Khammassi's Derivation

Problem Formulation: Consider the random vector $\mathbf{g} = [g_1, \ldots, g_N]^T \sim \mathrm{CN}(\mathbf{0}, \mathbf{\Sigma})$. The sought distribution is that of $[|g_1|, \ldots, |g_N|]$, which is correlated Rayleigh. Without loss of generality, we suppose that all the variances are equal: $\forall i \in \{1, \ldots, N\},\ \Sigma_{ii} = \sigma_{ii}^2 = \sigma^2$.

Major Steps:

1. Approximate $\mathbf{g}$ by $\hat{\mathbf{g}} = \hat{\mathbf{g}}(\epsilon)$ which converges in distribution to $\mathbf{g}$ as ϵ tends to zero.
2. Calculate the CDF of $\hat{\mathbf{g}}$; this is the approximate CDF of $\mathbf{g}$.

Approximation of $\mathbf{g}$

1. Express $\mathbf{g}$ in terms of standard complex normal vector: SInce $\mathbf{g} \sim \mathrm{CN}(\mathbf{0}, \mathbf{\Sigma})$, it is equal in distribution to $\mathbf{\Sigma}^{1/2}\mathbf{h}$ where $\mathbf{h} \sim \mathrm{CN}(\mathbf{0}, \mathbf{I}_N)$. Unitarily eigendecompose the /cvariance matrix: $\mathbf{\Sigma} = \mathbf{U}\mathbf{D}\mathbf{U}^T$, $\mathbf{U}\mathbf{U}^T = \mathbf{I}_N$. Hence $\mathbf{g} = \mathbf{U}\mathbf{D}\mathbf{U}^T\mathbf{h}$. Now, since $\mathbf{U}^T$ is unitary, $\mathbf{U}^T\mathbf{h}$ is a standard complex normal randm vector. Let its k-th component be $a_k + jb_k$, where a_k and b_k are i.i.d. $\mathrm{N}(0, \frac{1}{2})$. Let $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_N]$, where each eigenvector is written as $\mathbf{u_k} = [u_{1,k}, \ldots, u_{N,k}]^T$. Let $\mathbf{D} = \mathrm{diag}(s_1, \ldots, s_N)$. Assume $s_1 \geq s_2 \geq \ldots \geq s_N$. Then the k-th component of $\mathbf{g}$ can be written as

$$g_k = \sum_{\ell=1}^{N} u_{k,\ell}\sqrt{s_\ell}(a_\ell + jb_\ell)$$

2. Express the variance in terms of eigenvalues and eigenvectors: Reconsider the equality $\mathbf{\Sigma} = \mathbf{U}\mathbf{D}\mathbf{U}^T$, and equate the (k, k) entries of both sides:

$$\sigma^2 = \sum_{\ell=1}^{N}\sum_{m=1}^{N} u_{k,\ell}d_{\ell,m}u_{k,\ell} = \sum_{\ell=1}^{N} s_\ell u_{k,\ell}^2$$

3. Define the approximative vector $\hat{\mathbf{g}}$: The k-th componnet is given by:

$$\hat{g}_k = \sqrt{\sigma^2 - \sum_{\ell=1}^{\epsilon-\mathrm{rank}} s_\ell u_{k,\ell}^2}(x_k + jy_k) + \sum_{\ell=1}^{\epsilon-\mathrm{rank}} u_{k,\ell}\sqrt{s_\ell}(a_\ell + jb_\ell)$$

   where $\epsilon - \mathrm{rank} = \#\{s_i : s_i > \epsilon\} = \max_{s_i > \epsilon} i$, and x_k, y_k, a_k, b_k are i.i.d. $\mathrm{N}(0, 1)$. Actuallly, this is a lower rank approximation, in which we used $\epsilon - \mathrm{rank} + 1$ complex variabes intead of N ones.
4. Prove that $\hat{\mathbf{g}}$ converges in distribution to $\mathbf{g}$: Firstly, it's clear that $\epsilon - \mathrm{rank} = N$ for $\epsilon < s_N$. So, using 2. and 1., $\hat{g}_k$ converges to g_k in distribution as ϵ tends to zero.

Calculation of the CDF

1. Conition over $\mathbf{a}, \mathbf{b}$ and compute the CDF: Fixing $\mathbf{a}, \mathbf{b}$, the components of $\hat{g}$ are independent, since (x_k, y_k) are independent. So each component is a non-central complex normal variable, whose modulus is thus a Rician variable. In particular, the mean of $\hat{g}_k$ is

$\sum_{\ell=1}^{\epsilon-\text{rank}} u_{k,\ell}\sqrt{s_\ell}(a_\ell + jb_\ell)$ and its variance is $\sigma^2 - \sum_{\ell=1}^{\epsilon-\text{rank}} s_\ell u_{k,\ell}^2$. Recall that the CDF of a Rician variable, which is the modulus of $\mathrm{CN}(\mu, \sigma^2)$ is given by $F(x) = 1 - Q(\frac{\mu}{\sigma/\sqrt{2}}, \frac{x}{\sigma/\sqrt{2}})$. Hence the CDF of $|\hat{g}_k|$, conditioned on $\mathbf{a}, \mathbf{b}$, is given by

$$F_{|\hat{g}_k|}(r_k) = 1 - Q\left(\frac{\sqrt{2}\sum_{\ell=1}^{\epsilon-\text{rank}} u_{k,\ell}\sqrt{s_\ell}(a_\ell + jb_\ell)}{\sqrt{\sigma^2 - \sum_{\ell=1}^{\epsilon-\text{rank}} s_\ell u_{k,\ell}^2}}, \frac{\sqrt{2}r_k}{\sqrt{\sigma^2 - \sum_{\ell=1}^{\epsilon-\text{rank}} s_\ell u_{k,\ell}^2}}\right)$$

2. <u>Integrate ove the distribution of $\mathbf{a}, \mathbf{b}$:</u> Use the formula

$$F_X(x) = \int F_{X|Y}(x) f_Y(y) dy$$

Now $(a_k + jb_k)$ are i.i.d. $\mathrm{CN}(0,1)$, so their joint pdf is given by

$$f(a_1, \ldots, a_{\epsilon-\text{rank}}, b_1, \ldots, b_{\epsilon-\text{rank}}) = \frac{1}{\pi^{\epsilon-\text{rank}}} \exp\left(\sum_{\ell=1}^{\epsilon-\text{rank}} (a_\ell^2 + b_\ell^2)\right)$$

Integrating over $\mathbf{a}, \mathbf{b}$, we get

$$F_{|\hat{g}_1|,|\hat{g}_2|,\ldots,|\hat{g}_N|}(r_1, r_2, \ldots, r_N)$$
$$= \int \cdots \int \prod_{l=1}^{\epsilon-\text{ rank}} \frac{1}{\pi} \exp\left(-(a_l^2 + b_l^2)\right)$$
$$\prod_{k=1}^{N}\left(1 - Q_1\left(\frac{\sqrt{2\left(\sum_{l=1}^{\epsilon-\text{ rank}} \sqrt{s_l} u_{k,l} a_l\right)^2 + 2\left(\sum_{l=1}^{\epsilon-\text{ rank}} \sqrt{s_l} u_{k,l} b_l\right)^2}}{\sqrt{\sigma^2 - \sum_{l=1}^{\epsilon-\text{ rank}} s_l u_{k,l}^2}}, \frac{\sqrt{2}r_k}{\sqrt{\sigma^2 - \sum_{l=1}^{\epsilon-\text{ rank}} s_l u_{k,l}^2}}\right)\right)$$
$$da_1 \ldots da_{\epsilon-\text{ rank}}\, db_1 \ldots db_{\epsilon-\text{ rank}}\,.$$