Ramirez-Espinoza Real

Quadratic Form

This method works for positive definite forms, i.e., linear ccombinations f independent non-central chhi-squares wity positive coefficients:

$$Y \sim \sum_{i=1}^{N} \lambda_i \chi_1^2(h_i^2)$$

Sequence of Gamma Variates

The sequence of randm variables ξ_m, $m \sin \mathrm{N}$ is given by their pdfs

$$f_{\xi_m}(u) = \frac{(m-1)^m}{\Gamma(m)} u^{m-1} e^{-(m-1)u} \quad u > 0$$

which corresponds to a $\Gamma(\alpha, \beta)$, for shape parameter $\alpha = m$ and rate parameter $(m-1)$.

Claim: The sequence ξ_m converges in diistribution to 1. **Proof:** The MGF is given by

$$M_{\xi_m}(s) = (1 - \frac{s}{m-1})^{-m}, \quad \Re(s) < m-1$$

which converges to e^s as m tends to infinity. The latter MGF is that of the constant random variable 1, hence, due to the uniqueness of the MGF (with tech conditions), the sequence ξ_m converges in MGF, hence in distribution, to 1.

Approximative Quadratic Form

The sequence of approximative random variables is given by:

$$Y_m = \frac{Y}{\xi_m}$$

CDF of the Approximative Random Variable

1. The CDF of the gamma variate ξ_m can be written as

$$F_{\xi_m}(u) = 1 - \sum_{k=1}^{m-1} \frac{(m-1)^k}{k!} u^k e^{-(m-1)u}$$

2. Ratio Formula:

$$F_{Y_m}(r) = \mathrm{P}\,(\frac{Y}{\xi_m} \leq r)$$

$$= \int \int_{\frac{y}{u} \leq r} f_Y(y) f_{\xi_m}(u) dy du$$

$$= \int_0^{\infty} f_{\xi_m}(u) \int_0^{ru} f_Y(y) dy du$$

$$= \int_0^{\infty} f_{\xi_m}(u) F_Y(ru) du$$

Note that we have utilised the fact that both RVs are positive. Changing variables, $t = ru$, we get:

$$F_{Y_m}(r) = \int_0^{\infty} F_Y(t) \frac{1}{y} f_{\xi_m}\,(\frac{t}{x})\, dt$$

Integrating by parts, for $U = F_Y(t)$ and $dV = \frac{1}{y} f_{\xi_m}\,(\frac{t}{x})\, dt$, we get

$$F_{Y_m}(y) = 1 - \int_0^{\infty} f_Y(t) F_{\xi_m}\,(\frac{t}{y})\, dt$$

3. Substituting 1. in 2. we get:

$$F_{Y_m}(y) = \sum_{k=0}^{m-1} \frac{(m-1)^k}{k! y^k} \mathrm{E}\,[Y^k e^{-(m-1)Y/y}]$$

4. MGF: Knowing that $\frac{d^k}{ds^k} \mathrm{E}[e^{sY}] = \mathrm{E}[Y^k e^{sY}]$, and subsituting at $s = -\frac{m-1}{y}$, we get:

$$F_{Y_m}(y) = \sum_{k=0}^{m-1} \frac{(m-1)^k}{k! y^k} M_Y^{(k)}\,(-\frac{m-1}{y})$$