ASSIGINMENT # 02

LINEAR ALGEBRA

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Q: What is a Matrix Determinant?

MATRIX DETERMINANT:-

Determinant of a matrix is a special number which is defined only for square matrix (matrices which have same number of rows and column). It represents matrix in terms of real number which can be used in solving problems of linear equations and finding inverse of the matrix. It is calculated by getting cofactor of first each element of first row or column and then multiplying withe element with determinant of corresponding cofactor and then adding them with alternate signs.

Cofactor :- Cofactor of an element is a matrix we can get by removing row and column of that element from that matrix.

For example;

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - cb$$

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|B| = a(ei - hf) - b(di - gf) + c(df - ge)$$

if $A = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$

then $|A| = 4 \times 8 - 3 \times 6$

$= 14$

if $B = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$

$$|B| = 6 \times (-2 \times 7 - 5 \times 8) - 1(4 \times 7 - 5 \times 2) + 1(4 \times 8 - (-2 \times 2))$$

$$= 6 \times (-54) - 1(18) + 1(36)$$

$$= -306$$

PROPERTIES OF DETERMINANTS :-

① If A is an identity matrix of order m×m then its determinant i.e; |A| is always equals to 1

Example :-

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A| = (1)(1) - (0)(0)$$

$$= 1 - 0 = 1$$

② If two matrixes are of same size, then the determinant of product of both matrices is equal to product of determina[nts] of individual matrices.

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1\times5)+(2\times7) & (1\times6)+(2\times8) \\ (3\times5)+(4\times7) & (3\times6)+(4\times8) \end{bmatrix}$$

$$= \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$|AB| = (19\times50) - (43\times22)$$

$$= 950 - 946$$

$$= 4$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$|A| = 1 \times 4 - 3 \times 2$

$= 4 - 6$

$= -2$

$|B| = 5 \times 8 - 7 \times 6$

$= 40 - 42$

$= -2$

$|A||B| = -2 \times -2 = 4$

Hence $|AB| = |A||B|$.

(3) If any matrix is upper or lower trianglar matrix, then its determinant is product of all its diagonal entries.

Example:-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 3 \end{bmatrix}$$

$|A| = 1(2 \times 3 - 5 \times 0) - 0(4 \times 3 - 6 \times 0) + 0(4 \times 5 - 2 \times 6)$

$= 6 - 0 = 6$

$|A| = 1 \times 2 \times 3 = 6$ (Product of diagonal entries)

(4) If any square matrix has zero rows or zero columns then its determinant is always $(\text{zero})^n$

Example:-

$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $|A| = 1 \times 0 - 2 \times 0$

$= 0 - 0 = 0$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad |B| = 0 \times 2 - 0 \times 1$$

$$= 0 - 0 = 0.$$

(5) If the enhies of rows or columns of a mahix are same, then its determinant is always equals to zero.

Example:-

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$|A| = 1 \times 2 - 1 \times 2$$

$$= 2 - 2 = 0$$

(6) If each element of any row or column consists of two or more terms, then the determinant can be expressed as sum of two (or more) determinants

Example:-

$$C = \begin{bmatrix} 2+3 & 6 \\ 4+5 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 9 & 8 \end{bmatrix}$$

$$|C| = 5 \times 8 - 6 \times 9 = 40 - 54 = -14$$

$$A = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 6 \\ 5 & 8 \end{bmatrix}$$

$$|A| = 2 \times 8 - 4 \times 6 \qquad |B| = 3 \times 8 - 5 \times 6$$

$$= 16 - 24 \qquad = 24 - 30$$

$$= -8 \qquad = -6$$

$$|A| + |B| = -8 - 6 = -14$$

Hence $|C| = |A| + |B|$.

⑦ If we interchange any two rows of a square matrix then the sign of its determinant changes.

Example :-

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} , \quad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$|A| = 1\times 4 - 2\times 3$ $\qquad$ $|B| = 3\times 2 - 4\times 1$

$= 4-6$ $\qquad$ $= 6-4$

$= -2$ $\qquad$ $= 2$

⑧ If a determinant becomes zero on putting $x = \alpha$ then $(x-\alpha)$ is a factor of the determinant.)

Example :-

$$A = \begin{bmatrix} x & 1 & 2 \\ x+1 & 3 & 4 \\ x+2 & 5 & 6 \end{bmatrix} \quad \text{if} \quad x = 2$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}, \; |A| = 2(3\times 6-5\times 4) - 1(3\times 6 - 4\times 4) + 2(3\times 5 - 4\times 3) = 2(18-20) - 1(18-16) + 2(15-12)$$

$= 2(-2) - 1(2) + 2(3) = -4-2+6$

$= -6+6 = 0$

hence $(x-1)$ is a factor of determinant

⑨ If A is a square matrix, then if its row is multiplied by a constant k, then the constant can be taken out of determinant.

Example :-

$A = \begin{bmatrix} 3 & 6 \\ 2 & 3 \end{bmatrix}$ $\qquad$ $A = 3 \times \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$|A| = 3 \times 3 - 2 \times 6$

$= 9 - 12$

$= -3$

$|A| = \cancel{3} \times ((1 \times 3) - (2 \times 2))$

$= 3 \times (3 - 4)$

$= 3 \times (-1) = -3$

(10) If matrix A^{-1} is inverse of a matrix then $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

Example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $|A| = 1 \times 4 - 2 \times 3 = 4 - 6 = -2$

$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{(1)(4) - (3)(2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

$|A^{-1}| = (-1/2)(-2) - (3/2)(1) = 1 - 3/2 = -0.5$

$|A|^{-1} = \frac{1}{|A|} = \frac{1}{-2} = -0.5$

11) If the matrix A^T is the transpose of matrix A then

$|A^T| = |A|$

Example :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$|A| = 1 \times 4 - 3 \times 2$

$= -2$

$|A^T| = 1 \times 4 - 2 \times 3$

$= -2$