COMPLEX ANALYSIS

Complex analysis

Need to calculate square root of -ve Real number.

⇒ cartesian form

conjugate

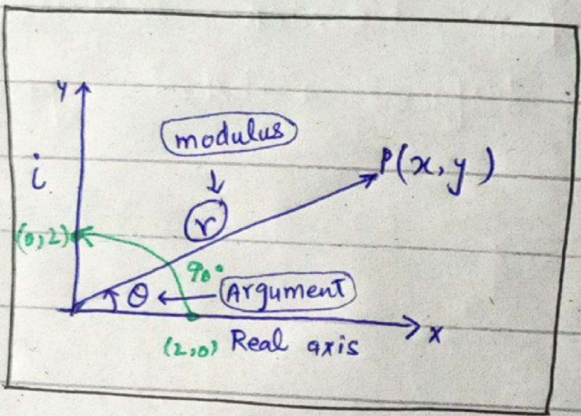
$$z = x + iy \Rightarrow \bar{z} = x + (-i)y = z^*$$

⇒ polar form

$$z = r\cos\theta + i\, r\sin\theta = re^{i\theta}$$

Graphical representation

Complex Argand plane



proporties

i. $z^* = z$ Real no.

ii. $z^* = -z$ pure i no.

iii. $\text{Real}(z) = x = \frac{1}{2}(z + z^*)$

iv. $\text{imaginary}(z) = \frac{1}{2i}(z - z^*) = y$

Modulus of complex number

$$|z| = r = \sqrt{(\text{real no.})^2 + (\text{imaginary no.})^2}$$

Argument

$$\theta = \tan^{-1}\left(\frac{\text{imaginary part}}{\text{real part}}\right)$$

θ may have infinite possible values.

function of complex ~~func~~ variable

$f(z) = u(x,y) + i\, v(x,y)$ (cartesian form)

$f(z) = u(r,\theta) + i\, v(r,\theta)$ (polar form)

Types

(i) single value function if $f(z)$ have only one value corresponding to each value of z. eg. z^2, $\frac{1}{z}$, $\sin(z)$

(ii) Multi value function $f(z)$ have multiple value corresponding to each value of z. eg: $\sqrt{z}$, $\sqrt[3]{z}$, $\ln(z)$

2 values, 3 value, Infinite value

How $\ln(z)$ is an ∞-valued function?

$\ln(r.e^{i\theta})$

$f(z) = \ln(z) \Rightarrow \ln[r.e^{i(\theta+2\pi n)}]$

$\because e^{i2n\pi} = \cos(2n\pi) + i\sin(2n\pi) = 1$

$\Rightarrow \ln(r) + \ln(e^{i(\theta+2\pi n)})$

$\because \ln e^x = x$

$f(z) = \ln(z) = \ln(r) + i(\theta + 2n\pi)$

for $n=0 \Rightarrow f(z) = \ln(r) + i\theta$

for $n=1 \Rightarrow f(z) = \ln(r) + i(\theta + 2\pi)$

$\vdots$

for $n=N \Rightarrow f(z) = \ln(r) + i(\theta + 2N\pi)$

Domain of def of function

The value of z for which the complex function f(z) is define.

example 1

$f(x) = 2x^3 + 1$

define for all 'z'

example 2

$f(z) = \frac{z^2+1}{z^2+4}$

not define at $z = \pm 2i$

complex exponential function

$f(z) = e^z = e^x \cos y + i e^x \sin y$ (cartesian form)

$$\boxed{\begin{aligned} e^z &= e^{x+iy} \\ &= e^x \cdot e^{iy} \\ &= e^x(\cos y + i \sin y) \end{aligned}}$$

Important points

i. e^z is single-value function.

ii. Domain of definiton of e^z is the entire finite complex plane.

iii. $e^z = 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 \ldots \frac{1}{n!} z^n$

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n}$$

iv. e^z is period function of z with $2\pi i$.

$$|e^z| = e^x \quad \text{and} \quad arg(e^z) = y$$

<u>proof</u>

we know

$$e^z = e^x(\cos y + i \sin y)$$

$$|e^z| = \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2}$$

$$= \sqrt{e^{2x}(\cos^2 y + \sin^2 y)}$$

$\boxed{\because \sin^2\theta + \cos^2\theta = 1}$

$$\boxed{|e^z| = \sqrt{e^{2x}(1)} = e^x}$$

we know

$$arg(e^z) = \tan^{-1}\left(\frac{\cancel{e^x} \sin y}{\cancel{e^x} \cos y}\right)$$

$$arg(e^z) = \tan^{-1}(\tan y)$$

$$\boxed{arg(e^z) = y}$$

Complex trignometery function

$\Rightarrow \sin z = \sin x \cosh y + i \cos x$ ~~tan~~ $\sinh y$

<u>Proof</u>

we know

cartesian

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}\left[e^{i(x+iy)} + e^{-i(x+iy)}\right]$$

$\boxed{i.i = 1}$

$$= \frac{1}{2i}\left[e^{ix} \cdot e^{-y} + e^{-ix} \cdot e^{y}\right]$$

$$= \frac{1}{2i}\left[e^{ix}\cdot e^{-y} - e^{-ix}\cdot e^{y}\right]$$

$$= \frac{1}{2i}\left[(\cos x + i\sin x)\, e^{-y} - (\cos x - i\sin x)\cdot e^{y}\right]$$

$$= \frac{1}{2i}\left[\cos x(\underbrace{e^{-y} - e^{y}}_{-2\sinh y}) + i\sin x\,(\underbrace{e^{-y} + e^{y}}_{2\cosh y})\right]$$

$$= \frac{1}{2i}\left[\cos x(-2\sinh y) + i\sin x\cdot(2\cosh y)\right]$$

$$= \frac{1}{2i}\left[2(-\cos x\cdot\sinh y) + 2(\sin x\ \cosh y)i\right]$$

$\boxed{\text{How?}}$

$$= \sin x\ \cosh y + i\cos x\ \sinh y$$

- $\sin z = \sin x\ \cosh y + i\cos x\ \sinh y$
- $\cos z = \sin x\ \cos hy - i\cos x\ \sinh y$

To find others solve

- $\tan z = \dfrac{\sin z}{\cos z}$, • $\cot z = \dfrac{\cos z}{\sin z}$
- $\sec = \dfrac{1}{\cos z}$, • $\csc = \dfrac{1}{\sin z}$

properties of complex trignometry function

i. ~~Sn~~ sin z & cos z are single-value functions.

ii. Domain for cos z & sin z is entire finite complex plane.

iv. $\cos z = 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 + \dots + \frac{1}{(2m)!} z^{2m}$

$$\boxed{\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} z^{2n}}$$

v. $\sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \dots + \frac{1}{(2m+1)} z^{2m+1}$

$$\boxed{\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}}$$

vi. sin z & cos z are periodic function with period 2π.

Differentiability of complex functions

i. $f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$

ii. $f(z)$ will be $f'(z)$ in nature

if limiting value will be independent of path along which $z \to 0$. ⁂

iii. Since z^n ($n =$ +ve integer) is differentiable at every point in the finite complex plane

So,

$$\sum_k a_k z^k \quad (k = +\text{ve integer})$$

will be differentiable in the finite complex plane.

iv. If the $f(z)$ is $f(z)$ in nature then the rule of diff for the real function are also valid for complex function.

for example

$$\sin 2z \;,\; \cos 5z \;,\; e^{3z} \;,\; e^{-z^2}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$2\cos 2z \;,\; -5\sin 5z \;,\; 3e^{3z} \;,\; -2z\, e^{z^2}$$

contains +ve integer power of z.

<u>Question</u> Check whether $f(z) = \bar{z}$ is diff or not.

we know

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{f}(\cancel{\bar{z}} + \Delta\bar{z}) - \cancel{f(\bar{z})}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z} \qquad \boxed{\frac{n}{0} = \infty}$$

one teardrop from my eye.

$$f'(z) = \lim_{z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \right]$$

Let we know

$z = x + iy$

$\Delta z = \Delta x + i\Delta y$

$$= \lim_{x \to 0} \left[\frac{\Delta x - im\Delta x}{\Delta x + im\Delta x} \right]$$

$\Delta y = m\Delta x + c$

$$= \lim_{x \to 0} \left(\frac{1 - im}{1 + im} \right)$$ only differential here

So, here 'm' is the slope of curve. that's mean $f(z) = \bar{z}$ is not path independent.

<u>Conclusion</u> $f(z) = \bar{z}$ is not differentiable in nature.

Complex Analytic Function

"Regular function" or "holomorphic function"

when a function said to be analytic?

A $f(z)$ said to be analytic at a point $z = z_0$ if

i. $f(z)$ is single valued (in nature)

ii. $f(z)$ is diff at $z = z_0$. as well as at each point in small neighbourdhood of the point $z = z_0$

A $f(z)$ is said to analytic in region R if :-

i. $f(z)$ is single-valued in nature.

ii. $f(z)$ is differentiable at all point in region R.

Important concept

① If $f(z)$ in analytic everywhere in region R, it said to be "entire function".

② The points where $f(z)$ is not analytical are know as singular points of complex function

③ Any $f(z)$ with +ve power values of z is analytical at every point in the finite complex plane

$$\sum_{k} a_k z^k \quad (k = +ve \text{ integer})$$

④ Any $f(z)$ which contains $\bar{z}$ or z^* will not be analytic in the entire finite complex plane.

Question #

Check whether $f(x) = (3x + 2iy - 5)^{10}$ is analytic or not.

$$f(x,y) = (3x + 2iy - 5)^{10}$$

In form of $x + iy$

$$f(x,y) = (x + 2x + 2iy - 5)^{10}$$

$$f(z) = (\underset{\downarrow}{x} + 2\underbrace{(x+iy)}_{z} - 5)^{10}$$

$$f(z) = \left(\frac{1}{2}(z + z^*) + 2z + 5\right)^{10}$$

$\Rightarrow$ Not analytic. (in nature)

⑤ If $f(z)$ & $g(z)$ are analytic in nature then $\boxed{f(z)g(z)}$, $\boxed{f(z) \pm g(z)}$ are also analytic in nature

Analytic in nature
$$\begin{cases} z^{10} + \cos(5z) & \text{// Addition} \\ \sin(z^2) - e^{-z^3} & \text{// substraction} \\ \cos(5z) \cdot \cosh(z) & \text{// multiplication} \end{cases}$$

⑥ Also, $\frac{f(z)}{g(z)}$ analytic if $g(z) \neq 0$

⇒ The points where $g(z) = 0$ are called singular points.

Question

Check wheter $\phi(z) = \frac{z^2-1}{(z^2+4)^2(z-3)^5}$ analytic or not.

$g(z) = (z^2+4)^2 (z-3)^5$

↓ ↓

$\pm 2i$, 3 (singular points)

$\phi(z)$ is diff except $\pm 2i$, 3.

Question

Check whether $\phi(z) = \frac{\cot z}{z^2}$

analatic or not

$\Rightarrow \frac{\cos z}{\sin z (z^2)}$

$\sin z = 0$

$z = n\pi$ $(0, \pm 1, \pm 2, \ldots)$

function is analytic except

$z = n\pi$ (where 'n' is integer)

Having No Regrets that
I really want

Cauchy Riemann (C-R) Equations

C-R eqs in cartesian coordinates
Suppose that the real part $u(x,y)$ and imaginary part $v(x,y)$ of the complex function $f(z)$ are ① continous and have ② continous partial derivatives in region R. The $f(z)$ will be analytic in region R. if

$$③\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad ④\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R eqs in ~~car~~ polar coordinates
Suppose that the real part $u(r,\theta)$ & imaginary part $v(r,\theta)$ of $f(z)$ are ① continous & ② have continous partial derivative in region R. then $f(z)$ will be Analytic in R
if

$$③\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad ④\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Important notes

While checking the continuity of real part u and imaginary part v and it's partial derivative, remeber the following

i) polynomial function are continous in nature.

ii) functions that are continous in nature

* sin, * cos, * cosh, * sinh, * exponential

Using C-R check function analytic or not

Question

$$f(z) = z^3$$

$$\Rightarrow f(z) = (x + iy)^3$$

$$= (x)^3 + 3(x^2)(iy) + 3(x)(iy)^2 + (iy)^3$$

$$\Rightarrow \quad = \underbrace{(x^3 - 3xy^2)}_{u} + \underbrace{(3x^2y - y^3)}_{v} i$$

$$\boxed{\begin{array}{ll} \dfrac{\partial v}{\partial x} = 6xy , & \dfrac{\partial u}{\partial y} = -6xy \\ \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 , & \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 \end{array}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \checkmark$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy$$

All condition satisfied $\therefore f(z)$ is analytic in nature

Question

$$f(z) = |z|^2$$

$$f(z) = |z|^2 = \sqrt{x^2+y^2}$$

$$= x^2+y^2 \Rightarrow \underbrace{(x^2+y^2)}_{u} + \underbrace{(0)}_{v} i$$

$$\boxed{\begin{array}{ll} \frac{\partial u}{\partial x} = 2x & \frac{\partial v}{\partial y} = 0 \\ \frac{\partial v}{\partial x} = 0 & \frac{\partial u}{\partial y} = 2y \end{array}}$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow 2x = 0 \Rightarrow x = 0 \\ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \Rightarrow 2y = 0 \Rightarrow y = 0 \end{array}\right\}$$

last 2 conditions not satisfied. except origin.

① removal , ② essential , ③ non-essential

Essential singular points

Example

$$f(z) = z^2 \cos\left(\frac{1}{z}\right)$$ → not defined at $z=0$

Non-essential singular point

Laurent series expansion

Problem session on
Analytic function