ESTIMATION
IN
THE BLACK-SCHOLES MODEL

Implementation of an application
thanks to a mathematical model

December 17, 2016

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End-of-studies Project
2016 - 2017

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1 Introduction

Our end-of-studies project at ESME Sudria specialiazed in Financial Banking is about estimation in Black-Scholes model. Actually, this project is about answering to the trader needs of simulating the price of financial assets. We create an application that can be used to do it. We decided to implement it in C#, the most used language in banking. This application is built thanks to financial mathematics equations. So the main goal is to manage those different skills for one project.

2 Historical Background

There are four important dates in the History of financial Mathematics:

- 1900 : Bachelier wrote *The theory of speculation*. He is a French mathematician of the 19th century. Moreover, he is the first person to model the stochastic process (now called Brownian motion). He used the Brownian motion to evaluate stock options. *The theory of speculation* is historically the first paper to use advanced mathematics in the study of finance so he is considered as a pioneer in this field.
- 1923 : Wiener, an American Mathematician, published *Differential space*. Wiener gave the first construction of Brownian motion process. We used this model for our project.
- 1973 : Fischer Black and Myron Scholes in *The pricing of options and corporate liabilities* introduced their methodology of asset pricing. The formula is still used by operators on the European markets and has radically modified the way to apprehend and manage the risks. *The Pricing of Options and Corporate Liabilities* was first published in the Journal of Political Economy in The University of Chicago Press.
- 1995 : Martin Bibby and Michael Sorensen wrote a thesis named *Martingale estimating functions for discretely observed diffusion processes*. They are two Danish mathematician who defined the estimation of the parameters in the Black Scholes Model.

3 The Brownian Motion

The Brownian motion was discovered in 1827 by the botanist Robert Brown, while looking at particles trapped in cavities inside pollen grains that were bathing in water. With his microscope, he noted that the particles moved through the water; but he was not able to determine the mechanisms that caused this motion.
In mathematics, the Brownian Motion is a Gaussian stochastic process, it is an estimation that depends on time and on randomness. Its distribution is Gaussian and its path is continuous, which means there is no jump. The following covariance hides the fact that the increments are independent.

$$cov(B_s, B_t) = t \wedge s$$

4 Black-Scholes Model

The Black-scholes model is a mathematical model that allows to pricing an asset. It is based on a differential stochastic equation as :

$$\begin{cases} \mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \\ S_0 > 0 \end{cases}$$

- S_0 : Initial price,
- μ : Drift,
- σ : Volatility
- B : Brownian Motion

4.1 Proof of the solution of the Black-Scholes equation

1. Firstly, let $\widehat{b}$, $\widehat{\sigma}$: $\mathbb{R} \mapsto \mathbb{R}$ the functions defined as :

$$\begin{aligned} \widehat{\mu}(x) &= \mu(x), \qquad \forall x \in \mathbb{R} \\ \widehat{\sigma}(x) &= \sigma(x), \qquad \forall x \in \mathbb{R} \end{aligned}$$

For any x, y $\in \mathbb{R}$,

$$\mid \widehat{\mu}(y) - \widehat{\mu}(x) \mid + \mid \widehat{\sigma}(y) - \widehat{\sigma}(x) \mid = \mid \mu + \sigma \mid\mid y - x \mid$$

and,

$$|\widehat{\mu}(x)| + |\widehat{\sigma}(x)| = |\mu+\sigma| \, |x|$$

We set a constant C :

$$C = |\mu+\sigma| > 0$$

So, here :

$$|\widehat{\mu}(t,y) - \widehat{\mu}(t,x)| + |\widehat{\sigma}(t,y) - \widehat{\sigma}(t,x)| \leq C \, |y-x|$$

and,

$$|\widehat{\mu}(t,x)| + |\widehat{\sigma}(t,x)| \leq C(1+|x|)$$

So for any x, y $\in \mathbb{R}$ and t $\in$ [0,T], the equation has only one solution $(S_t)_{t\in[0,T]}$.

2. We set

$$f(x) = \exp(x)$$

and

$$X_t = \theta t + \sigma B_t$$

From the Itô formula applied to the process X and the function f, we find:

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}<X>_s \\ &= f(X_0) + \int_0^t \exp(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \exp(X_s)\,\mathrm{d}<X>_s \end{aligned}$$

But :

$$X_0 = \sigma B_0 = 0,$$

$$\begin{aligned} \mathrm{d}X_s &= \mathrm{d}(\theta s + \sigma B_s) \\ &= \theta \mathrm{d}s + \sigma \mathrm{d}B_s \end{aligned}$$

and

$$\begin{aligned}\mathrm{d} < X >_s &= \mathrm{d} < \theta s + \sigma B_s > \\ &= \sigma^2 \mathrm{d} < B >_s \\ &= \sigma^2 \mathrm{d}s\end{aligned}$$

Thus,

$$\begin{aligned}f(X_t) &= 1 + \int_0^t \exp(X_s)(\theta \mathrm{d}s + \sigma \mathrm{d}B_s) + \frac{1}{2}\int_0^t \exp(X_s)\sigma^2 \mathrm{d}s \\ &= 1 + \left(\theta + \frac{\sigma^2}{2}\right)\int_0^t f(X_s)\mathrm{d}s + \sigma \int_0^t f(X_s)\mathrm{d}B_s\end{aligned}$$

Let S_t be :

$$\begin{aligned}S_t &= S_0 \exp(X_t) \\ &= S_0 f(X_t)\end{aligned}$$

and let $\mu \in \mathbb{R}$ be :

$$\mu = \theta + \frac{\sigma^2}{2}$$

So :

$$\begin{aligned}S_0 f(X_t) &= S_0 + \mu \int_0^t S_0 f(X_s)\mathrm{d}s + \sigma \int_0^t S_0 f(X_s)\mathrm{d}B_s \\ S_t &= S_0 + \mu \int_0^t S_s \mathrm{d}s + \sigma \int_0^t S_s \mathrm{d}B_s\end{aligned}$$

That is equivalent to :

$$\begin{cases} \mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \\ S_0 > 0 \end{cases}$$

We demonstrated the uniqueness so necessarily the solution of the linear stochastic differential equation is:

$$S_t = S_0 \exp(\theta t + \sigma B_t)$$

3. We replace the value :

$$\begin{aligned}X_t &= \ln(S_t) \\ &= \ln(S_0 \exp(\theta t + \sigma B_t)) \\ &= \ln(S_0) + \theta t + \sigma B_t\end{aligned}$$

4.2 Estimation of the volatility

4.2.1 Definition

The volatility is a measure of risk based on the deviation of the asset return, or in other words, on the dispersion of returns. It indicates the pricing behavior of the security and helps estimate the fluctuations that may happen in a short period of time. A low volatility means that there are few variations in a longer time span. Rather, a high volatility means that the price fluctuates rapidly, therefore, there is more risks.

4.2.2 Estimation

The volatility is estimated thanks to the Least squares method. It represents the sum of the squared gaps between the theoretical model and the observations. We try to find the value of the volatility that has the smallest gap. We need a large number of subdivisions in order to have a relevant estimation of the volatility. The expression of the estimator is :

$$\widehat{\sigma_n}^2 = \frac{1}{n-1}\left(\sum_{k=0}^{n-1}\frac{\mid X_{tk+1} - X_{tk}\mid^2}{t_{k+1}-t_k} - \frac{\mid X_T - X_0\mid^2}{T}\right)$$

4.3 The risk-free rate

4.3.1 Definition

The risk-free rate is the theoretical rate of an investment over a specified time period with zero risk.

4.3.2 Estimation

The risk-free rate of the asset is not directly expressed in the Black-Scholes model. However, it depends on two of its parameters: the volatility and the drift. Its expression is :

$$\theta = \mu + \frac{\sigma^2}{2}$$

Its estimation is built with the Ergodic theorem, is a strong law of large numbers generalization. But in our model the states are not independent. Regardless of the starting point of the model, the estimator will converge on average to a single value. Therefore, the idea is to converge a continuous average to an expectation of a certain random variable. To obtain a good estimation of the Risk-free rate parameter, a large time interval is required.

Thus the expression of our estimator is:

$$\widehat{\theta}_T = \frac{X_T - X_0}{T}$$

4.4 The drift

The drift is the trends that controls the deterministic component of the process, that means that depends on time.

$$\theta = \mu + \frac{\sigma^2}{2}$$

5 Application

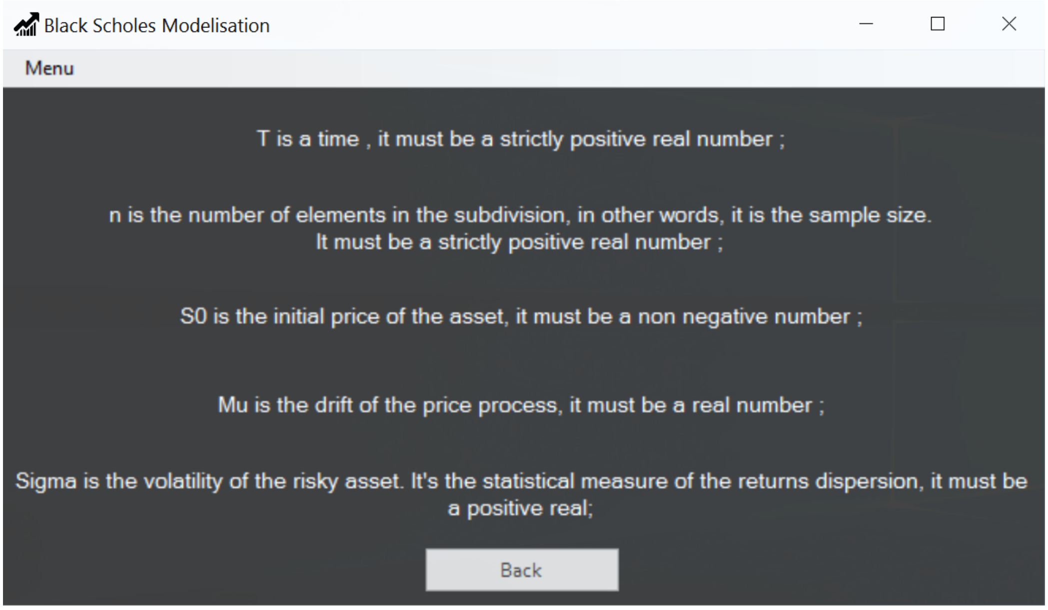
We decided to implement our application in C#, the most used language in the field of banking. We built this application according to the MVC architecture (Model View Controller).

This application will contain the possibility of modeling theoretically the price of an asset, the estimations of the volatility and the risk-free rate.

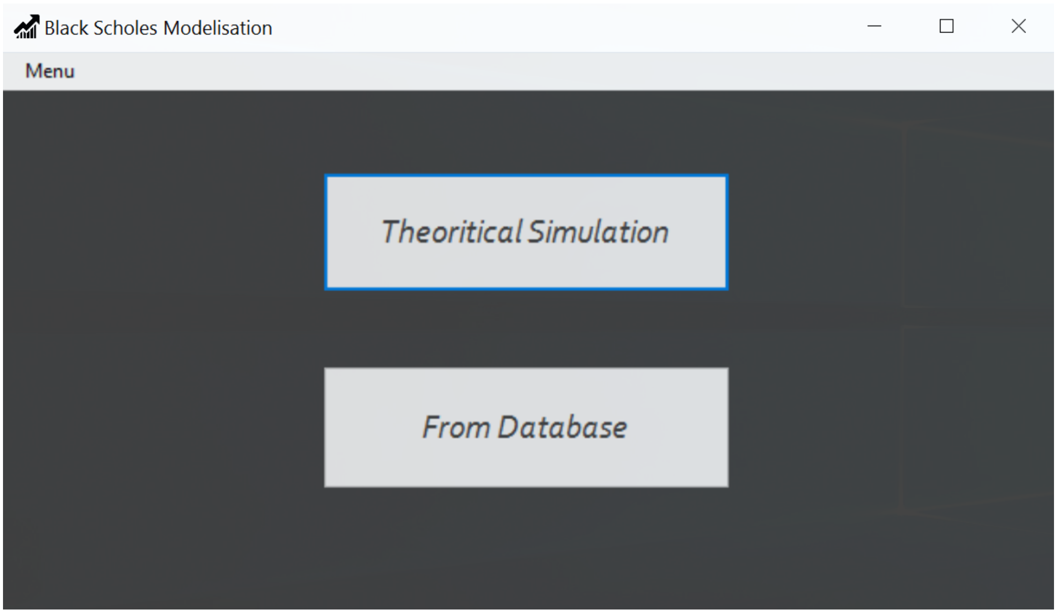
We plan to create several pages:

- Homepage

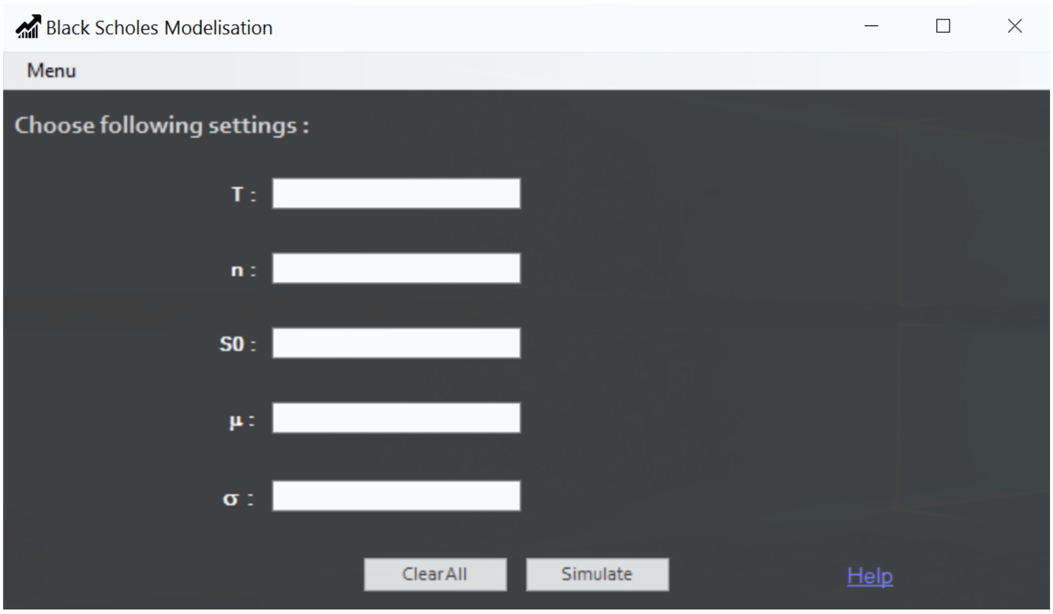
- Help page



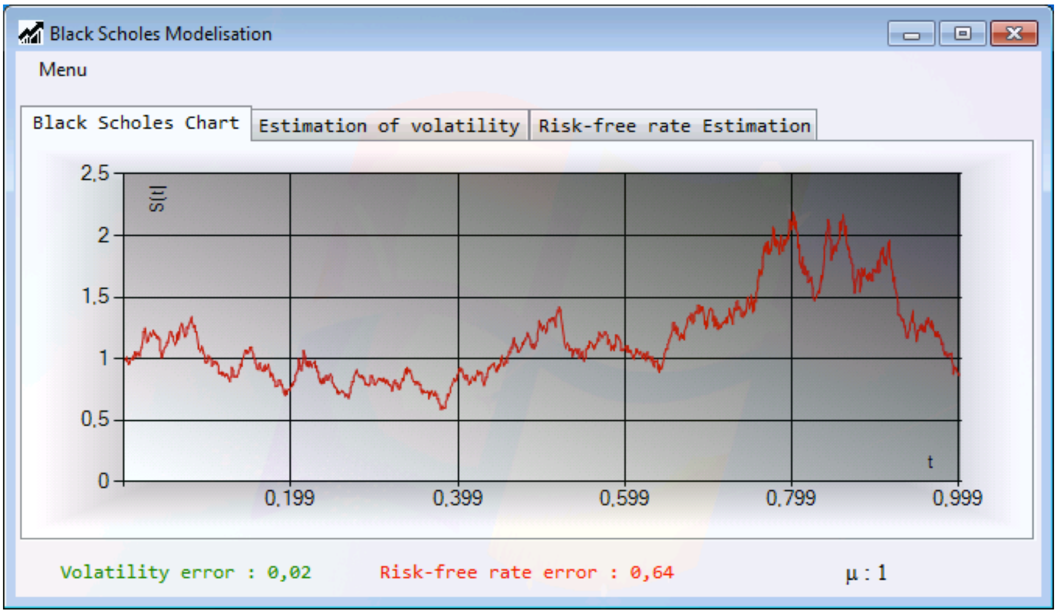
- Choice between theoretical simulation or simulation from real data



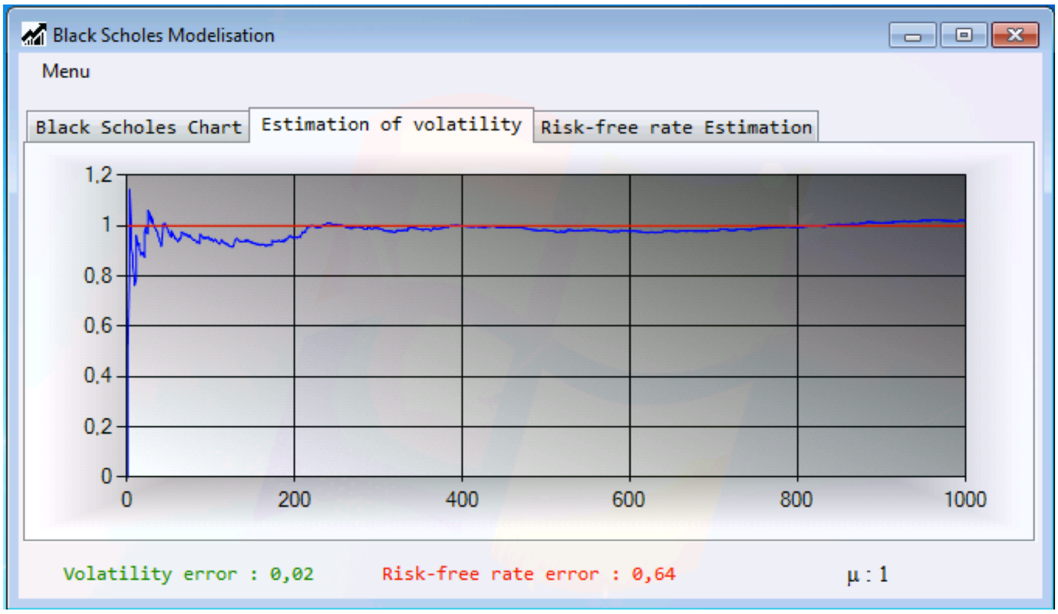
- Following parameters to be entered by the user:
  - S_0 : initial value,
  - μ : the drift,
  - σ : the volatility
  - T : the maturity
  - n : number of observations



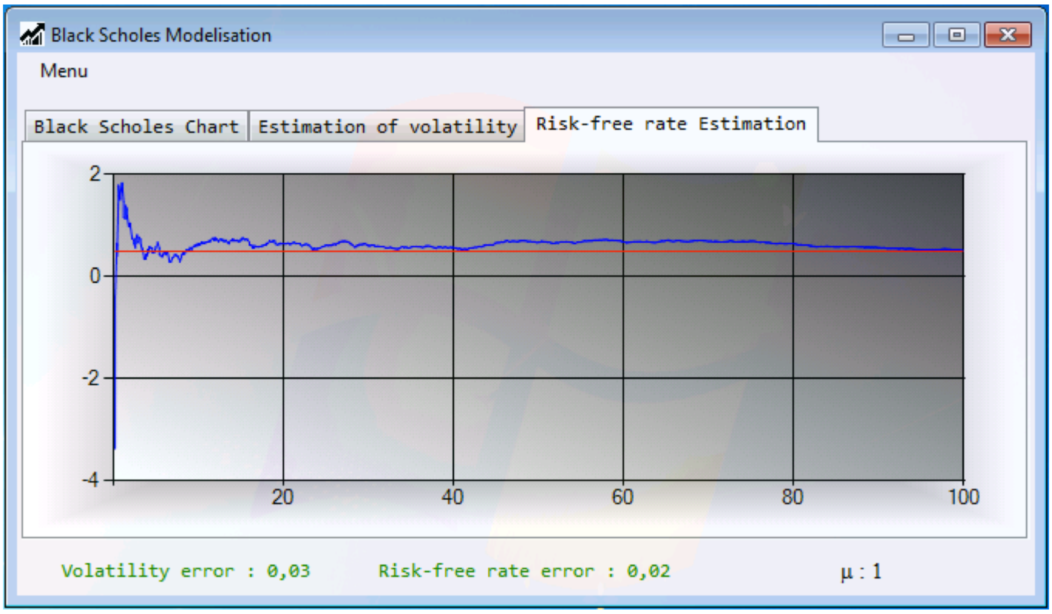
- Display of :
  - The Black-Scholes path



  - Estimation of the volatility



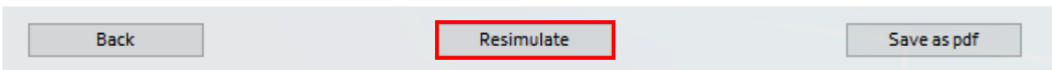
– Estimation of the risk-free rate



– Relevance of the estimators

– Value of the drift and the risk-free rate

– Resimulate : It allows to make a new simulation while keeping the same parameters. Every time we push on it, we can observe new charts. It shows well the randomness of the process and so the influence of the Brownian motion.



We built the mathematics way to create and display the graphics such as:

1. Calculation of a random variable of the normal centered reduced law
2. Calculation of the expression of the Brownian motion
3. Calculation of the risk-free rate
4. Calculation of a path of the Black Scholes model