Question -1 : A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 4% of items produced by Machine I are defective and 5% produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II.

Solution:

Let A_1 be the event that the items are produced by Machine-I, A_2 be the event that items are produced by Machine-II. Let B be the event of drawing a defective item. Now we are asked to find the conditional probability P (A_2 / B). Since A_1 , A_2 are mutually exclusive and exhaustive events, by Bayes' theorem,

We have,

$P(A_1) = 0.40$, $P(B / A_1) = 0.04$

$P(A_2) = 0.60$, $P(B / A_2) = 0.05$

$$P(A_2 / B) = \frac{P(A_2)\, P(B / A_2)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2)}$$

$$P(A_2 / B) = \frac{(0.60)(0.05)}{(0.40)(0.04)+(0.60)(0.05)} = \frac{15}{23}.$$

Question -2:

A construction company employs 2 executive engineers. Engineer-1 does the work for 60% of jobs of the company. Engineer-2 does the work for 40% of jobs of the company. It is known from the past experience that the probability of an error when engineer-1 does the work is 0.03, whereas the probability of an error in the work of engineer-2 is 0.04. Suppose a serious error occurs in the work, which engineer would you guess did the work?

Solution

Let A_1 and A_2 be the events of job done by engineer-1 and engineer-2 of the company respectively. Let B be the event that the error occurs in the work.

We have to find the conditional probability

$P(A_1 / B)$ and $P(A_2 / B)$ to compare their errors in their work.

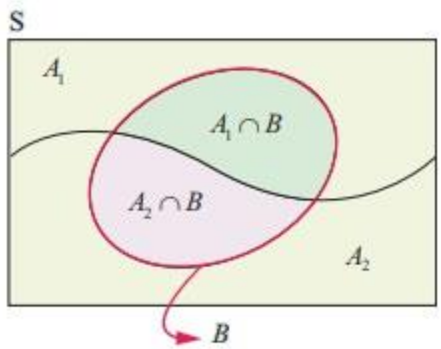
From the given information, we have

$P(A_1) = 0.60$, $P(B / A_1) = 0.03$

$P(A_2) = 0.40$, $P(B / A_2) = 0.04$

A_1 and A_2 are mutually exclusive and exhaustive events.

Applying Bayes' theorem,



$$P(A_1 / B) = \frac{P(A_1)\, P(B / A_1)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2)}$$

$$= \frac{(0.60)(0.03)}{(0.60)(0.03) + (0.40)(0.04)}$$

$$P(A_1 / B) = \frac{9}{17}.$$

$$P(A_2 / B) = \frac{P(A_2)\, P(B / A_2)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2)}$$

$$P(A_2 / B) = \frac{(0.40)(0.04)}{(0.60)(0.03) + (0.40)(0.04)}$$

$$P(A_2 / B) = \frac{8}{17}.$$

Since $P(A1/B) > P(A2/B)$, the chance of error done by engineer-1 is greater than the chance of error done by engineer-2. Therefore one may guess that the serious error would have been be done by engineer-1.

Question- 3 :

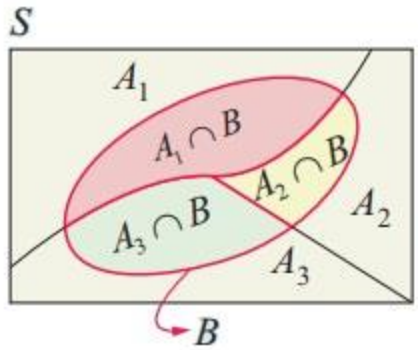
The chances of X, Y and Z becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus scheme will be introduced if X, Y and Z become managers are 0.3, 0.5 and 0.4 respectively. If the bonus scheme has been introduced, what is the probability that Z was appointed as the manager?

Solution

Let A_1, A_2 and A_3 be the events of X, Y and Z becoming managers of the company respectively. Let B be the event that the bonus scheme will be introduced.

We have to find the conditional probability $P(A_3 / B)$.

Since A_1, A_2 and A_3 are mutually exclusive and exhaustive events, applying Bayes' theorem



We have
$$P(A_3 / B) = \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)}$$

$$P(A_1) = \frac{4}{9}, \quad P(B/A_1) = 0.3$$

$$P(A_2) = \frac{2}{9}, \quad P(B/A_2) = 0.5$$

$$P(A_3) = \frac{3}{9}, \quad P(B/A_3) = 0.4$$

$$P(A_3 / B) = \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)}$$

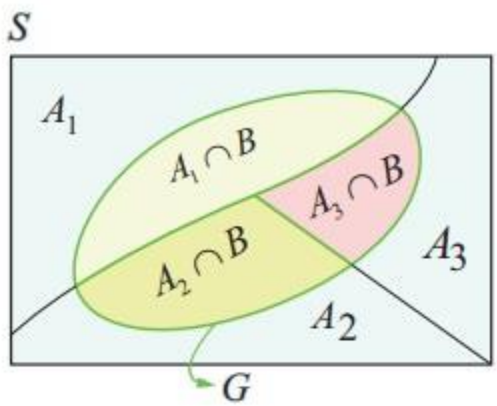
$$P(A_3 / B) = \frac{\left(\frac{3}{9}\right)(0.4)}{\left(\frac{4}{9}\right)(0.3) + \left(\frac{2}{9}\right)(0.5) + \frac{3}{9}(0.4)}$$
$$= \frac{12}{34} = \frac{6}{17}.$$

Question -4 :

A consulting firm rents car from three agencies such that 50% from agency *L*, 30% from agency *M* and 20% from agency *N*. If 90% of the cars from *L*, 70% of cars from M and 60% of the cars from N are in good conditions (i) what is the probability that the firm will get a car in good condition? (ii) if a car is in good condition, what is probability that it has come from agency N?

Solution

Let A_1 , A_2 , and A_3 be the events that the cars are rented from the agencies X, Y and Z respectively.



Let G be the event of getting a car in good condition.

We have to find

(i) the total probability of event G that is, $P(G)$

(ii) find the conditional probability A_3 given G that is, $P(A_3 / G)$ We have

$P(A_1) = 0.50$, $P(G / A_1) = 0.90$

$P(A_2) = 0.30$, $P(G / A_2) = 0.70$

$P(A_3) = 0.20$, $P(G / A_3) = 0.60$.

(i) Since A_1 , A_2 and A_3 are mutually exclusive and exhaustive events and G is an event in S, then the total probability of event G is P(G).

$$P(G) = P(A_1) P(G / A_1) + P(A_2) P(G / A_2) + P(A_3) P(G / A_3)$$

$$P(G) = (0.50)(0.90) + (0.30)(0.70) + (0.20)(0.60)$$

$$P(G) = 0.78.$$

(ii) The conditional probability A_3 given G is $P(A_3 / G)$

By Bayes' theorem,

$$P(A_3/G) = \frac{P(A_3)\,P(G/A_3)}{P(A_1)\,P(G/A_1)\;+P(A_2)\,P(G/A_2)+P(A_3)\,P(G/A_3)}$$

$$P(A_3/G) = \frac{(0.20)(0.60)}{(0.50)(\,0.90)+(0.30)(0.70)+(0.20)\;(0.60)}$$

$$= \frac{2}{13}.$$

Question -5:

A rare genetic disease is discovered. Although only one in a million people carry it, you consider getting screened. You are told that the genetic test is extremely good; it is 100% sensitive (it is always correct if you have the disease) and 99.99% specific (it gives a false positive result only 0.01% of the time). Having recently learned Bayes' theorem, you decide not to take the test. Why?

(From Durbin et.al. "Biological Sequence Analysis", Cambridge University Press, 1998)

Solution:

Bayes' Theorem states that for events X and Y:

$$P(X|Y)=P(Y|X)*P(X)/P(Y).$$

We want to know the probability of being healthy(X) given the positive test(PT) results(Y).

According to the Bayes' Theorem,

$$P(healthy|PT)=P(PT|healthy)*P(healthy)/P(PT).$$

From the problem we know that

$$P(healthy)=1-0.000001=0.999999$$

and getting a false positive

$$P(PT|healthy)=0.0001.$$

The only unknown in the formula above is the probability of having a positive test P(PT). It can be calculated using the definition of *marginal* probability

$$P(Y)=P(Y|Z_1)*P(Z_1)+...+P(Y|Z_n)*P(Z_n),$$

where Z_i, i=1...n are *all* possible events. In our case there are only two possible events: "being healthy" and "being sick". Therefore

$$P(PT)=P(PT|healthy)*P(healthy)+P(PT|sick)*P(sick).$$

From the problem we know that

$$P(PT|sick)=1.0$$

(test is always correct in presence of the disease) and

$$P(sick)=0.000001.$$

Substituting the numbers into the formula we get

$$P(PT)=0.0001*0.999999+1.0*0.000001=0.000101.$$

Finally,

$$P(healthy|PT)=0.0001*0.999999/0.000101=0.990098,$$

that is very close to 1.

So, the probability of still being healthy given that the results of the test turned positive is above 99%. That is a good reason for not taking the test.

Question -6:

Of the one million squirrels on MIT's campus most are good-natured. But one hundred of them are pure evil! An enterprising student in Course 6 develops an "Evil Squirrel Alarm" which she offers to sell to MIT for a passing grade. MIT decides to test the reliability of the alarm by conducting trials.

- When presented with an evil squirrel, the alarm goes off 99% of the time.
- When presented with a good-natured squirrel, the alarm goes off 1% of the time.

(a) If a squirrel sets off the alarm, what is the probability that it is evil?

(b) Should MIT co-opt the patent rights and employ the system?

Solution:

One solution

(This is a base rate fallacy problem)

We are given:

$$P(\text{nice}) = 0.9999, \qquad P(\text{evil}) = 0.0001 \text{ (base rate)}$$

$$P(\text{alarm} \mid \text{nice}) = 0.01, \quad P(\text{alarm} \mid \text{evil}) = 0.99$$

$$\begin{aligned} P(\text{evil} \mid \text{alarm}) &= \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm})} \\ &= \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm} \mid \text{evil}) P(\text{evil}) + P(\text{alarm} \mid \text{nice}) P(\text{nice})} \\ &= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.01)(0.9999)} \\ &\approx 0.01 \end{aligned}$$

Evil Squirrels Solution

answer: (a) This is the same solution as in the slides above, but in a more compact notation. Let E be the event that a squirrel is evil. Let A be the event that the alarm goes off. By Bayes' Theorem, we have:

$$
\begin{aligned}
P(E \mid A) &= \frac{P(A \mid E)P(E)}{P(A \mid E)P(E) + P(A \mid E^c)P(E^c)} \\
&= \frac{.99\frac{100}{1000000}}{.99\frac{100}{1000000} + .01\frac{999900}{1000000}} \\
&\approx .01.
\end{aligned}
$$

(b) No. The alarm would be more trouble than its worth, since for every true positive there are about 99 false positives.

Question -7:

Table Question: Dice Game

1. The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other.
2. The Roller selects one of the Randomizer's fists and covertly takes the die.
3. The Roller rolls the die in secret and reports the result to the table.

Given the reported number, what is the probability that the 6-sided die was chosen? (Find the probability for each possible reported number.)

answer: If the number rolled is 1-6 then $P(\text{six-sided}) = 4/7$.
If the number rolled is 7 or 8 then $P(\text{six-sided}) = 0$.

Explanation on next page

Solution:

Dice Solution

This is a Bayes' formula problem. For concreteness let's suppose the roll was a 4. What we want to compute is $P(\text{6-sided}|\text{roll 4})$. But, what is easy to compute is $P(\text{roll 4}|\text{6-sided})$.
Bayes' formula says

$$\begin{aligned} P(\text{6-sided}|\text{roll 4}) &= \frac{P(\text{roll 4}|\text{6-sided})P(\text{6-sided})}{P(4)} \\ &= \frac{(1/6)(1/2)}{(1/6)(1/2)+(1/8)(1/2)} = 4/7. \end{aligned}$$

The denominator is computed using the law of total probability:

$$P(4) = P(4|\text{6-sided})P(\text{6-sided}) + P(4|\text{8-sided})P(\text{8-sided}) = \frac{1}{6}\cdot\frac{1}{2}+\frac{1}{8}\cdot\frac{1}{2}.$$

Note that any roll of 1,2,...6 would give the same result. A roll of 7 (or 8) would give clearly give probability 0. This is seen in Bayes' formula because the term $P(\text{roll 7}|\text{6-sided}) = 0$.

Question -8 :

EXAMPLE

If a single card is drawn from a standard deck of playing cards, the probability that the card is a king is 4/52, since there are 4 kings in a standard deck of 52 cards. Rewording this, if King is the event "this card is a king," the prior probability $P(\text{King}) = \frac{4}{52} = \frac{1}{13}$.

If evidence is provided (for instance, someone looks at the card) that the single card is a face card, then the posterior probability $P(\text{King} \mid \text{Face})$ can be calculated using Bayes' theorem:

$$P(\text{King} \mid \text{Face}) = \frac{P(\text{Face} \mid \text{King})}{P(\text{Face})}P(\text{King}).$$

Since every King is also a face card, $P(\text{Face} \mid \text{King}) = 1$. Since there are 3 face cards in each suit (Jack, Queen, King) , the probability of a face card is $P(\text{Face}) = \frac{3}{13}$. Combining these gives a likelihood ratio of $\frac{1}{\frac{3}{13}} = \frac{13}{3}$.

Using Bayes' theorem gives $P(\text{King} \mid \text{Face}) = \frac{13}{3}\frac{1}{13} = \frac{1}{3}$. □

Question-9:

EXAMPLE

A family has two children. Given that one of the children is a boy, what is the probability that both children are boys?

We assume that the probability of a child being a boy or girl is $\frac{1}{2}$. We solve this using Bayes' theorem. We let B be the event that the family has one child who is a boy. We let A be the event that both children are boys. We want to find $P(A \mid B) = \frac{P(B|A) \times P(A)}{P(B)}$. We can easily see that $P(B \mid A) = 1$. We also note that $P(A) = \frac{1}{4}$ and $P(B) = \frac{3}{4}$. So $P(A \mid B) = \frac{1 \times \frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$. □

Question -10:

EXAMPLE

A family has two children. Given that one of the children is a boy, and that he was born on a Tuesday, what is the probability that both children are boys?

Your first instinct to this question might be to answer $\frac{1}{3}$, since this is obviously the same question as the previous one. Knowing the day of the week a child is born on can't possibly give you additional information, right?

Let's assume that the probability of being born on a particular day of the week is $\frac{1}{7}$ and is independent of whether the child is a boy or a girl. We let B be the event that the family has one child who is a boy born on Tuesday and A be the event that both children are boys, and apply Bayes' Theorem. We notice right away that $P(B \mid A)$ is no longer equal to one. Given that there are 7 days of the week, there are 49 possible combinations for the days of the week the two boys were born on, and 13 of these have a boy who was born on a Tuesday, so $P(B \mid A) = \frac{13}{49}$. $P(A)$ remains unchanged at $\frac{1}{4}$. To calculate $P(B)$, we note that there are $14^2 = 196$ possible ways to select the gender and the day of the week the child was born on. Of these, there are $13^2 = 169$ ways which do not have a boy born on Tuesday, and $196 - 169 = 27$ which do, so $P(B) = \frac{27}{196}$. This gives is that $P(A \mid B) = \frac{\frac{13}{49} \times \frac{1}{4}}{\frac{27}{196}} = \frac{13}{27}$. □

Note: This answer is certainly not $\frac{1}{3}$, and is actually much closer to $\frac{1}{2}$.

Question-11:

- Pam put in 15 paintings, 4% of her works have won First Prize.
- Pia put in 5 paintings, 6% of her works have won First Prize.
- Pablo put in 10 paintings, 3% of his works have won First Prize.

What is the chance that Pam will win First Prize?

$$P(\text{Pam}|\text{First}) = \frac{P(\text{Pam})P(\text{First}|\text{Pam})}{P(\text{Pam})P(\text{First}|\text{Pam}) + P(\text{Pia})P(\text{First}|\text{Pia}) + P(\text{Pablo})P(\text{First}|\text{Pablo})}$$

Put in the values:

$$P(\text{Pam}|\text{First}) = \frac{(15/30) \times 4\%}{(15/30) \times 4\% + (5/30) \times 6\% + (10/30) \times 3\%}$$

Multiply all by 30 (makes calculation easier):

$$\begin{aligned} P(\text{Pam}|\text{First}) &= \frac{15 \times 4\%}{15 \times 4\% + 5 \times 6\% + 10 \times 3\%} \\ &= \frac{0.6}{0.6 + 0.3 + 0.3} \\ &= 50\% \end{aligned}$$

A good chance!

Pam isn't the most successful artist, but she did put in lots of entries.

Question- 12:

Example 1:Bag I contains 4 white and 6 black balls while another Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags and it is found to be black. Find the probability that it was drawn from Bag I.

Solution:Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a black ball.

Then,$P(E_1) = P(E_2) = \frac{1}{2}$

Also,$P(A|E_1) = P$(drawing a black ball from Bag I) = $\frac{6}{10} = \frac{3}{5}$

$P(A|E_2) = P$(drawing a black ball from Bag II) = $\frac{3}{7}$

By using Bayes' theorem, the probability of drawing a black ball from bag I out of two bags,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{5}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{3}{5}} = \frac{7}{12}$$

Question – 13:

Example 2:A man is known to speak truth 2 out of 3 times. He throws a die andreports that number obtained is a four. Find the probability that the number obtained is actually a four.

Solution:Let A be the event that the man reports that number four is obtained.

Let E_1 be the event that four is obtained and E_2 be its complementary event.

Then, $P(E_1)$ = Probability that four occurs = $\frac{1}{6}$

$P(E_2)$ = Probability that four does not occurs = $1 - P(E_1) = 1 - \frac{1}{6} = \frac{5}{6}$

Also, $P(A|E_1)$ = Probability that man reports four and it is actually a four = $\frac{2}{3}$

$P(A|E_2)$ = Probability that man reports four and it is not a four = $\frac{1}{3}$

By using Bayes' theorem, probability that number obtained is actually a four,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)} = \frac{\frac{1}{6} \times \frac{2}{3}}{\frac{1}{6} \times \frac{2}{3} + \frac{5}{6} \times \frac{1}{3}} = \frac{2}{7}$$