Differential Equations

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Quadrature Method

Quadrature Method is one of the pre-computer integration methods, in which we divide the integration scope into N rectangles and estimate the area under the curve using these rectangles. In the midpoint algorithm, we use the average of the left and right points to calculate the area of each rectangle:

$$S = \frac{f_{(x)} + f_{(x+h)}}{2}h$$

Now I have to show that this algorithm's local error is $O(h^3)$ and then I will show the overall error is $O(h^2)$. By an $O(h^n)$ algorithm I mean:

$$Error < Ch^n \qquad for\ some\ constant\ C > 0$$

Let's calculate the local error (for only one rectangle),We have to show:

$$error = |\int_a^{a+h} f_{(x)}dx - \frac{f_{(a)} + f_{(a+h)}}{2}h| < Ch^3$$

First, I will use Taylor's expansion on the integral term. (I refer to $(x-a)$ as x in the expansion, for simplicity in notation.)

$$f_{(x)} = f_{(a)} + f'_{(a)}x + f''_{(a)}/2x^2 + ...$$

$$\Rightarrow \int_a^{a+h} f_{(x)}dx \Rightarrow \int_a^{a+h} (f_{(a)} + f'_{(a)}x + f''_{(a)}/2x^2 + ...)dx =$$

$$f_{(a)}h + f'_{(a)}/2h^2 + f''_{(a)}/6h^3 + O(4)$$

Now I will use Taylor's expansion on the $f(a+h)$ term in the error equation:

$$f_{(a)} = f_{(a)} + f'_{(a)}h + f''_{(a)}/2h^2 + ...$$

By substituting both Taylor expansions into the error equation, we obtain:

$$Error = |f''_{(a)}/3 * h^3 + O''(4)| \Rightarrow Error < Ch^3$$

so i showed local Error is $O(3)$, Global error is sum of these local errors:

$$Global\ Error = |\sum local\ Error| \leq \sum |local\ Error| \leq N * Ch^3$$

We can use $N = \frac{b-a}{h}$:

$$Global\ Error \leq \frac{b-a}{h} Ch^3 = Dh^2 \quad ,D=C(b\text{-}a)$$

So Global Error is $O(2)$.