Diferansiyel Denklemler

Hafta 12

$$f(t) \;|\; F(s)$$

$$1 \;|\; \frac{1}{s} \;,\; s>0$$

$$t \;|\; \frac{1}{s^2} \;,\; s>0$$

$$e^{at} \;|\; \frac{1}{s-a} \;,\; s>a$$

$$\sin at \;|\; \frac{a}{s^2+\theta^2}$$

$$\cos at \;|\; \frac{s}{s^2+\theta^2}$$

$\boxed{1}$ *Lineerlik* Özelliği

$$L\{c_1f_1+c_2f_2\}=c_1L\{f_1\}+c_2L\{f_2\}$$

$$\boxed{ÖRNEK\ 1}\; L\{\cos^2 t\}=L\left\{\frac{1+\cos 2t}{2}\right\}$$

$$=\frac{1}{2}L\{1\}+\frac{1}{2}L\{\cos 2t\}$$

$$=\frac{1}{2}.\frac{1}{5}+\frac{1}{2}\frac{s}{s^2+4}=\frac{1}{2}\left(\frac{2s^2+4}{s(s^2+4)}\right)$$

$$=\frac{s^2+2}{s(s^2+4)}$$

$\boxed{2}$ 1. Öteleme Özelliği

$$L\{f(t)\}=F(s)\ \ ise$$

$$\boxed{L\{e^{at}f(t)\}=F(s-a)}\ \ geçerlidir.$$

$\boxed{ÖRNEK\ 2}$ $L\{e^{-2t}sin3t\}=?$

$$f(t)=sin3t\ \Rightarrow\ F(s)=\frac{3}{s^2+3^2}\ olduğu\ için$$

$$L\{e^{-2t}sin3t\}\underbrace{=}_{a=-2}F(s+2)=\frac{3}{(s+2)^2+3^2}=\frac{3}{s^2+4s+13}$$

$\boxed{3}$ 2. Öteleme Özelliği

$$L\{f(t)\}=F(s)\ ve\ \ g(t)=\{\,f(t-a),t>a\,|\,0,t\leq a\}$$

$$olsun.\ Bu\ durumda$$

$$\boxed{L\{s(t)\}=e^{-as}F(s)}\ \ 'dir.$$

$\boxed{ÖRNEK\ 3}$ $g(t)=\{\sin(t-\pi)\,,\ \ t>\pi\,|\ \ 0\,,\qquad t\leq\pi\}$

$$ise\ L\{g(t)\} =?$$

$$f(t) = \sin t \Rightarrow F(s) = \frac{1}{s^2+1}\ olduğu\ için\ (a = \pi)$$

$$L\{g(t)\} = e^{-\pi s}F(s) = \frac{e^{-\pi s}}{s^2+1}$$

$\boxed{4}$ t^n *ile çarpma* özelliği $(n \in N)$

$$L\{f(t)\} = F(s)\ ise$$

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\big(F(s)\big)}\ 'dir.$$

$\boxed{ÖRNEK\ 4}$ $L\{t \sin 3t\} =?$

$$n = 1, \quad f(t) = sin3t$$

$$F(s) = \frac{3}{s^2+9}$$

$$L\{t \sin 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= (-1)\left\{\frac{0.(s^2+9) - 3(2s)}{(s^2+9)^2}\right\} = \frac{6s}{(s^2+9)^2}$$

$\boxed{ÖRNEK\ 5}$ $L\{t^n\} \overset{n \in N}{\triangleq}\ ?$

$$L\{t^n.1\} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right)$$

$$f(t) = 1 \Rightarrow F(s) = \frac{1}{s}$$

$$\boxed{\frac{1}{x} \text{ in türevi nasıl bulunur ?}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2} = (-1)^1 \frac{1!}{x'}$$

$$\left(\frac{1}{x}\right)'' = \frac{(-1)(-2)}{x^3} = (-1)^2 \frac{2!}{x^3}$$

:: bu şekilde devam edersek ::

$$\left(\frac{1}{x}\right)^{(n)} = (-1)^n \frac{n!}{x^{n+1}}$$

Soruya dönersek ...

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

$\boxed{5}$ 5. *Türevinin Dönüşümü*

$$L\{f'(t)\} =?$$

$$L\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt = \int_0^\infty \underbrace{e^{-st}}_{u}\,\underbrace{f'(t)dt}_{dv}$$

$$= \lim_{R\to\infty}\left(\overbrace{e^{-sR}f(R)}^{0} - \overbrace{e^{-s.0}}^{1} f(0)\right) + s.\lim_{R\to\infty}\left(\int_0^{\infty} f(t)e^{-st}dt\right)$$

$$= 0 - f(0) + s.F(s)$$

$$ise$$

$$L\{f'(t)\} = s.L\{f(t)\} - f(0) = s.F(s) - f(0)$$

$$L\{f''(t)\} = L\left\{\left(f'(t)\right)'\right\} = s.L\{f'(t)\} - f'(0)$$

$$= s[L\{f(t)\} - f(0)] - f'(0)$$

$$\underbrace{\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}}_{2.\ Türevinin\ Laplace\ Dönüşümü}$$

$$\underbrace{\boxed{L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)}}_{3.\ Türevinin\ Laplace\ Dönüşümü}$$

$$\underbrace{\boxed{L\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{(n-1)}f(0) - sf^{(n-2)}(0) - \cdots - f^{(n-1)}(0)}}_{n.\ Türevinin\ Laplace\ Dönüşümü}$$

$\boxed{ÖRNEK\ 6}$ $y'' + y = e^{2t}\ ,\ \ y(0) = 0,\ \ y'(0) = 1$

Başlangıç değer probleminin Laplace Dönüşümünü alalım.

1\. *Adım: Her tarafın Laplace Dönüşümünü al!*

$$L\{y'' + y\} = L\{e^{2t}\}$$

Lineerlik dönüşümlerimiz var o zaman ayıracağız.

$$L\{y''\} + L\{y\} = L\{e^{2t}\}$$

$$\left(s^2 L\{y\} - sy(0) - y'(0)\right) + L\{y\} = \frac{1}{s-2}$$

$$L\{y\} = Y \; dersek$$

$$(s^2+1)Y(s) = 1 + \frac{1}{s-2} = \frac{s-1}{s-2}$$

$$\boxed{Y(s) = \frac{s-1}{(s-2)(s^2+1)}}$$

$$Şimdi \; Y \; den \; y \; (küçük \; y'ye) \; nasıl \; döneriz?$$

Ters Laplace Dönüşümü

$$L\{f(t)\} = F(s) \; ise$$

$$f(t)'ye \; F(s)'nin \; ters \; laplace \; dönüşümü \; denir \; ve$$

$$f(t) = L^{-1}\{F(s)\}$$

$$yazılır.$$

$$L\{f(t)\} = F(s) \Leftrightarrow f(t) = L^{-1}\{F(s)\}$$

$\boxed{ÖRNEK\ 7}$ $L\{1\} = \frac{1}{s}$ $idi,$ $L^{-1}\left\{\frac{1}{s}\right\} = 1 \; olur.$

$$L\{\sin 5t\} = \frac{5}{s^2+25} \ \ idi, \ \ L^{-1}\left\{\frac{5}{s^2+25}\right\} = \sin 5t \ \ olur.$$

$$L\{t^2\} = \frac{2}{s^3} \ \ idi, \ \ L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}L^{-1}\left\{\frac{2}{s^3}\right\}$$

$$= \frac{1}{2}t^2$$

Özellikler

$\boxed{1}$ *Lineerlik* Özelliği

$$L^{-1}\{k_1F_1(s)+k_2F_2(s)\} = k_1L^{-1}\{F_1(s)\} + k_2L^{-1}\{F_2(s)\}$$

$\boxed{2}$ 1. Öteleme Özelliği

$$L^{-1}\{F(s)\} = f(t) \ \ ise$$

$$L^{-1}\{F(s-a)\} = e^{at}f(t) \ \ 'dir.$$

$\boxed{ÖRNEK\ 1}$ $L^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t \ \ idi.$

$$L^{-1}\left\{\frac{2}{s^2-4s+8}\right\} = L^{-1}\left\{\frac{2}{(s-2)^2+2^2}\right\} = e^{2t}\sin 2t$$

$\boxed{3}$ 2. Öteleme Özelliği

$$L^{-1}\{F(s)\} = f(t)\ ise$$

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a), & t > a \\ 0 & , \ t \leq a \end{cases}$$

$$\boxed{ÖRNEK\ 2}$$

$$L^{-1}\left\{ e^{-\pi s} \underbrace{\frac{1}{s^2+1}}_{F(s)} \right\} = \begin{cases} \sin(t-\pi), & t > \pi \\ 0 & , \ t \leq \pi \end{cases}$$

$$f(t) = \sin t$$

$$\boxed{4}\ L^{-1}\{F(s)\} = f(t)\ ise$$

$$\boxed{L^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)}$$

$$\boxed{ÖRNEK\ 3}\ L^{-1}\left\{\frac{1}{s-1}\right\} = e^t\ \ idi$$

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s-1}\right)\right\} = L^{-1}\left\{-\frac{1}{(s-1)^2}\right\} = (-1)^1 L^{-1}\left\{\left\{\frac{1}{(s-1)^2}\right\}\right\} = -te^t$$

$$\boxed{ÖRNEK\ 4}\ L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{(-1)^1\frac{d}{ds}\left(\underbrace{\underbrace{\frac{1}{s^2+1}}_{F(s)}}_{f(t)=\sin t}\right)\right\}$$

$$= \frac{1}{2}(-1)^1\, t\sin t$$

$$Basit\ bir\ teorem\ var. Daha\ kolay\ bulabiliriz.$$

$$Önceki\ diferansiyel\ denklem\ örneğinde\ şöyle\ bulmuştuk.$$

$$L\{y(t)\} = Y(s) = \frac{s-1}{(s-2)(s^2+1)}$$

$$Her\ tarafın\ Ters\ Laplace\ Dönüşümünü\ alırsak$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)} = L^{-1}\left\{\frac{s-1}{(s-2)(s^2+1)}\right\}$$

$$y(t) = L^{-1}\left\{\frac{A}{s-2}+\frac{Bs+C}{s^2+1}\right\}$$

$$A,B,C\ =?$$

$$A(s^2+1)+(Bs+C)(s-2) = s-1$$

$$\underbrace{(A+B)}_{0}s^2+\underbrace{(-2B+C)}_{1}s+\underbrace{(A-2C)}_{-1} = s-1$$

$$A=\frac{1}{5},\quad B=-\frac{1}{5},\quad C=\frac{3}{5}$$

$$y(t) = \frac{1}{5} L^{-1} \left\{ \frac{1}{s-2} \right\} - \frac{1}{5} L^{-1} \left\{ \frac{s}{s^2+1} \right\} + \frac{3}{5} L^{-1} \left\{ \frac{1}{s^2+1} \right\}$$

$$\boxed{y(t) = \frac{1}{5} e^{2t} - \frac{1}{5} \cos t + \frac{3}{5} \sin t}$$