Monte Carlo Integration

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1. Shelep Algorithm

I follow the instructionin the text for impementing this algorithm,its pretty straightforward, the result is as follows:
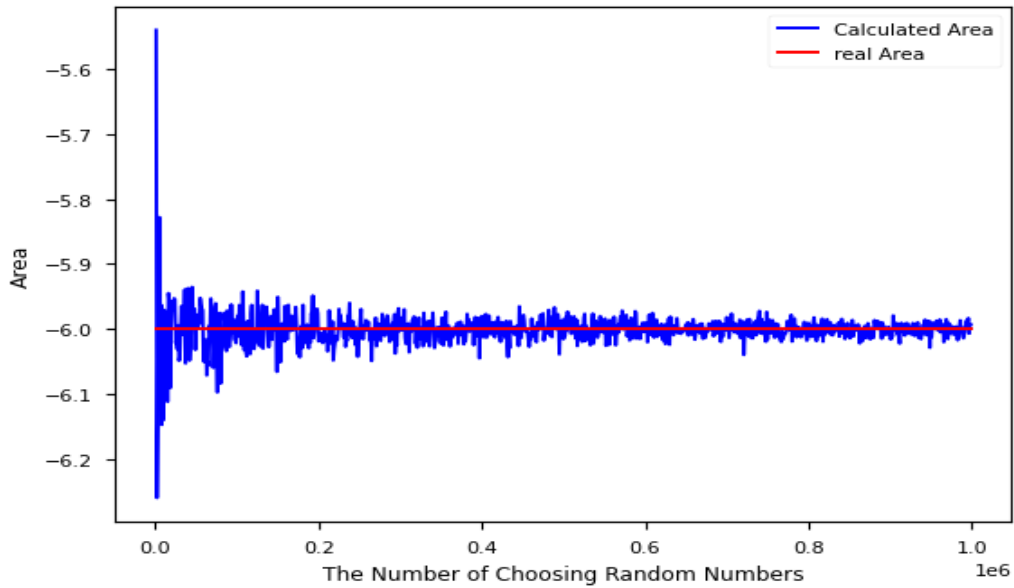

Figure 1: Calculated Area versus Real Area

2.Monte Carlo Integral

In the first part of this problem, we have to calculate $\langle f \rangle$ with simple sampling. Here, I used a uniform distribution and chose 100,000 points for each sampling. I repeated the process for different total numbers of

independent samples, from 100 to 10,000 with a step of 100. At each total sample size, I calculated the statistical error and the real error (the real value of the integral is approximately 0.882). The results are as follows:

	Real Error	Statistical Error
100	0.001465	9.3e-05
200	0.001532	7.1e-05
300	0.001766	6.2e-05
400	0.001814	5.7e-05
500	0.001645	4.6e-05
600	0.001679	4.3e-05
700	0.001734	4.1e-05
800	0.001714	3.8e-05
900	0.00167	3.5e-05
1000	0.001717	3.4e-05
1100	0.001788	3.4e-05
1200	0.001683	3.1e-05
1300	0.001762	3.1e-05
1400	0.001835	3e-05
1500	0.001768	2.9e-05
1600	0.001733	2.7e-05
1700	0.001717	2.6e-05
1800	0.001703	2.5e-05
1900	0.001697	2.4e-05

Figure 2: Real Error and Statistical Error with Normal Sampling

As expected, the statistical error decreases when the number of samples increases. The runtime for creating this table was 54.55 seconds.

For smart sampling, the process is the same as above, but here we have to generate numbers with the distribution $g(x) = e^{-x}$. To create data that follows this distribution, I used the algorithm introduced in the previous chapter:

$$\int P(x)dx = \int g(y)dy$$

Here we want $g(y) = e^{-y}$, and it must be 0 outside of the scope of integration (outside the interval $(0, 2)$). For this, after integration (the process is the same as in the text in the previous chapter), we have to transform the values which are generated via a uniform distribution in the range $(0, 1 - e^{-2})$, so that after the transformation below, the range of inputs will lie inside the integration scope:

$$X = -\ln(1 - x)$$

the results are as follows:

	Real Error	Statistical Error
100	0.000919	0.000134
200	0.000936	9.6e-05
300	0.000936	8e-05
400	0.000921	6.7e-05
500	0.000986	6.2e-05
600	0.000928	5.4e-05
700	0.000952	5.2e-05
800	0.000993	5e-05
900	0.000903	4.4e-05
1000	0.000967	4.4e-05
1100	0.000932	4.1e-05
1200	0.000951	3.9e-05
1300	0.000924	3.7e-05
1400	0.000923	3.6e-05
1500	0.000922	3.4e-05
1600	0.000953	3.4e-05
1700	0.000943	3.3e-05
1800	0.000958	3.3e-05
1900	0.000964	3.2e-05

Figure 3: Real Error and Statistical Error with Smart Sampling

Runtime for this approach is about 44.97 seconds, and the results did not improve much compared to the previous method because I chose a large enough number of data points for the integration, and the function $f(x) = e^{-x^2}$ does not exhibit any sharp behavior. So this outcome is somewhat expected. However, the result from the second approach was produced using an order of magnitude fewer points, yet the results are in the same range. Therefore, we can expect that for a function with sharper behavior, the second approach would be much faster and more accurate.

3.Multiple Integral

First, let's define the problem completely, then solve it. We have to calculate the center of mass (CM) of a sphere whose density has a linear dependency on Z, and it decreases as Z decreases. At the top of the sphere, the density is twice that at the bottom. So the general dependency for the density is as follows: (ρ_0 is the density at the top of the sphere, R is the radius of the sphere)

$$\rho = \frac{3}{4}\rho_0 + \frac{\rho_0}{4R}Z = \frac{\rho_0}{4}(3 + \cos(\theta))$$

For simplicity, I set $\rho_0 = R = 1$, so:

$$\rho = \frac{1}{4}(3 + \cos(\theta)) \Rightarrow M = \int \rho \, dV = \pi$$

For the center of mass we have:

$$\vec{R}_{\text{CM}} = \frac{1}{M} \int \vec{r} \rho \, dV$$

Now i will use Monte Carlo ith simple Sampling for calculating this integral,(Check the code for more details) the result is as follows:

$$\vec{R}_{\text{CM}} = 1/M * 4/3\pi R^3(\langle \vec{r} \rho \rangle) = 4/3(\langle x\rho \rangle \hat{x} + \langle y\rho \rangle \hat{y} + \langle z\rho \rangle \hat{z}$$

The result is as follows:

$$X_{CM} = -2.17e - 05 \qquad Y_{CM} = -1.49e - 05 \qquad Z_{CM} \approx 0.0667$$

which are very close to theorical expectations:

$$X_{CM} = 0 \qquad Y_{CM} = 0 \qquad Z_{CM} \approx 1/15 = 0.0666$$