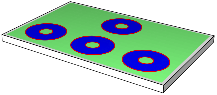
Physical properties of entangled Majorana fermion states on textured surfaces of topological insulators

Maike Schön

February 26, 2019



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Content

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3. ***Fusion***

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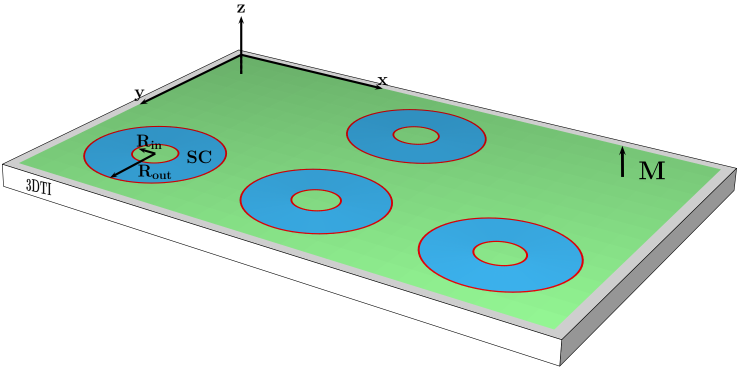
1. ***Existence* of topological protected Majorana zero modes located at the boundaries**
2. ***Controlled* quantum state *manipulations***
   (**braiding** of Majorana bound states)
3. ***Fusion***
4. ***Read out***
   (method for initialization and read out of the quantum bit states)

The architecture



SC: superconducting ring (blue)
3DTI: three dimensional topological insulator thin film (white)
M: ferromagnetic dopant (green)

Question:
Is that possibly a
new potential quantum bit architecture?

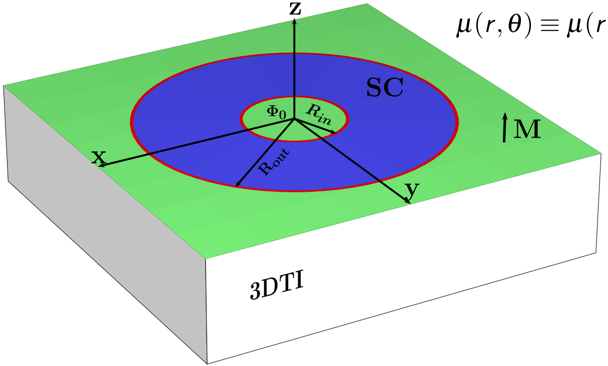
Subsystem of consideration effective (2D) surface Hamiltonian

$$\mathcal{H}_{\text{surface}} = \frac{i}{2}\left[\sigma\mathbf{p}, \sigma\mathbf{n}\right] = \frac{i}{2}\mathbf{p}\mathbf{n} + \frac{1}{2}\left(\mathbf{n}\left(\mathbf{p}\times\sigma\right) + \left(\mathbf{p}\times\sigma\right)\mathbf{n}\right)$$

(regime of low-energy excitations)

electrochemical potential:

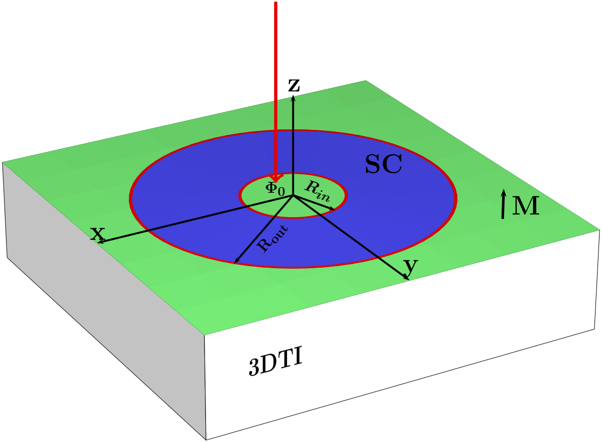
$$\mu(r,\theta) \equiv \mu(r)$$



effective surface Hamiltonian of the 3DTI

$$\mathcal{H}_{\text{3DTI}} = v_F\left(\boldsymbol{\sigma}\times\boldsymbol{p}\right)\hat{\boldsymbol{e}}_z\tau_z - \mu\sigma_0\tau_z = v_F\left(\sigma_x p_y - \sigma_y p_x\right)\tau_z - \mu\sigma_0\tau_z$$

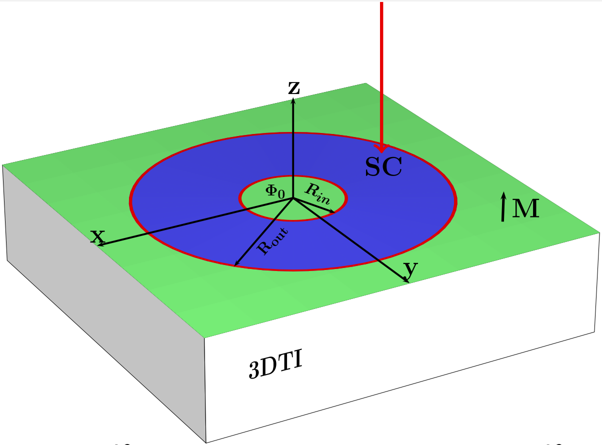
Subsystem of consideration magnetic flux quantum $\Phi_0 = \frac{h}{2e}$



$\Phi_0 = \frac{h}{2e}$ created by external magnetic field $\boldsymbol{B} = |B|\,\hat{\boldsymbol{e}}_z$ (set $\hbar = 1$)

Subsystem of consideration

proximitized super conductor $\Delta(r,\theta)$

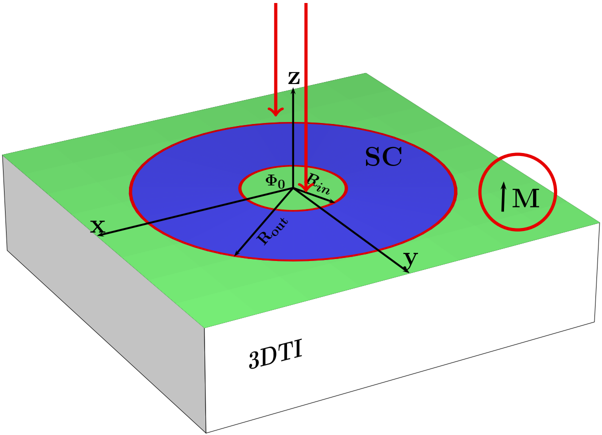


$$\Delta(r,\theta) = \Delta_0 e^{i\theta}\Theta(R_{out}-r)\Theta(r-R_{in}) = \vartheta(r)e^{i\theta} \text{ (blue)}$$

effective proximity induced Hamiltonian:

$$\mathscr{H}_\Delta = \frac{1}{2}\left[\Delta\left(\tau_x + i\tau_y\right) + \Delta^*\left(\tau_x - i\tau_y\right)\right]$$

Subsystem of consideration doped magnetic field $M(r)$

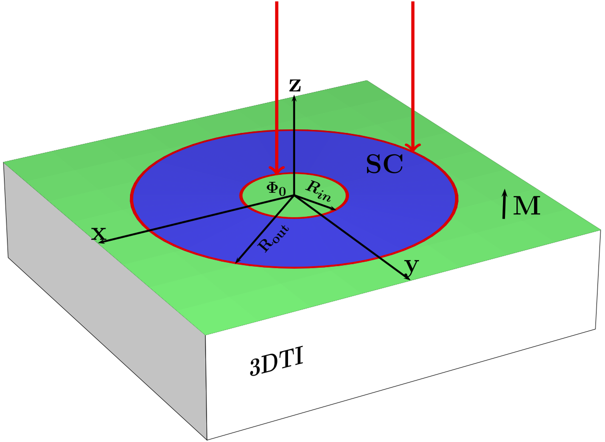


$$M(r,\theta) \equiv M(r) = M_0[\Theta(r-R_{out})+\Theta(R_{in}-r)] \text{ (green)}$$

effective magnetic Hamiltonian: $\mathscr{H}_{\mathrm{M}} = \boldsymbol{M}(r)\boldsymbol{\sigma}\tau_0 = M(r)\sigma_z\tau_0$

Subsystem of consideration

geometric boundaries R_{in} and R_{out}



Goal: find states of zero energy at the boundaries R_{in} and R_{out} (red)

The full Hamiltonian matrix for the low energy surface states:

$$\mathscr{H} = \mathscr{H}_{\mathrm{3DTI}} + \mathscr{H}_{\Delta} + \mathscr{H}_{\mathrm{M}} = \begin{pmatrix} -\mu + M & v_F p_+ & \Delta & 0 \\ v_F p_- & -\mu - M & 0 & \Delta \\ \Delta^* & 0 & \mu + M & -v_F p_+ \\ 0 & \Delta^* & -v_F p_- & \mu - M \end{pmatrix}$$

The total Hamiltonian: $H = 1/2 \Psi^\dagger \mathscr{H} \Psi$

(in polar coordinates)
$p_+ = e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right)$, $p_- = -e^{+i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right)$, $p_+ = p_-^*$

$\psi = \left(\psi_\uparrow, \psi_\downarrow\right)^T$ (spin space)
$\Psi = ((\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger))^T$ (Nambu spinor space)

Important property of the Hamiltonian matrix: $\mathscr{H}$

particle-hole symmetry: $\Xi\mathscr{H}\Xi = -\mathscr{H}$,
with particle-hole symmetry operator $\Xi = \sigma_y \tau_y \hat{C}$

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$\Rightarrow \Xi \Psi_0 = \Psi_0$

$\Rightarrow \Psi_0 = (a, b, b^*, -a^*)^T = \Psi_0(r, \theta)$

with $a(r, \theta) \rightarrow \psi_\uparrow$ and $b(r, \theta) \rightarrow \psi_\downarrow$

Important properties of the Hamiltonian matrix: $\mathscr{H}$

- **hermitian:** $\mathscr{H}^\dagger = \mathscr{H}$
  $\Rightarrow$ **real eigenenergies**

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- **real energy spectrum in *k*-space**

$$\varepsilon(\boldsymbol{k}) = \sqrt{v_F^2 k^2 + |\Delta_0|^2 + \mu^2 + M^2 \pm 2\sqrt{v_F^2 k^2 \mu^2 + M^2(|\Delta_0|^2 + \mu^2)}}$$

different topological phases in different geometrical regions

magnetic region

$$\varepsilon_M(\boldsymbol{k}) = -\mu \pm \sqrt{v^2 k^2 + M_0^2} \Rightarrow \varepsilon_M(\boldsymbol{k}=0) = -\mu \pm |M_0|$$

$$\begin{array}{llcll} \text{(i)} & |M_0| & < & |\mu| & \text{(topological region)} \\ \text{(ii)} & |M_0| & > & |\mu| & \text{(topological trivial region)} \end{array}$$

superconducting region:

$$\varepsilon(\boldsymbol{k}) = \sqrt{(\mu \pm v_F |\boldsymbol{k}|)^2 + |\Delta_0|^2}$$

$$|\Delta_0| \neq 0 \quad \text{(topological region)}$$

assume: $\mu(r) \equiv \mu \equiv const.$ **after adjustment**

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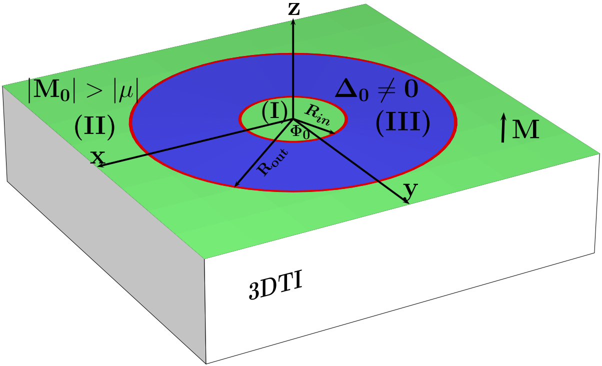
chemical potential

assume: $\mu(r) \equiv \mu \equiv const.$ **after adjustment**

solve for each region separately

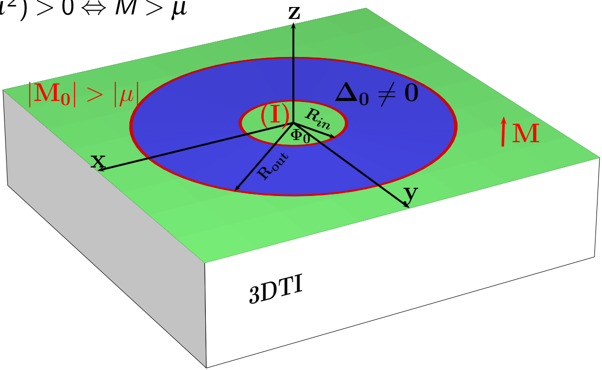


for each region reduce to a set of two **Bessel differential equations**

in the following $f(r) \rightarrow \psi_\uparrow$ and $g(r) \rightarrow \psi_\downarrow$

solutions for different regions

$$a = \sqrt{(M^2 - \mu^2)} > 0 \Leftrightarrow M > \mu$$

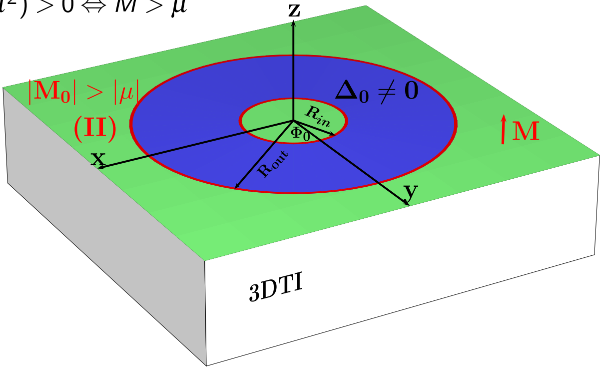


for region (I):

$$f_<^{out}(r) = c_<^{out} I_0(ar)$$
$$g_<^{out}(r) = -b c_<^{out} I_1(ar)$$

solutions for different regions

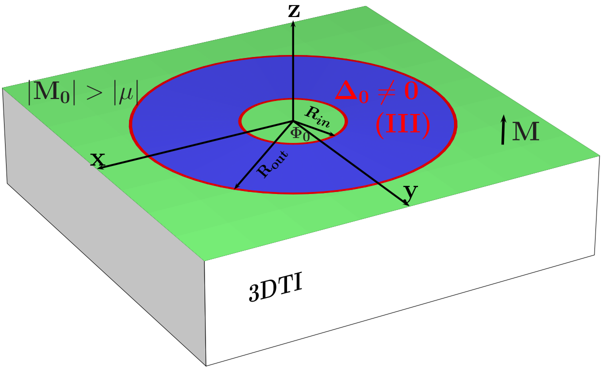
$a = \sqrt{(M^2 - \mu^2)} > 0 \Leftrightarrow M > \mu$



for region (II):

$$f_{>}^{out}(r) = c_{>}^{out} K_0(ar)$$
$$g_{>}^{out}(r) = bc_{>}^{out} K_1(ar)$$

solutions for different regions



for region (III):

$$f^{in}(r) = c_1^{in} J_0(\mu r) + c_2^{in} Y_0(\mu r)$$
$$g^{in}(r) = c_1^{in} J_1(\mu r) + c_2^{in} Y_1(\mu r)$$

Wave functions of zero energy at the boundaries

$$\Psi^{R_{in}}{}_0(r,\theta) = e^{-V(r)} \begin{pmatrix} f_<(r) \\ g_<(r)e^{i\theta} \\ g_<(r)e^{-i\theta} \\ -f_<(r) \end{pmatrix}$$

$$\Psi^{R_{out}}{}_0(r,\theta) = e^{-V(r)} \begin{pmatrix} f_>(r) \\ g_>(r)e^{i\theta} \\ g_>(r)e^{-i\theta} \\ -f_>(r) \end{pmatrix}$$

outer boundary: $V(r) = \int_r^{R_{out}} \mathrm{d}r' \vartheta(r')$,

inner boundary: $V(r) = \int_{R_{in}}^{r} \mathrm{d}r' \vartheta(r')$,

with $\vartheta(r) = \Delta_0 \Theta(R_{out} - r)\Theta(r - R_{in})$

When they are localized at the boundaries?

$$\Psi^{R_{in}}{}_0(r,\theta) = e^{-V(r)} \begin{pmatrix} f_<(r) \\ g_<(r)e^{i\theta} \\ g_<(r)e^{-i\theta} \\ -f_<(r) \end{pmatrix}$$

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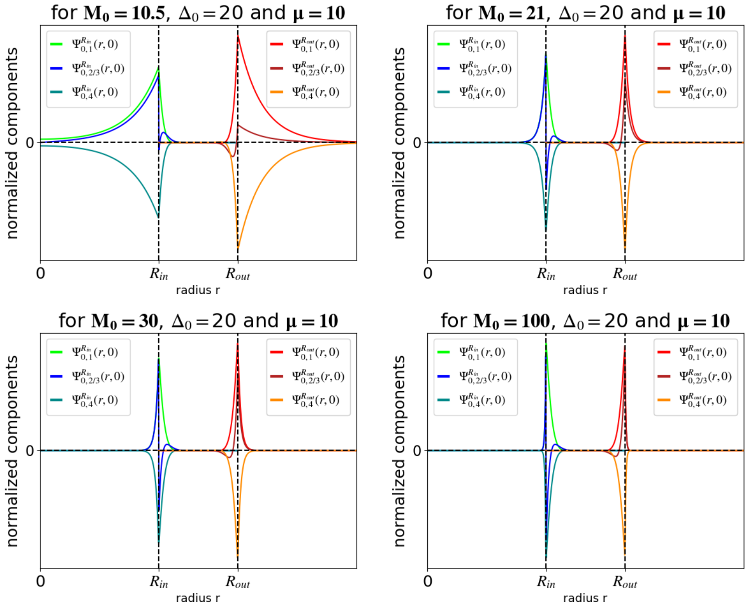
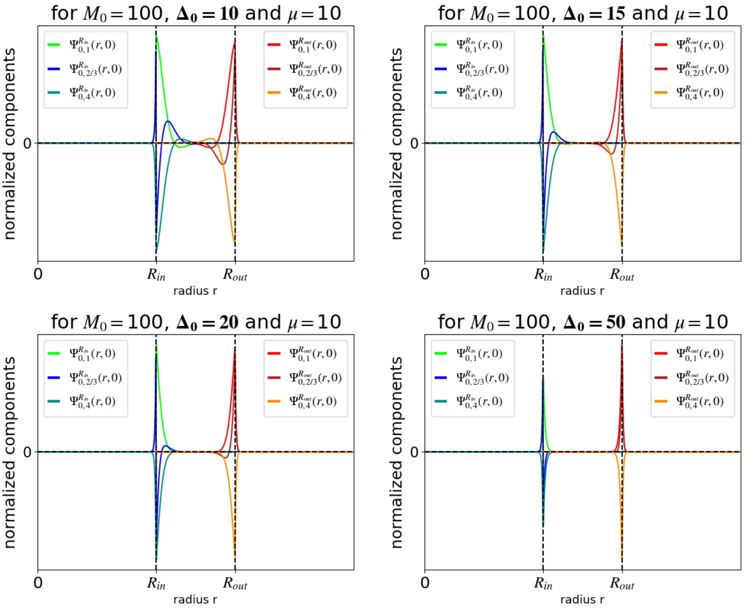
with $\vartheta(r) = \Delta_0 \Theta(R_{out} - r)\Theta(r - R_{in})$

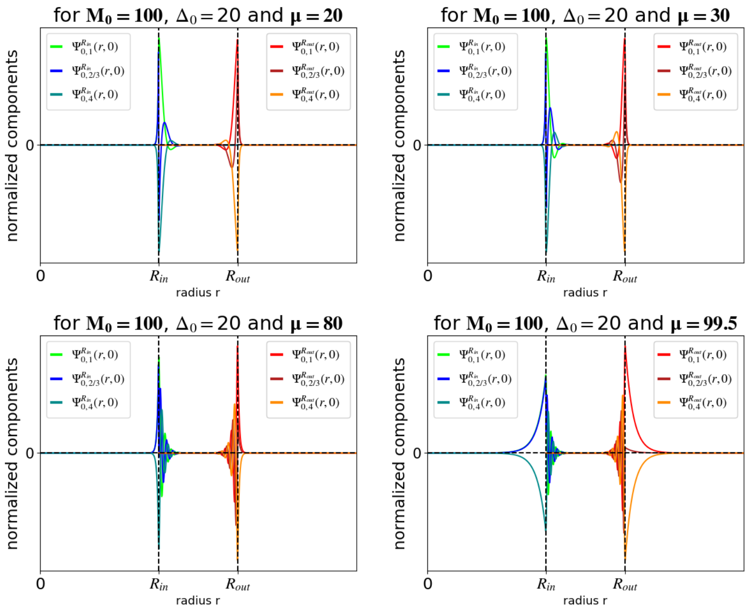
Edge state components for different magnetic fields relative to μ



Edge state components for different values for gap strength Δ_0 (SC)



Edge state components for different values of the chemical potential μ



Majorana zero mode operators

$$\Psi = ((\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger))^T$$

$$\gamma_0^\alpha = \int \mathrm{d}\boldsymbol{r} (\Psi_0^\alpha(\boldsymbol{r}))^\dagger \Psi(\boldsymbol{r}) \ ,$$

$$(\gamma_0^\alpha)^\dagger = \int \mathrm{d}\boldsymbol{r} \Psi^\dagger(\boldsymbol{r}) \Psi_0^\alpha(\boldsymbol{r}) \ ,$$

for real functions $f(r)$, $g(r)$ it is

$$\gamma_0^{R_{in}} = \left(\gamma_0^{R_{in}}\right)^\dagger \quad \text{and} \quad \gamma_0^{R_{out}} = \left(\gamma_0^{R_{out}}\right)^\dagger$$

$$\Psi_0^{R_{in}}(r,\theta) = e^{-V(r)} \begin{pmatrix} f_<(r) \\ g_<(r)e^{i\theta} \\ g_<(r)e^{-i\theta} \\ -f_<(r) \end{pmatrix} , \ \Psi_0^{R_{out}}(r,\theta) = e^{-V(r)} \begin{pmatrix} f_>(r) \\ g_>(r)e^{i\theta} \\ g_>(r)e^{-i\theta} \\ -f_>(r) \end{pmatrix}$$

Majorana operators of higher excitation states

$\Psi^{R_{in}}_{\pm k}$, $\Psi^{R_{out}}_{\pm k}$ with energy $\pm E_k$

$$\gamma_k^\alpha = \int \mathrm{d}\boldsymbol{r}(\Psi_k^\alpha(\boldsymbol{r}))^\dagger \Psi(\boldsymbol{r}) \ ,$$
$$(\gamma_k^\alpha)^\dagger = \gamma_{-k}^\alpha$$

Majorana operators in real-space

$$\gamma^\alpha(\theta) = C \int \mathrm{d}k e^{ik\theta} \gamma_k^\alpha = (\gamma^\alpha(\theta))^\dagger$$

next higher low energy excitations restrained by the ring size

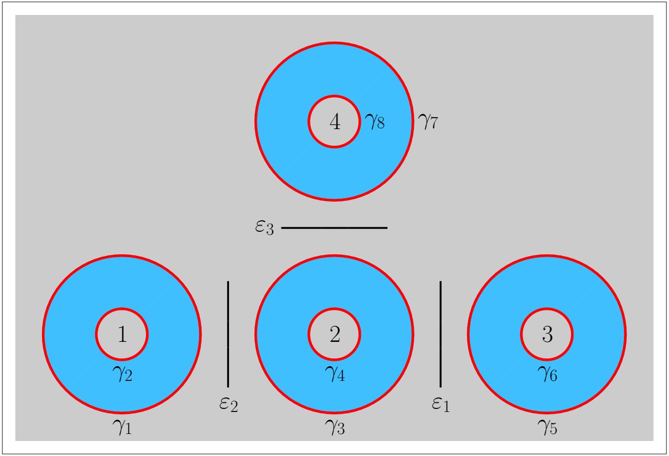
$$E^{in} \propto \frac{v(M_0, \mu, \Delta_0)}{R_{in}}, \ E^{out} \propto \frac{v(M_0, \mu, \Delta_0)}{R_{out}}$$

Questions

How to provide ***Controlled*** **quantum state** ***manipulations***?

What are the ***exchange statistics***
of the present Majorana bound states?

schematic view: basis setup for an exchange process of two Majorana zero modes



$$\varepsilon_i(t) = 1 \quad \text{, for gate } i \text{ is switched on,}$$
$$\varepsilon_i(t) = 0 \quad \text{, for gate } i \text{ is switched off.}$$

adiabatically exchange of two Majorana zero modes

Throughout the exchange process:

stay in the degenerated ground state manifold!

Avoid ground state excitations!

The speed of the exchange process is limited by the energy gap.

adiabatically exchange of two Majorana zero modes

Throughout the exchange process:

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adiabatic time evolution process from and back to an initial systems parameter set

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$\Rightarrow$ **unitary evolution of the systems ground state**

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$\Rightarrow$ **braiding**

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Throughout the exchange process:

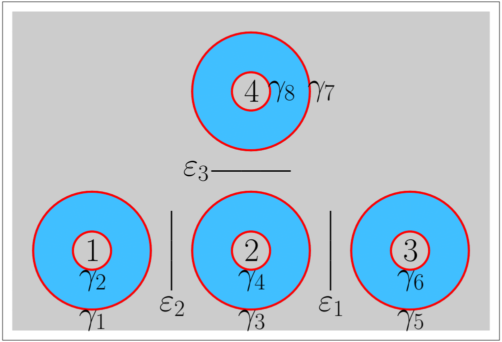
stay in the degenerated ground state manifold!

adiabatic time evolution process from and back to an initial systems parameter set
$\Rightarrow$ **unitary evolution of the systems ground state**

$\Rightarrow$ **braiding**

$\rightarrow$ use the concept of **Berry phase**

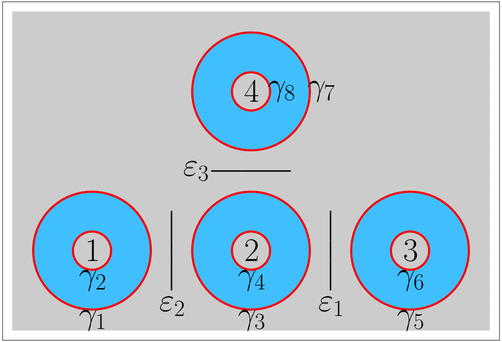
effective system Hamiltonian



$$
\begin{aligned}
H_{eff} &= i\left(n_1\varepsilon_1(t)\gamma_3\gamma_5 + n_2\varepsilon_2(t)\gamma_1\gamma_3 + n_3\varepsilon_3(t)\gamma_7\gamma_3\right) \\
&= \alpha(t)(f_2 f_1 + f_2^\dagger f_1) - \beta(t)(f_2^\dagger f_1^\dagger + f_2 f_1^\dagger) + \varepsilon_3(t)(1 - 2f_2^\dagger f_2),
\end{aligned}
$$

$\alpha(t) = \varepsilon_1(t) - i\varepsilon_2(t)$, $\beta(t) = \varepsilon_1(t) + i\varepsilon_2(t)$ and we choose $n_i = 1$ for all $i = 1, 2, 3$

$2^4 = 16$ fold degenerated ground state manifold



$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_5), \quad f_2 = \frac{1}{2}(\gamma_3 + i\gamma_7),$$
$$f_3 = \frac{1}{2}(\gamma_2 + i\gamma_6), \quad f_4 = \frac{1}{2}(\gamma_4 + i\gamma_8),$$

effective system Hamiltonian

description of
the time evolution of the system state
in the ground state manifold

$$
\begin{aligned}
H_{eff} &= i\left(n_1\varepsilon_1(t)\gamma_3\gamma_5 + n_2\varepsilon_2(t)\gamma_1\gamma_3 + n_3\varepsilon_3(t)\gamma_7\gamma_3\right) \\
&= \alpha(t)(f_2 f_1 + f_2^\dagger f_1) - \beta(t)(f_2^\dagger f_1^\dagger + f_2 f_1^\dagger) + \varepsilon_3(t)(1 - 2f_2^\dagger f_2),
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effective system Hamiltonian

the time evolution of the system state

$$
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steps of reduction

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steps of reduction

- matrix representation (16×16) matrix

effective system Hamiltonian

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steps of reduction

- matrix representation (16×16) matrix
- all fermion operators consisting of inner boundary modes **commute** with H_{eff}!
  $\Rightarrow$ reduction to a (4×4) matrix
  $|00\rangle$, $|11\rangle = f_1^\dagger f_2^\dagger |00\rangle$, $|10\rangle = f_1^\dagger |00\rangle$, $|01\rangle = f_2^\dagger |00\rangle$

effective system Hamiltonian

$$H_{eff} = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} |00\rangle & |11\rangle & |10\rangle & |01\rangle \end{array} \\ \begin{pmatrix} \varepsilon_3 & \varepsilon_1 - i\varepsilon_2 & 0 & 0 \\ \varepsilon_1 + i\varepsilon_2 & -\varepsilon_3 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & \varepsilon_1 + i\varepsilon_2 \\ 0 & 0 & \varepsilon_1 - i\varepsilon_2 & -\varepsilon_3 \end{pmatrix} \end{array} = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix}$$

steps of reduction

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- all fermion operators consisting of inner boundary modes **commute** with H_{eff}!
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  $|00\rangle\,,\ |11\rangle = f_1^\dagger f_2^\dagger\,|00\rangle\,,\ |10\rangle = f_1^\dagger\,|00\rangle\,,\ |01\rangle = f_2^\dagger\,|00\rangle$

effective system Hamiltonian

$$H_{eff} = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} |00\rangle & |11\rangle & |10\rangle & |01\rangle \end{array}$$

$$H_{eff} = \begin{pmatrix} \varepsilon_3 & \varepsilon_1 - i\varepsilon_2 & 0 & 0 \\ \varepsilon_1 + i\varepsilon_2 & -\varepsilon_3 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & \varepsilon_1 + i\varepsilon_2 \\ 0 & 0 & \varepsilon_1 - i\varepsilon_2 & -\varepsilon_3 \end{pmatrix} = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix}$$

steps of reduction

- matrix representation (16×16) matrix

- all fermion operators consisting of inner boundary modes
  commute with H_{eff}!
  $\Rightarrow$ reduction to a (4×4) matrix
  $|00\rangle\,,\ |11\rangle = f_1^\dagger f_2^\dagger\,|00\rangle\,,\ |10\rangle = f_1^\dagger\,|00\rangle\,,\ |01\rangle = f_2^\dagger\,|00\rangle$

- **parity conservation**
  $\Rightarrow$ either *even* or *odd* parity subspace $\rightarrow (2 \times 2)$ matrix

effective system Hamiltonian

$$H_{even} = \varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3 = \boldsymbol{\varepsilon}_{even}\cdot\boldsymbol{\sigma},$$
$$H_{odd} = \varepsilon_1\sigma_1 - \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3 = \boldsymbol{\varepsilon}_{odd}\cdot\boldsymbol{\sigma}$$

steps of reduction

- matrix representation (16×16) matrix
- all fermion operators consisting of inner boundary modes **commute** with H_{eff}!
  $\Rightarrow$ reduction to a (4×4) matrix
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  $\Rightarrow$ either *even* or *odd* parity subspace $\rightarrow$ (2×2) matrix

mapping on a sphere

reduced **time evolution of the system state**

$$H_{even} = \begin{pmatrix} \varepsilon \ \cos\theta & \varepsilon \ \sin\theta\, e^{-i\phi} \\ \varepsilon \ \sin\theta\, e^{i\phi} & -\varepsilon \ \cos\theta \end{pmatrix}$$

$\varepsilon = \left|\boldsymbol{\varepsilon}_{even}\right|$

eigenvalues: $\lambda_\pm = \pm\sqrt{\varepsilon_1(t)^2 + \varepsilon_2(t)^2 + \varepsilon_3(t)^2}$

mapping of system state time evolution onto the
time evolution of the gate vector

normalized gate vector: $\hat{\boldsymbol{\varepsilon}}(t) = \frac{\boldsymbol{\varepsilon}(t)}{|\boldsymbol{\varepsilon}(t)|}$, for all time: $\boldsymbol{\varepsilon}(t) \neq 0$

$\rightarrow$ **unit vector moving on a sphere**

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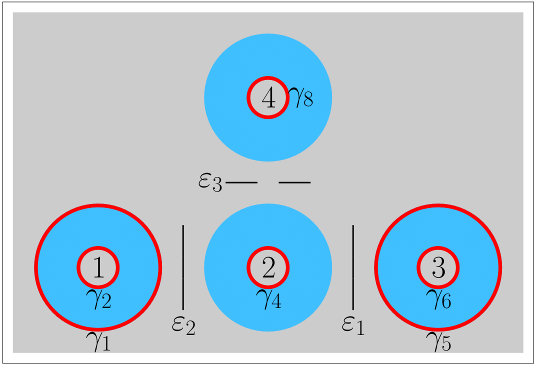
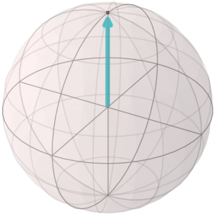
Consider now the time evolution...

$t = t_0$



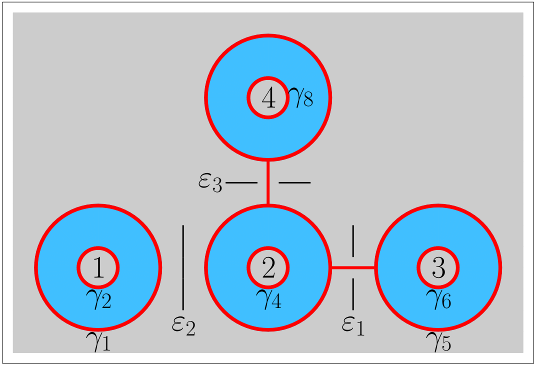


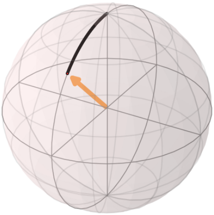
$$\hat{\boldsymbol{\varepsilon}}(t_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

initial Hamiltonian

$$H_{eff}(t_0) = i\gamma_7\gamma_3$$

$t = t_1$

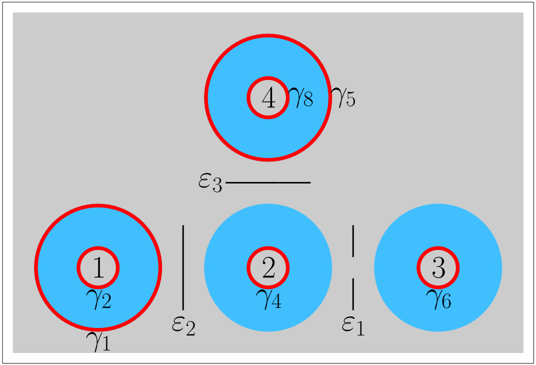


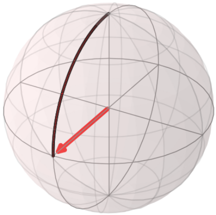


$$\hat{\boldsymbol{\varepsilon}}(t_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$H_{eff}(t_1) = i(\gamma_7\gamma_3 + \gamma_3\gamma_5)$$

$t = t_2$

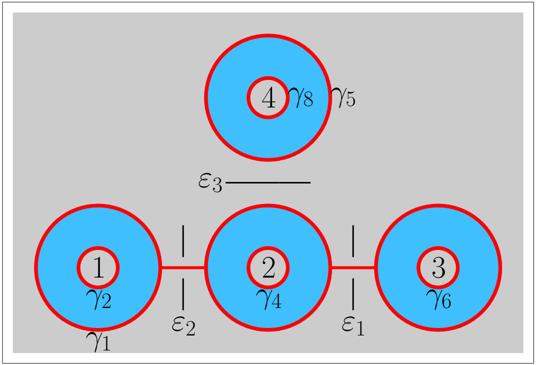


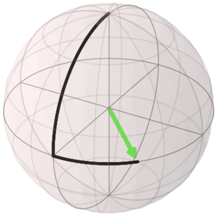


$$\hat{\boldsymbol{\varepsilon}}(t_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$H_{eff}(t_2) = i\gamma_3\gamma_5$$

$t = t_3$

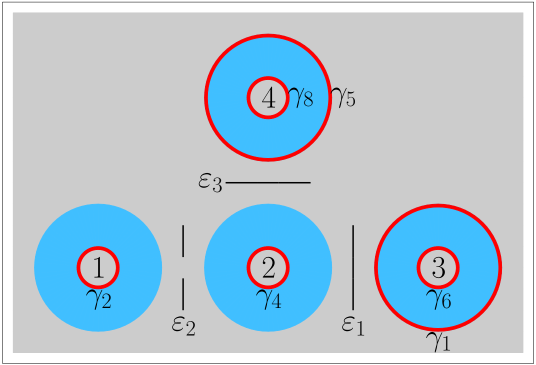


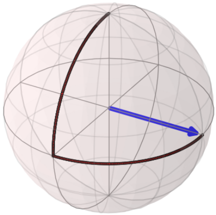


$$\hat{\boldsymbol{\varepsilon}}(t_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$H_{eff}(t_3) = i(\gamma_1\gamma_3 + \gamma_3\gamma_5)$$

$t = t_4$

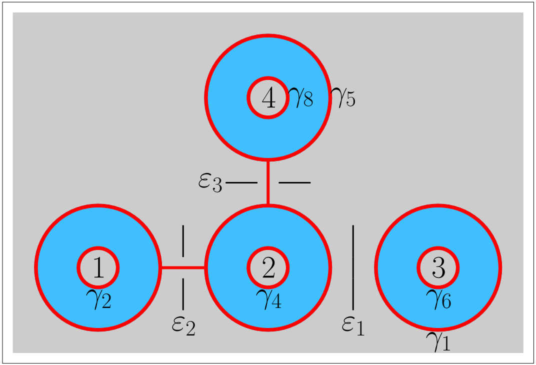


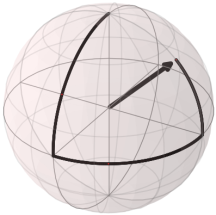


$$\hat{\boldsymbol{\varepsilon}}(t_4) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$H_{eff}(t_4) = i\gamma_1\gamma_3$$

$t = t_5$

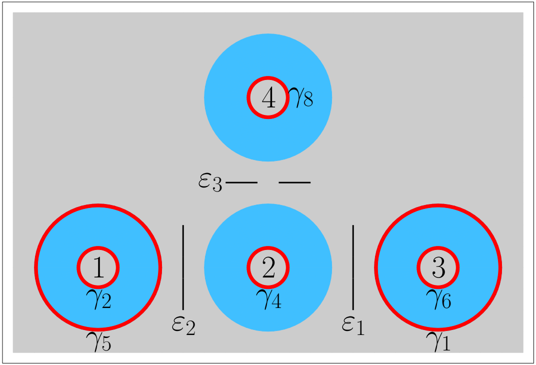


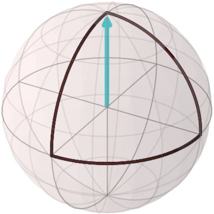


$$\hat{\boldsymbol{\varepsilon}}(t_5) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$H_{eff}(t_5) = i(\gamma_1\gamma_3 + \gamma_7\gamma_3)$$

$t = t_6$



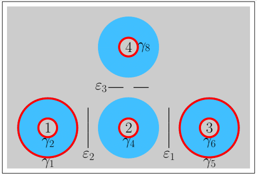


$$\hat{\boldsymbol{\varepsilon}}(t_6) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

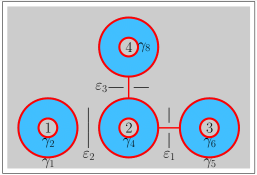
final Hamiltonian

$$H_{eff}(t_6) = i\gamma_7\gamma_3 = H_{eff}(t_0)$$

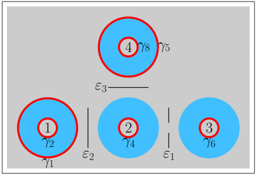
sketch of the braid process



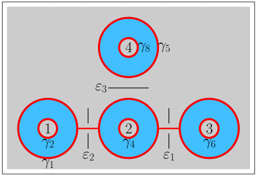
$t = t_0$



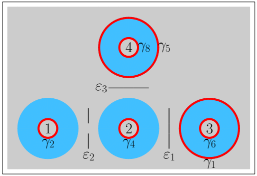
$t = t_1$



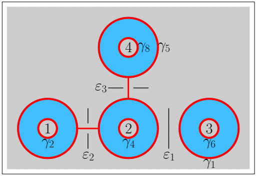
$t = t_2$



$t = t_3$



$t = t_4$



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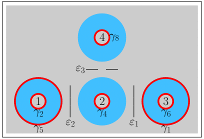
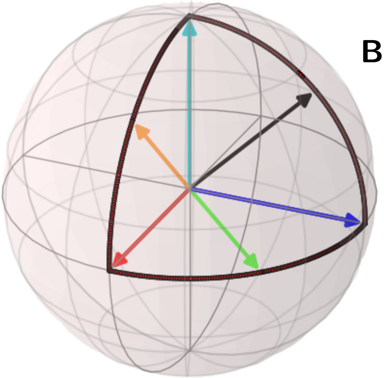


$t = t_6$

Berry phase



Berry curvature: $\mathscr{F}_{\theta\phi} = \mp\frac{1}{2}\sin\theta$

Berry phase:

$$i\varphi_{even} = -i\int_0^\theta \int_0^\phi \mathrm{d}\theta' \mathrm{d}\phi' \mathscr{F}_{\theta'\phi'} = \pm\frac{i}{2}\Omega$$

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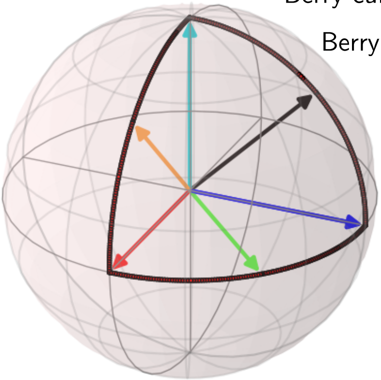

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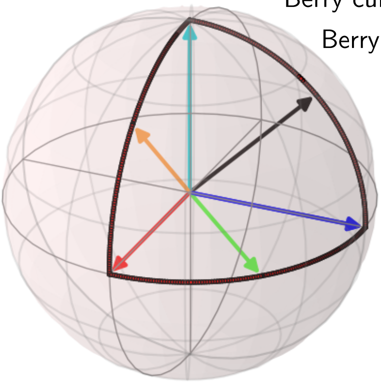

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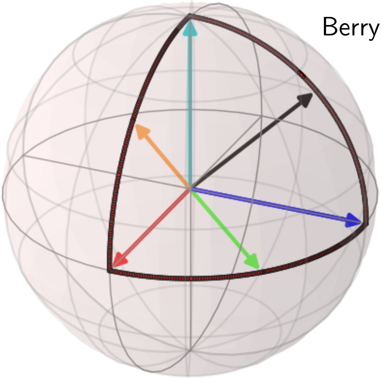
results

for the even and odd parity space

$$\boldsymbol{e^{i\varphi_{even}} = e^{i\frac{\pi}{4}}}$$

$$\boldsymbol{e^{i\varphi_{odd}} = e^{-i\frac{\pi}{4}}}$$

Berry phase



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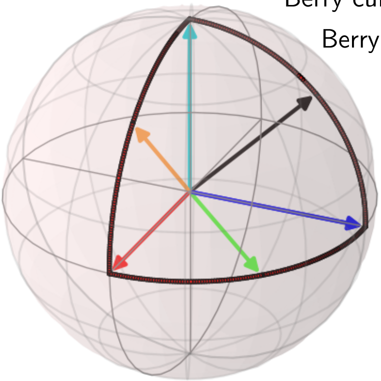

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braid matrix: $\boldsymbol{U_{15} = e^{\frac{\pi}{4}\gamma_1\gamma_5}}$

acting on system states: $\boldsymbol{U|\Psi\rangle = e^{\pm i\frac{\pi}{4}}|\Psi\rangle}$

with $\boldsymbol{U_{15}\gamma_1 U_{15}^{\dagger} = -\gamma_5}$ and $\boldsymbol{U_{15}\gamma_5 U_{15}^{\dagger} = \gamma_1}$

fusion

anyon types of mzm

$$\sigma \times \sigma = I + \Psi$$

vacuum channel

particle channel

split up the ground state degeneracy

- **gating (adjust μ) $\rightarrow$ overlap of 2 outer Majorana wave functions**

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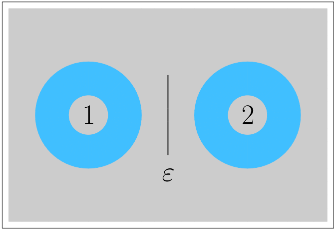
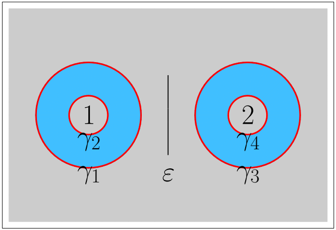
- **parity-to-charge conversion** $\rightarrow$ inner and outer Majoranas to charge state

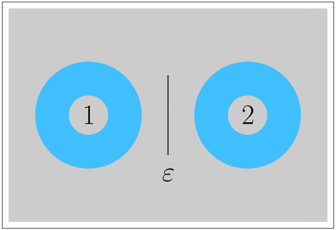
trivial fusion protocol (parity-to-charge conversion between t_1 and t_2)



t_0: $|Q_1, Q_2\rangle$



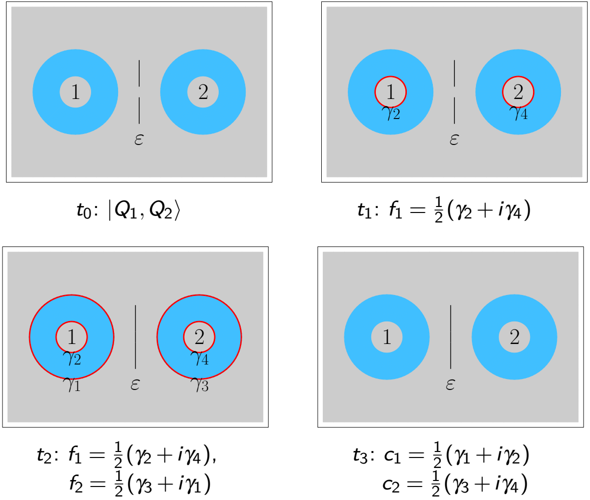
t_1: $f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$
$f_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$



t_2: $f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$
$f_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$

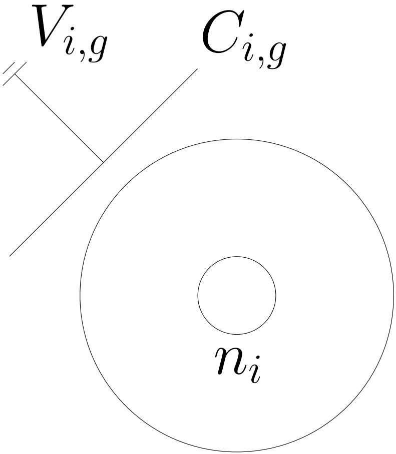
non-trivial fusion protocol $\begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix}$

(parity-to-charge conversion between t_2 and t_3)



parity-to-charge conversion

Fuse inner and outer Majoranas of one ring



Goal: read out the information stored in a quantum bit after processing manipulations.

- **initialize the quantum bit in a well defined state**
- create pairs of Majorana zero mode out of the vacuum (degenerated ground state manifold)

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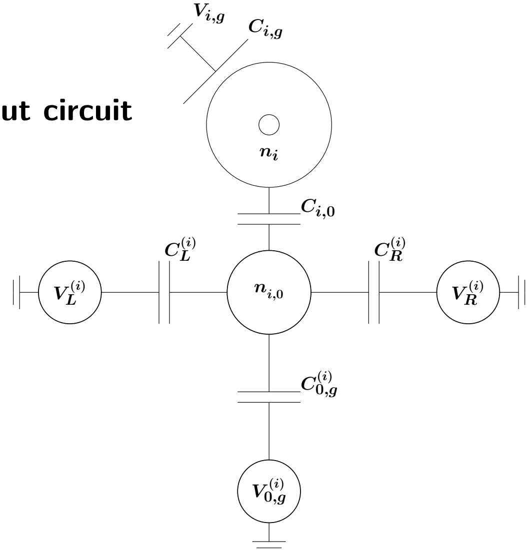
- **read out** the final system state

initialization: *parity-to-charge conversion* (side gate at each ring)

One read out circuit



$E_{el}(n_i, n_{i,0})$

read out: *charge sensing* (**S**ingle **E**lectron **T**ransistor)

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One read out circuit

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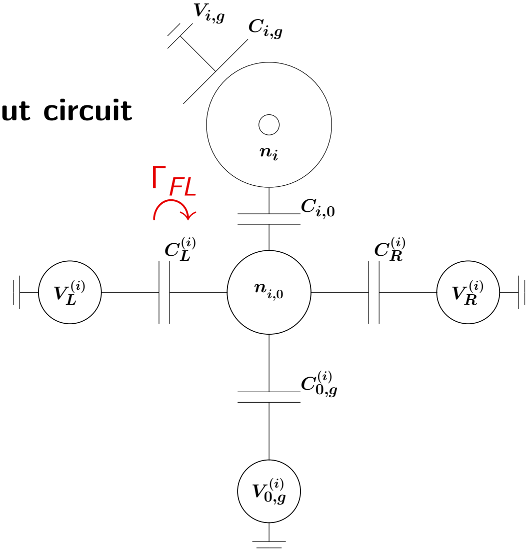

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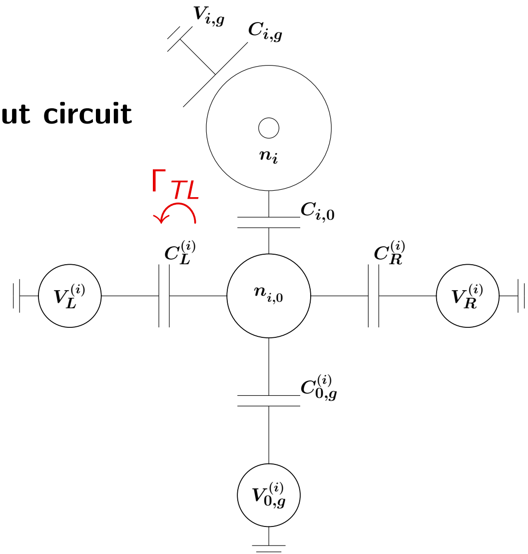

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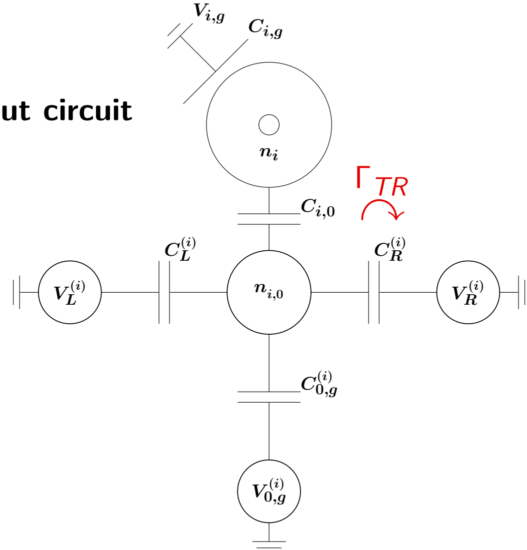

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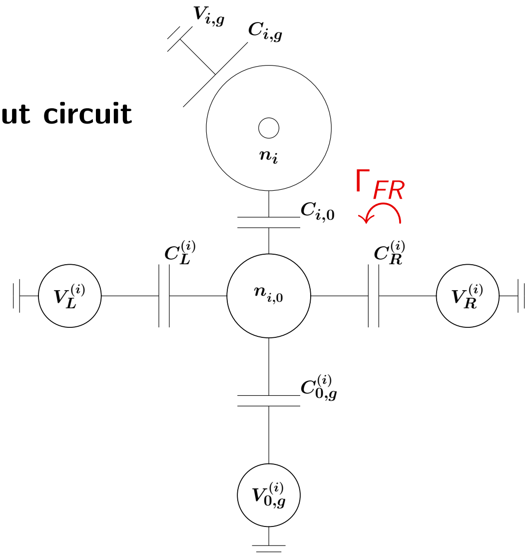

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read out: *charge sensing* (**S**ingle **E**lectron **T**ransistor)

Quantum mechanical solution of the master equation

System of two read out circuits:

$$\left|a\right\rangle, \left|b\right\rangle, \left|c\right\rangle, \left|d\right\rangle \in \Big\{ \left|0_{qb}0_1 0_2\right\rangle, \left|0_{qb}0_1 1_2\right\rangle, \left|0_{qb}1_1 0_2\right\rangle, \left|0_{qb}1_1 1_2\right\rangle, \left|1_{qb}0_1 0_2\right\rangle, \left|1_{qb}0_1 1_2\right\rangle, \left|1_{qb}1_1 0_2\right\rangle, \left|1_{qb}1_1 1_2\right\rangle \Big\}$$

general notation: $|a\rangle = |\alpha \; n_{1,0} \; n_{2,0}\rangle$, $|b\rangle = |\beta \; m_{1,0} \; m_{2,0}\rangle$,...

Goal: determine the tunneling rates Γ in the SET and its dependence on the quantum bit state

Hamiltonian of each circuit

$$H_i = H_{i,S} + H_{i,B} + H_{i,I}$$

$$\begin{aligned}
H_{i,B} &= H_{i,L} + H_{i,R} + H_{\text{i,island}} \ , \\
H_{i,L} &= \sum_k c^\dagger_{i,L,k} c_{i,L,k} \varepsilon_{i,L,k} + \mu_{i,L} \sum_k c^\dagger_{i,L,k} c_{i,L,k} \ , \\
H_{i,R} &= \sum_k c^\dagger_{i,R,k} c_{i,R,k} \varepsilon_{i,R,k} + \mu_{i,R} \sum_k c^\dagger_{i,R,k} c_{i,R,k} \ , \\
H_{\text{i,island}} &= \sum_k d^\dagger_{i,k} d_{i,k} \varepsilon_{i,k}
\end{aligned}$$

$$H_{i,S} = E_{00}^{(C,i)} (\hat{n}_{i,0} - n_{0,g}^{(i)})^2 + E_{11}^{(C,i)} (c_i^\dagger c_i)^2 + 2E_{10}^{(C,i)} (\hat{n}_{i,0} - n_{0,g}^{(i)}) c_i^\dagger c_i$$

$$H_{i,I} = \sum_{\sigma=L,R} \left[\sum_{q,k} \tilde{t}^{i,\sigma}_{q,k} d^\dagger_{i,q} c_{i,\sigma,k} e^{i\varphi_i} + \sum_{q,k} (\tilde{t}^{i,\sigma}_{k,q})^* c^\dagger_{i,\sigma,k} d_{i,q} e^{-i\varphi_i} \right]$$

The Makovian master equation

Approximations

weak bath interaction $\rightarrow$ *interaction representation*

$$\rho(t) = e^{-iH_I t}\rho'(0)e^{iH_I t}$$

no first order dynamics $\rightarrow$ reduced eq. of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i[H_I(t), \rho(t)]\ ,\ \textit{with}\ H_I(t) = e^{iH_0 t}H_I e^{-iH_0 t}$$

Born approximation: bath constant in time $\rightarrow \rho(t) = \rho_S(t) \otimes \rho_B$

Markovian master equation

(systems damping time $\gg$ time reservoir keeps its correlation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\int_0^\infty \mathrm{d}s\ \mathrm{tr}_B\left([H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]]\right)$$

The final Makovian master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S^{ab}(t) = -\Gamma_{\text{out}}^{a,b}\,\rho_S^{ab}(t) + \Gamma_{\text{in,-}}^{a-1,b-1}\rho_S^{a-1,b-1}(t) + \Gamma_{\text{in},+}^{a+1,b+1}\rho_S^{a+1,b+1}(t)$$

notation: $a \pm 1 = |\alpha\ n_{1,0} \pm 1\ n_{2,0}\rangle$, $b \pm 1 = |\beta\ m_{1,0} \pm 1\ m_{2,0}\rangle$

$\Gamma_{\text{in},\pm}^{a\pm 1,b\pm 1} \equiv \Gamma_{\text{in},\pm}(\alpha,\beta,n_{1,0},m_{1,0},t)$, $\Gamma_{\text{out}}^{a,b} \equiv \Gamma_{\text{out}}(\alpha,\beta,n_{1,0},m_{1,0})$

$\rightarrow$ **current** through the SET depends on the fixed qubit charge state

Rate solutions

<u>*total out-rate:*</u> $\Gamma^{a,b}_{\text{out}} = \Gamma^{a,b}_{\text{out},+} + \Gamma^{a,b}_{\text{out,-}}$

$$\Gamma^{a,b}_{\text{out},\pm} = \sum_{\sigma=R,L} \alpha_0^\sigma \quad \left\{ \pi\cdot \left[\frac{\Delta^\pm_{a,\sigma}}{1-e^{-\beta_{\text{th}}\Delta^\pm_{a,\sigma}}} + \frac{\Delta^\pm_{b,\sigma}}{1-e^{-\beta_{\text{th}}\Delta^\pm_{b,\sigma}}} \right] \right.$$
$$\left. + \frac{i}{2}\left[\Delta^\pm_{a,\sigma} \ln\left(\frac{\Lambda}{\left|\Delta^\pm_{a,\sigma}\right|} \right) - \Delta^\pm_{b,\sigma} \ln\left(\frac{\Lambda}{\left|\Delta^\pm_{b,\sigma}\right|} \right) \right] \right\}$$

<u>*total in-rate:*</u> $\Gamma^{a,b}_{\text{in}} = \Gamma^{a+1,b+1}_{\text{in},+} + \Gamma^{a-1,b-1}_{\text{in,-}}$

$$\Gamma^{a\pm1,b\pm1}_{\text{in},\pm} = e^{i(\Delta E^\pm_a - \Delta E^\pm_b)t} \quad \sum_{\sigma=R,L} \alpha_0^\sigma \left\{ \pi\cdot \left[\frac{\Delta^\pm_{a,\sigma}}{e^{\beta_{\text{th}}\Delta^\pm_{a,\sigma}}-1} + \frac{\Delta^\pm_{b,\sigma}}{e^{\beta_{\text{th}}\Delta^\pm_{b,\sigma}}-1} \right] \right.$$
$$\left. - \frac{i}{2}\left[\Delta^\pm_{a,\sigma} \ln\left(\frac{\Lambda}{\left|\Delta^\pm_{a,\sigma}\right|} \right) - \Delta^\pm_{b,\sigma} \ln\left(\frac{\Lambda}{\left|\Delta^\pm_{b,\sigma}\right|} \right) \right] \right\}.$$

Rate solutions $\alpha_0^\sigma = |\tilde{t}^\sigma|^2 \nu_0^\sigma \nu_0^I = \frac{h}{4\pi^2 e^2 R_{\tilde{t}}^\sigma} > 0$

<u>*total out-rate:*</u> $\Gamma_{\text{out}}^{a,b} = \Gamma_{\text{out},+}^{a,b} + \Gamma_{\text{out,-}}^{a,b}$

$$\Gamma_{\text{out},\pm}^{a,b} = \sum_{\sigma=R,L} \alpha_0^\sigma \quad \left\{ \pi \cdot \left[\frac{\Delta_{a,\sigma}^{\pm}}{1 - e^{-\beta_{\text{th}} \Delta_{a,\sigma}^{\pm}}} + \frac{\Delta_{b,\sigma}^{\pm}}{1 - e^{-\beta_{\text{th}} \Delta_{b,\sigma}^{\pm}}} \right] + \frac{i}{2} \left[\Delta_{a,\sigma}^{\pm} \ln \left(\frac{\Lambda}{|\Delta_{a,\sigma}^{\pm}|} \right) - \Delta_{b,\sigma}^{\pm} \ln \left(\frac{\Lambda}{|\Delta_{b,\sigma}^{\pm}|} \right) \right] \right\}$$

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$$\Gamma_{\text{in},\pm}^{a\pm1,b\pm1} = e^{i(\Delta E_a^{\pm} - \Delta E_b^{\pm})t} \quad \sum_{\sigma=R,L} \alpha_0^\sigma \left\{ \pi \cdot \left[\frac{\Delta_{a,\sigma}^{\pm}}{e^{\beta_{\text{th}} \Delta_{a,\sigma}^{\pm}} - 1} + \frac{\Delta_{b,\sigma}^{\pm}}{e^{\beta_{\text{th}} \Delta_{b,\sigma}^{\pm}} - 1} \right] - \frac{i}{2} \left[\Delta_{a,\sigma}^{\pm} \ln \left(\frac{\Lambda}{|\Delta_{a,\sigma}^{\pm}|} \right) - \Delta_{b,\sigma}^{\pm} \ln \left(\frac{\Lambda}{|\Delta_{b,\sigma}^{\pm}|} \right) \right] \right\} .$$

Rate solutions

$$\Delta^{\pm}_{k,\sigma} = \Delta E^{\pm}_{k} \pm \mu_{\sigma}, \quad \Delta E^{\pm}_{k} = E_{el}(\kappa, n^0_1, n^0_2) - E_{el}(\kappa, n^0_1 \pm 1, n^0_2)$$

<u>*total out-rate*:</u> $\Gamma^{a,b}_{\text{out}} = \Gamma^{a,b}_{\text{out},+} + \Gamma^{a,b}_{\text{out,-}}$

$$\Gamma^{a,b}_{\text{out},\pm} = \sum_{\sigma=R,L} \alpha^{\sigma}_0 \quad \left\{ \pi \cdot \left[\frac{\Delta^{\pm}_{a,\sigma}}{1 - e^{-\beta_{\text{th}} \Delta^{\pm}_{a,\sigma}}} + \frac{\Delta^{\pm}_{b,\sigma}}{1 - e^{-\beta_{\text{th}} \Delta^{\pm}_{b,\sigma}}} \right] + \frac{i}{2} \left[\Delta^{\pm}_{a,\sigma} \ln\left(\frac{\Lambda}{|\Delta^{\pm}_{a,\sigma}|} \right) - \Delta^{\pm}_{b,\sigma} \ln\left(\frac{\Lambda}{|\Delta^{\pm}_{b,\sigma}|} \right) \right] \right\}$$

<u>*total in-rate*:</u> $\Gamma^{a,b}_{\text{in}} = \Gamma^{a+1,b+1}_{\text{in},+} + \Gamma^{a-1,b-1}_{\text{in,-}}$

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Rate solutions approximation: *upper energy cut-off* $\Lambda \gg |\Delta^{\pm}_{k,\sigma}|$

<u>*total out-rate:*</u> $\Gamma^{a,b}_{\text{out}} = \Gamma^{a,b}_{\text{out},+} + \Gamma^{a,b}_{\text{out,-}}$

$$\Gamma^{a,b}_{\text{out},\pm} = \sum_{\sigma=R,L} \alpha_0^{\sigma} \quad \left\{ \pi \cdot \left[\frac{\Delta^{\pm}_{a,\sigma}}{1 - e^{-\beta_{\text{th}} \Delta^{\pm}_{a,\sigma}}} + \frac{\Delta^{\pm}_{b,\sigma}}{1 - e^{-\beta_{\text{th}} \Delta^{\pm}_{b,\sigma}}} \right] + \frac{i}{2} \left[\Delta^{\pm}_{a,\sigma} \ln \left(\frac{\Lambda}{|\Delta^{\pm}_{a,\sigma}|} \right) - \Delta^{\pm}_{b,\sigma} \ln \left(\frac{\Lambda}{|\Delta^{\pm}_{b,\sigma}|} \right) \right] \right\}$$

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Results

- **total out-rate is *time independent***

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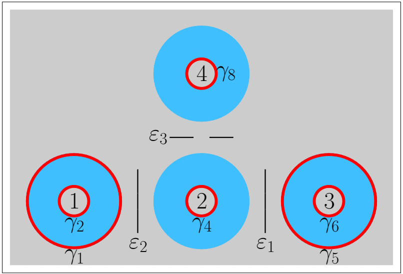
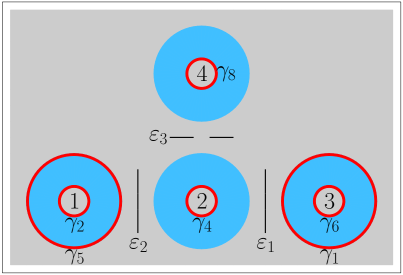
- total out-rate is *time independent*
- **in-rate** *depends exponentially on time* as long as the energy difference between two different rings charge states is not zero
- the real part of the out-rate correspond to the classical rate function

Fusion inner and outer Majoranas after braiding





$$t_0\colon\ f_1 = \tfrac{1}{2}(\gamma_1 + i\gamma_2),\quad f_2 = \tfrac{1}{2}(\gamma_6 + i\gamma_5)$$

$$t_6\colon\ c_1 = \tfrac{1}{2}(\gamma_5 + i\gamma_2)\quad c_2 = \tfrac{1}{2}(\gamma_6 + i\gamma_1)$$

$$\begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix}$$

Reduced density matrix in the long time limit

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initial state after one braiding:

$$|00\rangle \to |\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$$

$$\hat{\rho}_{in}(0) = |\Psi_{in}\rangle\langle\Psi_{in}| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \xrightarrow{t\to+\infty} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\rho}_\infty$$

$\to$ measuring **long** enough:
get **both** possible qubit state **outcomes with equal probability**

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Measure the braid outcome

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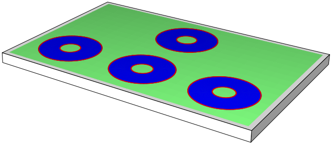
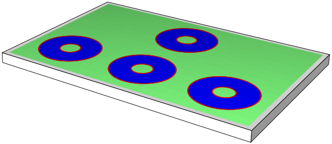
Conclusion

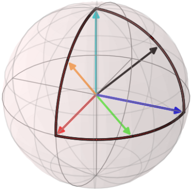
- **Existence of localized Majorana zero modes**



Conclusion

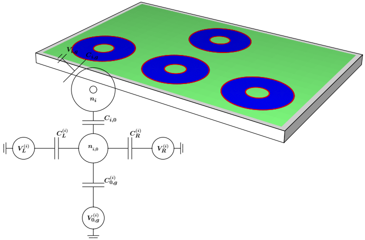
- Existence of **localized Majorana zero modes**
- **Braiding** of Majorana zero modes is possible

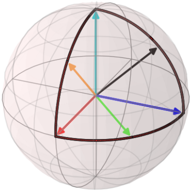




Conclusion

- Existence of **localized Majorana zero modes**
- **Braiding** of Majorana zero modes is possible
- **Measuring** of the outcome of braiding is realizable





References and total Master thesis

For references and total Master thesis download:

https://github.com/MaiSchoen/master-thesis/blob/master/MasterThesisMaikeSchoen_v1.pdf