Differential Equations

Mahyar Albalan

403201223

Quadrature Method

Quadrature Method is one of the pre-computer integration methods, in which we divide the integration scope into N rectangles and estimate the area under the curve using these rectangles. In the midpoint algorithm, we use the average of the left and right points to calculate the area of each rectangle:

$$S = \frac{f_{(x)} + f_{(x+h)}}{2}h$$

Now I have to show that this algorithm's local error is $O(h^3)$ and then I will show the overall error is $O(h^2)$. By an $O(h^n)$ algorithm I mean:

$$Error < Ch^n \qquad for\ some\ constant\ C > 0$$

Let's calculate the local error (for only one rectangle),We have to show:

$$error = |\int_a^{a+h} f_{(x)}dx - \frac{f_{(a)} + f_{(a+h)}}{2}h| < Ch^3$$

First, I will use Taylor's expansion on the integral term. (I refer to $(x - a)$ as x in the expansion, for simplicity in notation.)

$$f_{(x)} = f_{(a)} + f'_{(a)}x + f''_{(a)}/2x^2 + ...$$

$$\Rightarrow \int_a^{a+h} f_{(x)}dx \Rightarrow \int_a^{a+h} (f_{(a)} + f'_{(a)}x + f''_{(a)}/2x^2 + ...)dx =$$

$$f_{(a)}h + f'_{(a)}/2h^2 + f''_{(a)}/6h^3 + O(4)$$

Now I will use Taylor's expansion on the $f(a+h)$ term in the error equation:

$$f_{(a)} = f_{(a)} + f'_{(a)}h + f''_{(a)}/2h^2 + ...$$

By substituting both Taylor expansions into the error equation, we obtain:

$$Error = |f''_{(a)}/3 * h^3 + O''(4)| \Rightarrow Error < Ch^3$$

so i showed local Error is $O(3)$, Global error is sum of these local errors:

$$Global\ Error = |\sum local\ Error| \leq \sum |local\ Error| \leq N * Ch^3$$

We can use $N = \frac{b-a}{h}$:

$$Global\ Error \leq \frac{b-a}{h} Ch^3 = Dh^2 \quad ,D{=}C(b{-}a)$$

So Global Error is $O(2)$.

Euler's Method

For this question, I have to solve an RC circuit using Euler's method. The differential equation governing this problem is:

$$\dot{Q} = -\frac{Q}{RC} \quad ; \quad Q_{(0)}{=}q$$

analytical solution to this equation is:

$$Q_{(t)} = q * exp(-t/(RC))$$

now i will use Euler's method to solve this DE:

$$Q_{(\delta t)} = q - q/(RC) = q(1 - 1/(RC))$$

$$Q_{(2\delta t)} = Q_{(\delta t)}(1 - 1/(RC))$$

$$...$$

By using this method N times, we can obtain the DE's solution. Here is the result:

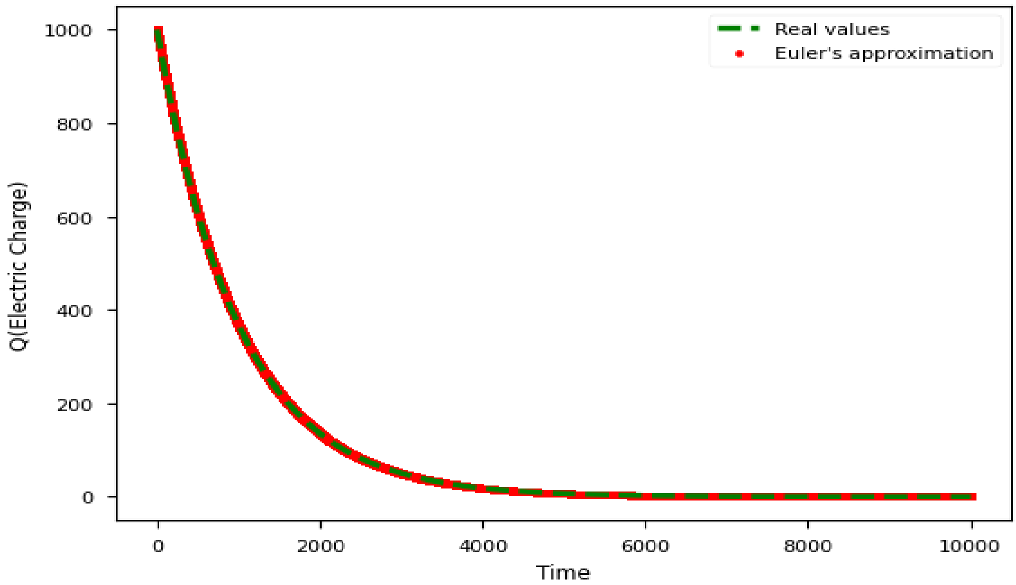

Figure 1: Euler's method vs real solution

in the top plot, i used the following initial conditions:

$$RC = 1000; \quad Q_0 = 1000; \quad T_0 = 0; \quad \delta t = 0.01; \quad N = 1,000,000$$

As you can see, Euler's method result is very close to the analytical solution, which is expected! I calculated the MSE of the result from Euler's method, and the result is as follows:

$$MSE = 6.25 * 10^{-7}$$

which again confirms good fit to real solution.