi) = a\cos(2\pi\lambda_0 n + \phi)$ with $\lambda_0 = \frac{f_0}{f_s}$

$$X_s(\lambda) = \sum_{n=-\infty}^{\infty} x_s[n]e^{-j2\pi\lambda n} = \sum_{n=-\infty}^{\infty} a\cos(2\pi\lambda_0 n + \phi)e^{-j2\pi\lambda n}$$

$$X_s(\lambda) = \frac{a}{2}\sum_{n=-\infty}^{\infty} e^{j2\pi\lambda_0 n}e^{-j2\pi\lambda n}e^{j\phi} + e^{-j2\pi\lambda_0 n}e^{-j2\pi\lambda n}e^{-j\phi}$$

$$X_s(\lambda) = e^{j\phi}\frac{a}{2}\sum_{n=-\infty}^{\infty} e^{-j2\pi(\lambda-\lambda_0)n} + \sum_{n=-\infty}^{\infty} e^{-j2\pi(\lambda+\lambda_0)n}$$

$$\boxed{X_s(\lambda) = \frac{a}{2}e^{j\phi}\sum_{k=-\infty}^{\infty}\delta(\lambda - \lambda_0 - k) + \frac{a}{2}e^{-j\phi}\sum_{k=-\infty}^{\infty}\delta(\lambda + \lambda_0 - k)}$$

Using the scaling property of the Dirac delta function : $|a|\delta(at) = \delta(t)$, and since $\lambda = \frac{f}{f_s}$ and $\lambda_0 = \frac{f_0}{f_s}$, we can reach the following :

$$\mathcal{F}_{dc}x_s(\lambda) = \mathcal{F}_{dc}x_s(\frac{f}{f_s}) = \frac{a}{2}e^{j\phi}\sum_{k=-\infty}^{\infty}\delta(\frac{f - f_0 - kf_s}{f_s}) + \frac{a}{2}e^{-j\phi}\sum_{k=-\infty}^{\infty}\delta\frac{f + f_0 - kf_s}{f_s})$$

$$\mathcal{F}_{dc}x_s(\frac{f}{f_s}) = \frac{af_s}{2}e^{j\phi}\sum_{k=-\infty}^{\infty}\delta(f - f_0 - kf_s) + \frac{af_s}{2}e^{-j\phi}\sum_{k=-\infty}^{\infty}\delta(f + f_0 - kf_s)$$

$$\boxed{\frac{1}{f_s}\mathcal{F}_{dc}x_s(\frac{f}{f_s}) = \frac{a}{2}e^{j\phi}\sum_{k=-\infty}^{\infty}\delta(f - f_0 - kf_s) + \frac{af_s}{2}e^{-j\phi}\sum_{k=-\infty}^{\infty}\delta(f + f_0 - kf_s)}$$

c) The Fourier Transform of rect_{Nt} is the Dirichlet kernel $D_{N_t}(\lambda)$ as defined in the course notes.

$$y[n] = x_s[n]\text{rect}_{N_t} \iff Y(\lambda) = X_s(\lambda) * D_{N_t}(\lambda)$$

$$\boxed{Y(\lambda) = \frac{a}{2}e^{j\phi}\sum_{k=-\infty}^{\infty}D_{N_t}(\lambda - \lambda_0 - k) + \frac{a}{2}e^{-j\phi}\sum_{k=-\infty}^{\infty}D_{N_t}(\lambda + \lambda_0 - k)}$$

d) Since $f_s >> 2f_0$, therefore the Shannon condition is satisfied and the periodic spectrums do not overlap, we can perform all our calculations on just one period of the spectrum $Y(\lambda)$ without loss of generalisation. We can chose any portion of the spectrum that is equal to 1 *(since $Y(\lambda)$ is 1-periodic)*. So for reasons of symmetry, I will chose the period centred around 0, that corresponds to $k = 0$ in the previous equation.

From here on out, I will consider that :

$$Y(\lambda) = \frac{a}{2}e^{j\phi}D_{N_t}(\lambda - \lambda_0) + \frac{a}{2}e^{-j\phi}D_{N_t}(\lambda + \lambda_0)$$

To sample $Y(\lambda)$, we need to take its value at frequencies multiple of $\frac{1}{N_f}$, so if we multiply $Y(\lambda)$ with a discrete comb of repeated deltas with period equal to $\frac{1}{N_f}$. In other words, replace λ with $\frac{n}{N_f}$ where$n \in \mathbb{Z}$ in $Y(\lambda)$ and we will manage to get the desired sampled spectrum :

$$Y_{N_f}[n] = Y(\lambda)\mathbf{1}_{\uparrow N_f}[n] = Y(\lambda)\sum_{k=-\infty}^{\infty}\delta[n - \frac{k}{N_f}] = Y(\frac{k}{N_f})$$

$$Y_{N_f}[n] = \frac{a}{2}e^{j\phi}D_{N_t}(\frac{n - n_0}{N_f}) + \frac{a}{2}e^{-j\phi}D_{N_t}(\frac{n + n_0}{N_f})$$

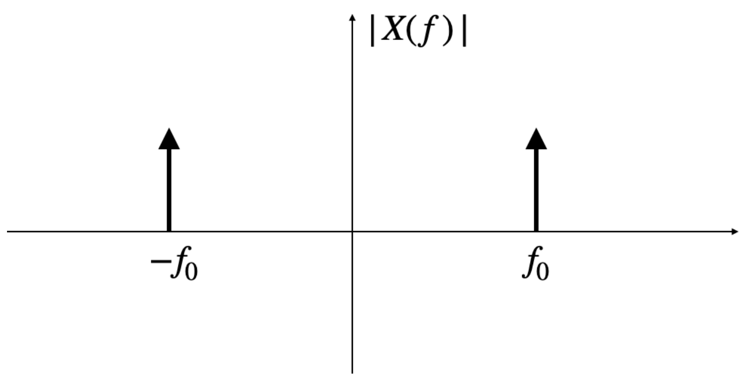

Figure 1: Plot of the Spectrum of $x(t)$.

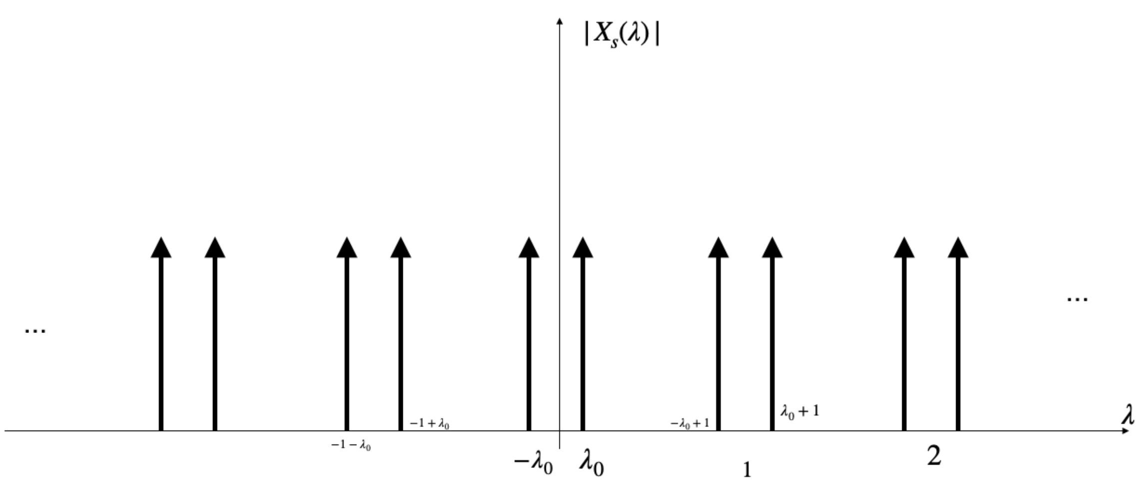

Figure 2: Plot of the Spectrum of $x_s[n]$.

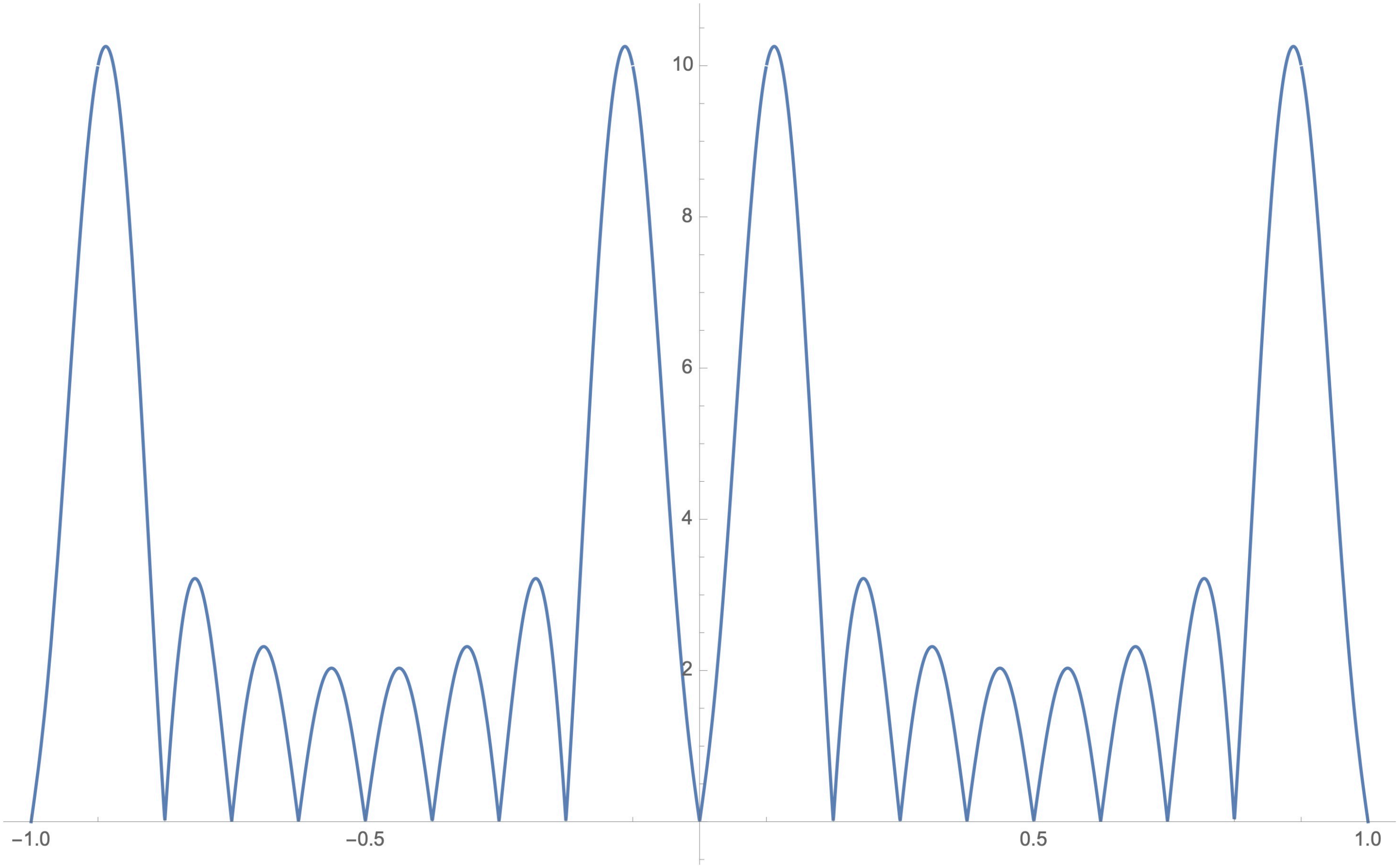

Figure 3: Plot of the Spectrum of $y[n]$ over 2 periods.

Numerical Implementation

a) If $N_f = N_t$, we can recognise the formula for the Discrete Fourier Transform, where the result of this computation is the discrete spectrum of the discrete time signal $y[n]$, and it allows us to obtain the spectral samples directly from the time samples.

b) If $N_f \geq N_t$, then we can zero samples after the $N_t - 1$ sample of the original signal until we can have $N_f = N_t$ and then, we'll be able to calculate the Discrete Fourier Transform and obtain the spectral samples.