Derivation of the optimization gradients

Technical report of:
"Full-pose Trajectory Tracking of Overactuated Multi-Rotor Aerial Vehicles with Limited Actuation Abilities"
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Abstract

This document is a technical attachment to [1] containing the derivation of the gradients of the optimization problem presented in the paper.

.1. Gradient of the multiobjective optimization

First of all, let us rewrite the control input as a fixed part that does not depend on the optimization variables, and a variable part as follows:

$$\boldsymbol{u}_c = \boldsymbol{A}_1\boldsymbol{v} + \boldsymbol{A}_2\sqrt{1-\|\boldsymbol{C}\boldsymbol{v}\|^2} + \boldsymbol{B}\boldsymbol{x} + \hat{\boldsymbol{u}} \tag{1}$$

where $\hat{\boldsymbol{u}}$ is the fixed part of the control input. The derivative of $\boldsymbol{u}_c$ w.r.t. the optimization variables can be written as follows:

$$\frac{\delta \boldsymbol{u}_c}{\delta \boldsymbol{v}} = \boldsymbol{A}_1 - \boldsymbol{A}_2\frac{\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{v}}{\sqrt{1-\|\boldsymbol{C}\boldsymbol{v}\|^2}} \tag{2}$$

$$\frac{\delta \boldsymbol{u}_c}{\delta \boldsymbol{x}} = \boldsymbol{B} \tag{3}$$

Objective function The objective function has three parts, where the first two parts are function of one of the variables, while the third depends on the two variables.

$$\alpha/2\frac{\delta(\|\boldsymbol{v}_r-\boldsymbol{v}\|^2)}{\delta \boldsymbol{v}} = \alpha(\boldsymbol{v}_r-\boldsymbol{v}) \tag{4}$$

$$\beta/2\frac{\delta(\|\boldsymbol{x}\|_{\boldsymbol{Q}})}{\delta \boldsymbol{x}} = \beta\boldsymbol{Q}\boldsymbol{x} \tag{5}$$

$$\gamma/2\frac{\delta(\|\boldsymbol{u}_c-\boldsymbol{u}_c(t-1)\|^2)}{\delta \boldsymbol{v}} = \gamma(\boldsymbol{u}_c-\boldsymbol{u}_c(t-1))\frac{\delta \boldsymbol{u}_c}{\delta \boldsymbol{v}} \tag{6}$$

$$\gamma/2\frac{\delta(\|\boldsymbol{u}_c-\boldsymbol{u}_c(t-1)\|^2)}{\delta \boldsymbol{x}} = \gamma(\boldsymbol{u}_c-\boldsymbol{u}_c(t-1))\frac{\delta \boldsymbol{u}_c}{\delta \boldsymbol{x}} \tag{7}$$

Cost function As discussed [1], in this paper we assume that actuation limits can be written as a combination of linear and quadratic inequality constraints. In what follows we will provide gradients for such constraints, while assuming that constraints for each propeller are independent from the constraints for other propellers. As such, it is convenient to write the corresponding propeller control as follows:

$$\boldsymbol{u}_i = \boldsymbol{A}_{1,i}\boldsymbol{v} + \boldsymbol{A}_{2,i}\sqrt{1-\|\boldsymbol{C}\boldsymbol{v}\|^2} + \boldsymbol{B}_i\boldsymbol{x} + \hat{\boldsymbol{u}}_i \tag{8}$$

where $\boldsymbol{A}_{1,i}$, $\boldsymbol{A}_{2,i}$ and $\boldsymbol{B}_i$ are the rows of the corresponding matrices contributing to $\boldsymbol{u}_i$.
A very common quadratic constraint involves the maximum thrust generated by each propeller, which is directly related to the norm of the propeller's controls.

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The squared norm of the propeller thrust can be written as follows:

$$\begin{aligned}\boldsymbol{u}_i^T \boldsymbol{u}_i = \boldsymbol{v}^T \boldsymbol{A}_{1,i}^T \boldsymbol{A}_{1,i} \boldsymbol{v} + \boldsymbol{A}_{2,i}^T \boldsymbol{A}_{2,i} (1 - \|\boldsymbol{C}\boldsymbol{v}\|^2) + \\ 2\sqrt{1 - \|\boldsymbol{C}\boldsymbol{v}\|^2} \boldsymbol{A}_{2,i}^T \boldsymbol{A}_{1,i} \boldsymbol{v} + 2\boldsymbol{v}^T \boldsymbol{A}_{1,i}^T \hat{\boldsymbol{u}}_i + \\ 2\sqrt{1 - \|\boldsymbol{C}\boldsymbol{v}\|^2} \boldsymbol{A}_{2,i}^T \hat{\boldsymbol{u}}_i + \boldsymbol{x}^T \boldsymbol{B}_i^T \boldsymbol{B}_i \boldsymbol{x} + \|\hat{\boldsymbol{u}}_i\|^2\end{aligned} \quad (9)$$

Note that propeller commands allocated from the allocation matrix ($\boldsymbol{A}_{1,i}\boldsymbol{v}$, $\boldsymbol{A}_{2,i}\sqrt{1 - \|\boldsymbol{C}\boldsymbol{v}\|^2}$ and $\hat{\boldsymbol{u}}_i$) are orthogonal to the propeller commands from the null space of the allocation matrix. As such, all cross products between the first and second group have been omitted from Eq. (9).

In this paper, we will focus on two types of inequality constraints as follows:

$$\boldsymbol{J}_1 : \boldsymbol{D}\boldsymbol{u}_i \leq \boldsymbol{b} \quad (10)$$

$$\boldsymbol{J}_2 : \boldsymbol{u}_i^T \boldsymbol{u} \leq \|u_{i,max}\|^2 \quad (11)$$

The derivatives of the two constraints can be written as follows:

$$\frac{\delta \boldsymbol{J}_1}{\delta \boldsymbol{v}} = \boldsymbol{D}\frac{\delta \boldsymbol{u}_i}{\delta \boldsymbol{v}} \quad (12)$$

$$\frac{\delta \boldsymbol{J}_1}{\delta \boldsymbol{x}} = \boldsymbol{D}\frac{\delta \boldsymbol{u}_i}{\delta \boldsymbol{x}} \quad (13)$$

$$\begin{aligned}\frac{\delta \boldsymbol{J}_1}{\delta \boldsymbol{v}} =& 2\boldsymbol{A}_{1,i}^T \boldsymbol{A}_{1,i} \boldsymbol{v} - 2\boldsymbol{A}_{1,i}^T \boldsymbol{A}_{1,i} \boldsymbol{C}^T \boldsymbol{C} \boldsymbol{v} + \\ & 2\frac{\boldsymbol{v}^T \boldsymbol{C}^T \boldsymbol{C}}{\sqrt{1 - \|\boldsymbol{C}\boldsymbol{v}\|^2}} \boldsymbol{A}_{2,i}^T \boldsymbol{A}_{1,i} \boldsymbol{v} - 2\sqrt{1 - \|\boldsymbol{C}\boldsymbol{v}\|^2} \boldsymbol{A}_{1,i}^T \boldsymbol{A}_{2,i} \\ & 2\boldsymbol{A}_{1,i}^T \boldsymbol{u}_i^T - +2\frac{\boldsymbol{v}^T \boldsymbol{C}^T \boldsymbol{C}}{\sqrt{1 - \|\boldsymbol{C}\boldsymbol{v}\|^2}} \boldsymbol{A}_{2,i}^T \boldsymbol{u}_i\end{aligned} \quad (14)$$

$$\frac{\delta \boldsymbol{J}_1}{\delta \boldsymbol{x}} = 2\boldsymbol{Q}_i \boldsymbol{x} \quad (15)$$

References

[1] M. Hamandi, I. Al-Ali, L. Seneviratne, A. Franchi, and Y. Zweiri, “Full-pose trajectory tracking of overactuated multi-rotor aerial vehicles with limited actuation abilities,” *In revision to IEEE Robotics and Automation Letters*, 2023.