Set Theory: (Week 1- Lecture1)

Video segment 1:

Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. There are some set expressions that is given below:

The following notations is more common in Discrete Mathematics.

- The empty set, denoted ∅, is the set that has no element.
- N := {1, 2, 3, 4……..}, the set of Natural numbers;
- W := {0, 1, 2, . . .}, the set of whole numbers
- Z := {0, 1, −1, 2, −2, . . .}, the set of Integers;
- Q := { p q : p, q ∈ Z, q 6= 0}, the set of Rational numbers;
- R := the set of Real numbers; and
- C := the set of Complex numbers. We start with the basic set theory.

A set is typically expressed by curly braces, { } enclosing its elements. If A is a set and a is an element of it, we write a ∈ A. The fact that a is not an element of A is written as a 6∈ A. For instance, if A is the set {1, 4, 9, 2}, then 1 ∈ A, 4 ∈ A, 2 ∈ A and 9 ∈ A. But 7 6∈ A, π 6∈ A, the English word 'four' is not in A, etc.

- The set A = {1, 2, 3, 4} in the predicate notation can be written as i. A = {x ∈ N : 1 ≤x ≤ 4} where, x is natural number.
- The set B= $\{2^1, 2^2, 2^3, 2^4\}$ in the predicate notation can be written as i. A = {x ∈ N : $x=2^k$}
  Where, k ∈ N is natural number and 1 ≤k ≤ 4.

Set Intervals:

Closed Set Intervals:

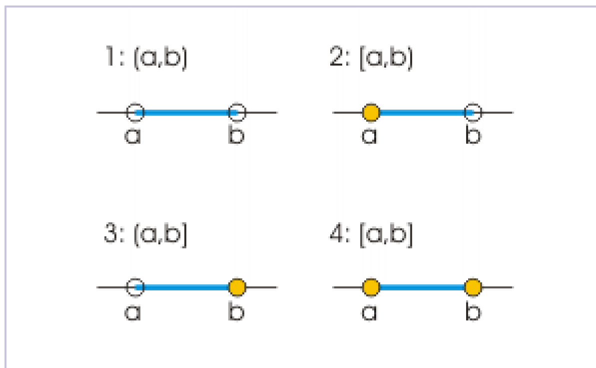
closed interval [*a*, *b*]

A closed interval is an interval that includes all of its limit points. If the endpoints of the interval are finite numbers a and b, then the interval {x : a ≤ x ≤ b} is denoted [a, b].

Open Set Intervals:

An open interval does not include its endpoints and is indicated with parentheses. In case of (a,b), the interval will represent as {x : a < x < b)

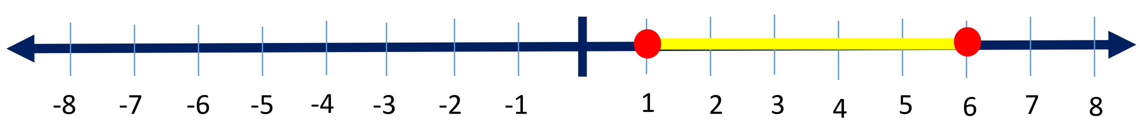
To indicate that only one endpoint of an interval is included in that set, both symbols will be used. For example, the interval of numbers between 1 and 5, including 1 but excluding 5, is written as [1,5). The interval will represent as {x : 1≤ x < 5)



Example of Open and closed ended intervals:**Closed Set intervals:**

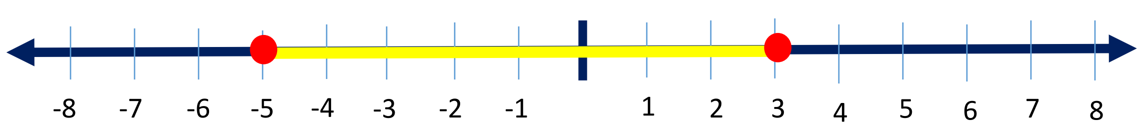
Example 1: [1,6]

Answer: The expression is {x : 1≤ x ≤ 2} .



Example 2: [-5,3]

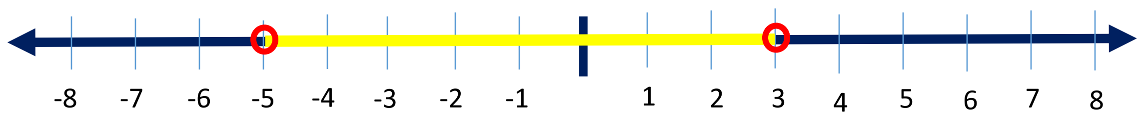
Answer: The expression is {x : -5≤ x ≤ 3}



Open Set intervals:

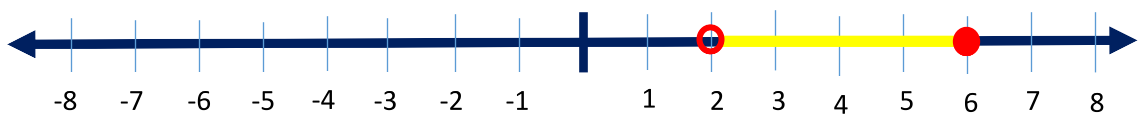
Example 3: (-5,3)

Answer: The expression is {x : -5< x < 3}



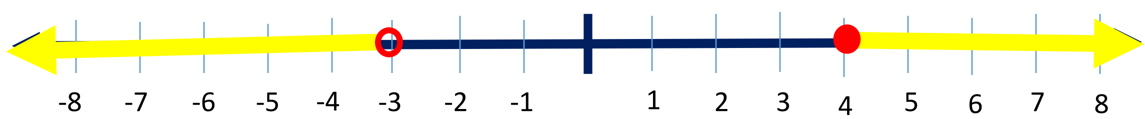
Example 4: (2,6]

Answer: The expression is {x : 2< x ≤ 6}



Example 5: (-∞,3) U [4, ∞)

Answer: The expression is {x : x < 3 and x ≥4 }

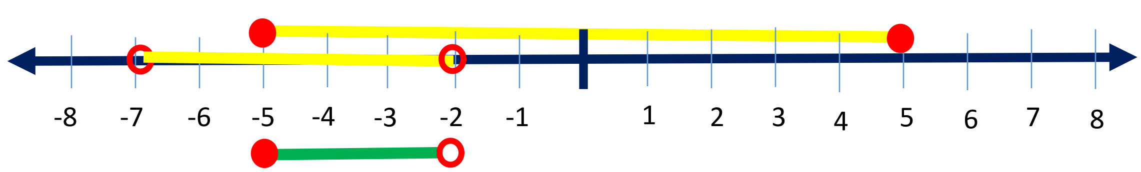


Example 5: (-∞,3) ∩ [4, ∞)

Answer: The answer will be empty set.

Example 5: (-7,-2) ∩ [-5, 5]

Answer: The expression is {x : -7<x <- 2 ∩ -5≤ x ≤ 5 } or {x : -5≤ x < -2 }



-5≤ x < -2 or [-5,-2)

Video segment 2:

Empty Set:

The set S that contains no element is called the empty set or the null set and is denoted by { } or ∅. A set that has only one element is called a singleton set.

Universal Set:

A **Universal Set** is the set of all elements under consideration, denoted by U. All other sets are subsets of the universal set.

Let , A={1,3,5}

B={3,6}

C={7,8,9}

Let U be the universal set Then, the complement Universal set will be U={1, 2, 3, 5, 6, 7, 8, 9}

Equal Set:

Two sets A and B can be equal only if each element of set A is also the element of the set B. Also if two sets are the subsets of each other, they are said to be equal. This is represented by:

$A = B$

$A \subset B$ and $B \subset A \Leftrightarrow A = B$

If the condition discussed above is not met, then the sets are said to be unequal. This is represented by

$A \neq B$

If $P = \{1, 3, 9, 5, -7\}$ and $Q = \{5, -7, 3, 1, 9,\}$, then $P = Q$. It is also noted that no matter how many times an element is repeated in the set, it is only counted once. Also, the order doesn't matter for the elements in a set. So, to rephrase in terms of cardinal number, we can say that

SubSet:

Suppose X is the set such that whenever $x \in X$, then $x \in Y$ as well. Here, X is said to be a subset of the set Y , and is denoted by $X \subseteq Y$.

Example: the set {1, 2, 3, 4, 5}

A **subset** of this is {1, 2, 3}. Another subset is {3, 4} or even another is {1}, etc.

But {1, 6} is **not** a subset, since it has an element (6) which is not in the parent set.

Common Symbols Used in Set Theory

In the examples C = {1, 2, 3, 4} and D = {3, 4, 5}

Symbol	Meaning	Example
{ }	Set: a collection of elements	{1, 2, 3, 4}
A ∪ B	Union: in A or B (or both)	C ∪ D = {1, 2, 3, 4, 5}
A ∩ B	Intersection: in both A and B	C ∩ D = {3, 4}
A ⊆ B	Subset: every element of A is in B.	{3, 4, 5} ⊆ D
A ⊂ B	Proper Subset: every element of A is in B, but B has more elements.	{3, 5} ⊂ D
A ⊄ B	Not a Subset: A is not a subset of B	{1, 6} ⊄ C
A ⊇ B	Superset: A has same elements as B, or more	{1, 2, 3} ⊇ {1, 2, 3}
A ⊃ B	Proper Superset: A has B's elements and more	{1, 2, 3, 4} ⊃ {1, 2, 3}
A ⊅ B	Not a Superset: A is not a superset of B	{1, 2, 6} ⊅ {1, 9}
A^c	Complement: elements not in A	D^c = {1, 2, 6, 7} When $\mathbb{U}$ = {1, 2, 3, 4, 5, 6, 7}
A − B	Difference: in A but not in B	{1, 2, 3, 4} − {3, 4} = {1, 2}
a ∈ A	Element of: *a* is in A	3 ∈ {1, 2, 3, 4}
b ∉ A	Not element of: *b* is not in A	6 ∉ {1, 2, 3, 4}
Ø	Empty set = {}	{1, 2} ∩ {3, 4} = Ø
$\mathbb{U}$	Universal Set: set of all possible values (in the area of interest)	
P(A)	Power Set: all subsets of A	P({1, 2}) = { {}, {1}, {2}, {1, 2} }
A = B	Equality: both sets have the same members	{3, 4, 5} = {5, 3, 4}
A×B	Cartesian Product (set of ordered pairs from A and B)	{1, 2} × {3, 4} = {(1, 3), (1, 4), (2, 3), (2, 4)}
|A|	Cardinality: the number of elements of set A	|{3, 4}| = 2
|	Such that	{ *n* | *n* > 0 } = {1, 2, 3,...}
:	Such that	{ *n* : *n* > 0 } = {1, 2, 3,...}
∀	For All	∀x>1, x^2>x
∃	There Exists	∃ x | x^2>x
∴	Therefore	*a*=*b* ∴ *b*=*a*
$\mathbb{N}$	Natural Numbers	{1, 2, 3,...} or {0, 1, 2, 3,...}
$\mathbb{Z}$	Integers	{..., −3, −2, −1, 0, 1, 2, 3, ...}
$\mathbb{Q}$	Rational Numbers	

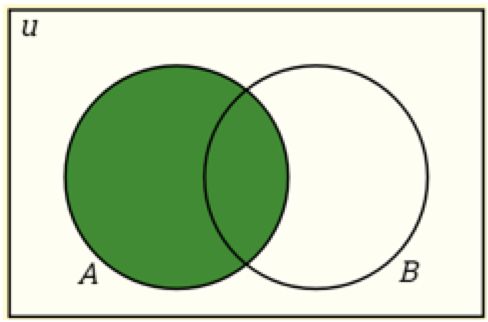
Venn Diagram:

A Venn Diagram is a pictorial representation of the relationships between sets.

We can represent sets using **Venn diagrams.** In a Venn diagram, the sets are represented by shapes; usually circles or ovals. The elements of a set are labeled within the circle. In the figure A and B set is represented in a different relationship.

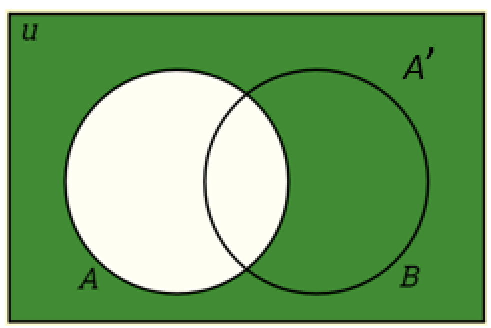
The following diagrams show the set operations and Venn Diagrams for Complement of a Set, Disjoint Sets, Subsets, Intersection and Union of Sets.

The set of all elements being considered is called the **Universal Set (U)** and is represented by a rectangle. Here, Colored portion represent the element of Set A.



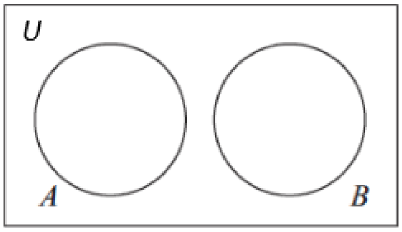
Set *A*

The **complement of A, A'**, is the set of elements in U but not in A. A' $=\{x \mid x \in U \text{ and } x \notin A\}$. Colored section is A’.

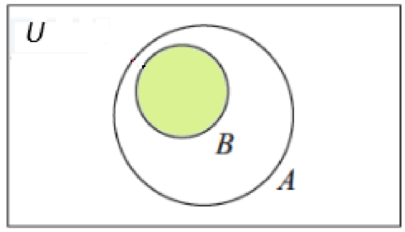


A′ the complement of *A*

Set A and B are **disjoint sets** if they do not share any common elements.
B is a **proper subset** of A. This means B is a subset of A, but B ≠ A.

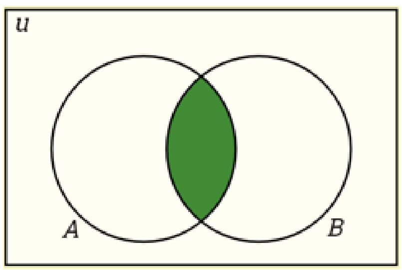


A and *B* are disjoint sets

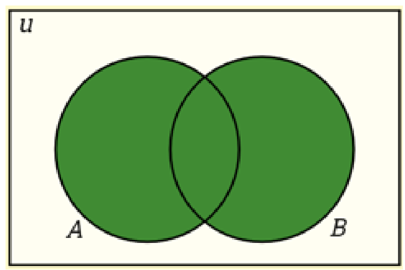


B is proper subset of *A* $B \subset A$

The **intersection of A and B** is the set of elements in both set A and set B. A ∩ B = {x| x ∈ A and x ∈ B}
The **union of A and B** is the set of elements in set A or set B. A ∪ B = {x| x ∈ A or x ∈ B}
A ∩ ∅ = ∅
A ∪ ∅ = A

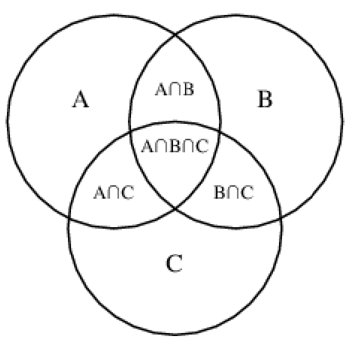


Both A and B $A \cap B$
A intersect *B*



Either A or B $A \cup B$
A union *B*

The order-three diagram consists of three symmetrically placed mutually intersecting circles comprising a total of eight regions. The regions labeled A, B, and C consist of members which are only in one set and no others, the three regions labelled $A \cap B$, $A \cap C$, and $B \cap C$ consist of members which are in two sets but not the third, the region $A \cap B \cap C$ consists of members which are simultaneously in all three, and no regions occupied represents $\emptyset$



Video segment 3:

Cardinality of set:

Total number of elements of a set is known as cardinality or size of the set. If a set A={2,4,5,8,10} then the cardinality of set A is 5. Which is denoted as |A| or n(A). So, |A|=5.

Finite set

Finite sets are the sets having a finite/countable number of members. Finite sets are also known as **countable sets** as they can be counted. The process will run out of elements to list if the elements of this set have a finite number of members.

Examples of finite sets:

P={ 0, 3, 6, 9, …, 99}

Q={ a : a is an integer, 1<a<10}

A set of all English Alphabets (because it is countable).

Another example of a Finite set:

A set of months in a year.
M = {January, February, March, April, May, June, July, August, September, October, November, December}

n (M) = 12

It is a finite set because the number of elements is countable.

Cardinality of Finite Set

If 'a' represents the number of elements of Set A, then the cardinality of a finite set is **n(A)=a**.

So, the cardinality of the above set is 26, because the number of elements(alphabets) is 26.

Hence, n (A) = 26.

Similarly, for a set containing the months in a year will have a cardinality of 12.

So, this way we can list all the elements of any finite set and list them in the *curly braces* or in *Roster form.*

Properties of Finite sets

The following finite set conditions are always finite.

- A subset of Finite set.
- The union of two finite sets
- The power set of a finite set

Infinite set

If a set is not finite, it is called an **infinite set** because the number of elements in that set is not countable and also we cannot represent it in Roster form. Thus, infinite sets are also known as **uncountable sets**.

So, to represent the elements of an Infinite set are represented by 3 dots (ellipse) to represent the infinity of that set.

Examples of Infinite Sets

- A set of all whole numbers.W= {0, 1, 2, 3, 4,…}
- A set of all points on a line.
- A set of all triangles.
- The set of leaves on a tree
- The set of hair on the head
- The set of integers.

Cardinality of Infinite Sets

The cardinality of a set is n (A) = x, where x is the number of elements of a set A. The cardinality of an infinite set is n (A)= infinite as the number of elements is unlimited in it.

Properties of Infinite Sets

- The union of two infinite sets is infinite
- The power set of an infinite set is infinite
- The superset of an infinite set is also infinite

The power set of an infinite set is unlimited because its elements are also unlimited.

Power Set:

In set theory, the power set (or powerset) of a Set A is defined as the set of all subsets of the Set A including the Set itself and the null or empty set. It is denoted by P(A). Basically, this set is the combination of all subsets including null set, of a given set.

If the given set has n elements, then its Power Set will contains 2^n elements.

Here, Let us say Set A = { a, b, c }

Number of elements: 3

Therefore, the subsets of the set are:

{ } which is the null or the empty set

{ a }

{ b }

{ c }

{ a, b }

{ b, c }

{ c, d }

{ a, b, c }

The power set P(A) = { { } , { a }, { b }, { c }, { a, b }, { b, c }, { c, d }, { a, b, c } }

Now, the Power Set should have $2^3 = 8$ elements.

The number of elements of a set is written as |A|, If A has n elements then it can be written as

$|P(A)| = 2^n$

Properties

- It is much larger than the original set.
- The powerset of a countable finite set is uncountable.
- For a set of natural numbers, we can do one-to-one mapping of the resulted set, P(S), with the real numbers.
- P(s) of set S, if operated with the union of sets, the intersection of sets and complement of sets, denotes the example of Boolean Algebra.

Power Set of a Empty Set

An empty set has zero element. Therefore, powerset of a empty set{ }, can be mentioned as;

- A set containing a null set.
- It contains zero or null elements.
- Empty set is the only subset.

Cartesian Products:

If A and B are two non-empty sets, then their Cartesian product A × B is the set of all ordered pair of elements from A and B.

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

Suppose, if A and B are two non-empty sets, then the Cartesian product of two sets, A and set B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ which is denoted as A × B.

For Example;

1. If A = {7, 8} and B = {2, 4, 6}, find A × B.

Solution:

A × B = {(7, 2); (7, 4); (7, 6); (8, 2); (8, 4); (8, 6)}

The 6 ordered pairs thus formed can represent the position of points in a plane, if a and B are subsets of a set of real numbers.

2. If A × B = {(p, x); (p, y); (q, x); (q, y)}, find A and B.

Solution:

A is a set of all first entries in ordered pairs in A × B.

B is a set of all second entries in ordered pairs in A × B.

Thus A = {p, q} and B = {x, y}

3. If A and B are two sets, and A × B consists of 6 elements: If three elements of A × B are (2, 5) (3, 7) (4, 7) find A × B.

Solution:

Since, (2, 5) (3, 7) and (4, 7) are elements of A × B.

So, we can say that 2, 3, 4 are the elements of A and 5, 7 are the elements of B.

So, A = {2, 3, 4} and B = {5, 7}

Now, A × B = {(2, 5); (2, 7); (3, 5); (3, 7); (4, 5); (4, 7)}

Thus, A × B contain six ordered pairs.

Video segment 4:

Set Operations:

Union:

The **union** of sets ***A*** and ***B***, denoted by $A \cup B$, is the set defined as

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$

Example 1: If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Example 2: If $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Note that elements are not repeated in a set.

Intersection:

The **intersection** of sets ***A*** and ***B***, denoted by $A \cap B$, is the set defined as

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$

Example 3: If $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cap B = \{1, 2\}$.

Example 4: If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \cap B = \emptyset$.

Difference:

The **difference** of sets ***A*** from ***B***, denoted by $A - B$, is the set defined as

$$A - B = \{ x \mid x \in A \wedge x \notin B \}$$

Example 5: If $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A - B = \{3\}$.
Example 6: If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A - B = \{1, 2, 3\}$.

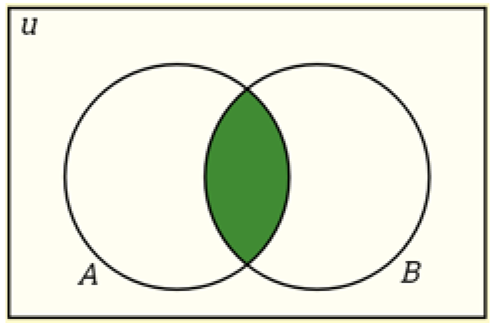
Note that in general $A - B \neq B - A$

Complement:

For a set ***A***, the difference $U - A$, where ***U*** is the universe, is called the **complement** of ***A*** and it is denoted by $\bar{A}$.
Thus $\bar{A}$ is the set of everything that is not in ***A***.

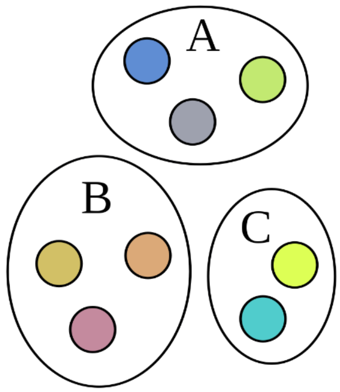
Joint set:

If at least one element of two or more sets are common then those are called by joint set.
A={1,2,6}A={1,2,6} and B={6,12,42}B={6,12,42}. There is a common element '6', hence these sets are overlapping sets.



Disjoint set are those sets whose intersection with each other results in a null set. In Set theory, sometimes we notice that there are no common elements in two sets or we can state that the intersection of the sets is an empty set or null set. This type of set is called a **disjoint set**. For example, if we have X = {a, b, c} and Y = {d, e, f}, then we can say that the given two sets are disjoint since there are no common elements in these two sets X and Y.

Here is a disjoint set collection is represented.



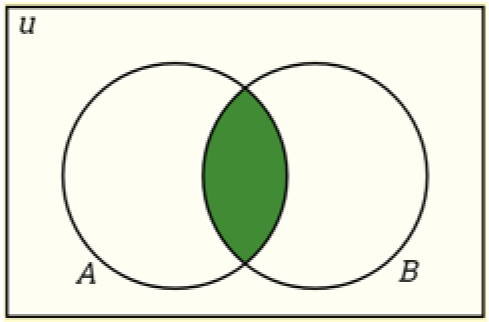
Inclusion and Exclusion Principle:

The inclusion-exclusion principle can be expressed as follows:To compute the size of a union of multiple sets, it is necessary to sum the sizes of these sets **separately**, and then subtract the sizes of all **pairwise** intersections of the sets, then add back the size of the intersections of **triples** of the sets, subtract the size of **quadruples** of the sets, and so on, up to the intersection of **all** sets.

For two set In combinatorics, a branch of mathematics, the **inclusion–exclusion principle** is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

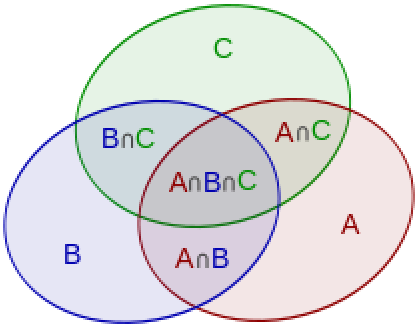
The formulation using Venn diagrams

Let the diagram show three sets A, B



$$S(A\cup B)=S(A)+S(B)-S(A\cap B)$$

Let the diagram show three sets A, B and C:



Then the area of their union A∪B∪CA∪B∪C is equal to the sum of the areas A, Band C less double-covered areas A∩BA∩B, A∩CA∩C, B∩CB∩C, but with the addition of the area covered by three sets A∩B∩CA∩B∩C:

$$S(A\cup B\cup C)=S(A)+S(B)+S(C)-S(A\cap B)-S(A\cap C)-S(B\cap C)+S(A\cap B\cap C)$$

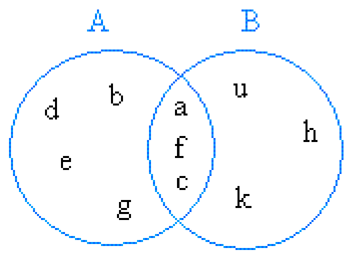
Video Segment 5:

Set Difference Example:

Given set A and set B the set difference of set B from set A is the set of all element in A, but not in B.

We can write A − B

Example #1.



Take a close look at the figure above. Elements in A **<u>only</u>** are b, d, e,g.

Therefore, A − B = { b, d, e, g}

Notice that although elements a, f, c are in A, we did not include them in A − B because we must not take anything in set B.

Sometimes, instead of looking at a the Venn Diagrams, it may be easier to write down the elements of both sets

Then, we show in bold the elements that are in A, but not in B

A = {**b**

, **d**, **e**, **g**, a, f, c}

B = { k, h, u, a, f, c}

Example #2.

Find B − A

Notice that this time you are looking for anything you see in B **<u>only</u>**

Elements that are in B only are shown in bold below

Let A = {1 orange, 1 pinapple, 1 banana, 1 apple}

Let B = {1 orange, **1 apricot**, 1 pinapple, 1 banana, **1 mango**, 1 apple, **1 kiwifruit** }

B − A = {1 apricot, 1 mango, 1 kiwifruit}

Example #3.

Find A − B

B = { 1, 2, 4, 6}

A = {1, 2, 4, 6, **7**, **8**, **9** }

What I see in A that are not in B are 7, 8, and 9

A − B = { 7, 8, 9}

Example #4.

Find B − A

A = { x / x is a number bigger than 6 and smaller than 10}

B = { x / x is a positive number smaller than 15}

A = {7, 8, 9} and B = { **1**, **2**, **3**, **4**, **5**, **6**, 7, 8, 9, **10**, **11**, **12**, **13**, **14**}

Everything you see in bold above are in B only.

B − A = {1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14}

Complement Example:

Examples :

1) If A = { 1, 2, 3, 4} and U = { 1, 2, 3, 4, 5, 6, 7, 8} then find A complement (A').

Solution :
A = { 1, 2, 3, 4} and Universal set = U = { 1, 2, 3, 4, 5, 6, 7, 8}

Complement of set A contains the elements present in universal set but not in set A.

Elements are 5, 6, 7, 8.

∴ A complement = A’ = { 5, 6, 7, 8}.

2) If B = { x | x is a book on Algebra in your library} . Find B’.

Solution : B’ = { x | x is a book in your library and x ∉ B }

3) If A = { 1, 2, 3, 4, 5 } and U = N , then find A’.

Solution :
A = { 1, 2, 3, 4, 5 }

U = N

⇒ U = { 1, 2, 3, 4, 5, 6, 7, 8, 9,10,… }

A’ = { 6, 7, 8, 9, 10, … }

4) If A = { x | x is a multiple of 3, x ∉ N }. Find A’.

Solution :
As a convention, x ∉ N in the bracket indicates N is the universal set.

N = U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,11, … }

A = { x | x is a multiple of 3, x ∉ N }

A = { 3, 6, 9, 12, 15, … }

So, A’ = { 1, 2, 4, 5, 7, 8, 10,11, … }

De Morgan’s Law proof: Follow the book.

Generalized Union and intersection:

As we saw earlier, union, intersection and Cartesian product of sets are associative For example $(A \cup B) \cup C = A \cup (B \cup C)$

To denote either of these we often use $A \cup B \cup C$ **.**

This can be generalized for the union of any finite number of sets as $A_1 \cup A_2 \cup \ldots \cup A_n$ **,** which we write as

$\cup_{i=1}^{n} A_i$

This **generalized union** of sets can be rigorously defined as follows:

Definition ($\cup_{i=1}^{n} A_i$) **:**

Basis Clause: For $n = 1$, $\cup_{i=1}^{n} A_i = A_1$.

Inductive Clause: $\cup_{i=1}^{n+1} A_i = (\cup_{i=1}^{n} A_i) \cup A_{n+1}$

Similarly the **generalized intersection** $\cap_{i=1}^{n} A_i$ and **generalized Cartesian product** $\times_{i=1}^{n} A_i$ can be defined.

Based on these definitions, De Morgan's law on set union and intersection can also be generalized as follows:

Theorem (Generalized De Morgan)

$\overline{\cup_{i=1}^{n} A_i} = \cap_{i=1}^{n} \overline{A_i}$, and

$\overline{\cap_{i=1}^{n} A_i} = \cup_{i=1}^{n} \overline{A_i}$