Digital Signal Processing

Lab 05: Analyzing Continuous-Time Systems

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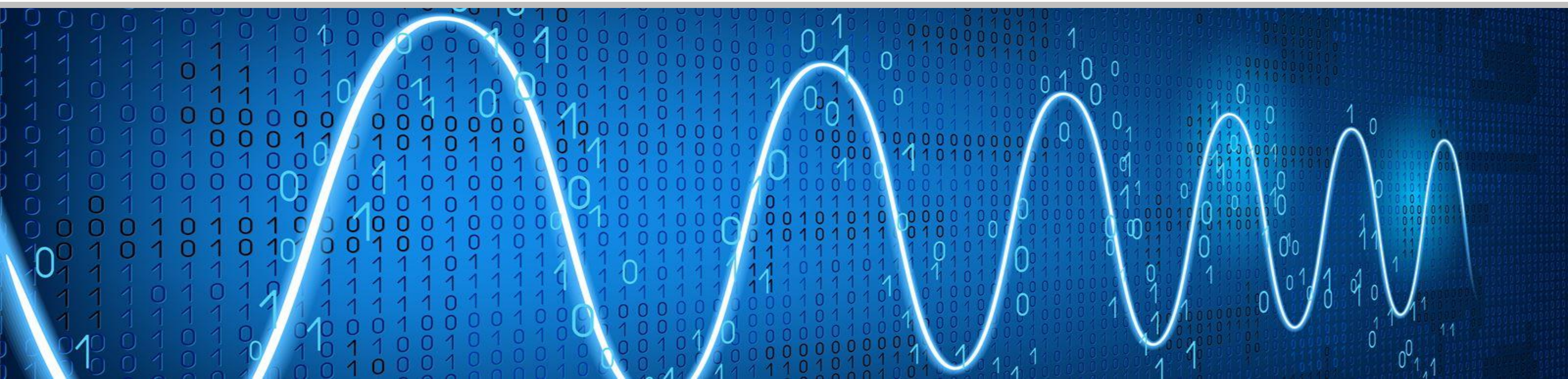


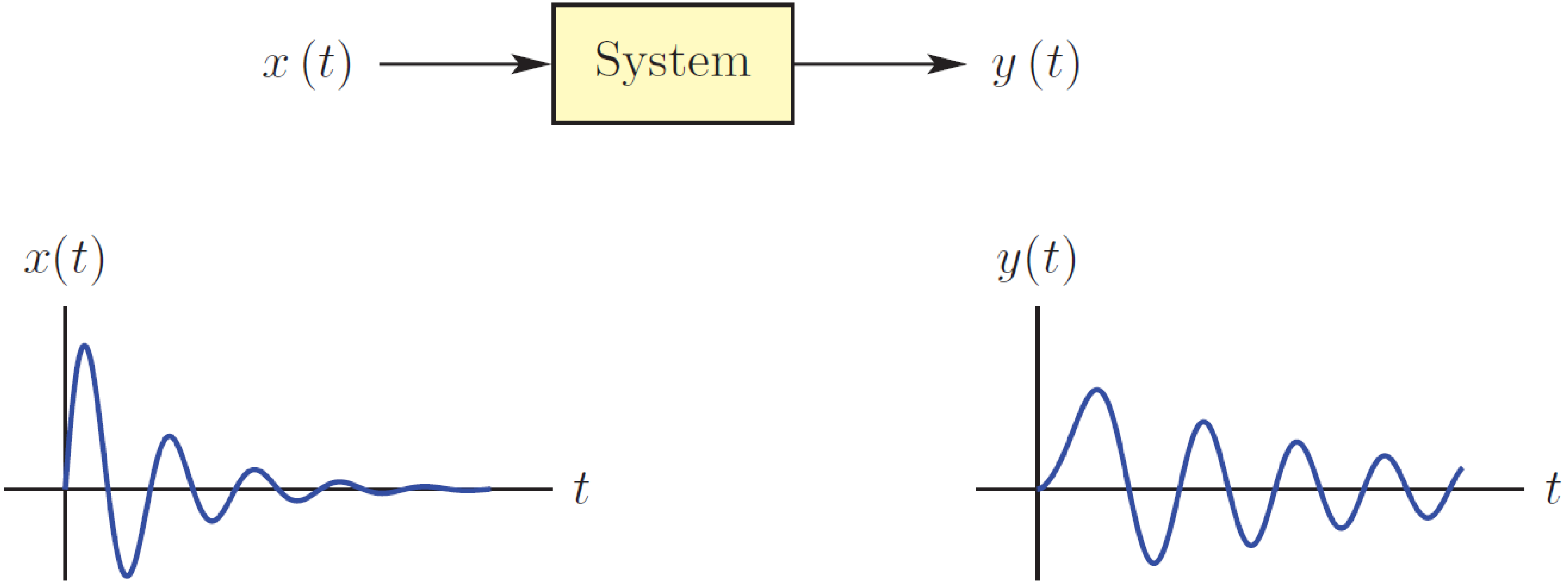
Analyzing Continuous-Time Systems in the Time Domain

The purpose of this lab is to

- Develop the notion of a continuous-time system.
- Discuss the concepts of linearity and time invariance.
- Learn how to compute the output signal for a linear and time-invariant system using convolution.
- Understand the graphical interpretation of the steps involved in carrying out the convolution operation.
- Learn the concepts of causality and stability.

System

- A system is any physical entity that takes in a set of one or more physical signals and, in response, produces a new set of one or more physical signals.



System

- The mathematical model of a system is a function, formula or algorithm to approximately recreate the same cause-effect relationship between the mathematical models of the input and the output signals.
- The relationship between the input and the output signals of a continuous-time system will be mathematically modeled as

$$y(t) = Sys\{x(t)\}$$

- The operator $Sys\ \{\ldots\}$ represents the transformation applied to $x(t)$.

System

- A system that amplifies its input signal by a constant gain factor K to yield an output signal

$$y(t) = Kx(t)$$

- A system that delays its input signal by a constant time delay τ to produce

$$y(t) = x(t - \tau)$$

- A system that produces an output signal that is proportional to the square of the input signal as in

$$y(t) = K[x(t)]^2$$

Linearity

- A system is said to be linear if the mathematical transformation

$$y(t) = Sys\{x(t)\}$$

satisfies the following two equations for any two input signals $x_1(t)$, $x_2(t)$ and any arbitrary constant gain factor α_1.

$$Sys\{x_1(t) + x_2(t)\} = Sys\{x_1(t)\} + Sys\{x_2(t)\}$$

$$Sys\{\alpha_1\, x_1(t)\} = \alpha_1\, Sys\{x_1(t)\}$$

Linearity

- The additivity rule can be stated as follows:

  The response of a linear system to the sum of two signals is the same as the sum of individual responses to each of the two input signals.

$$Sys\{x_1(t) + x_2(t)\} = Sys\{x_1(t)\} + Sys\{x_2(t)\}$$

- The homogeneity rule can be stated as follows:

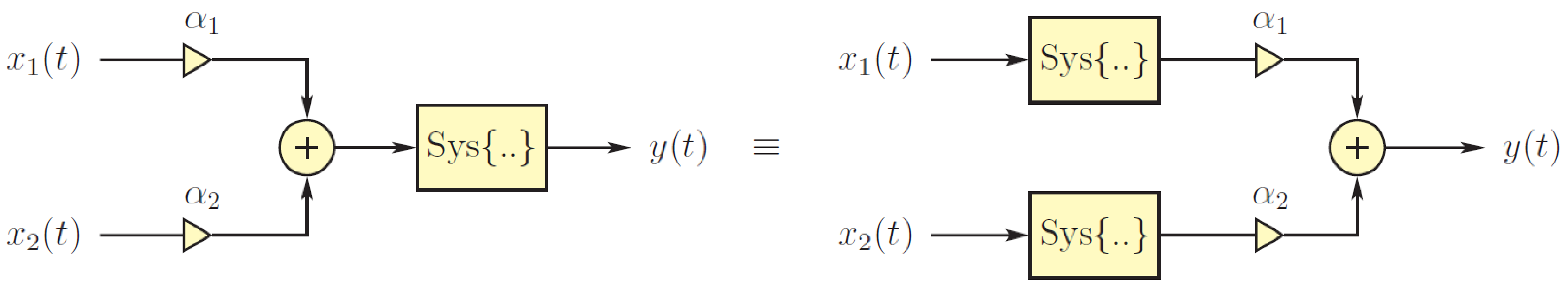
  Scaling the input signal of a linear system by a constant gain factor causes the output signal to be scaled with the same gain factor.

$$Sys\{\alpha_1\, x_1(t)\} = \alpha_1\, Sys\{x_1(t)\}$$

Linearity: Superposition Principle

- The two criteria can be combined into one equation which is referred to as the superposition principle.

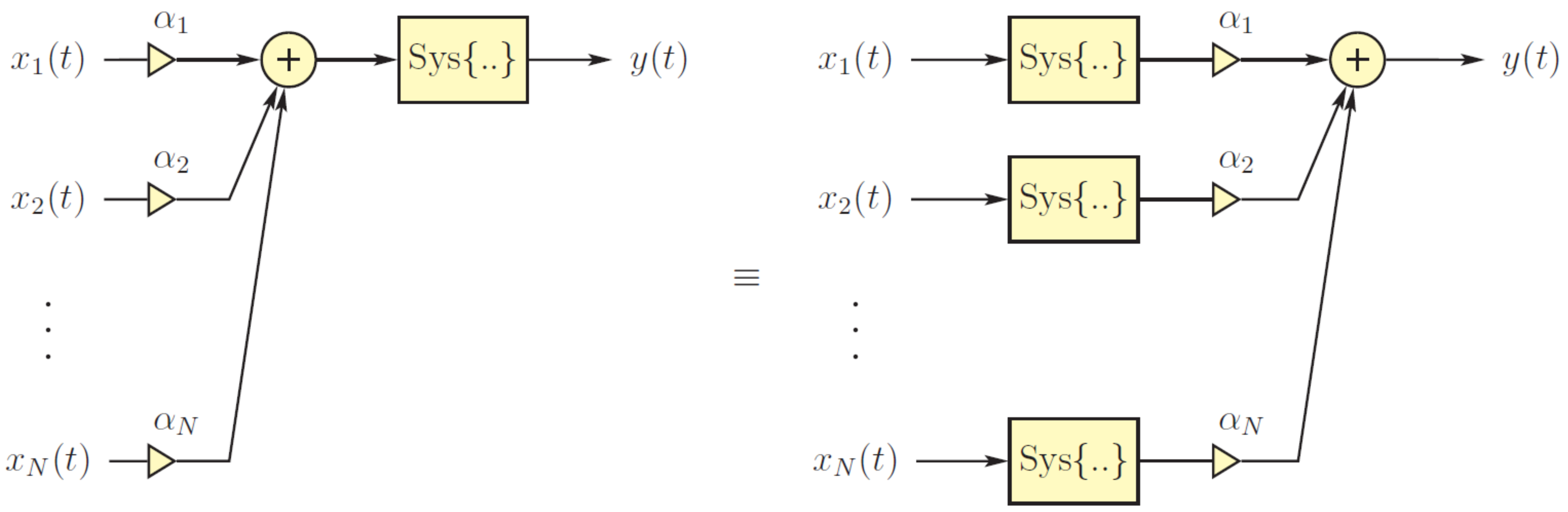
$$Sys\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\} = \alpha_1\, Sys\{x_1(t)\} + \alpha_2\, Sys\{x_2(t)\}$$



Linearity: Superposition Principle

- The generalized form of the superposition principle can be expressed verbally as follows:

$$y(t) = \mathrm{Sys}\left\{ \sum_{i=1}^{N} \alpha_i\, x_i(t) \right\} = \sum_{i=1}^{N} \alpha_i\, y_i(t)$$



Example 2.1

Four different systems are described below through their input-output relationships.

For each, determine if the system is linear or not:

a. $y(t) = 5\,x(t)$

b. $y(t) = 5\,x(t) + 3$

c. $y(t) = 3\,[x(t)]^2$

d. $y(t) = \cos(x(t))$

Example 2.1 (a) – Solution

a. If two input signals $x_1(t)$ and $x_2(t)$ are applied to the system individually, they produce the output signals $y_1(t) = 5\,x_1(t)$ and $y_2(t) = 5\,x_2(t)$ respectively. Let the input signal be $x(t) = \alpha_1\,x_1(t) + \alpha_2\,x_2(t)$. The corresponding output signal is found using the system definition:

$$
\begin{aligned}
y(t) =& 5\,x(t) \\
=& 5\,\left[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)\right] \\
=& \alpha_1\,\left[5\,x_1(t)\right] + \alpha_2\,\left[5\,x_2(t)\right] \\
=& \alpha_1\,y_1(t) + \alpha_2\,y_2(t)
\end{aligned}
$$

Superposition principle holds; therefore this system is linear.

Example 2.1 (b) – Solution

b. If two input signals $x_1(t)$ and $x_2(t)$ are applied to the system individually, they produce the output signals $y_1(t) = 5\,x_1(t) + 3$ and $y_2(t) = 5\,x_2(t) + 3$ respectively.

We will again use the combined input signal $x(t) = \alpha_1\,x_1(t) + \alpha_2\,x_2(t)$ for testing. The corresponding output signal for this system is

$$\begin{aligned} y(t) &= 5\,x(t) + 3 \\ &= 5\,\alpha_1\,x_1(t) + 5\,\alpha_2\,x_2(t) + 3 \end{aligned}$$

The output signal $y(t)$ cannot be expressed in the form $y(t) = \alpha_1\,y_1(t) + \alpha_2\,y_2(t)$.

Superposition principle does not hold true in this case.

The system is not linear.

Example 2.1 (c) – Solution

c. Using two input signals $x_1(t)$ and $x_2(t)$ individually, the corresponding output signals produced by this system are $y_1(t) = 3\,[x_1(t)]^2$ and $y_2(t) = 3\,[x_2(t)]^2$ respectively. Applying the linear combination $x(t) = \alpha_1\,x_1(t) + \alpha_2\,x_2(t)$ to the system produces the output signal

$$\begin{aligned} y(t) =& 3\,[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)]^2 \\ =& 3\,\alpha_1^2\,[x_1(t)]^2 + 6\,\alpha_1\alpha_2\,x_1(t)\,x_2(t) + 3\,\alpha_2^2\,[x_2(t)]^2 \end{aligned}$$

Superposition principle does not hold true in this case.

The system is not linear.

Example 2.1 (d) – Solution

d. The test signals $x_1(t)$ and $x_2(t)$ applied to the system individually produce the output signals $y_1(t) = \cos[x_1(t)]$ and $y_2(t) = \cos[x_2(t)]$ respectively. Their linear combination $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$ produces the output signal

$$y(t) = \cos[\alpha_1 x_1(t) + \alpha_2 x_2(t)]$$

Superposition principle does not hold true in this case.

The system is not linear.

Time Invariance

- A system is said to be time-invariant if its behavior characteristics do not change in time.
- Consider a continuous-time system with the input-output relationship

$$Sys\{x(t)\} = y(t)$$

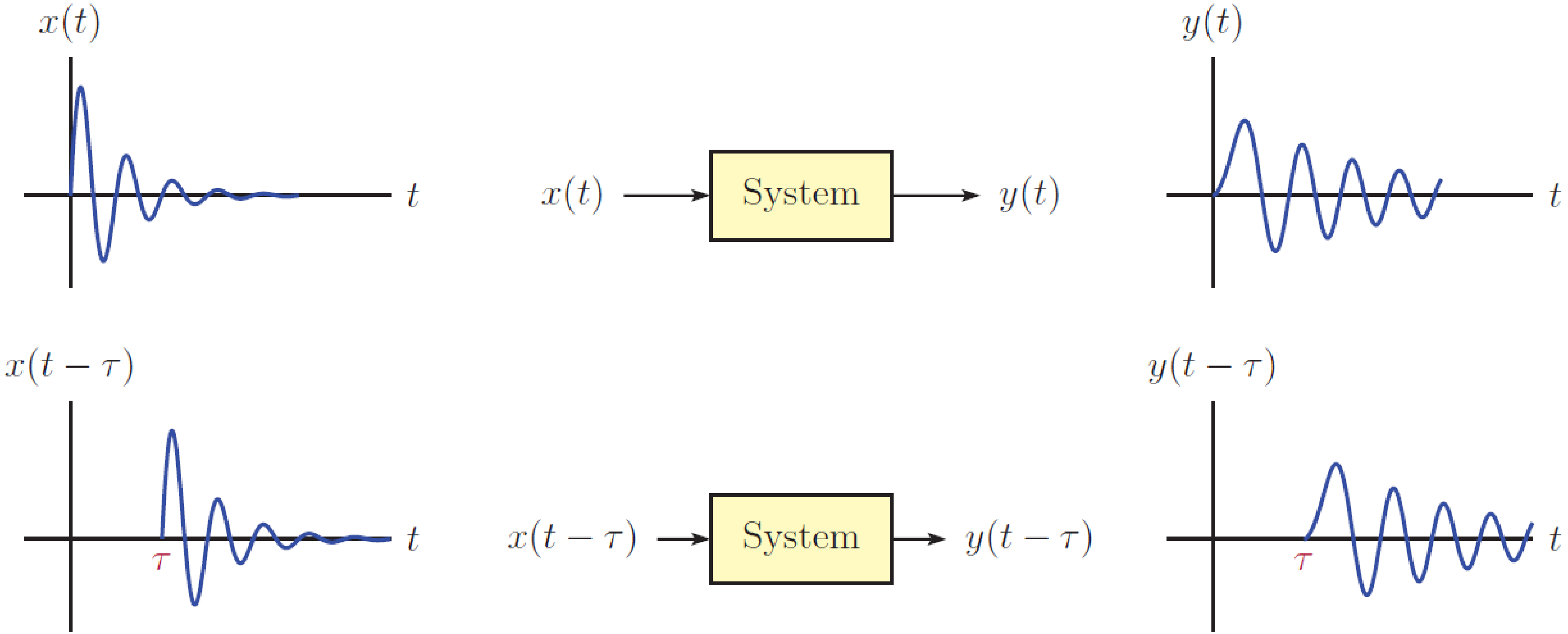
- If the input signal applied to a time-invariant system is time-shifted by τ seconds, the only effect of this delay should be to cause an equal amount of time shift in the output signal, but to otherwise leave the shape of the output signal unchanged.

$$Sys\{x(t-\tau)\} = y(t-\tau)$$

Time Invariance

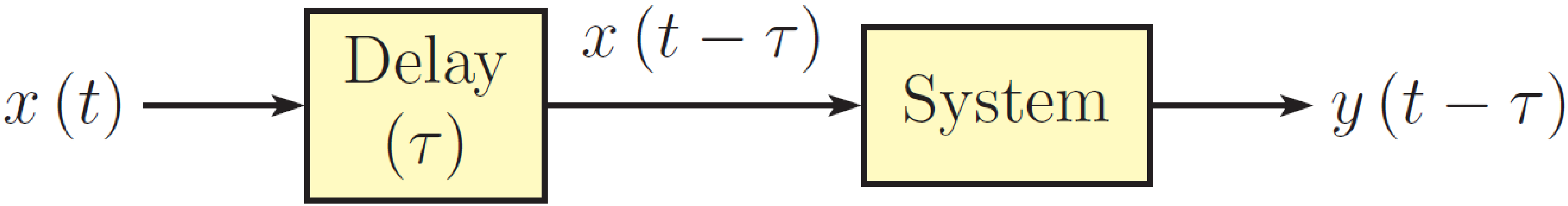
- Condition for time-invariance:

$$Sys\{x(t)\} = y(t) \quad \text{implies that} \quad Sys\{x(t-\tau)\} = y(t-\tau)$$

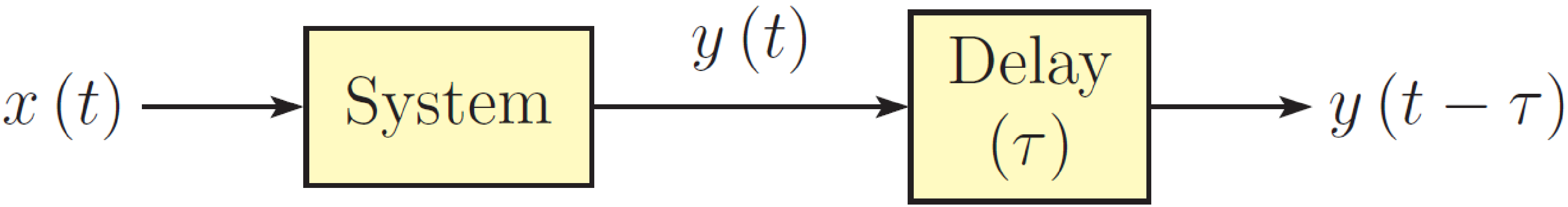


Time Invariance

- Alternatively, the relationship described can be characterized by the equivalence of the two system configurations shown in Figure.



(a)



Example 2.2

Three different systems are described below through their input-output relationships.

For each, determine whether the system is time-invariant or not:

a. $y(t) = 5\,x(t)$

b. $y(t) = 3\cos(x(t))$

c. $y(t) = 3\cos(t)\;x(t)$

Example 2.2 (a) – Solution

a. For this system, if the input signal $x(t)$ is delayed by τ seconds, the corresponding output signal would be

$$\mathrm{Sys}\{x(t-\tau)\} = 5\,x(t-\tau) = y(t-\tau)$$

This system is time-invariant.

Example 2.2 (b) – Solution

b. Let the input signal be $x\left(t-\tau\right)$. The output of the system is

$$\text{Sys}\{x\left(t-\tau\right)\} = 3\,\cos\left(\,x\left(t-\tau\right)\,\right) = y\left(t-\tau\right)$$

This system is time-invariant.

Example 2.2 (c) – Solution

c. Again using the delayed input signal $x\,(t-\tau)$ we obtain the output

$$\text{Sys}\{x\,(t-\tau)\} = 3\,\cos\,(t)\;x\,(t-\tau) \neq y\,(t-\tau)$$

In this case the system is not time-invariant since the time-shifted input signal leads to a response that is not the same as a similarly time-shifted version of the original output signal.

Problem 2.1

A number of systems are specified below in terms of their input-output relationships.

For each case, determine if the system is linear and/or time-invariant.

a. $y(t) = |x(t)| + x(t)$

b. $y(t) = t\,x(t)$

c. $y(t) = e^{-t}\,x(t)$

Problem 2.1 (a) – Solution

a. $y(t) = |x(t)| + x(t)$

$$y_1(t) = \text{Sys}\{x_1(t)\} = |x_1(t)| + x_1(t)$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = |x_2(t)| + x_2(t)$$

Using $x(t) = \alpha_1\, x_1(t) + \alpha_2\, x_2(t)$ as input we obtain

$$\begin{aligned} y(t) &= \text{Sys}\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\} \\ &= |\alpha_1\, x_1(t) + \alpha_2\, x_2(t)| + \alpha_1\, x_1(t) + \alpha_2\, x_2(t) \\ &\neq \alpha_1\, y_1(t) + \alpha_2\, y_2(t) \end{aligned}$$

The system is not linear.

$$\text{Sys}\{x_1(t-\tau)\} = |x_1(t-\tau)| + x_1(t-\tau) = y_1(t-\tau)$$

The system is time-invariant.

Problem 2.1 (b) – Solution

b. $y(t) = t\,x(t)$

$$y_1(t) = \text{Sys}\left\{x_1(t)\right\} = t\,x_1(t)$$

$$y_2(t) = \text{Sys}\left\{x_2(t)\right\} = t\,x_2(t)$$

Using $x(t) = \alpha_1\,x_1(t) + \alpha_2\,x_2(t)$ as input we obtain

$$\begin{aligned} y(t) &= \text{Sys}\left\{\alpha_1\,x_1(t) + \alpha_2\,x_2(t)\right\} \\ &= t\left[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)\right] \\ &= \alpha_1\,t\,x_1(t) + \alpha_2\,t\,x_2(t) \\ &= \alpha_1\,y_1(t) + \alpha_2\,y_2(t) \end{aligned}$$

The system is linear.

$$\text{Sys}\left\{x_1(t-\tau)\right\} = t\,x_1(t-\tau) \neq y_1(t-\tau)$$

The system is not time-invariant.

Problem 2.1 (c) – Solution

c. $y(t) = e^{-t}\,x(t)$

$$y_1(t) = \text{Sys}\{x_1(t)\} = e^{-t}\,x_1(t)$$

$$y_2(t) = \text{Sys}\{x_2(t)\} = e^{-t}\,x_2(t)$$

Using $x(t) = \alpha_1\,x_1(t) + \alpha_2\,x_2(t)$ as input we obtain

$$\begin{aligned} y(t) &= \text{Sys}\{\alpha_1\,x_1(t) + \alpha_2\,x_2(t)\} \\ &= e^{-t}\left[\alpha_1\,x_1(t) + \alpha_2\,x_2(t)\right] \\ &= \alpha_1\,e^{-t}\,x_1(t) + \alpha_2\,e^{-t}\,x_2(t) \\ &= \alpha_1\,y_1(t) + \alpha_2\,y_2(t) \end{aligned}$$

The system is linear.

$$\text{Sys}\{x_1(t-\tau)\} = e^{-t}\,x_1(t-\tau) \neq y_1(t-\tau)$$

The system is not time-invariant.

Problem 2.2

2.2. Consider the cascade combination of two systems shown in Fig. P.2.2(a).

Figure P. 2.2

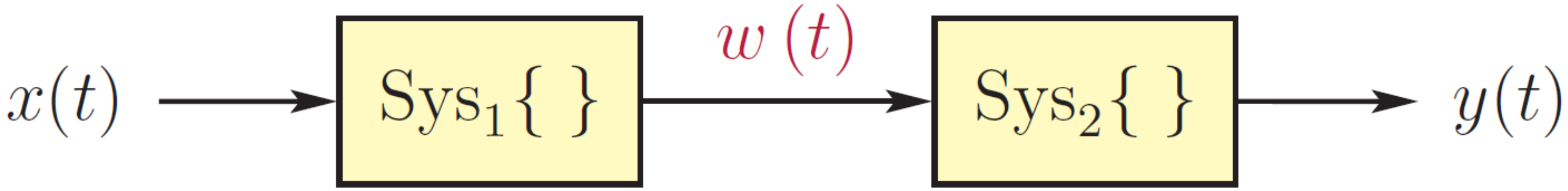
a. Let the input-output relationships of the two subsystems be given as

$$\mathrm{Sys}_1\{x(t)\} = 3\,x(t) \quad \text{and} \quad \mathrm{Sys}_2\{w(t)\} = w(t-2)$$

Write the relationship between $x(t)$ and $y(t)$.

b. Let the order of the two subsystems be changed as shown in Fig. P.2.2(b). Write the relationship between $x(t)$ and $\bar{y}(t)$. Does changing the order of two subsystems change the overall input-output relationship of the system?

Problem 2.2 (a) – Solution

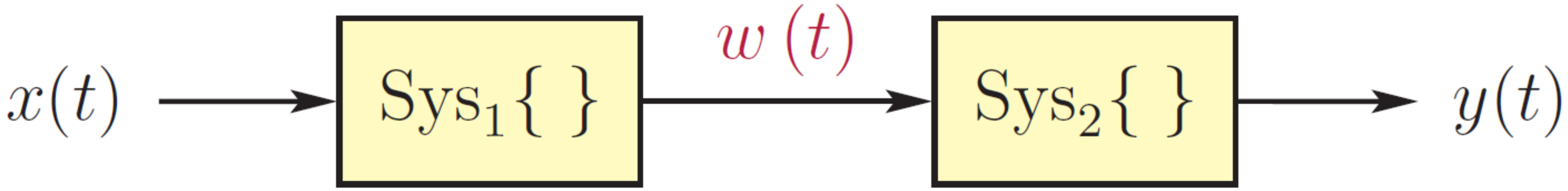


$$\mathrm{Sys}_1\{x(t)\} = 3\,x(t) \quad \text{and} \quad \mathrm{Sys}_2\{w(t)\} = w(t-2)$$

$$w(t) = 3\,x(t)$$

$$y(t) = w(t-2) = 3\,x(t-2)$$

Problem 2.2 (b) – Solution



$$\text{Sys}_1\{x(t)\} = 3\,x(t) \quad \text{and} \quad \text{Sys}_2\{w(t)\} = w(t-2)$$

$$\bar{w}(t) = x(t-2)$$

$$\bar{y}(t) = 3\,\bar{w}(t) = 3\,x(t-2)$$

Input-output relationship of the system does not change when the order of the two subsystems is changed.

Problem 2.3 (b)

2.3. Repeat Problem 2.2 with the following sets of subsystems:

Figure P. 2.2

a. Let the input-output relationships of the two subsystems be given as

$$\text{Sys}_1\{x(t)\} = 3\,x(t) \quad \text{and} \quad \text{Sys}_2\{w(t)\} = w(t) + 5$$

Write the relationship between $x(t)$ and $y(t)$.

b. Let the order of the two subsystems be changed as shown in Fig. P.2.2(b). Write the relationship between $x(t)$ and $\bar{y}(t)$. Does changing the order of two subsystems change the overall input-output relationship of the system?

Problem 2.3 (b) – Solution

Using the first configuration:

$$w(t) = 3x(t)$$

$$y(t) = w(t) + 5 = 3x(t) + 5$$

Using the second configuration:

$$\bar{w}(t) = x(t) + 5$$

$$\bar{y}(t) = 3\bar{w}(t) = 3\left[x(t) + 5\right] = 3x(t) + 15$$

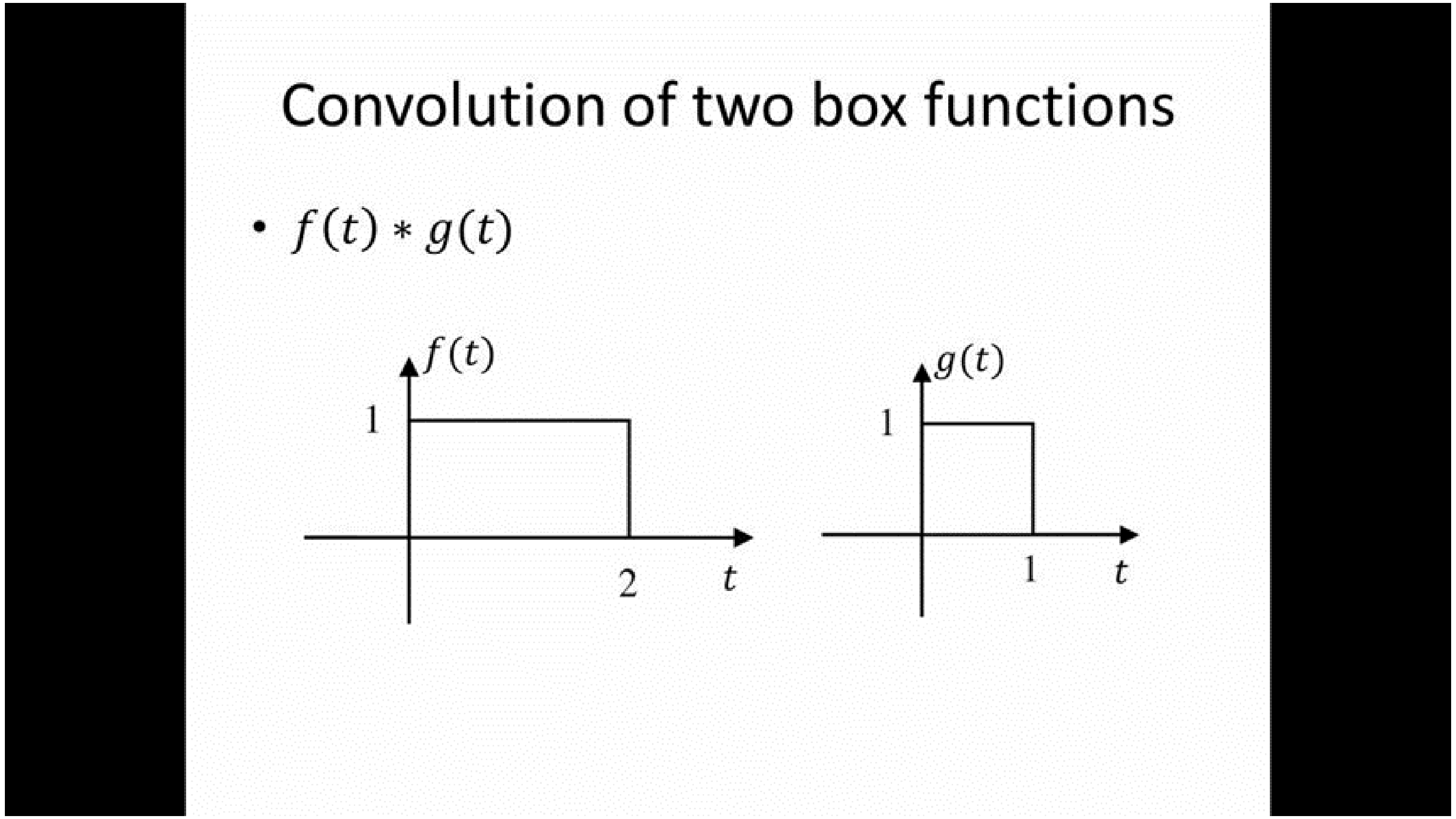
Input-output relationship of the system changes when the order of the two subsystems is changed.

CTLTI Systems

- We will work with continuous-time systems that are both linear and time-invariant.
- A number of time- and frequency-domain analysis and design techniques will be developed for such systems.
- For simplicity, we will use the acronym CTLTI to refer to continuous-time linear and time-invariant systems.

Convolution

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.



https://www.youtube.com/watch?v=C1N55M1VD2o

Convolution

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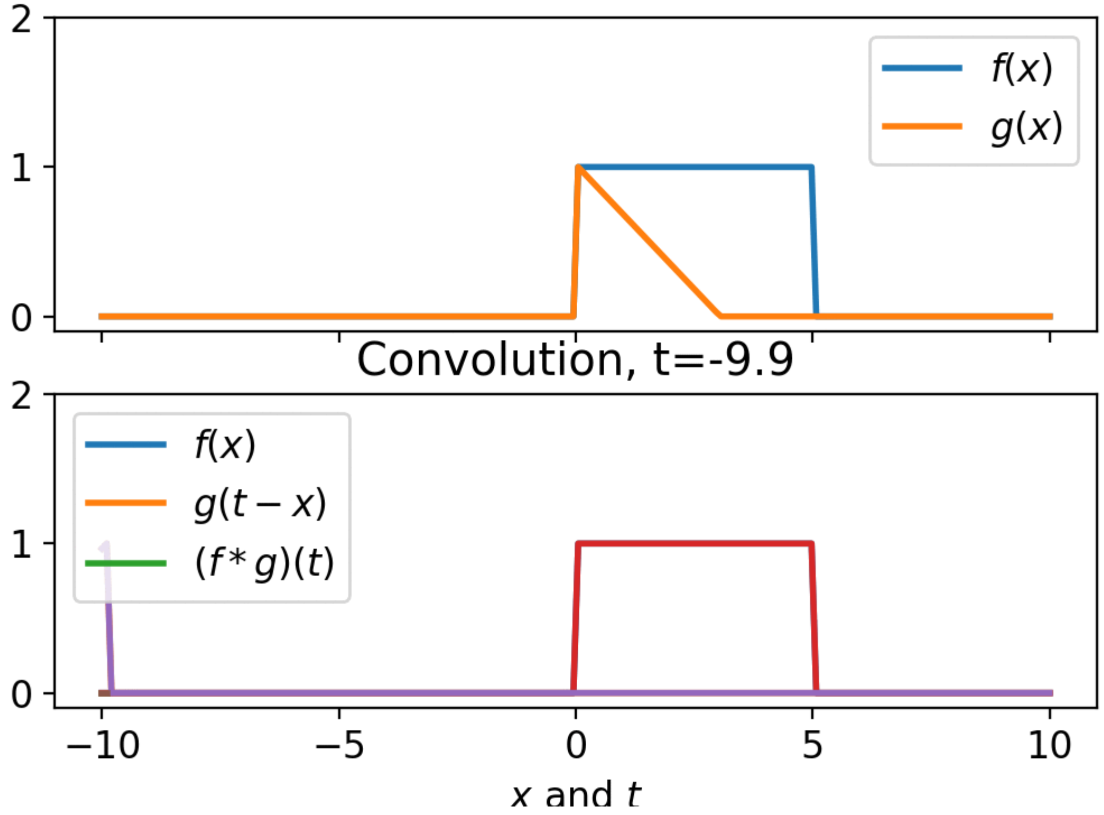
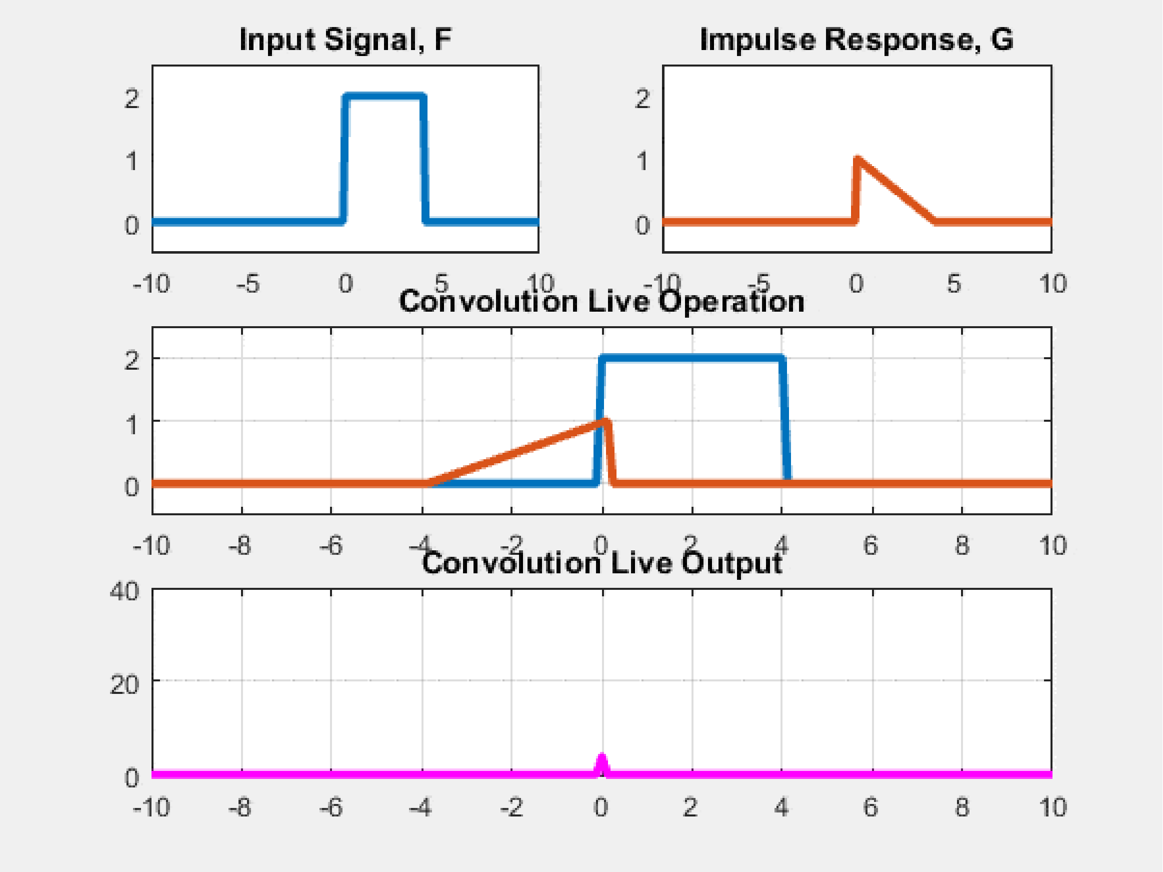


https://mahfuzdotsite.files.wordpress.com/2018/12/convolution_anim1.gif

Convolution

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.



http://quincyaflint.weebly.com/uploads/2/6/5/0/26500868/convolution-1-fast_orig.gif

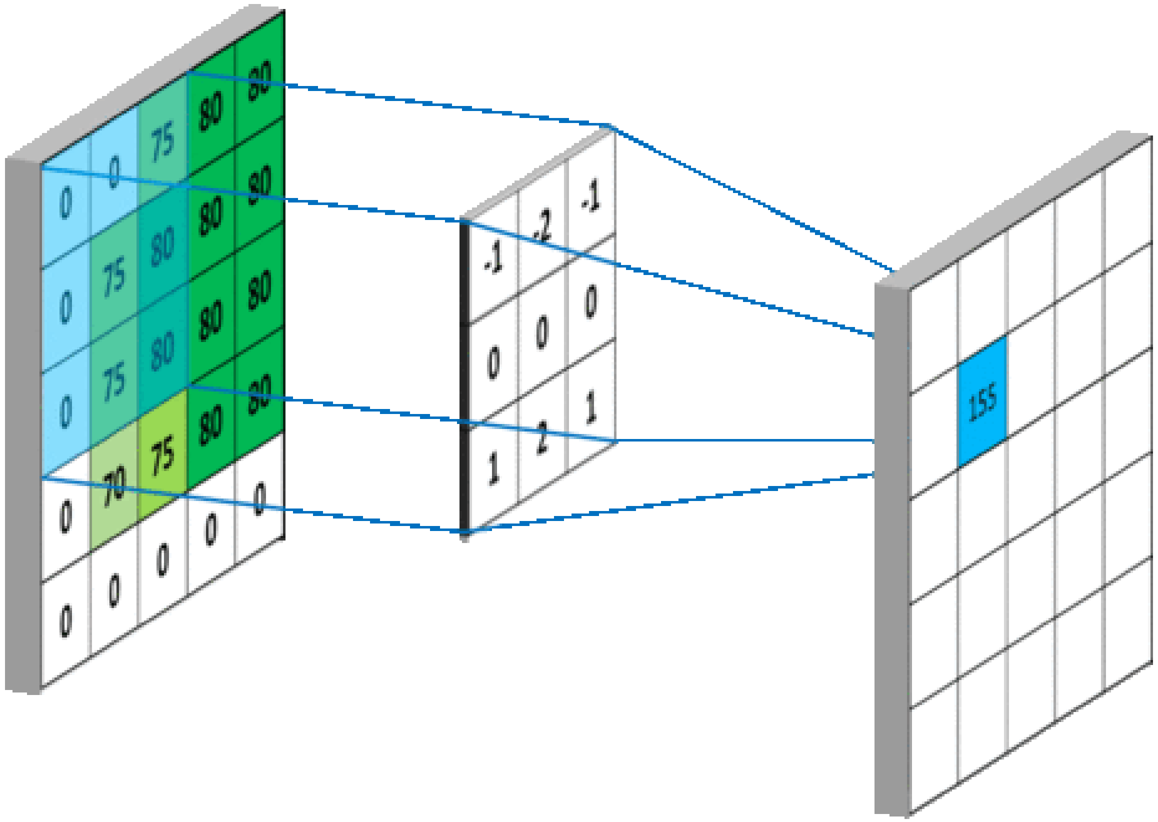
Convolution

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

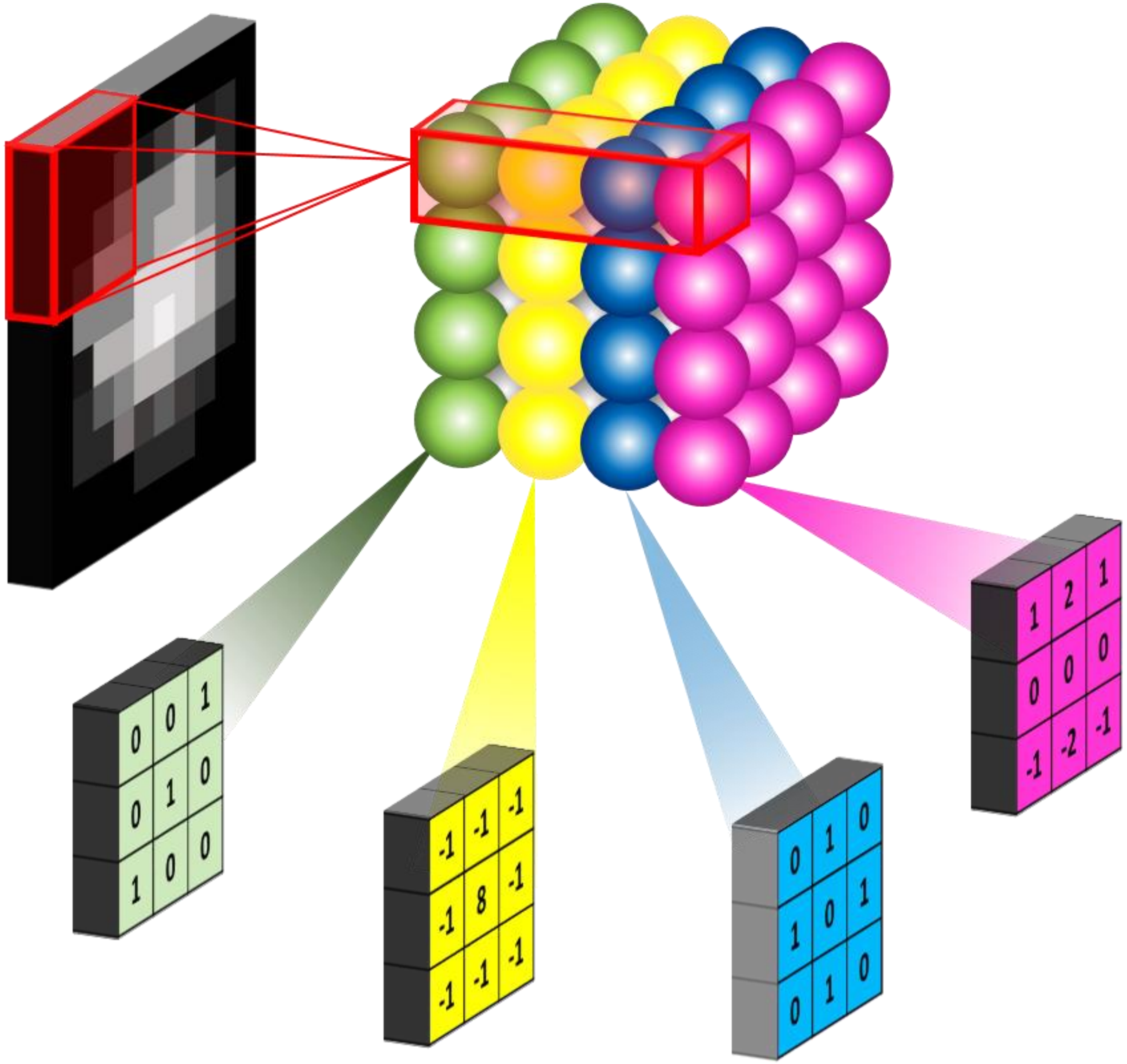
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Convolutional Neural Networks

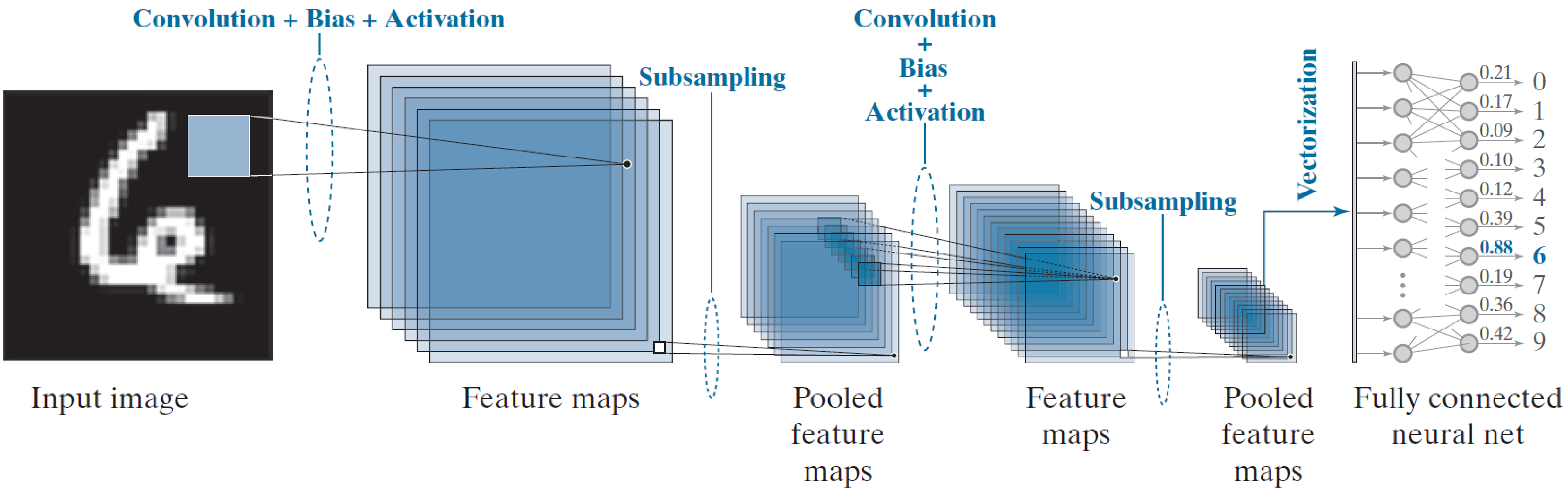


https://mlnotebook.github.io/post/CNN1/

Convolutional Neural Networks



Convolutional Neural Networks



Convolution Operation for CTLTI Systems

- The output signal $y(t)$ of a CTLTI system is obtained by convolving the input signal $x(t)$ and the impulse response $h(t)$ of the system.
- This relationship is expressed in compact notation as

$$y(t) = x(t) * h(t)$$

where the symbol $*$ represents the convolution operator.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

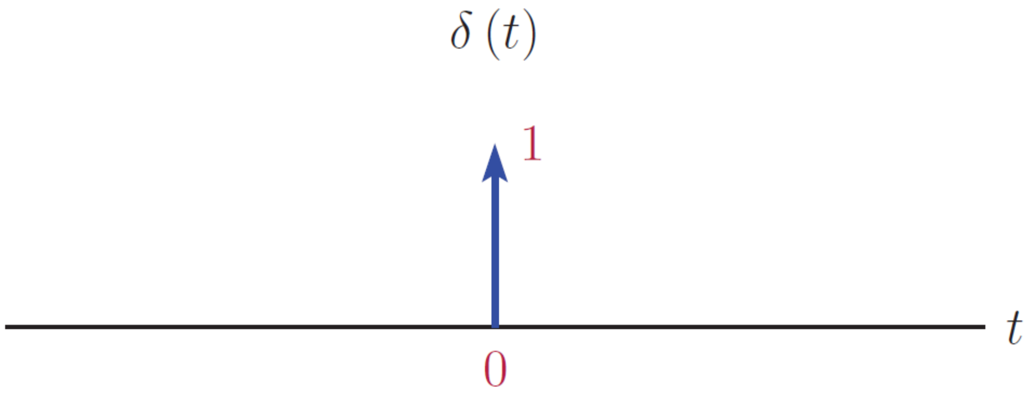
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

Unit-Impulse Function

- The unit-impulse function plays an important role in mathematical modeling and analysis of signals and linear systems.

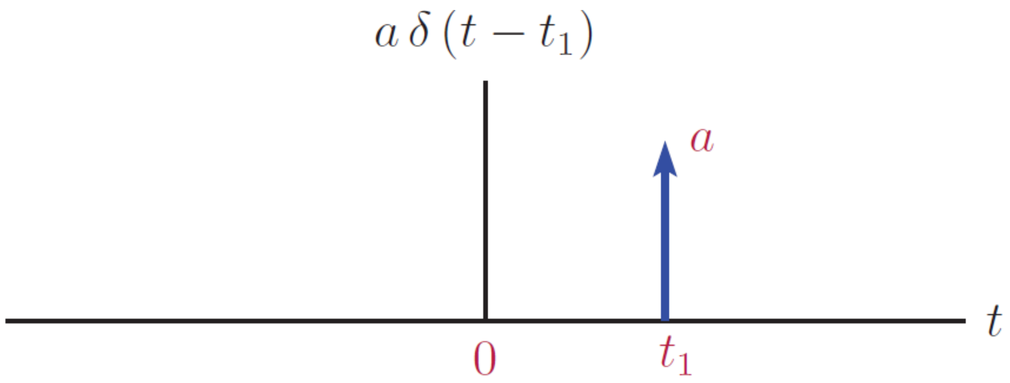
$$\delta(t) = \begin{cases} 0\,, & \text{if } t \neq 0 \\ \text{undefined}\,, & \text{if } t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



$$a\,\delta(t - t_1) = \begin{cases} 0\,, & \text{if } t \neq t_1 \\ \text{undefined}\,, & \text{if } t = t_1 \end{cases}$$

$$\int_{-\infty}^{\infty} a\,\delta(t - t_1)\, dt = a$$



Unit-Impulse Function: Shifting Property

- The shifting property of the unit-impulse function is given by

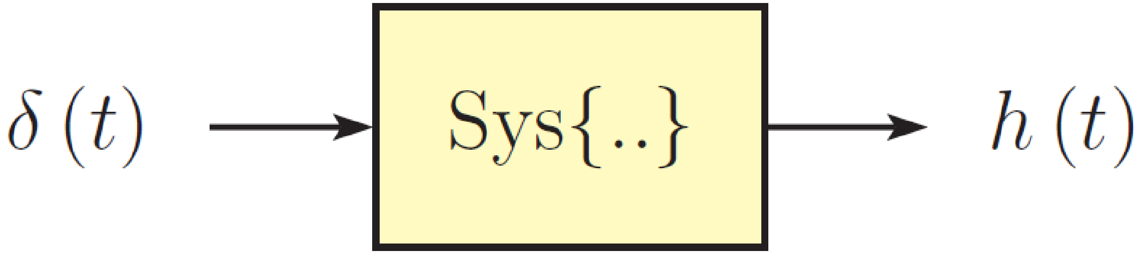
$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\, \delta(t-\lambda)\, d\lambda$$

$$x(\lambda)\delta(t-\lambda) = 0, \qquad t \neq \lambda$$

$$\begin{aligned}\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)\,d\lambda &= \int_{-\infty}^{\infty} x(t)\delta(t-\lambda)\,d\lambda \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-\lambda)\,d\lambda \\ &= x(t)\end{aligned}$$

Impulse Response

- In previous sections, we have explored that a CTLTI system can be described by constant-coefficient ordinary differential equation.
- An alternative description of a CTLTI system can be given in terms of its impulse response $h(t)$ which is simply the forced response of the system under consideration when the input signal is a unit impulse.



Problem 2.22

2.22. Using the convolution integral given by Eqns. (2.153) and (2.154) prove each of the relationships below:

a. $x(t) * \delta(t) = x(t)$

b. $x(t) * \delta(t - t_0) = x(t - t_0)$

Problem 2.22 – Solution

a. $$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(t)$$

b. $$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-t_0-\tau)\, d\tau = x(t-t_0)$$

Problem 2.23

2.23. The impulse response of a CTLTI system is

$$h(t) = \delta(t) - \delta(t-1)$$

Determine sketch the response of this system to the triangular waveform shown in Fig. P.2.23.

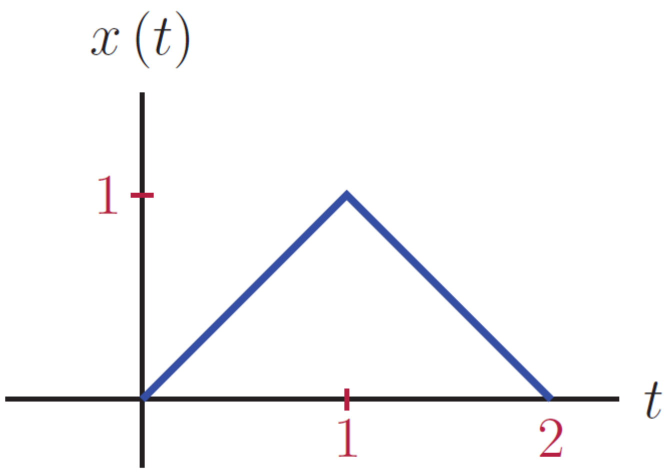


Figure P. 2.23

Problem 2.23 – Solution

$$h(t) = \delta(t) - \delta(t-1)$$

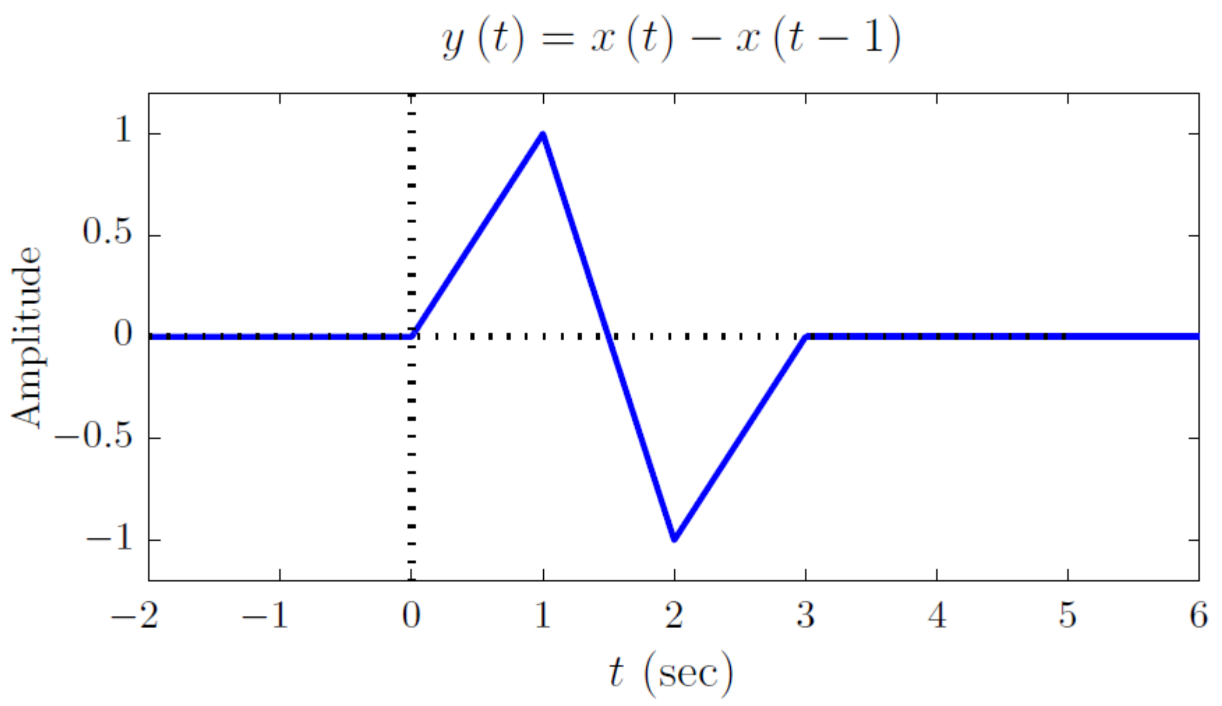
$$\begin{aligned}
y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda)\; \delta(t-\lambda) - \delta(t-1-\lambda)\; d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)\, d\lambda - \int_{-\infty}^{\infty} x(\lambda)\delta(t-1-\lambda)\, d\lambda \\
&= x(t)\int_{-\infty}^{\infty} \delta(t-\lambda)\, d\lambda - x(t-1)\int_{-\infty}^{\infty} \delta(t-1-\lambda)\, d\lambda \\
&= x(t) - x(t-1)
\end{aligned}$$

Problem 2.23 – Solution

$$x(t)=\begin{cases} t, & 0<t<1 \\ -t+2, & 1<t<2 \\ 0, & otherwise \end{cases}$$

$$x(t-1)=\begin{cases} t-1, & 1<t<2 \\ -t+3, & 2<t<3 \\ 0, & otherwise \end{cases}$$

$$y(t)=x(t)-x(t-1)=\begin{cases} t, & 0<t<1 \\ -2t+3, & 1<t<2 \\ t-3, & 2<t<3 \\ 0, & otherwise \end{cases}$$



Problem 2.23 – Another Solution

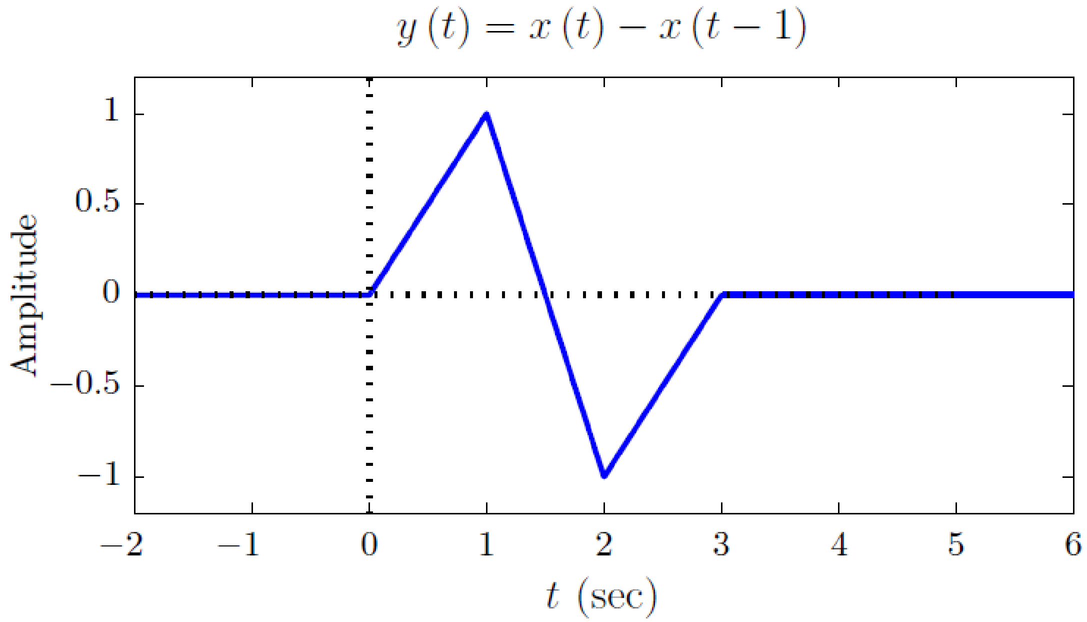
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} [\delta(\tau) - \delta(\tau-1)]\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau)\, x(t-\tau)\, d\tau - \int_{-\infty}^{\infty} \delta(\tau-1)\, x(t-\tau)\, d\tau$$

Using the sifting property of the unit-impulse function, we have

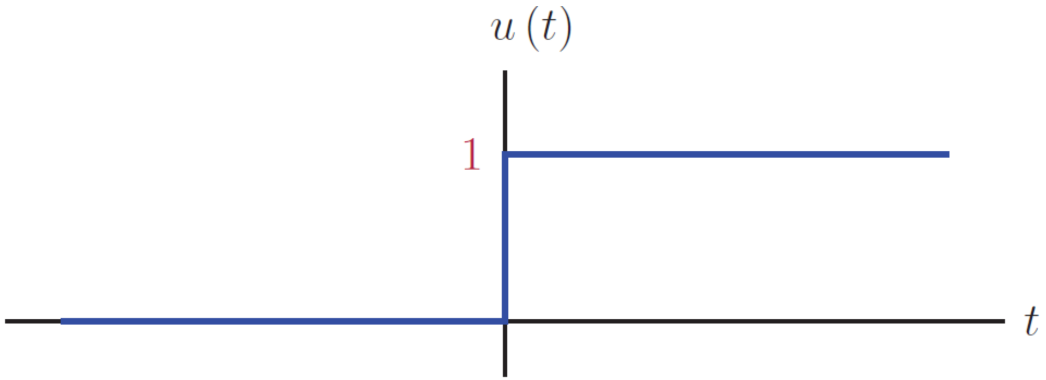
$$y(t) = x(t) - x(t-1)$$



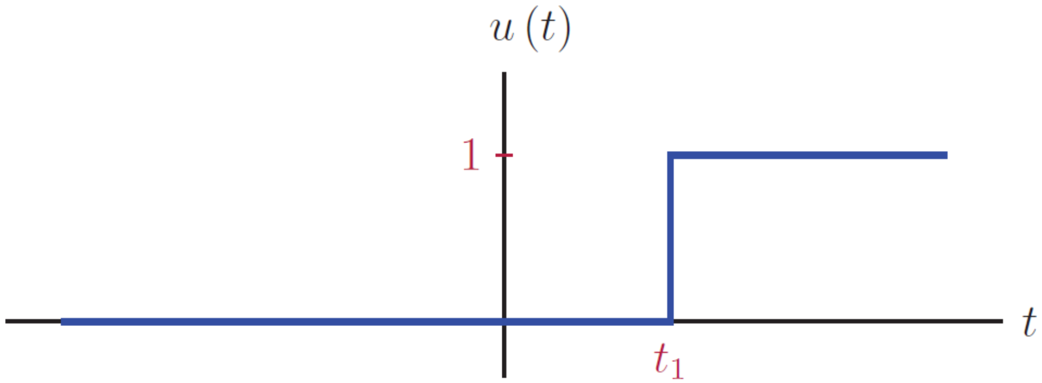
Unit-Step Function

- The unit-step function is useful in situations where we need to model a signal that is turned on or off at a specific time instant.

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



$$u(t - t_1) = \begin{cases} 1, & t > t_1 \\ 0, & t < t_1 \end{cases}$$



Problem 2.26

2.26. For each pair of signals $x(t)$ and $h(t)$ given below, find the convolution $y(t) = x(t) * h(t)$. In each case sketch the signals involved in the convolution integral and determine proper integration limits.

a. $x(t) = u(t)$, $h(t) = e^{-2t}\, u(t)$

c. $x(t) = u(t-2)$, $h(t) = e^{-2t}\, u(t)$

e. $x(t) = e^{-t}\, u(t)$, $h(t) = e^{-2t}\, u(t)$

Problem 2.26 (a) – Solution

a. $x(t) = u(t)\ , \quad h(t) = e^{-2t}\,u(t)$

a.

$$y(t) = \int_{-\infty}^{\infty} u(\lambda)\, e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_{0}^{\infty} e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda$$

<u>Case 1: $t < 0$</u>

$$y(t) = 0$$

<u>Case 2: $t \geq 0$</u>

$$\int_{0}^{t} e^{-2(t-\lambda)}\, d\lambda = \frac{1}{2}\left(1 - e^{-2t}\right)$$

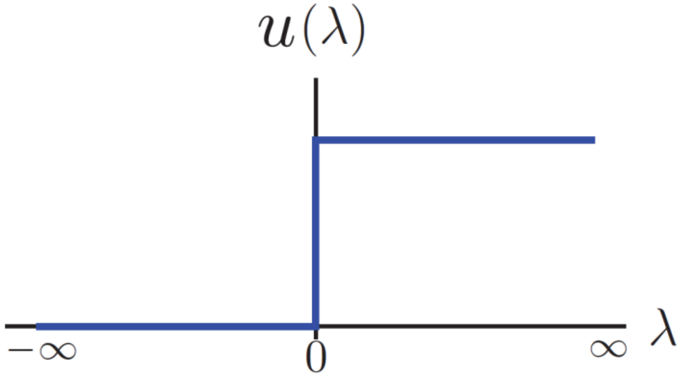
Problem 2.26 (a) – Explanation

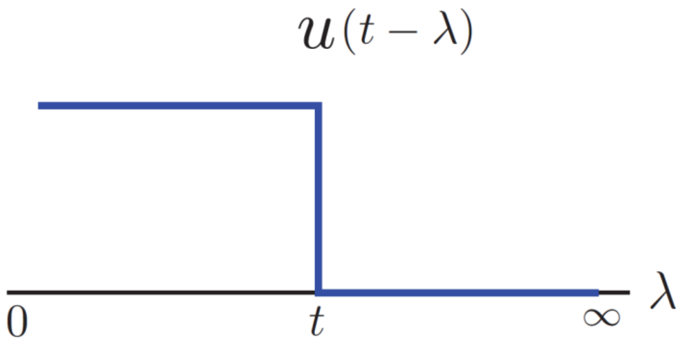
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} u(\lambda) e^{-2(t-\lambda)} u(t-\lambda)\, d\lambda$$

$$= \int_{0}^{\infty} e^{-2(t-\lambda)} u(t-\lambda)\, d\lambda$$

$$= \int_{0}^{t} e^{-2(t-\lambda)}\, d\lambda$$

$$= \frac{1}{2} e^{-2(t-\lambda)} \Big|_{0}^{t} = \frac{1}{2}(1 - e^{-2t})$$





Problem 2.26 (c) – Solution

c. $x(t) = u(t-2)\ , \quad h(t) = e^{-2t}\,u(t)$

c.

$$y(t) = \int_{-\infty}^{\infty} u(\lambda-2)\, e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda = \int_{2}^{\infty} e^{-2(t-\lambda)}\, u(t-\lambda)\, d\lambda$$

<u>Case 1: $t < 2$</u>

$$y(t) = 0$$

<u>Case 2: $t \ge 2$</u>

$$\int_{2}^{t} e^{-2(t-\lambda)}\, d\lambda = \frac{1}{2}\left(1 - e^{-2(t-2)}\right)$$

Problem 2.26 (e) – Solution

e. $$x(t) = e^{-t}\,u(t)\ , \quad h(t) = e^{-2t}\,u(t)$$

e.

$$y(t) = \int_{-\infty}^{\infty} e^{-\lambda}\,u(\lambda)\,e^{-2(t-\lambda)}\,u(t-\lambda)\,d\lambda = \int_{0}^{\infty} e^{-2t+\lambda}\,u(t-\lambda)\,d\lambda$$

Case 1: $t < 0$

$$y(t) = 0$$

Case 2: $t \geq 0$

$$y(t) = \int_{0}^{t} e^{-2t+\lambda}\,d\lambda = e^{-t} - e^{-2t}$$

Causality in Continuous-Time Systems

- A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.
- The system with input-output relationship is causal since the output signal can be computed based on current and past values of the input signal.

$$y(t) = x(t) + x(t - 0.01) + x(t - 0.02)$$

- Conversely, the system is non-causal since the computation of the output signal requires anticipation of a future value of the input signal.

$$y(t) = x(t) + x(t - 0.01) + x(t + 0.01)$$

Causality in Continuous-Time Systems

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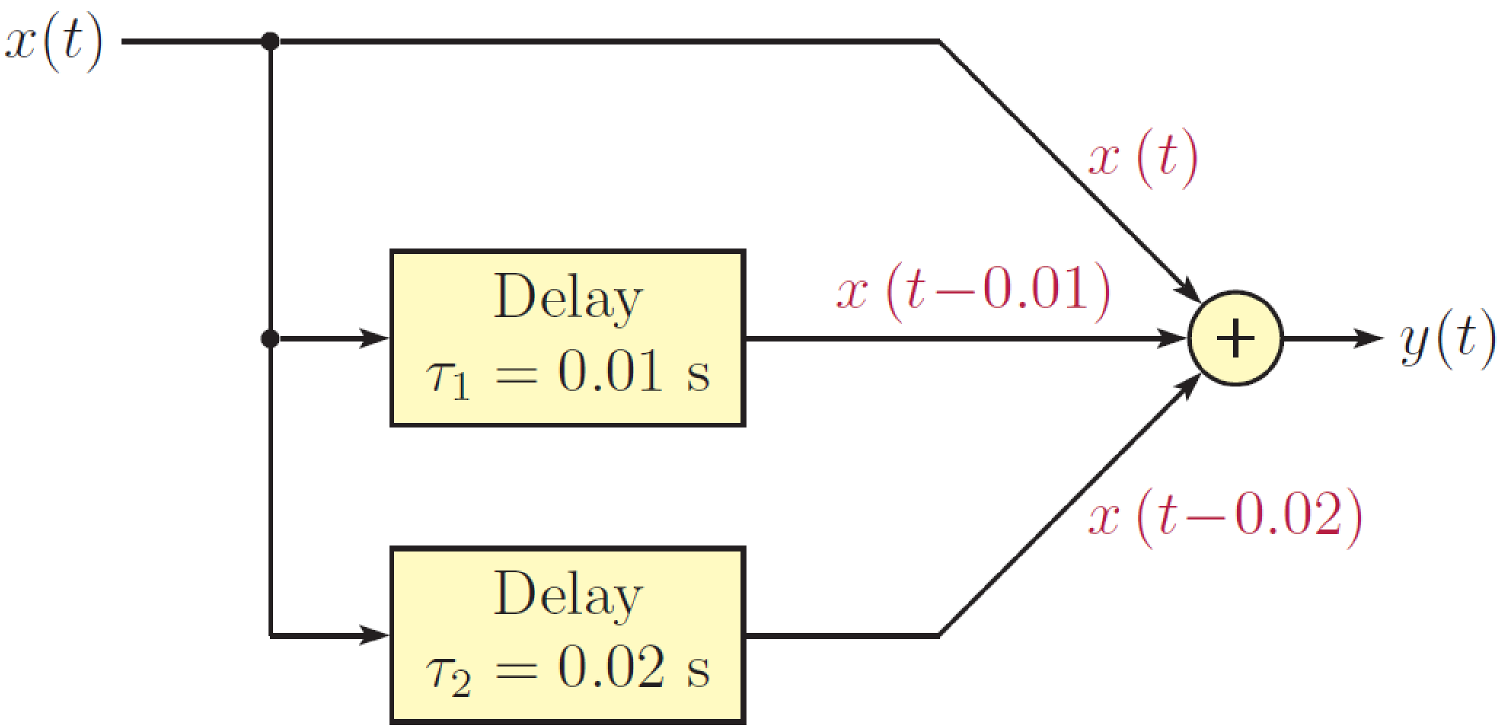


Stability in Continuous-Time Systems

- A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal is guaranteed to produce a bounded output signal.
- An input signal $x(t)$ is said to be bounded if an upper bound B_x exists for all values of t such that

$$|x(t)| < B_x < \infty \quad \text{implies that} \quad |y(t)| < B_y < \infty$$

Causality and Stability in CTLTI Systems

- The impulse response of a causal CTLTI must be equal to zero for all negative values of its argument (t).

$$h(t) = 0 \qquad \text{for all } t < 0$$

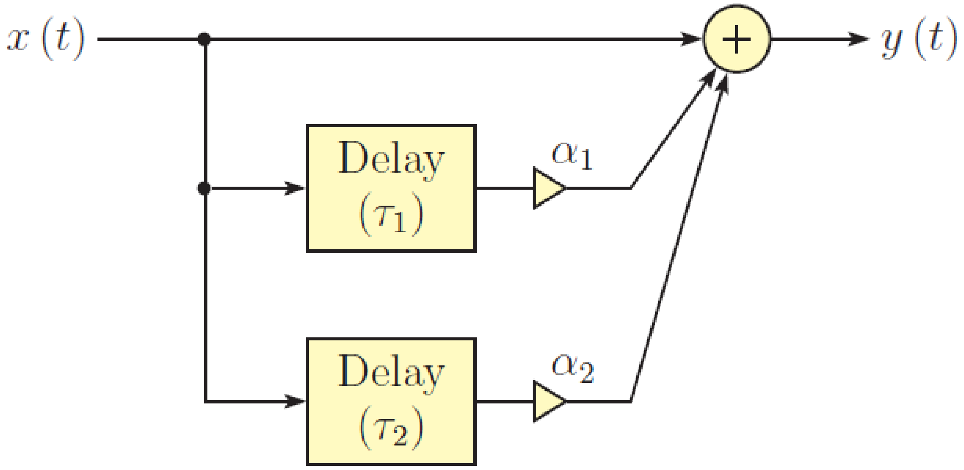
- For a CTLTI system to be stable, its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} \left| h(\lambda) \right| d\lambda < \infty$$

Problem 2.30

2.30. The system shown in Fig. P.2.30 represents addition of echos to the signal $x(t)$:

$$y(t) = x(t) + \alpha_1\, x(t - \tau_1) + \alpha_2\, x(t - \tau_2)$$



Comment on the system's

a. Linearity
b. Time invariance
c. Causality
d. Stability

Problem 2.30 – Solution

a. Let the input signal to the system be $x_1(t)$.

$$y_1(t) = \text{Sys}\left\{x_1(t)\right\} = x_1(t) + \alpha_1\, x_1(t-\tau_1) + \alpha_2\, x_1(t-\tau_2)$$

Similarly, if the input signal is $x_2(t)$

$$y_2(t) = \text{Sys}\left\{x_2(t)\right\} = x_2(t) + \alpha_1\, x_2(t-\tau_1) + \alpha_2\, x_2(t-\tau_2)$$

The response of the system to the input signal $x(t) = \beta_1\, x_1(t) + \beta_2\, x_2(t)$ is

$$\begin{aligned}\text{Sys}\left\{\beta_1\, x_1(t) + \beta_2\, x_2(t)\right\} &= \beta_1\left[x_1(t) + \alpha_1\, x_1(t-\tau_1) + \alpha_2\, x_1(t-\tau_2)\right] + \beta_2\left[x_2(t) + \alpha_1\, x_2(t-\tau_1) + \alpha_2\, x_2(t-\tau_2)\right]\\ &= \beta_1\, y_1(t) + \beta_2\, y_2(t)\end{aligned}$$

The system is linear.

Problem 2.30 – Solution

b. The response to $x_1(t-a)$ is

$$\text{Sys}\{x_1(t-a)\} = x_1(t-a) + \alpha_1 x_1(t-\tau_1-a) + \alpha_2 x_1(t-\tau_2-a) = y_1(t-a)$$

The system is time-invariant.

c. The system is causal provided that $\tau_1 > 0$ and $\tau_2 > 0$.

d. The system is stable provided that $\alpha_1,\ \alpha_2 < \infty$.

Problem 2.31

2.31. For each system described below determine if the system is causal and/or stable.

a. $$y(t) = \text{Sys}\{x(t)\} = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

b. $$y(t) = \text{Sys}\{x(t)\} = \int_{t-T}^{t} x(\lambda)\, d\lambda\,, \qquad T > 0$$

c. $$y(t) = \text{Sys}\{x(t)\} = \int_{t-T}^{t+T} x(\lambda)\, d\lambda\,, \qquad T > 0$$

Problem 2.31 (a) – Solution

a. $$y(t) = \mathrm{Sys}\{x(t)\} = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

a. Let $x(t) = \delta(t)$.

$$h(t) = \mathrm{Sys}\{\delta(t)\} = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore

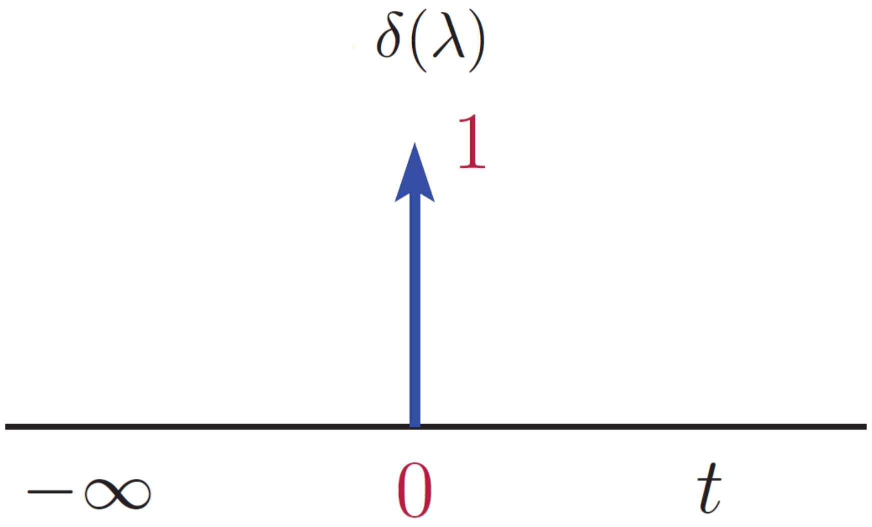
$$h(t) = u(t)$$

Since $h(t) = 0$ for $t < 0$, the system is causal. However, since $h(t)$ is not absolute summable, the system is not stable.

$$\int_{0}^{\infty} |h(\lambda)|\, d\lambda = \int_{0}^{\infty} d\lambda = \lambda\Big|_{0}^{\infty} = \infty$$

Problem 2.31 (a) – Explanation

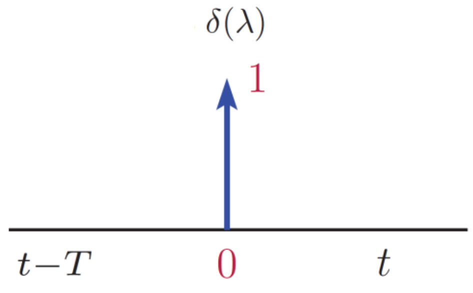
a. $$y(t) = \mathrm{Sys}\{x(t)\} = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$



$$h(t) = \mathrm{Sys}\{\delta(t)\} = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 2.31 (b) – Solution

b. $y(t) = \text{Sys}\{x(t)\} = \int_{t-T}^{t} x(\lambda)\, d\lambda\ , \qquad T > 0$



b. Let $x(t) = \delta(t)$.

$$h(t) = \text{Sys}\{\delta(t)\} = \int_{t-T}^{t} \delta(\lambda)\, d\lambda = \begin{cases} 1\,, & 0 < t < T \\ 0\,, & \text{otherwise} \end{cases}$$

Therefore

$$h(t) = \Pi\left(\frac{t - T/2}{T}\right)$$

Since $h(t) = 0$ for $t < 0$, the system is causal. Also, since $h(t)$ is absolute summable, the system is stable.

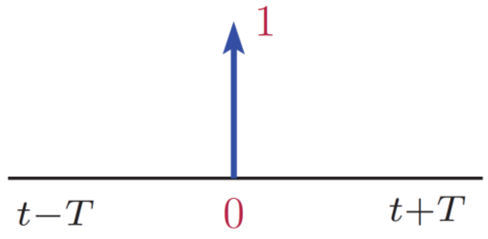
$$\int_0^\infty |h(\lambda)|\, d\lambda = \int_0^T d\lambda = \lambda\Big|_0^T = T$$

Problem 2.31 (c) – Solution

c. $$y(t) = \mathrm{Sys}\{x(t)\} = \int_{t-T}^{t+T} x(\lambda)\, d\lambda\,, \qquad T > 0$$

c. Let $x(t) = \delta(t)$.

$$h(t) = \mathrm{Sys}\{\delta(t)\} = \int_{t-T}^{t+T} \delta(\lambda)\, d\lambda = \begin{cases} 1\,, & -T < t < T \\ 0\,, & \text{otherwise} \end{cases}$$



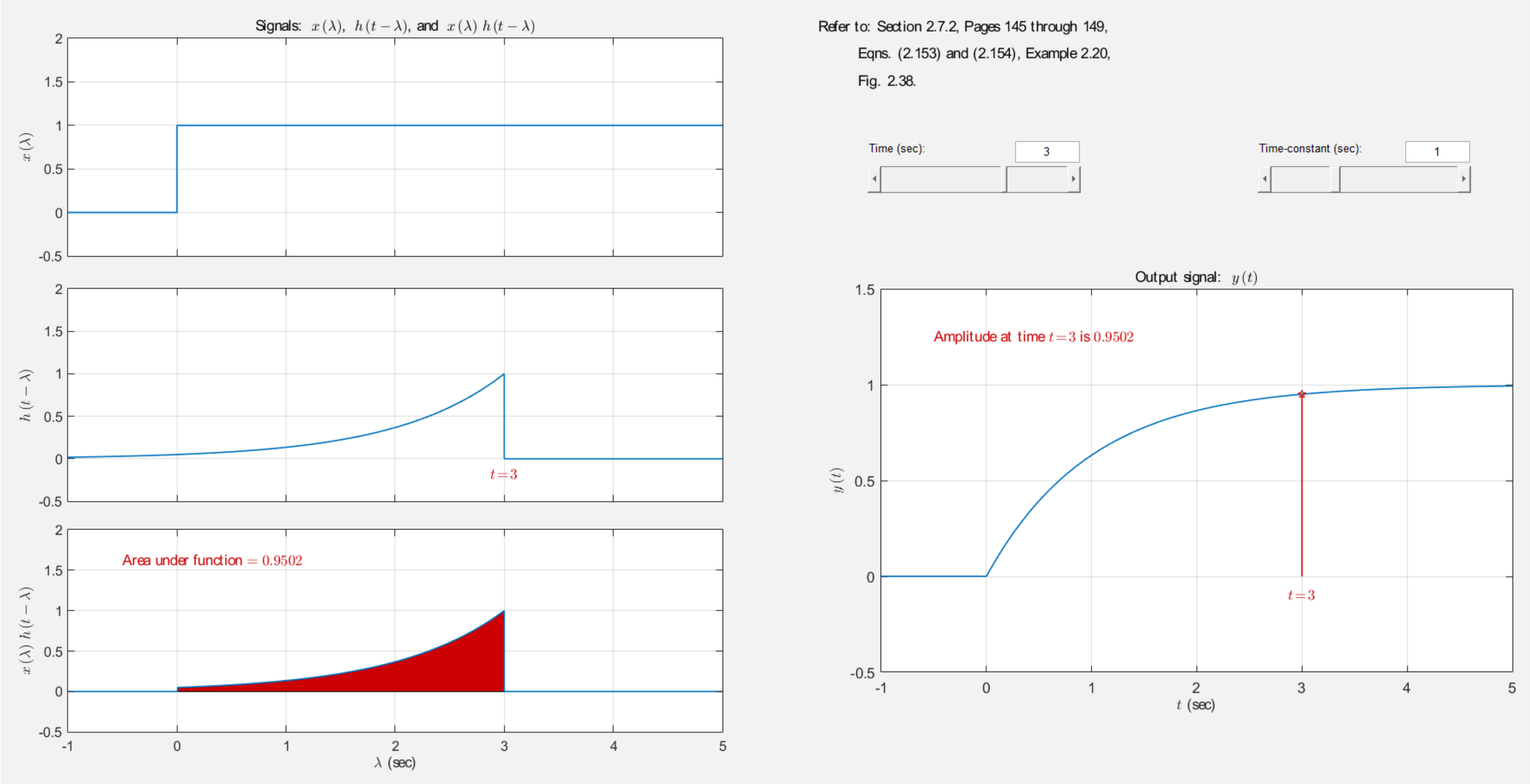
Therefore

$$h(t) = \Pi\left(\frac{t}{2T}\right)$$

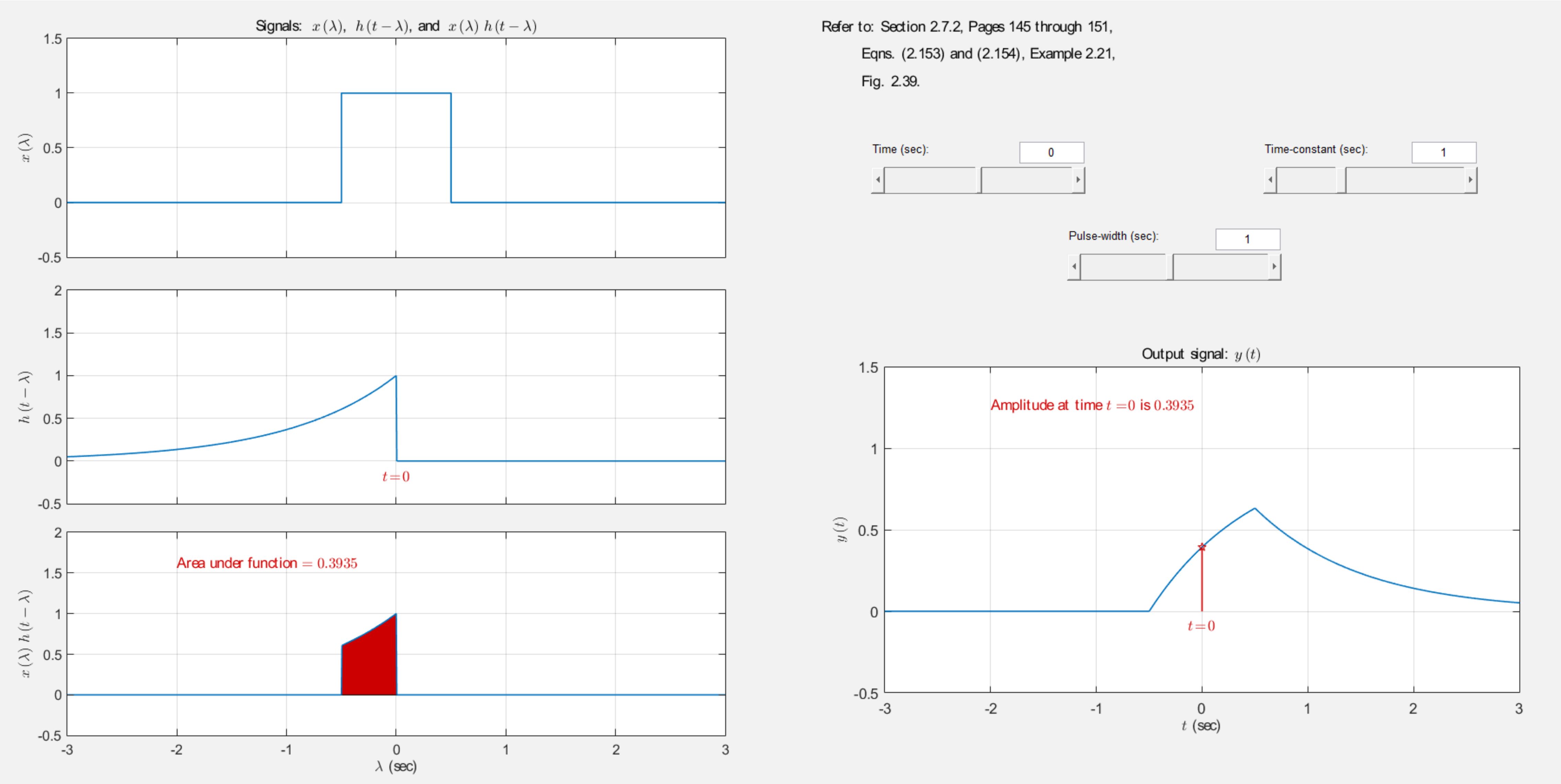
Since $h(t)$ has nonzero values for some $t < 0$, the system is not causal. It is stable, however, since $h(t)$ is absolute summable.

$$\int_0^{\infty} |h(\lambda)|\, d\lambda = \int_{-T}^{T} d\lambda = \lambda\Big|_{-T}^{T} = T + T = 2T$$

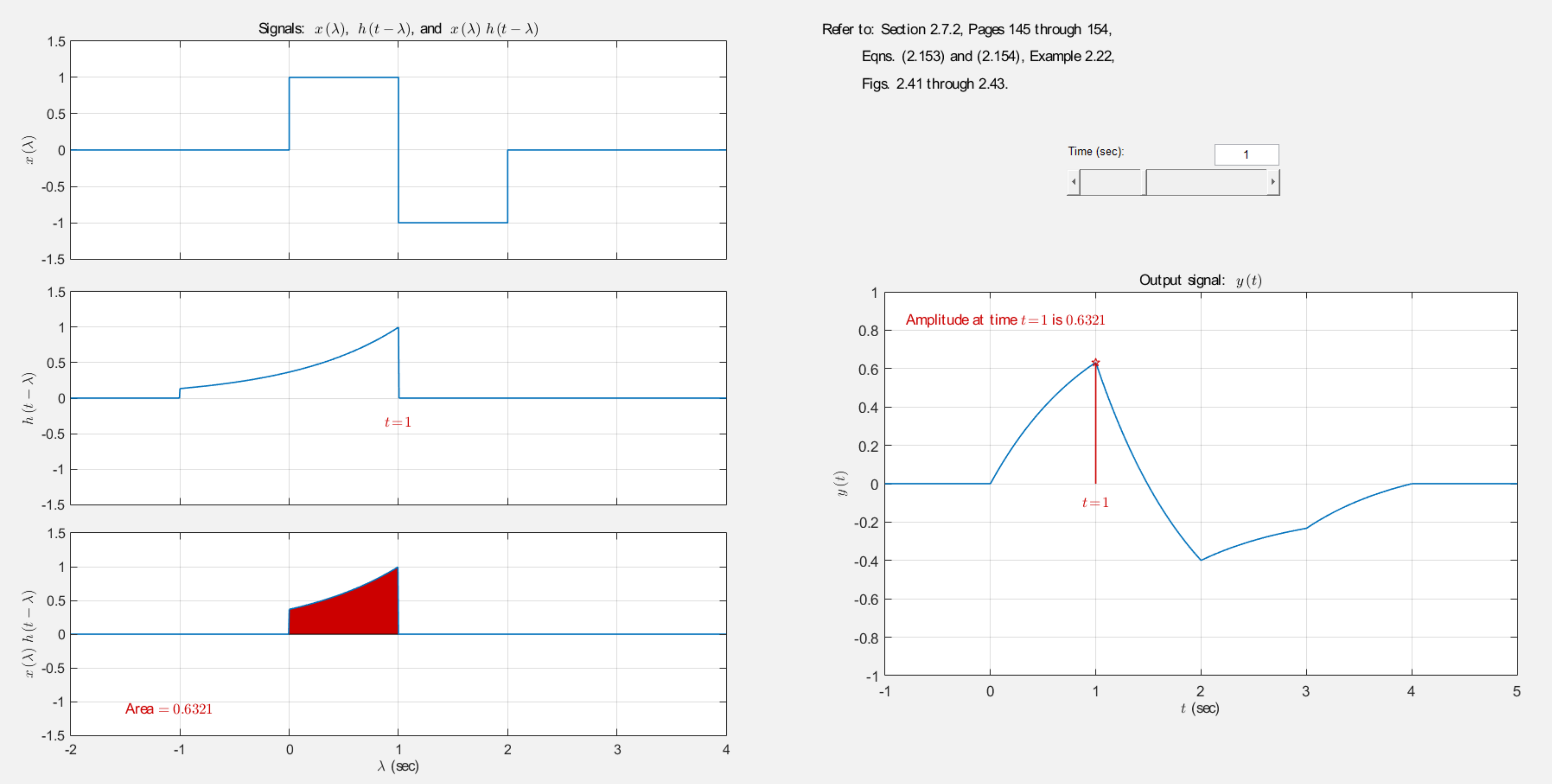
Interactive Demo: conv_demo1



Interactive Demo: conv_demo2



Interactive Demo: conv_demo3



Interactive Demo: conv_demo4