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Physical properties of entangled Majorana fermion states on textured surfaces of topological insulators

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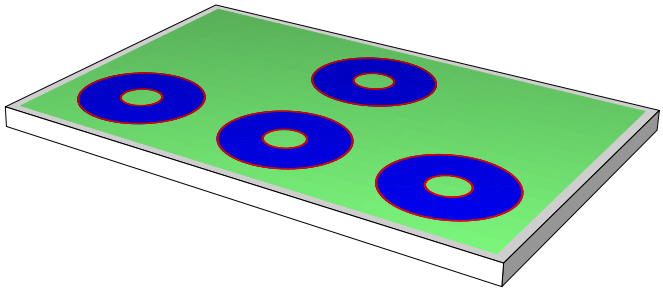

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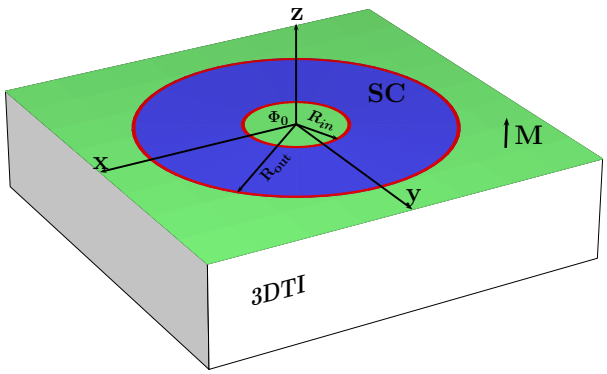
Abstract

In this Master thesis we describe and investigate a new potential qubit architecture, which allows a varied recombination of its constituents and therefore a lot of creativity in construction. Nevertheless here we limit ourselves to one possible combination and application to realize a potential Majorana quantum bit and give the reader only a few outlines what else could be possible with this platform for qubits. We consider the realization of a new Majorana quantum bit architecture based on topological protected Majorana bound states localized at the boundaries between the edges of superconducting rings and magnetic material, both layered on the surface of a three dimensional topological insulator. In the quantum bit architecture of consideration are combined four of such ring structures in such a way, that the basic operations for quantum information processing could be realized - initialization, manipulation, fusion and read out. Here we want to proof and gain confidence that under some strict conditions this new architecture is able to provide quantum information processing. We want to establish what these conditions are and how quantum information processing will be ensured by such a device.



Contents

1. Introduction

First let us think briefly about the question: *Why Quantum Computers?* The last few decades we live in the prediction of Moor's Law, increasing the number of transistors in dense integrated circuits linear and continuously. Increasing the number of transistors in a integrated circuit was and is of high interest, because a transistor as the fundamental building block of electronic devices is the device for controlling the path of electrons in a computer, and so for realizing electronic switches being *on* or *off*. Being *on* or *off* means having *one* or *zero*, the two states of a modern classical computer. So increasing the number of transistors enables a higher complexity and a larger number of calculations, which can be done parallel, means a higher number of parallel information processing, which leads to faster computations.
But increasing the transistors without enlarge the size of the integrated circuit means to make them smaller. But decreasing the size of an transistor is limited, due to the quantum nature of the electrons, which should be controlled by such a transistor. Therefore think of a *single-atom transistor*, where the gate between source and drain is a single atom. So to improve further the performance of computers in our future we have to leave the classical regime.
The idea of an quantum computer offers the possibility of nearly perfect parallel computing, being much more faster then every existing classical computer with highest possible dense of classical transistors. A quantum computer is not limited by the quantum nature of an electron, rather it tries to take advantage of the quantum physical description of particles as wave functions. We know that if we want to use the physics valid in a quantum mechanical regime we have to shrink the temperature of the system of consideration so that the energy scale of the electron wave function becomes larger compared to the temperature and we have to scale the size of some electronic devices to nano scales, which are scales of the wave length of electrons. Therefore quantum computing operates in quantum system at nano up to mesoscopic scales and very low temperatures. By creating peculiar environments where these quantum systems are embedded, it is possible to take advantage of the properties of particles emerging in these regimes as quasi excitations. Here we start with the promising properties of these quantum excitations and the way to manipulate and detect them. Finally we give a brief overview of some of the physics, which are important to create environments predicted to exhibit such useful quasi excitations.

1.1. Quantum bits and quantum information processing

1.1.1. Majorana fermions in condensed matter physics

Majorana fermions are realized as quasiparticle excitations at the boundaries of different topological phases of matter. If these quasiparticle excitations have zero energy they are called Majorana zero modes. And if these Majorana zero modes are localized at the boundaries they are called Majorana bound states. Due to their exotic exchange statistics they are anyons, and further they are predicted to exhibit non-abelian exchange statistics.

The annihilation (γ_i) and creation ($\gamma_i^\dagger$) operators of Majorana fermion particles have the following properties

$$\begin{aligned} &\text{(i)} \quad \gamma_i = \gamma_i^\dagger && \text{(being their own antiparticles)} \\ &\text{(ii)} \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij} && \text{(obeying fermionic anticommutation relations),} \end{aligned} \tag{1.1}$$

where the first property concludes that $\gamma_i\gamma_i = 1 = \gamma_i^\dagger\gamma_i^\dagger$ and the second property leads to $\gamma_i\gamma_j = -\gamma_j\gamma_i$. Further Majorana fermions can only occur in pairs, where each pair of Majorana fermions combine to one ordinary fermion with $f_{ij} = \frac{1}{2}(\gamma_i + i\gamma_j)$ and therefore it is $\gamma_i = f_{ij}^\dagger + f_{ij}$ and $\gamma_j = i(f_{ij}^\dagger - f_{ij})$. Due to the property (i) the wave function belonging to the Majorana state which is created by such an Majorana fermion operator has to be real. They exhibit no characteristics which would make them distinguishable like mass or charge. The emergent of $2N$ Majorana zero modes leads to 2^N-fold ground state degeneracy. This all results from the fact that they were first predicted and defined by the Italian Ettore Majorana as real solutions of a modified form of the Dirac equation [1], which leads to spin-$1/2$ particles being their own antiparticles. But at this time these kind of particles were still a mathematical construction and not validated in reality. After a long time of researcher tried to verify these as real particles they were found in the field of condensed matter physics as the quasiparticle excitations mentioned above. In the following we will briefly explain the non-abelian braiding statistics and what are the fusion rules of these particles.

1.1.2. Braiding and Berry phase

In three dimensions we distinct quantum particles due to their quantum exchange statistics between *bosons* and *fermions*. By exchanging their common wave function gets a phase of (± 1). In effective two dimensional systems, where their exchange paths are restricted to a flat surface, exchanging quasiparticles can exhibit much more exotic statistics. These quasiparticles are called anyons and divided in two sub classes. The abelian anyons, for which exchanging

two of them leads to a phase factor of the common wave function, and the non-abelian anyons, for which exchanging two of them leads to an unitary operation represented by multiplying the wave function by an unitary matrix. While multiplication with a phase always commutes, the multiplication by an unitary matrix has not to be commutable. That is why a wave function describing such particles having non-abelian exchange statistics needs to be a part of a degenerate subspace of states, so that the exchange of particles within this degenerate subspace do not change the energy of the system. Because this has to be in agreement with the present exchange symmetry to preserve the physics when two non-abelian anyons are exchanged, it has to be set an important constraint onto the statistical evolution, that the degenerate states have to be non-distinguishable if one looks at each of the non-abelian anyons individually. This means that a local measurement of the statistics is impossible [2].

If the adiabatic exchange of two non-abelian anyons leads to a multiplication by a matrix, which rotates the system within its ground state manifold we call this kind of exchange process *braiding*. Important is that this exchange process is adiabatic in this sense, that the system remains in its instantaneous ground state without crossing other energy levels, and is therefore a process which fulfils the *adiabatic theorem* of quantum mechanics.

That is why we can connect the system in its initial state before the braid process $|\Psi_{\text{init}}\rangle$ and its final wave function after the braid process $|\Psi_{\text{final}}\rangle$ with the unitary transformation U by $|\Psi_{\text{init}}\rangle \rightarrow U\,|\Psi_{\text{init}}\rangle = |\Psi_{\text{final}}\rangle$. The matrix U is often called the *braid matrix* and is a non trivial matrix, which acts as a rotation on the wave function $|\Psi_{\text{init}}\rangle$ and therefore the operations done by the braid matrix U are non-commutative and this means that it makes a different in which order one executes a series of exchanging non-abelian anyons. Here we consider in the following the exchange of Majorana zero modes, which are particles which belong to the class of non-abelian anyons.

D. A. Ivanov has derived a way to prove the non-abelian statistics of half-quantum vortices [3], which we want to explain briefly in its basics steps in the following, because in this thesis we want to follow the way of Ivanov to prove the non-abelian statistics for boundary modes in the here given regime. D. A. Ivanov considers a system with half-quantum vortices each exhibiting one Majorana modes of zero energy, which are stable with respect to any local perturbation [3]. For the spatially exchanging two of these Majorana zero modes he considers the motion of two half-quantum vortices, which are considered as the anyons, for which he proves non-abelian statistics under exchange them. He assumes an adiabatically motion of these vortices and therefore that the motion presents an unitary evolution in the ground state manifold. He introduces *braiding* as a permutation of the half-quantum vortices which brings the system back to its original state, but with a possible different ordering of the present vortices. To be a little bit more concrete this means by denoting the vortices exhibiting Majorana zero modes γ_i at position x_i as $v(x_i)$ one can see, that an permutation of two of this vortices $v(x_i)$ with γ_i and $v(x_j)$ with γ_j ends up

in the original system state due to the ground state degeneracy and the indistinguishability of these two vortices. such that the permuted vortex at its new position $v(x_j)$ exhibits the Majorana zero mode γ_i and the vortex $v(x_i)$ exhibits the Majorana zero mode γ_j. Ivanov describes this as a braid interchange and induces the braid operation as its mathematically description. He represents the generator of the so called *braid group T* for a braid interchange of two Majorana zero modes γ_i and γ_j as $\tau(T) = e^{\frac{\pi}{4}\gamma_i\gamma_j}$, which can be written as a matrix in the degenerated ground state

$$U_{ij} = e^{\frac{\pi}{4}\gamma_i\gamma_j}, \tag{1.2}$$

where its structure then depends how we choose initially the Majorana fermion pairing to one original fermion, and which of them we will braid. One peculiar thing of the braid matrix is that they do not commute

$$U_{ij}U_{km} \neq U_{km}U_{ij} \tag{1.3}$$

and that the permutation of γ_i and γ_j can be represented by this braid operation independent how much of the other vortices has to be permute therefore in between, i.e. it is for example

$$U_{ij} = U_{ik}^{\dagger}U_{kj}^{\dagger}U_{ik}\ , \tag{1.4}$$

where the operation $U_{ik}^{\dagger}$ represents the permutation in the opposite direction. To underline his result he gives as an example for an four fold degenerated ground state manifold with an initial pairing of the four Majorana zero modes to the two original fermions given as $\Psi_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $\Psi_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$. For this pairing one gets the ground state basis $(|0\rangle, \Psi_1^{\dagger}|0\rangle, \Psi_2^{\dagger}|0\rangle, \Psi_1^{\dagger}\Psi_2^{\dagger}|0\rangle)$ and therefore D. A. Ivanov writes the braid operators in matrix representation as [3]

$$\tau(T_1) = \begin{pmatrix} e^{-i\frac{\pi}{4}} & & & \\ & e^{i\frac{\pi}{4}} & & \\ & & e^{-i\frac{\pi}{4}} & \\ & & & e^{i\frac{\pi}{4}} \end{pmatrix},\ \tau(T_3) = \begin{pmatrix} e^{-i\frac{\pi}{4}} & & & \\ & e^{-i\frac{\pi}{4}} & & \\ & & e^{i\frac{\pi}{4}} & \\ & & & e^{i\frac{\pi}{4}} \end{pmatrix},$$

$$\tau(T_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}. \tag{1.5}$$

Strictly speaking one has to consider the exchange statistics of anyons, which are objects (most of geometrical nature) exhibiting Majorana zero modes, but in the following we will use the terms non-abelian anyons and Majorana zero modes in a synonymous way and we will describe the exchange statistics of these Majorana zero modes, although in general it is the exchange

statistics of the boundaries exhibiting the localized Majorana zero modes.

Berry phase

An important concept of determining the phase, which gets the systems ground state wave function by an adiabatic time evolution process from and again back to an initial set of system parameters, is the quantum mechanical concept of the *Berry phase*. Because the braid process belongs to this kind of time evolution processes and this concept gives us an important tool to determine the adiabatic phase change of the systems ground state by exchanging Majorana fermions in space, we want to give a little bit more detailed repetition of the quantum mechanical concept of the Berry phase, although it is discussed in many literatures like the two used here [4,5]. Think of a system which is described by a set of time dependent variables $\{x_a(t)\}$ describing the degrees of freedom of the system. Let us write these set of variables into a vector $\boldsymbol{x}(t)$. The Hamiltonian describing this system can be written as

$$\hat{H} \equiv \hat{H}(\boldsymbol{x}(t); \boldsymbol{\lambda}(t)) \ , \tag{1.6}$$

where $\boldsymbol{\lambda}$ is a vector with its N components are the parameters of the system $\{\lambda_a(t)\}$. Assume that the system is given in one of its ground states $|\text{g.s.}\rangle$. When we change adiabatically the parameters of the system, i.e. the system parameters $\boldsymbol{\lambda}(t)$ change slowly enough that the system stays in one of its instantaneous ground states for all times $|\text{g.s.}(\boldsymbol{\lambda}(t))\rangle$ for a Hamiltonian changing in time with $\boldsymbol{\lambda}(t)$, so that it is $\hat{H}(t)\,|\text{g.s.}(t)\rangle = \varepsilon(t)\,|\text{g.s.}(t)\rangle$ throughout the entire observation time. So the change of system parameters will follow the *adiabatic theorem*. This theorem tells you, that the velocity with which you change these parameters is limited by the difference of the ground state and the lowest energy excitations, so that after the process of changing the system parameters you will end up in the ground state where you have started without crossing any other energy levels. Interesting is now the dynamical phase corresponding to the time evolution of $|\text{g.s.}(t)\rangle$. We can describe the time dependency of the time dependent wave function of the system $|\Psi(t)\rangle$ by the Schrödinger equation $i\hbar\partial_t\,|\Psi(t)\rangle = \hat{H}(\boldsymbol{\lambda}(t))\,|\Psi(t)\rangle$. In the following we set $\hbar = 1$ For an instantaneous ground state we have a instantaneous basis $|n(t)\rangle$ with $|0(t)\rangle \equiv |\text{g.s.}(t)\rangle$, $\varepsilon_{n=0}(t) = \varepsilon(t)$ and $\hat{H}(t)\,|n(t)\rangle = \varepsilon_n(t)\,|n(t)\rangle$, so that each wave function of the system can be represented in this instantaneous basis of the system $|\Psi(t)\rangle = \sum_{n=0}^{+\infty} c_n(t)\,|n(t)\rangle$, where the $c_n(t)$ are time dependent complex coefficients. Plug in this into the Schrödinger equation and using the approximation that for $n > 0$ it has to be $c_n(t) \ll 1$, which is a good approximation in the adiabatic theorem, we find that it is

$$\varepsilon(t)c_0(t) \simeq i\dot{c}_0(t) + i\,\langle\text{g.s.}(t)|\,\frac{\partial}{\partial t}\,|\text{g.s.}(t)\rangle\,c_0(t) \ . \tag{1.7}$$

At this point one can introduce the phase

$$c_0(t) \equiv e^{-i\phi(t)} \tag{1.8}$$

and we get the relation

$$\varepsilon(t) = \dot{\phi}(t) + i \langle \text{g.s.}(t)| \frac{\partial}{\partial t} |\text{g.s.}(t)\rangle , \tag{1.9}$$

so that we can write

$$|\Psi(t)\rangle = e^{-i\phi(t)} |\text{g.s.}(t)\rangle \ , \tag{1.10}$$

where $\phi(t)$ is determined through equation (1.7) as

$$\phi(t) = \int_0^t \mathrm{d}t' \varepsilon(t') - i \int_0^t \mathrm{d}t' \left\langle \text{g.s.}(t')\right| \frac{\partial}{\partial t'} \left|\text{g.s.}(t')\right\rangle \ , \tag{1.11}$$

where $e^{-i\int_0^t \mathrm{d}t' \varepsilon(t')}$ is known as the dynamical phase, that existing independently of the time evolution of the parameters $\boldsymbol{\lambda}(t)$ is given or not and for any eigenstate with energy $\varepsilon(t)$ [4], and which can be normalized to one by setting the ground state energy to zero, the other phase term is of greater interest here. To see this we assume the peculiar requirement, that the path of time evolution for the set of system parameters is closed, i.e. that the parameters set of the system ends up after time t in the same parameter set as it was initially. The here interesting phase term leads then to the *Berry phase* φ by take into account, that the ground state dependency of time is given through the time dependency of the system parameters $\boldsymbol{\lambda}(t)$ we can write $\langle \text{g.s.}(t')| \frac{\partial}{\partial t'} |\text{g.s.}(t')\rangle = \langle \text{g.s.}(\boldsymbol{\lambda}(t'))| \partial/\partial\lambda^a(t') |\text{g.s.}(t')\rangle \dot{\lambda}^a(t)$. Using the fact that we integrate over a closed path C for all $\mathrm{d}\lambda_a$ we get for the integral of the interesting phase term the expression

$$\varphi = -\sum_{a=1}^{N} \oint_C \mathrm{d}\lambda^a \langle \text{g.s.}| \frac{\partial}{\partial \lambda^a} |\text{g.s.}\rangle \ . \tag{1.12}$$

Finally we use the definition for the *Berry connection*, which is given by

$$\mathscr{A}_a(\boldsymbol{\lambda}) = -i \langle \text{g.s.}| \frac{\partial}{\partial \lambda^a} |\text{g.s.}\rangle \ , \tag{1.13}$$

to express the Berry phase as

$$\varphi = -i \sum_{a=1}^{N} \oint_C \mathrm{d}\lambda^a \mathscr{A}_a(\boldsymbol{\lambda}) \ . \tag{1.14}$$

It is peculiarly that the phase does not depend on time, but much more it depends only on the closed path formed by the systems parameter evolution. So in that sense we can say the phase is a pure geometrical phase and this is what makes this phase so interesting and so useful by determining the phase appearing by the exchange process of two identical particles of a system

staying in its degenerated ground state. Another important quantity related to the theory is the *Berry curvature*, which is defined by

$$\mathcal{F}_{kl}(\boldsymbol{\lambda}) \equiv \frac{\partial \mathcal{A}_k}{\partial \lambda^l} - \frac{\partial \mathcal{A}_l}{\partial \lambda^k} \tag{1.15}$$

connected by the *Stoke's theorem* with the Berry phase. The simplest way to see this is, if we consider the Berry connection as a 1-form $A = \sum_{a=1}^{N} \mathcal{A}_a(\boldsymbol{\lambda})\, \mathrm{d}\lambda^a$, because then the Berry curvature is the differential of the Berry connection $F \equiv \mathrm{d}A = \frac{1}{2} \sum_{k,l=1}^{N} \mathcal{F}_{kl}(\boldsymbol{\lambda})\, \mathrm{d}\lambda^k \wedge \mathrm{d}\lambda^l$. Using that the closed curve C is the boundary of the two dimensional surface S, one gets the Berry phase related to its Berry curvature

$$\varphi = \sum_{k>l}^{N} \int_S \mathcal{F}_{kl}(\boldsymbol{\lambda})\, \mathrm{d}\lambda^k \wedge \mathrm{d}\lambda^l. \tag{1.16}$$

One can see, that the Berry curvature is invariant under gauge transformation of the Berry connection and therefore the physics remain unchanged by multiplying the ground state with an arbitrary phase depending on the parameters λ, so that we have an arbitrariness in choosing the phase of the reference state [4].
For the Majorana zero modes means that in the adiabatic limit it does not matter how in detail they are exchanged, but only which pair of Majorana zero modes was exchanged.

1.1.3. Fusion and Ising anyons

As well as it is possible to create two anyons out of the vacuum or out of an other anyon particle, it is possible to bring them together and fuse them to an other anyon or back to the vacuum. The possibilities of fuse anyons depend on which class of anyons they belong to, each anyon class has its own *fusion rules* [2]. In the following we will give as example the fusion rules of the class of *Ising anyons*.
To fuse to anyons in practise one has for example the possibility either to hybridize the wave functions of two anyons by bring the spatially together or to remove the ground state degeneracy of the Majorana zero mode system as it is done by the method of *parity-to-charge conversion* [6].

Ising anyons

In the Ising anyon model one has three different kinds of particles. First the anyon of non-abelian exchange statistics called σ, second the ordinary Dirac fermion ψ and third the vacuum

I representing the neutral particle. The given fusion rules for this three kind of particles are

$$\begin{aligned} \sigma \times \sigma &= \mathrm{I} + \psi \\ \sigma \times \psi &= \sigma \\ \psi \times \psi &= \mathrm{I} \end{aligned} \tag{1.17}$$

and the F matrix or also called fusion matrix for the Ising anyons is given by

$$F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.18}$$

in the basis of $|\sigma, \sigma \to \mathrm{I}\rangle$, $|\sigma, \sigma \to \psi\rangle$ [2]. The boundaries exhibiting Majorana zero modes belong to this class of Ising anyons.

1.1.4. Applicability of Majorana zero modes for quantum computing

Unlike classical computer, a register in quantum computer consisting of $2N$ Majorana zero modes forming a 2^N-fold ground state degeneracy, can store superpositions of different states. Algorithms for quantum computer are significant different from these for classical computer. Perform an algorithm in a quantum computer based on Majorana zero modes means performing different series of braiding the existing Majorana zero modes. Different sequences of braiding will lead to different algorithms and increasing the complexity of these algorithms requires an increasing number of Majorana zero modes and the length of the braid sequences [7]. The goodness of quantum memories based on Majorana zero modes is predicted to be high, due to the topological protection of the state in which the system of Majorana zero modes stays in and therefore the algorithms based on braiding them are protected, too. Further the fermion parity degrees of freedom are shared non-locally by these well separated Majorana zero modes encoding the system state and therefore it has the advantage, that the system state is protected against local perturbations and as long as the Majorana zero modes are kept far enough from each other the quantum state is protected for decoherence. This means that the quantum memory state as well as the algorithms running on them are protected against all local perturbation. The only exception, which could perturb such a system is given by the probabilistic scattering of quasiparticles into the quantum bit system, which is known as *quasiparticle poisoning* [7] .

1.2. Read out methods

To realize a quantum computer it is not enough to know how we can transform manipulations, a further important point is the method to measure a certain quantum state of a given system. One

big challenge is surely to not distort the information of the system state during the measurement. Another one would be to detect that the system stays in a superposition of states. Measuring Majorana zero modes holds the extra hurdle in itself, that theses particles have no charge. Some methods have been proposed in the recent centuries are the measuring by *interference* [8] and the *charge sensing*. Important is, that one needs first a method to convert a pair of Majorana zero modes into an ordinary Dirac fermion. For the conversion into a Dirac fermion it was proposed for example to use the *parity-to-charge* conversion [6] and for charge sensing of a single electron charge e on structures resembling Coulomb islands it lends itself to use a single electron transistor. Here we want only give a short introduction in the physic of a single electron transistor, which we will use later. For a detailed description of the *parity-to-charge* conversion we refer to the review '*Milestones Toward Majorana-Based Quantum Computing*' [6].

1.2.1. The physics of single electron transistors

1.2.1.1. The coulomb blockade

The coulomb blockade regime is a regime with dominating interaction effects finding its application in quantum transport. To combine the so called coulomb blockades into coulomb blockade circuits offers many interesting and useful possibilities such as the single electron transistor which is often used for measuring charges in mesoscopic systems like in our case. Due to systems suitable for quantum computing are mesoscopic systems, the used electronics include the importance of Coulomb blockades. Simplified coulomb blockades are very small ($\sim \mu m$) metallic islands for which the charging energy E_C, which is the energy is to be used to add a single electron to this island, is relative high and charge quantization on the island predominates. Because for a metallic island of a suitable small size the charging energy is much bigger than the quantum level spacings of the atoms in the metallic island, the classical concept of charging energy E_C provides a good description [7]. An estimation is that the spatial extension L of such an island has to be greater than the distance between the atoms $a \sim nm = 10^{-9}m$, which is given for a length of about $10^{-4} - 10^{-7}m$ (mesoscopic size). In this classical description the charging energy of a single Coulomb island is given by Coulombs law as $E_C = \frac{e^2}{2C}$, while C is the self-capacitance of the island and is proportional to the spatial extension L of the island. This is why the charging energy increases with decreasing island size and decreases for increasing island extensions. In such a regime the electrical current is characterized by a *single-electron transfer*, which is determined by the *master equation*. While the lower critical island size is estimated to be the distance of the atoms in the island by comparing the charging energy with the atomic quantum level spacing, the upper critical island size can be estimated by considering the conditions to be in a coulomb blockade regime, such as the condition, that the charging energy should be bigger than the thermal energy k_BT to prevent island occupations by thermal excitations. Or

for example the condition, that the charging energy has to be bigger than a applied bias energy eV_g, where e is the elementary charge and V_g is an applied gate voltage, so that in a circuit the applied gate voltages have to be smaller than the charging energy divided by the elementary charge to prevent that the island is flooded and the circuit behaves like a normal electrical circuit (no charge blockade any longer). Due to the smallness of the voltages are technically limited the island size is restricted in this way too. At last we mention one example, which takes into account, that coulomb blockade circuits only exist by coupling electrodes, voltages and other islands only capacitively to each other, the tunnelling conductance σ_t has to be smaller than the half of the quantum conductance $\sigma_0 = \frac{2e^2}{h}$ to ensure that the tunnel probability onto the island is less than unity.

Using the classical description of charging energy, one gets for the electrostatic Coulomb energy of an island, which is occupied by N charges ($Q_{island} = N \cdot e$) $E(N) = E_C N^2$, so that one needs to invest the energy of $E(N+1) - E(N) = E_C \cdot (2N+1)$ if one wants to add another electron [7]. This means, that the energy to add an extra electron to the island increases linear with the number of electrons N just sitting on the island and therefore without the presence of strong enough thermal fluctuations and without an increasing applied voltage the electron transport over such an island can be blocked. That is why it is called a coulomb blockade. In the here considered system it will be a good estimation to say that for islands of the size of a few micro meter $L \sim 10^{-4} - 10^{-6} m$ for which we have an charging energy of about $E_C \sim 0.001 - 0.1 eV$, such that the classical description with the charging energy is a suitable approximation in this case, but we have to be careful with the size of applied voltages which will make the read out to a very sensitive and challenging measurement process.
The importance for quantum computing and the realization of solid state quantum bits becomes more clear if we think of connections of superconducting Coulomb islands and normal as well as superconducting electrodes, for which the combination of Josephson effect and coulomb blockade leads to quantum systems offering the possibility of states being in a quantum superposition of charge states. This is what we use later to describe the fusion and read out process of our system of consideration and while in circuits without the possibility of a superposition of charge states the electron transport can be determined by solving the *mater equation* with a classical approach, we have to apply later the quantum mechanical approach to solve the *master equation* for our read out circuit.

1.2.1.2. Coulomb blockade and Josephson junction circuits

The charging energy E_C of an circuit consisting of coulomb blockades and Josephson junctions can be formally derived by the use of the Lagrange formalism. Therefore we set up the Lagrangian

using the classical Coulomb approach such that the Lagrangian of our system can be written as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{island}} + \mathcal{L}_{\text{island,gate}} + \mathcal{L}_{\text{gate}} \\ &= \frac{1}{2}\sum_{\alpha<\beta} C^{(cb)}_{\alpha,\beta}(V_\alpha - V_\beta)^2 + \frac{1}{2}\sum_{\alpha,i} C^{(el)}_{\alpha,i}(V_\alpha - V_i)^2 + \frac{1}{2}\sum_{i<j}\tilde{C}_{i,j}(V_i - V_j)^2 \,,\end{aligned} \tag{1.19}$$

where the first term describes only the capacitive coupling of different Coulomb islands to each other, so the $C^{(cb)}_{\alpha,\beta}$ are the capacities between the Coulomb islands, which are labelled by the Greek indices. The second term represents the interaction between the Coulomb island and applied voltages from gate electrodes or transport electrodes, which also has to be coupled capacitively with capacitances $C^{(el)}_{\alpha,i}$, here marked by an extra upper *el* and the labels i, j for all electrodes. The third term represents the pure interaction between all electrodes, which are all capacitively coupled to each other via $\tilde{C}_{i,j}$. This third term is decoupled and unimportant for the energy description of the Coulomb islands and can be ignored for the further derivation of the charging energy for the Coulomb islands. Using $\dot{\varphi}_\alpha(t) = eV_\alpha(t)$ and $\hat{n}_\alpha \cdot e = q_\alpha$, which commute with each other $[\hat{\varphi}_\alpha, \hat{n}_\beta] = i\delta_{\alpha,\beta}$ and we have that $\hat{n}_\alpha = -i\frac{\partial \mathcal{L}}{\partial_{\varphi_\alpha}}$ [9], we can find the Hamiltonian from the Lagrangian above without the term $\mathcal{L}_{gate}$ with the use of the Legendre transformation and a new representation of the rest of the Lagrangian in matrix representation, where we write all capacitances into the capacitance matrix, so that its elements are given by

$$C_{\alpha\beta} = \begin{cases} \sum_{\eta\neq\alpha} C^{(cb)}_{\alpha,\eta} + \sum_i C^{(el)}_{\alpha,i} \,, & \text{for } \alpha = \beta, \\ -C^{(cb)}_{\alpha,\beta} \,, & \text{for } \alpha \neq \beta \end{cases} . \tag{1.20}$$

Due to the different applied voltages of the electrodes each island has an extra induced charge by these electrodes which we can write as

$$q_\alpha = -\sum_i C^{(el)}_{\alpha,i} V_i \,. \tag{1.21}$$

Multiplying by the inverse capacitance matrix $\hat{C}^{-1}$ one can resolve the problem and gets as result the Hamiltonian

$$H = \frac{e^2}{2}\sum_{\alpha,\beta}(\hat{C}^{-1})_{\alpha\beta}(N_\alpha - q_\alpha/e)(N_\beta - q_\beta/e) - \frac{1}{2}\sum_{\alpha,i} C^{(el)}_{\alpha,i} V_i^2 \,, \tag{1.22}$$

where N_α is the number of charges on the island labelled by α. The last term in (1.22) does not depend on the number of charges of any island, and because we are only interest in the difference of the energies of different charge states given by the charging numbers N_α we can drop the last

term. So we get for the elements of the charging energy matrix $\hat{E}^{(C)}$

$$E^{(C)}_{\alpha\beta} = \frac{e^2}{2}(\hat{C}^{-1})_{\alpha\beta} \ , \tag{1.23}$$

where the diagonal elements of the charging matrix denote the energy cost which one has to pay by adding an excess electron to the given island and the off-diagonal entries represent a repulsive interaction between the quantized charges of the different capacitively coupled island. A less formal derivation of Coulomb blockade circuits charging energy is given in the book *Quantum transport* by Y. V. Nazarov [7].

1.2.1.3. The single electron transistor

The single electron transistor represents the simplest coulomb blockade circuit. It consists of a single Coulomb island, which is capacitively coupled to two transport electrodes V_1 and V_2 with capacitances C_1 and C_2 and one gate Voltage V_g with C_g. With the above derived expression for the charging energy one sees directly that for a single given island the matrices above reduce to a single quantity and with (1.23) and (1.20) we get for the charging energy of an single electron transistor

$$E^{(C)} = \frac{e^2}{2C_{tot}} \ , \tag{1.24}$$

where $C_{tot} = C_1 + C_2 + C_g$ is the total capacitance of the circuit. For the electrostatic energy of interest we get with (1.22) the following expression

$$E(N) = E^{(C)}(N - q/e)^2 \ . \tag{1.25}$$

If one wants to add or extract one electron of the island by using the transport voltages V_i one can write the energy differences of all four possible tunnelling processes

$$\begin{aligned} &\text{from electrode i to island:} \quad \Delta E^{ad}_i(N) \quad = E(N+1) - E(N) - eV_i \ , \\ &\text{from island to electrode i:} \quad \Delta E^{ex}_i(N) \quad = E(N-1) - E(N) + eV_i \ , \end{aligned} \tag{1.26}$$

where for a given charge state N all four single-electron processes are forbidden if

$$\Delta E^{ad}_i(N) > 0 \quad \text{and} \quad \Delta E^{ex}_i(N) > 0, \text{ for all } i = 1,2 \tag{1.27}$$

is fulfilled [7]. In addition considering the charging energy it is obviously that changing the gate voltage V_g has only influence on the size of the charging energy itself. Lowering the gate voltage means an increasing charging energy. Considering the energy differences for the transport processes we can see that it is possible to block the single-electron transfer for a certain charge

number N on the island by adjusting the gate voltage. So on the one hand the gate voltage enables to control the single-electron transfer and on the other hand it enables to adjust the number of charges N on the island to a certain value.

1.2.2. Equation of motion in quantum systems

1.2.2.1. Density matrix

Wave functions are only able to describe pure states $|\psi_n\rangle$ and a state of a system can be determined by a series of measurements. A reduced system coupled to an environment forming a mixed state of superpositions of pure states cannot be determined by any single measurement. Such mixed states can be represented and described by the density operator $\hat{\rho} = \sum_n p_n |\psi_n\rangle \langle\psi_n|$ which represents an statistical operator with p_n being the probabilities to find the system in the pure state $|\psi_n\rangle$. The *diagonal elements* of the density operator in matrix representation correspond to the probability to find the system in state $|\psi_n\rangle$ and its time evolution describes the temporal stability of finding the system in the corresponding state. The *off-diagonal elements* describe the coherent superposition between the different states and its time evolution $\rho_{ij}(t)$ $(i \neq j)$ describes the coherent dynamics and interference effects. Further this density matrix has to fulfil the properties of a statistical operator, which are here

$$\begin{aligned} &\text{(i)} && \mathrm{tr}\hat{\rho} = \sum_i \rho_{ii} = 1 && (\textit{normalization}) \\ &\text{(ii)} && \rho_{ii} > 0 && (\textit{positive probabilities}) \\ &\text{(iii)} && \rho_{ii}\rho_{jj} \geq \left|\rho_{ij}\right|^2 . \end{aligned} \tag{1.28}$$

The representation of the density operator leads to the description of the expectation value of an observable as $\langle\hat{O}\rangle = \mathrm{tr}\left(\hat{\rho}\hat{O}\right)$.
Averaging the density operator over a subsystem yields the *reduced density operator* or in matrix representation the *reduced density matrix*.

1.2.2.2. The Makovian master equation

The *time evolution* of the density operator can be described by the *unitary time evolution operator* $\hat{\mathcal{U}}$

$$\begin{aligned} \hat{\mathcal{U}}(t) &= e^{-i\hat{H}t}\hat{\mathcal{U}}(0)e^{i\hat{H}t} , \\ \hat{\rho}(t) &= \hat{\mathcal{U}}(t)\hat{\rho}(0)\hat{\mathcal{U}}^{\dagger}(t) , \end{aligned} \tag{1.29}$$

where $\hat{H}$ is the Hamiltonian describing the system and we set $\hbar = 1$. Then the *Liouville-von Neumann equation* describes the time evolution of the density operator by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = -i\left[\hat{H}(t), \hat{\rho}(t)\right] = \hat{\mathscr{L}}\cdot\hat{\rho}(t)\ , \tag{1.30}$$

where $\hat{\mathscr{L}} = \left[\hat{H}(t),\ \cdot\ \right]$ is called the *Liouville operator*.
For the special kind of problems, where the system exhibiting only few discrete states is small compared to its environment and this system is taken out of equilibrium by coupling to its environment exhibiting nearly or even continuous states, a method to reduce and to solve the corresponding Liouville-von Neumann equation was discussed by R. K. Wangsness and F. Bloch (1953) [10], U. Fano (1954) [11] and F. Bloch (1956) [12]. Here we want to use their developed theory and we give a brief summary of what we will use of it to solve later the Master equation of our given system.
The system of the kind above is describable as

$$H = H_S + H_I + H_B\ , \tag{1.31}$$

where H_S is the Hamiltonian which describes the small system, H_I is the interaction Hamiltonian describing the interaction of the small system with the large environment, which is called the 'bath' and H_B is the Hamiltonian which describes this large environment 'bath', the so called 'bath' Hamiltonian. We have omitted the hat for the operators and we will omit them in the following. The interaction between system and bath is assumed to be only weak. Because we assume that the initial system state at time $t = 0$ (time at which the interaction begins) and the density operator at time $t = 0$ is also well known as $\rho'(0)$ and we can write the density operator at time t as

$$\rho'(t) = e^{-iHt}\rho'(0)e^{iHt} \tag{1.32}$$

and its time derivative is given by the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho'(t) = -i[H, \rho'(t)]\ . \tag{1.33}$$

It is used the assumption of a weak-interacting system with the surrounding 'bath' to rewrite the density operator in the *interaction representation*

$$\rho(t) = e^{-iH_It}\rho'(0)e^{iH_It}_{,} \tag{1.34}$$

where we have used that we can write our Hamiltonian as sum of the non-interacting part $H_0 = H_B + H_S$ with pure 'bath' and system Hamiltonians and the interacting part H_I such that $H = H_0 + H_I$. One can check that the reduced equation of motion in the weak-interacting regime

is

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i[H_I(t), \rho(t)] \ , \tag{1.35}$$

where it is

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} \tag{1.36}$$

the time dependent equivalent of the interaction Hamiltonian. Taking the trace over the states of the reservoir one can show under the assumption that the initial state at time $t = 0$ is given such that the interaction does not generate any first-order dynamics, such that one has $\mathrm{tr}_B([H_I(t), \rho(0)])$ the reduced equation of motion can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \mathrm{tr}_B\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\right) = -\int_0^t \mathrm{d}s \ \mathrm{tr}_B\left([H_I(t), [H_I(s), \rho(s)]]\right) \ , \tag{1.37}$$

where $\rho(t)$ is still the density operator of the full system, while $\rho_S(t)$ is the reduced density operator by tracing over the 'bath'. Further we have assumed that the interaction is weak in such a way that the influence on the bath is small and we can assume that the environment remains in equilibrium while only the small system is affected by the interaction. This makes the *Born approximation* applicable treating the 'bath' approximately constant in time and we write the total density operator as $\rho(t) = \rho_S(t) \otimes \rho_B$. A further approximation is given by the assumption, that the time in which the system is damped is much longer than that in which the reservoir maintains its correlation, so that the influence of the past time on the systems time evolution is lost and the system should not be able to preserve memory. This last assumption is known as the Makovian approximation. Now one can derive with the use of the Markov approximation the well known *Markovian master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\int_0^\infty \mathrm{d}s \ \mathrm{tr}_B\left([H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]]\right) \ , \tag{1.38}$$

where the only part depending on the integration variable s is $H_I(t-s) = e^{iH_0(t-s)} H_I e^{-iH_0(t-s)}$. This equation of motion describes the transition from an initial state $\rho(0)$ to the long asymptotic limit $\rho_\infty \equiv \lim_{t\to\infty} \rho(t)$ when the system reaches its stationary solution $\frac{\mathrm{d}}{\mathrm{d}t}\rho_\infty = 0$.
In general the probabilistic time evolution of a time dependent system state can be described by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = -\sum_{n'} \Gamma^{out}_{n\to n'} p_n(t) + \sum_{n'} \Gamma^{in}_{n'\to n} p_{n'}(t) \ , \tag{1.39}$$

where $\Gamma^{in(out)}_{n\to n'}$ is the in-rate (out-rate) at state n to charge state n'. Comparing the Makovian master equation with this classical probabilistic master equation one can identify the corresponding in and out rates.
Later we want derive the Makovian master equation for a given Coulomb circuit with two Quantum dots for the read out of a given quantum bit. One of these Quantum dots will be the Coulomb island of a single electron transistor, which we want to use for the read out of the charge state of the second Quantum, which is a part of the quantum bit system. There we will be interested in the tunnelling rates of charges onto and from the Coulomb island of the single electron transistor in dependence of the quantum bit charge state.

1.3. Topology in condensed matter physics

The area of topology in condensed matter physics provides direct access to Majorana zero modes. For a detailed introduction we refer to the literature stated in the following. Here we only mention a few important terms for a basic understanding and some of the stuff we have used for our calculation presented in this thesis.
Because beside *symmetry* and *bulk-edge correspondence topology* is one of the three main actors in our story let us first give a brief statement, what *topological* in this context of physical area means [13].
Generally *topology* is about all the stuff, which cannot be transformed continuously or to be less sloppy, it is the area, which studies the immutability of the properties of given objects under continuous transformations.
In our context, the object of interest are systems which cannot be transformed continuously into other systems without the existence of zero energy excitations appearing somewhere on the way of this continuously transformation. The absence of an zero energy excitation is called a gap. We can group all systems with such a gap into classes, so called *topological classes*, and we can identify what are the differences between those classes. Then it is sufficient to determine to which kind of topological class our system of consideration belongs to, to gain important informations about its characteristics.

1.3.1. Topological invariant and topological phase transition

If we consider a set of gapped Hamiltonians with the same number of filled energy levels, then it is clear that as long as each Hamiltonian stays gapped we can continuously transform the Hamiltonians into each other. But if one of these Hamiltonians is gapless then one only does a continuously transformation by crossing zero energy. By crossing zero energy the number of filled energy levels changes. The state of constant number of filled energy levels is what

is called a *topological phase*. The number of filled energy levels then is what one calls a *topological invariant*. Crossing zero energy and changing the topological invariant is what is called a *topological phase transition*. That means, if two Hamiltonians have different topological invariants it is impossible to transform them continuously into each other without closing the gap and that means we need a topological phase transition.
This means, that if we have a boundary between a bulk associated with a topological nontrivial region and a bulk associated with a topological trivial region, it is necessary for the wave functions of our systems having a zero energy state at this boundary to enable a continuous transformation between the two Hilbert spaces of different topological phases. The existence of zero energy state due to geometrically conditioned topological phase transition is what we call a *bulk-boundary correspondence in topological phases* [13–15].
The following part represents a nice example to make the concepts of topology in physics tangible [15]. If one can establish the Hamilton matrix $\mathscr{H}$ for a given Hamiltonian H and present the Hamilton matrix as a vector of Pauli matrices

$$\mathscr{H} = \mathbf{h} \cdot \sigma = h_x \sigma_x + h_y \sigma_y + h_z \sigma_z, \tag{1.40}$$

where $\mathbf{h}$ have to be a vector depending on the system parameters. Then the energy levels of a given set of system parameters can be determined by

$$0 = \det H - \lambda \mathbf{1} = \det \begin{pmatrix} h_z - \lambda & h_x - i h_y \\ h_x + i h_y & -h_z - \lambda \end{pmatrix} \tag{1.41}$$

$$\Leftrightarrow \lambda = \pm \sqrt{h_x^2 + h_y^2 + h_z^2} = |\mathbf{h}| \ . \tag{1.42}$$

So crossing zero energy means, that it is $\lambda = 0 = |\mathbf{h}|$ for a distinct system parameter set $\mathbf{P}_0$. If one can now distinguish between different system parameter sets by their different numbers of energy levels and one has to cross $\mathbf{P}_0$ to go from one set of system parameters to the other, one has a topological phase transition at $\mathbf{P}_0$.

1.3.2. Topological classes

A topological classification in the $\mathbf{k}$-space topology is given by the quasiparticle excitation spectrum of the system. One distinguishes three main generic classes in the quantum vacuum of a $3+1$ fermionic system [16]. The energy spectra of one of these topology classes are determined by their excitation spectrum of the type $\varepsilon^2(\mathbf{k}) \propto F^2 + c\mathbf{k}^2$, where F can be a field quantity like the Zeeman field strange M due to the influence of an magnetic field or $|\Delta|$ the gap size due to the influence of a superconductor.
If the energy spectrum of fermionic quasiparticles has point nodes, it means, that the propagator

of these fermionic quasiparticles has singularities at points in the hypersurface. These points of singularities are the Fermi point nodes. The Green's function has the form

$$G = (ik_0 - H_k)^{-1}, \tag{1.43}$$

where H_k is the Hamiltonian operator of our system in $\mathbf{k}$-space and ik_0 is an induced imaginary frequency to avoid the conventional singularity for $\mathbf{k} = \mathbf{k_F}$, where $H_{\mathbf{k_F}} = 0$ is zero at these Fermi points $\mathbf{k_F}$. That means, that our Green's function has singularities on the hypersurface at the points $(k_0 = 0, \mathbf{k} = \mathbf{k_f})$.
The important point here is that one can write the Green's function as $G = |G| e^{i\phi}$. Then ϕ is called the phase of the Green's function. It turns out, that this phase changes by an integer multiple n of 2π, when one encircles a singularity of the Green's function one times on a closed path. This integer is the topological invariant for this topological class [16]. A change of the topological phase in this topological class means a change in this integer number n, and n cannot change without closing the gap. Changing the topological invariant is the equivalent of a topological phase transition.

1.3.3. Symmetries and their influence on topology

Symmetries determine the laws of conservation. But symmetries like the time-reversal symmetry can protect the non trivial topology of a given system, if its energy spectrum is invariant under time-reversal symmetry operations. This influence of the symmetry on the topology can cause interesting restraints leading to new promising properties, for example the surface states of an three dimensional topological insulator, which is time-reversal invariant harbour Dirac fermions with a lifted spin degeneracy and spins locked to their momentum. This *helical spin polarization* of the Dirac fermion state together with proximity-induced superconductivity enables for example the emergence of Majorana fermions [17].
Depending on the symmetry of the Hamiltonian of a topological system, its topological invariant has certain constraints. Another important symmetry is the particle-hole symmetry given in superconducting systems, which we want to discuss more detailed in the following.

1.3.4. Particle-hole symmetry

In general a system has particle-hole symmetry if the Hamiltonian is symmetric under the interchange of particle and hole operators. Let Ψ an operator in Nambu space, where its components consist of particle and hole operators and $\mathscr{H}$ the Hamiltonian matrix in this space. Then the particle-hole symmetry of the system Hamiltonian says that for each eigenstate Ψ_{E_k} with eigenenergy E_k it has to exist an eigenstate Ψ_{-E_k} with eigenenergy $-E_k$ [14]. This symmetry

can be expressed by the particle-hole operator Ξ with

$$\Xi\mathscr{H}\Xi = -\mathscr{H} \tag{1.44}$$

and therefore

$$\Xi\Psi_{E_k} = \Psi_{-E_k} \ , \tag{1.45}$$

where

$$H\Psi_{\pm E_k} = \pm E_k \Psi_{\pm E_k} \ . \tag{1.46}$$

This means, that if we have the operator $\gamma_k^\dagger$ which creates an particle with energy E_k in state Ψ_{E_k}, then it is identical to the operator γ_{-k} which annihilates an particle with energy $-E_k$ in state Ψ_{-E_k}. Which means that always $\gamma_{-k} = \gamma_k^\dagger$. For zero energy eigenstates the particle-hole operator transforms the eigenstates into itself

$$\Xi\Psi_0 = \Psi_0 \tag{1.47}$$

and we have the creation and annihilation operators of this zero energy state which is always

$$\gamma_0 = \gamma_0^\dagger \tag{1.48}$$

and if these operators are additionally fermionic operators they are Majorana operators. That is why the particle-hole symmetry leads to the existence of Majorana zero modes.

1.3.5. Vortices in topological superconductors

In Typ-II superconductors exist a phase where the paring gap potential becomes zero at distinct points in space, we call such a point a *node* of $|\Delta|$, where the order parameter is defined as a complex field of space $\Delta(\mathbf{r}) = |\Delta|\, e^{i\theta}$. One can place the node of $|\Delta|$ in the center of the Bravais unit cell and calculate the contour integral along the unit cell boundary.

$$\oint \left(\mathbf{A} - \frac{\hbar c}{2e} \nabla\theta \right) \mathrm{d}\mathbf{r} = 0, \tag{1.49}$$

where $\mathbf{A}$ is the vector potential of the given magnetic fiel $\mathbf{H} = \nabla \times \mathbf{A}$. This integral is zero due to the fact, that $\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\theta\right) = 0$ vanishes at the boundary. Further the lines of constant $|\Delta|$ are perpendicular to the boundary and the magnetic flux through the unit cell is equal to the magnetic flux quantum $\Phi_0 = \frac{\pi\hbar c}{|e|}$. So it is

$$1 = \Phi_0^{-1} \int_S \mathbf{H}\, \hat{\mathbf{n}} \mathrm{d}\sigma = \frac{1}{2\pi}\delta\theta, \tag{1.50}$$

which means that the variation of the phase of the order parameter along this contour is equal to 2π. In other words the phase of the order parameter acquires an increment of 2π after encircling the point where the order parameter vanishes one times. This is what we call an quantized vortex. A vortex is a topological object, which is in three dimensions a line for which the phase of the order parameter changes by an integer multiple of 2π by circulation around this line [18].
An example of the importance of vortices is the $p+ip$ superconductor. The $p+ip$ superconductors are a special type of superconductors exhibit vortices with zero energy Majorana excitations in the center of the core of each these vortices [19, 20]. For this one important property of an $p+ip$ superconductor is that, one can talk about a so to say 'spinless' system with only one active fermionic species. This leads to quasiparticle operators γ as a linear superposition of normal Dirac fermion operators and having the property of $\gamma^{\dagger} = \gamma$.

1.3.6. Topological insulators

Topological insulators refer to a novel class of solid-state systems with the unique property of bulk wave functions having topological character leading to extraordinary edge or surface states existing as gapless excitations being topological protected against essential kinds of disorder and interactions. This special class of insulators are called *topological* due to the nontrivial topology of the Hilbert space spanned by the electronic states of the wave functions of these systems. This means that at an interface between an topological insulator and an ordinary insulator (e.g. the vacuum) which has an Hilbert space with an trivial topology, it has to occur an topological phase transition by passing a zero energy state (closing the gap) at this interface. This topological phase transition gives us reason to expect fermion states of zero energy at such interfaces (compare with the previous paragraph about topology). When we have a system of topological insulators in which the third direction is suppressed for the wave functions we speak about two dimensional topological insulators having gapless edge states and for wave functions free in all three dimensions we speak about three dimensional topological insulators and their gapless surface states. So three dimensional topological insulators are known as materials behaving as insulators at their bulk but having topologically protected surface states whose excitations are gapless chiral Dirac fermions. Candidates for three dimensional topological insulators are materials like Bi_2Se_3, $Bi_2Te_3$and Bi_xSb_{1-x} [17].

1.3.6.1. Effective surface states

The microscopic Hamiltonian of a three dimensional topological insulator with strong spin-orbit interaction can be modelled by the three dimensional massive (mass M) Dirac Hamiltonian (Fermi velocity $v_F = 1$). Its 4×4 Hamiltonian matrix has the general form [21]

$$\mathscr{H}_{\text{3DD}} = \begin{pmatrix} -M & \sigma\mathbf{p} \\ \sigma\mathbf{p} & M \end{pmatrix} . \tag{1.51}$$

One can always find the effective low-energy surface Hamiltonian for a surface with surface normal $\mathbf{n}$ and takes a coordinate system in which our three dimensional topological insulator has one of three coordinates α with $\mathbf{n} = \mathbf{e}_\alpha$, where $\mathbf{e}_\alpha$ is the unity vector in α-direction and $-\mathbf{e}_\alpha$ points into the bulk. Because we are only interests in the edge states with zero energy with a fast decay into the bulk, we can solve the Schrödinger equation $H\Psi = 0$ for $\Psi = e^{M\alpha}(\psi, \chi)^T$, which gives us the general boundary condition $\psi = -i\sigma\mathbf{n}\chi$ [21]. Use this boundary condition in the representation of our three dimensional Dirac Hamiltonian for the states at the surface ($\alpha = 0$) $\Psi_0 = (\psi, -i\sigma\mathbf{n}\chi)$, one gets

$$H_{\text{3DD}} = \frac{1}{2}\Psi_0^\dagger \mathscr{H}_{3DD}\Psi_0^\dagger = \psi^\dagger \frac{i}{2}\left[\sigma\mathbf{p}, \sigma\mathbf{n}\right]\psi , \tag{1.52}$$

where

$$\mathscr{H}_{\text{surface}} = \frac{i}{2}\left[\sigma\mathbf{p}, \sigma\mathbf{n}\right] = \frac{i}{2}\mathbf{pn} + \frac{1}{2}\left(\mathbf{n}\left(\mathbf{p}\times\sigma\right) + \left(\mathbf{p}\times\sigma\right)\mathbf{n}\right) \tag{1.53}$$

represents the effective two dimensional surface Hamiltonian matrix in the regime of low-energy excitations [21].

For our system here we consider in the following a surface which lies in the xy-plane with its surface normal in z-direction $\mathbf{n} = \mathbf{e}_z$. Therefore the first term of the effective surface Hamiltonian becomes $\frac{i}{2}\mathbf{pn} = \frac{1}{2}\partial_z$. The Schrödinger equation of our system can be considered to be of the form

$$E\psi = H(x,y,z)\psi(x,y,z) = h_{xy}(x,y)\psi + h_z(z)\psi , \tag{1.54}$$

So that we are allowed to separate and to take the ansatz $\psi = \psi_{surface}\cdot\psi_z$. Assume that E_z are the eigenenergies for $h_z(z)\psi_z = E_z\psi_z$. We are only interested in the solution at the surface and not in the bulk, so that we only have to solve the Schrödinger equation

$$h_{xy}\psi_{\text{surface}} = (E - E_z)\psi_{\text{surface}} = E_{xy}\psi_{\text{surface}} . \tag{1.55}$$

So for high enough gaps into z-direction this energy states are effectivly gaped out, and this is why the first term of our surface Hamiltonian vanishes for surface states of a surface which lies in the xy-plane and is completely orthogonal to the z-directoin.

1.3.6.2. Combination of materials in a topological regime

A ferromagnetic three dimensional thin film topological insulator in proximity to an superconductor is a structure which combines ferromagnetism and superconductivity on the surface of a three dimensional topological insulator. The pure three dimensional topological insulator has an energy dispersion of the spin non-degenerated surface state forming a two dimensional Dirac cone. The spin of the massless Dirac fermions being situated in these surface states is locked to their momentum, such that one have a helical spin polarization. Further the non-trivial topology of this three dimensional topological insulator is protected by a time-reversal symmetry. Introducing a ferromagnetism into such a suitable topological insulator breaks the time-reversal symmetry and can lead to the anomalous Hall effect without the existence of Landau levels or an external magnetic field. By breaking the time-reversal symmetry the energy dispersion of the Dirac fermions gains a mass-gap. This gap lies around the Dirac point of the pure topological insulator.
So that it is predicted the possibility of creating at the boundary between regions with and without proximitized superconductor a boundary between a topological trivial and a topological non trivial region by adjusting the chemical potential of the three dimensional topological insulator [17].

2. The architecture

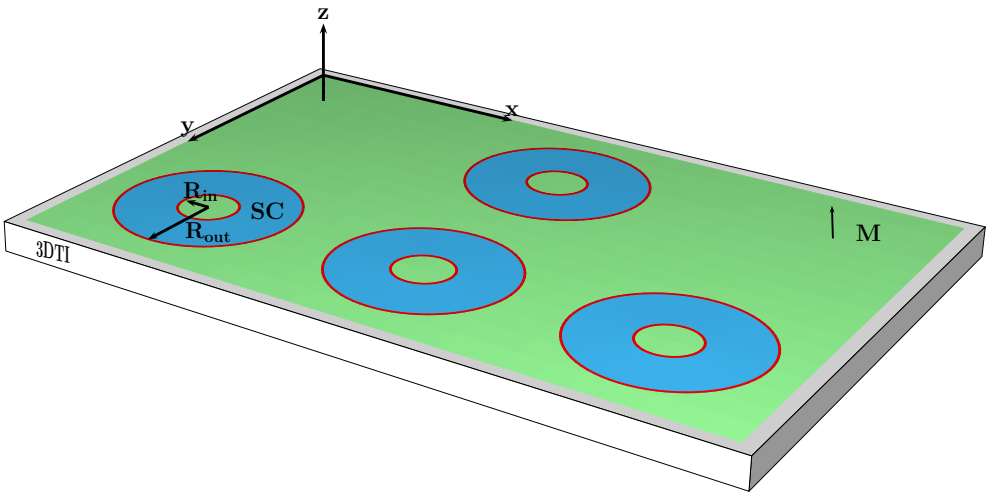


Figure 2.1.: A superconducting ring (SC,blue) and three dimensional topological insulator thin film (3DTI,white) with an ferromagnetic dopant (M,green) [22].

We want to present a new possible platform for a Majorana qubit, which belongs to one of the ideas for Majorana qubit platforms based on three dimensional topological insulators given by Yoichi Ando, who wants to build and evaluate basic architectures for such topological insulator based Majorana qubit platforms at the university of cologne. In this chapter we describe our system of consideration and give the reader an understanding of this potentially new Majorana qubit setup explaining briefly the aim of several used components and their composition.
It is constituted by a three dimensional topological insulator ferromagnetic thin film and an

equally thin ring shaped superconductor.

2.1. The components and their propose

A sketch of the new architecture for the design of a possible new Majorana qubit is shown in Figure 2.1. The ferromagnetic three dimensional topological insulator thin film is a three dimensional topological insulator thin film of an suitable material, which is doped by an appropriate ferromagnetic material. It is proposed to employ $Bi_{2-x}Sb_xTe_3$ with an applied ferromagnetic modulation-doping with Cr, because for such an doped heterostructure it was experimentally shown that it represents an quantum anomalous Hall insulator, if the chemical potential lies in the ferromagnetic order induced gap of the surface states [23], even for higher temperatures [24]. On the surface (on top of the three dimensional topological insulator thin film) are four equally thin s-wave superconducting rings. The four superconducting rings on the surface are well separated from each other by the ferromagnetic doped surface. Three of the four superconducting rings lie in a row and the fourth lies above the middle one, so that their arrangement give a T-shape (compare Figure 2.1).
The ferromagnetism produces a magnetic field at the surface of the magnetically doped three dimensional topological insulator film, which can be assumed to be smaller than the upper critical magnetic field for the used superconducting material [23, 24]. Respectively the superconducting material has to be chosen so that its upper critical magnetic field is much more larger as the magnetic field created by the magnetic dopants. By bringing the s-wave superconductor in proximity to the topological insulator film it is predicted that due to the proximity effect the s-wave superconductor leads effectively to a topological non-trivial phase of matter at the surface in two dimensions [25, 26]. Further for our intentions it is important, that we have a superconductor of type II. In this architecture we assume a superconductor ideally doped on the surface of the topological insulator, so that we get interface-free connection between the surface states and the superconducting bulk states, which ensures that we can neglect the effects due to unwanted reflections or absorptions with regard to the electron wave functions due to disorder. Further we will neglect the unwanted influences of the ferromagnetism to the superconducting gap by assuming very low temperatures and adiabatic variations of the crucial system parameters. It is expected to find chiral Majorana edge states at the boundaries of the superconducting region and the ferromagnetic surface region [26].
So effective we have ring shaped regions of topological superconductors and magnetic regions surrounding the superconducting regions.
For such structures exist many proposals, which for example are predicting Majorana fermions as quasiparticles of a chiral branch of edge states localized at the interface between the magnetic

and the superconducting region [26] usable for charge transport [27] or promise the possibility to detect Majorana fermions created by such interfaces using interferometry [8].
But the successful realization of such a device stays out until now.
Summarizing, in our system we consider the surface states of a three dimensional topological insulator, which are influenced differently in two different geometrical regions on the surface. One of the regions determine the energy spectrum by an superconducting gap Δ and the other region by an induced Zeeman field gap M. While the Hamiltonian of the superconducting region conserved time-reversal symmetry, the region of Zeeman field will breaks it. So the energy spectrum due to the two regions are different from each other in the number of occupation levels. That is why we expect two different topological phases in each region of the surface and therefore we expect to have an zero energy crossing at the boundary between these regions to perform an continuously transition between the Hamiltonians of the two different regions. We expect a topological phase transition at the boundary between these regions, where we expect our localized Majorana zero modes [13, 15].
Further our architecture needs the possibilities, first to shift the chemical potential by a back gate at the topological insulator bulk to ensure that the Fermi energy lies into the gap created by ferromagnetism. Second we need three gates on the surface in between superconducting rings to enable a controlled manipulation of the outer Majorana edge states by changing the hybridization of their wave functions [6, 15], which is a prerequisite for performing fusion and braiding operations, whereby fusion is essential for the manipulation and read out of a Majorana qubit. As gates one can perhaps use Josephson Junctions, but this has to checked.
Due to we have to realize not only Majorana bound states for realizing a quantum computer but rather Majorana bound state with zero energy, we follow the prediction, that these zero energy Majorana modes can occur at the inner and outer boundaries if the hole of the superconducting ring is threaded by an magnetic flux of an *odd* number of flux quantum Φ_0 determining the phase of the effective paring gap, such that located states with zero energy at the inner and outer boundary should become available [15]. For this one can apply a strong external magnetic field.
To enable the possibility for controlled fusion and readout it is supposed to add some gate voltages to each of the superconducting islands and connect the superconducting islands capacitively to single electron transistors and extra side gates for the use of the *parity-to-charge* approach [6].

2.2. Magnitudes of components

The used components of the architecture have to be between micro and nano meter to make a quantum mechanical regime available [1]. So we are in a mesoscopic regime. And for the same reason we need to stay the system at very low temperatures of only a few Kelvin. Due to the magnitude of our superconducting ring (an inner radius of a few 10nm) to create one flux quantum through the hole of the ring one will need a very strong external magnetic field. On the other hand the magnetic field produced by the ferromagnetic surface of the magnetically doped three dimensional topological insulator is not strong enough for such a small ring to induce vortices, which would destroy the controlled creation of zero energy modes [15].
The temperature is proposed to be a few 10mK, which is restricted in higher values due to the energy gap of the expected Majorana bound states at the inner and outer boundaries [15]. The thickness of the three dimensional topological insulator thin film has to be thick enough that the proximity-induced superconducting states are only on one surface of the topological insulator, but thin enough, that gating the chemical potential at this surface with a back-gate located at the opposite surface is possible. It was proposed to take a thickness of about 10nm.
Later we will elaborate some restriction on the magnitudes of several components of our system in order to fulfil necessary assumptions.

[1] For example, if the radius becomes essential larger as the wave length of an electron with Fermi energy, the quantum mechanical nature of electrons as waves gets lost and they start to behave classically.

3. Applicability for quantum information processing

To proof, if the system described in the previous chapter realizes a quantum bit which is able to provide quantum information processing and in this way could be a possible storage unit of a real quantum computer, we have to show four main properties that the system has to exhibit.

(1) *Existence of Majorana zero modes:*
we have to prove the existence of topological protected Majorana zero modes at the inner and outer boundaries of the superconducting rings.

(2) *Controlled quantum state manipulations:*
we have to show how manipulation of these Majorana bound states, and that braiding of the Majorana zero modes within this architecture could works.

(3) *Fusion:* we have to prove the possibility of fuse the Majorana bound states.

(4) *Read out:* we have to give a method for initialization and read out of the quantum bit states.

The last two points *fusion* and *read out* are strong related to each other, so we treat them together in one part. These are the important characteristics, which should be available in our architecture to be a candidate for a real Majorana qubit [7].

3.1. Majorana bound states

Here we consider a subsystem of our system described in the previous chapter.
This subsystem exactly matches the architecture described above with only one superconducting ring placed in the middle of the upper surface of the quantum anomalous Hall insulator. An external applied Magnetic field $\boldsymbol{B} = |B|\,\hat{\boldsymbol{e}}_z$ penetrating only the area of the inner hole of the superconducting ring creates an flux of exact one flux quantum $\Phi_0 = \frac{h}{2e}$. It was predicted, that the inner and outer boundaries of the superconducting ring and the magnetic material can exhibit a pair of Majorana zero mode, which are topological protected [15]. We choose the coordinate system in such a way, that the the upper surface of the thin film ferromagnetic three dimensional

topological insulator lies in the xy-plane with its surface normal in z-direction, on which is coated partly with the (only few atoms thin) superconducting ring. The rest of this surface is only doped with a magnetic insulating material. The upper surface provides an effective two dimensional electron system.
For simplicity we can choose the origin of our coordinate system to be in the center of our superconducting ring. A sketch of our subsystem is given by Figure 3.1

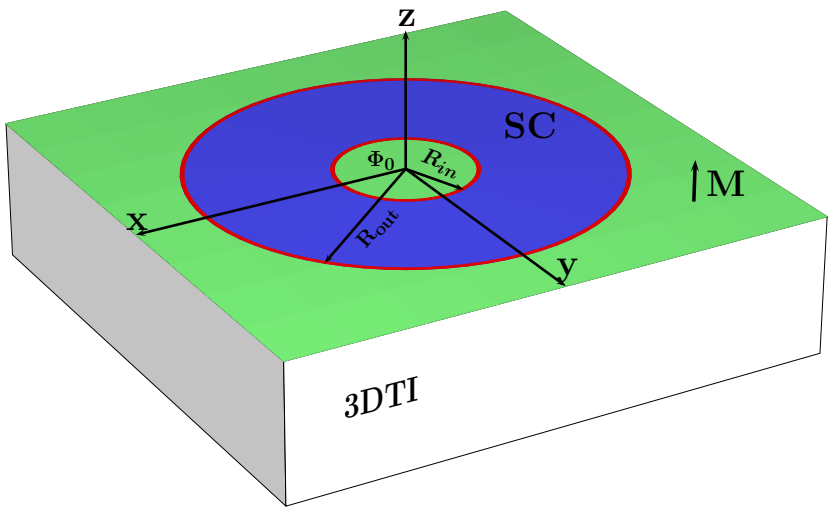


Figure 3.1.: A superconducting ring (SC,blue) and a magnetic material (M,green) layered on the surface of an three dimensional topological insulator (3DTI,white) represents the subsystem of consideration. The magnetic field of the ferromagnetic film at the surface points in z-direction. The hole of the superconducting ring is threaded by a magnetic flux quantum Φ_0. We want to determine the states of zero energy at the boundaries R_{in} and R_{out} (red) [22].

From the proposal of L. Fu and C. L. Kane [26, 27] we know, that it is possible to find located states at such boundaries between regions determined by a superconducting gap Δ and regions determined by an induced Zeeman field gap M. We want to elaborate the edge state wave functions for zero energy on the surface of this three dimensional topological insulator at the inner and outer ring shaped boundaries between the superconducting ring and the magnetic layer and calculate these wave functions by solving the Dirac equation at these boundaries.
Then we explore the peculiar characteristics of these wave functions. For example we will discuss their topological protection against local parameter transformations, their dispersion and how well they are located at the boundaries. We will examine their overlap due to the finite size

of the Ring and the scope between inner and outer radius. At the same time we determine under which conditions it is possible to find these peculiar edge states.

Consider again the Figure 3.1. The magnetic material creates an constant magnetic field on the surface, except at that area, where the s-wave superconducting ring is placed, where due to the proximity effect an effective induced paring gap has influence on the surface states of the quantum anomalous Hall insulator, which leads to non-trivial topological characteristics [15]. We assume that the magnetic field at the rest of the surface is weak and especially smaller than the upper critical field H_{c2} of the s-wave superconductor. We assume a rapid decay of the effective magnetic field in the superconducting region at the inner and outer boundaries. In reality the magnetic field at the boundaries depends on the London penetration depth for the here given superconductor of type II, but with our assumption, for simplicity, we can approximately describe this magnetic field by

$$M(r,\theta) \equiv M(r) = M_0[\Theta(r-R_{out}) + \Theta(R_{in}-r)], \tag{3.1}$$

where R_{out} and R_{in} are the outer and inner radius and r is the radial distance from the center of the superconducting ring. M_0 is the value of the constant effective magnetic field at the surface. The superconductor has the effective superconducting gap $\Delta(\boldsymbol{r}) = |\Delta| e^{\pm i\phi} = \Delta_0 e^{\pm i\phi}$. The real constant Δ_0, its modulus is a measure for the binding energy of the electrons creating a Cooper pair and the phase ϕ depending on the superconductor and the interface [26], is a measure for the coherence of the Cooper pair ensemble.

By applying an external magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{e}}_z$ to create a magnetic flux $\Phi = B \cdot \pi R_{in}^2 = n\Phi_0 = n\frac{h}{2e}$ through the hole of the superconducting ring one gets an superconducting pairing phase of $\phi = n\theta$, where θ is the angular of the polar coordinates. Here we assume that the magnetic field is chosen in such a way, that the hole of the ring is threaded by exact one flux quantum, which gives us a proximity-induced paring gap of $\Delta(\boldsymbol{r}) = \Delta_0 e^{i\theta}$. One could proof, that an even number of flux quantum yields a gauged wave function with anti-periodic boundary conditions having no zero energy states [15]. As a function on the surface it is given approximatively by

$$\Delta(r,\theta) = \Delta_0 e^{i\theta}\Theta(R_{out}-r)\Theta(r-R_{in}) = \vartheta(r)e^{i\theta}. \tag{3.2}$$

Further we have an electrochemical potential in the system given by

$$\mu(r,\theta) \equiv \mu(r)\ , \tag{3.3}$$

which represents an adjustable parameter (for example via back-gating of the three dimensional topological insulator). Primarily we want only to adjust our chemical potential in such a way, that our two magnetic regions are in a topological trivial phase, while the superconducting region

is in a topological non-trivial phase. For this, the chemical potential could be considered constant on the total surface for simplicity. In the following we set $\hbar = 1$.
With the surface normal $\hat{\boldsymbol{n}} \equiv \hat{\boldsymbol{e}}_z$ in the direction of the z-axis and for a low energy consideration, we get for the effective surface Hamiltonian matrix of the upper surface of the three dimensional topological insulator [21]

$$\mathscr{H}_{\text{3DTI}} = v_F \left(\boldsymbol{\sigma} \times \boldsymbol{p}\right) \hat{\boldsymbol{e}}_z \tau_z - \mu \sigma_0 \tau_z = v_F \left(\sigma_x p_y - \sigma_y p_x\right) \tau_z - \mu \sigma_0 \tau_z, \tag{3.4}$$

therefore compare with equation (1.53).
For the magnetic field at the surface we can write the Hamiltonian matrix

$$\mathscr{H}_{\text{M}} = \boldsymbol{M}(r) \boldsymbol{\sigma} \tau_0 = M(r) \sigma_z \tau_0, \tag{3.5}$$

where we have assumed, that the magnetic field points into z-direction $\boldsymbol{M} = M(r)\hat{\boldsymbol{e}}_z$.
The Hamiltonian matrix which describes the proximity-induced superconductor is given by

$$\mathscr{H}_{\Delta} = \frac{1}{2} \left[\Delta \left(\tau_x + i\tau_y\right) + \Delta^* \left(\tau_x - i\tau_y\right)\right], \tag{3.6}$$

where v_F is the Fermi velocity, with z^* we write the complex conjugate of an complex quantity z, σ_0 is the 2×2 unit matrix and σ_i, $i = x,y,z$ are the Pauli matrices in spin space (A.1) with electron fields $\psi = \left(\psi_\uparrow, \psi_\downarrow\right)^T$, and τ_0 is the unit matrix and τ_i, $i = x,y,z$ are the Pauli matrices in Nambu spinor space with the Nambu spinor states $\Psi = ((\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger))^T$, $\Psi^\dagger = ((\psi_\uparrow^\dagger, \psi_\downarrow^\dagger), (\psi_\downarrow, -\psi_\uparrow))$.
As long as the applied magnetic field is only in z-direction, effectively the external magnetic field has only influence on the phase of the proximity-induced paring gap. Let h_z is the effective magnetization in z-direction which only increases to higher values if the surface ferromagnetism and the applied magnetic both are parallel to $\hat{\boldsymbol{e}}_z$ add up to an effective magnetic field h_z, so that the region without superconductor the magnetic fields only add up each other to an higher effective magnetic field. In the region with the proximitized superconductor the vector field $\boldsymbol{A}$ in the xy-plane produced by the applied magnetic field has the only above described influence on the phase of the proximity-induced paring gap. Then the full Hamiltonian matrix for the low energy surface states of our given subsystem reads

$$\mathscr{H} = \mathscr{H}_{\text{3DTI}} + \mathscr{H}_{\Delta} + \mathscr{H}_{\text{M}} = \begin{pmatrix} -\mu + M & v_F p_+ & \Delta & 0 \\ v_F p_- & -\mu - M & 0 & \Delta \\ \Delta^* & 0 & \mu + M & -v_F p_+ \\ 0 & \Delta^* & -v_F p_- & \mu - M \end{pmatrix} \tag{3.7}$$

with the use of Polar coordinates ($\partial_x = cos\theta\partial_r - \frac{sin\theta}{r}\partial_\theta$ and $\partial_y = sin\theta\partial_r + \frac{cos\theta}{r}\partial_\theta$) we have defined

$$p_+ = e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right), \tag{3.8}$$

$$p_- = -e^{+i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right), \tag{3.9}$$

$$p_+ = p_-^* \quad . \tag{3.10}$$

It is obviously, that our Hamiltonian matrix is hermitian $\mathscr{H}^\dagger = \mathscr{H}$. The total Hamiltonian of the subsystem of consideration then reads $H = {}^1\!/_2\Psi^\dagger\mathscr{H}\Psi$.
At this point it is worth mentioning, that our Hamiltonian has particle-hole symmetry $\Xi\mathscr{H}\Xi = -\mathscr{H}$, with particle-hole symmetry operator $\Xi = \sigma_x\tau_y\hat{C}$, where $\hat{C}$ is the complex conjugation operator. How we mentioned in the introduction, the particle-hole symmetry enables the existence of Majorana operators in the system and if their existing fermionic eigenstates with zero energy for the given system Hamiltonian it would exists a Majorana zero mode operator γ_0. Due to the induced Zeeman field M time-reversal symmetry is broken in the region where M does not vanish. This broken time-reversal symmetry enables the ocurance of chiral edge states in our system between the superconducting and the magnetic gaped regions [26]. The Schrödinger equation is

$$\mathscr{H}\Psi_E = E\Psi_E, \tag{3.11}$$

which gives us stationary state solutions Ψ_E. We are interested in the zero eigenvalue $E = 0$ solution Ψ_0 for eq. (3.11), because we are searching for zero energy excitations in this system. So, due to the particle-hole symmetry, one condition is that for $\underline{E = 0}$:

$$\Xi\Psi_0 = \Psi_0 \Rightarrow \Psi_0 = (a, b, b^*, -a^*)^T = \Psi_0(r,\theta), \tag{3.12}$$

where $a \equiv a(r,\theta)$ and $b \equiv b(r,\theta)$ are functions of r and θ.
That is why we have to solve the following system of differential equations

$$\begin{aligned} &\text{(i)} \quad (-\mu + M)\,a + \quad v_F e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right) b \quad + \Delta b^* \quad = 0 \\ &\text{(ii)} \quad (-\mu - M)\,b - \quad v_F e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) a \quad - \Delta a^* \quad = 0\,. \end{aligned} \tag{3.13}$$

We solved this system of differential equations for the boundary conditions given by the geometry of our system. Because we are interested in states of zero excitation energy at the boundaries between the superconducting ring and the magnetic area, we solve the problem for these *inner*

and *outer* boundaries seperately. At the *outer* boundary we have to consider the cases

$$\begin{aligned} r &\geq R_{out} \ , \\ R_{in} &< r \leq R_{out} \end{aligned} \tag{3.14}$$

and at the *inner* boundary we have

$$\begin{aligned} r &\leq R_{in} \quad , \\ R_{in} &\leq r < R_{out} \quad . \end{aligned} \tag{3.15}$$

We start with the solutions for the *outer* boundary.
But let us first consider the energy spectrum in k-space. We have mentioned in the introduction, why we expect Majorana bound states at the boundaries between superconducting and magnetic regions, but let us make it more clear. At these boundaries our Dirac fermion surface states of the three dimensional topological insulator are exposed to the superconducting proximity effect with pairing gap amplitude of Δ_0 and to the magnetic field of strength M_0. Neither the Δ_0 of the superconductor nor the magnetic field nor the chemical potential μ vanishes at the boundaries. Beforehand we take a look at the given energy spectrum for each region. By diagonalizing the total Hamiltonian matrix in $\boldsymbol{k}$-space, we get its real energy spectrum

$$\varepsilon(\boldsymbol{k}) = \sqrt{v_F^2 k^2 + |\Delta_0|^2 + \mu^2 + M^2 \pm 2\sqrt{v_F^2 k^2 \mu^2 + M^2(|\Delta_0|^2 + \mu^2)}} \ . \tag{3.16}$$

If we consider the case for the region of the ring, where the effective Zeeman field is zero $M = 0$ one gets

$$\varepsilon(\boldsymbol{k}) = \sqrt{(\mu \pm v_F \, |\boldsymbol{k}|)^2 + |\Delta_0|^2} \ . \tag{3.17}$$

We have in the superconducting region a topological non-trivial phase of matter, as long as the superconducting gap does not vanish, i.e. $|\Delta_0| \neq 0$ and therefore $|\vartheta(r)| \neq 0$ in this region with $R_{in} \leq r \leq R_{out}$.
In the Zeeman field region without the influence of an superconducting gap we have the spectrum

$$\varepsilon_M(\boldsymbol{k}) = -\mu \pm \sqrt{v^2 k^2 + M_0^2} \Rightarrow \varepsilon_M(\boldsymbol{k} = 0) = -\mu \pm |M_0| \ . \tag{3.18}$$

We have to distinguish the two cases

$$\begin{aligned} &\text{(i)} \quad |M_0| \quad < \quad |\mu| \quad \text{(topological region)} \\ &\text{(ii)} \quad |M_0| \quad > \quad |\mu| \quad \text{(topological trivial region).} \end{aligned} \tag{3.19}$$

Because we expect a topological phase transition at the inner and outer boundaries, we have the

condition that the magnetic regions have to be in a topological trivial phase, which gives us the condition that it has to be $|M| > |\mu|$.
For the case of zero momentum $\boldsymbol{k} = 0$, in principle we have

$$\varepsilon(\boldsymbol{k}=0) = M \pm \sqrt{|\Delta_0|^2 + \mu^2}\,. \tag{3.20}$$

At the outer and inner boundary ($r = R_{out}$ and $r = R_{in}$) for $M_0^2 = \Delta_0^2 + \mu^2$ (or general $M^2 = \Delta^2 + \mu^2$) the energy becomes zero at the boundaries.
So we have the three topological distinct regions

$$\begin{aligned} &\text{(I)} \ \underline{r < R_{in}}: && |M_0| > |\mu|\ , \quad \Delta = 0 \\ &\text{(II)} \ \underline{r > R_{out}}: && |M_0| > |\mu|\ , \quad \Delta = 0 \\ &\text{(III)} \ \underline{R_{in} < r < R_{out}}: && \Delta_0 \neq 0\ , \quad M = 0. \end{aligned} \tag{3.21}$$

In the following we want to assume that the chemical potential is adjust to values $\mu > 0$. We set up the Hamiltonian for each of the three regions in which we have to distinct the superconducting region, which we want label with *in* for in the superconducting region and the magnetic regions, which we want to label with *out* for out of the superconducting region. For these outer regions we distinct the region (I) with the additional label $<$ and the region (II) with the additional label $>$.
First we have to solve each case separately from each other.
For the r dependent factor of the effective gap is $\vartheta(r) = V'(r)$ a real function. For states of zero energy excitations at the outer (inner) boundary we use the ansatz

$$\begin{aligned} \text{(i)}\ a_0(r,\theta) &= f(r)e^{-i\phi - V(r)} \\ \text{(ii)}\ b_0(r,\theta) &= g(r)e^{i(\theta+\phi) - V(r)}, \end{aligned} \tag{3.22}$$

for the components of the Nambu spinor wave function (3.13). Here we solve the systems of differential equations for the diffrent regions individually searching states of zero energy excitations. In the following we set $v_F = 1$ for the constant Fermi velocity.
For the inner region (III) we have to solve

$$\begin{aligned} \text{(i)} \quad & -\mu a + e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right) b + \vartheta(r) e^{i\theta} b^* = 0 \\ \text{(ii)} \quad & -\mu b - e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) a - \vartheta(r) e^{i\theta} a^* = 0 \end{aligned} \tag{3.23}$$

and for the outer regions (I) and (II) we have to solve the following system of differential

equations

$$\begin{aligned} \text{(i)} \quad & (-\mu + M_0)\, a + \quad e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right) b \quad = 0 \\ \text{(ii)} \quad & (-\mu - M_0)\, b - \quad e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right) a \quad = 0, \end{aligned} \tag{3.24}$$

where μ and M_0 are constants. For both we use our ansatz (3.22) and find only solutions for $\phi = m\pi \Rightarrow e^{\pm i\phi} = (-1)^m$, where m is an integer[1], so that we can set $\phi = 0$. For the inner region we get the reduced set of two differential equation depending only on the radius r ($r \geq 0$ and $\mu > 0$)

$$\begin{aligned} \text{(i)} \quad & f'(r) \quad = -\mu g(r) \\ \text{(ii)} \quad & (rg)'(r) \quad = r\mu f(r). \end{aligned} \tag{3.25}$$

This leads to the Bessel differential equations

$$\begin{aligned} \text{(i)} \quad & (r^2 - 1)g(r) + rg'(r) + r^2 g''(r) \quad = 0 \\ \text{(ii)} \quad & r^2\mu^2 f(r) + rf'(r) + r^2 f''(r) \quad = 0, \end{aligned} \tag{3.26}$$

for which the general solutions

$$\begin{aligned} f^{in}(r) &= c_1^{in} J_0(\mu r) + c_2^{in} Y_0(\mu r), \\ g^{in}(r) &= c_3^{in} J_1(\mu r) + c_4^{in} Y_1(\mu r) \end{aligned} \tag{3.27}$$

exist. The functions $J_n(x)$ and $Y_n(x)$ for $n = 0, 1$ are the Bessel functions of first and second kind respectively (A.2). Due to this solutions have to fulfil both equations in (3.25) we get the relations $c_3^{in} = c_1^{in}$ and $c_4^{in} = c_2^{in}$ between the constants.
For the outer region we know due to the topological criterion, that it has to be $M > \mu$ and we can set $a = \sqrt{(M-\mu)(M+\mu)} > 0$, where $a \in \mathbb{R}$ and we get for the outer region the following reduced system of differential equations

$$\begin{aligned} \text{(i)} \quad & f'(r) \quad = -(\mu + M)g(r) \\ \text{(ii)} \quad & (rg)'(r) \quad = -r(M-\mu)f(r). \end{aligned} \tag{3.28}$$

[1] This ambiguity in sign is one characteristic of Majorana wave functions. If Ψ_i is the Majorana wave function corresponding to the Majorana operator γ_i then $-\Psi_i$ is also a Majorana wave function corresponding to the Majorana operato preserving $\gamma_i = \gamma_i^\dagger$ and $\gamma_i^2 = 1$.

This leads to the Bessel differential equations

$$\begin{aligned} \text{(i)} \quad & -(r^2a^2+1)g(r)+rg'(r)+r^2g''(r) && =0 \\ \text{(ii)} \quad & -r^2a^2f(r)+rf'(r)+r^2f''(r) && =0. \end{aligned} \tag{3.29}$$

Again we find as general solutions superpositions of Bessel functions with restriction to the constants due to the equations in (3.28)

$$\begin{aligned} f^{out}_{</>}(r) &= c^{out}_{1,</>}I_0(ar)+c^{out}_{2,</>}K_0(ar) \\ g^{out}_{</>}(r) &= -bc^{out}_{1,</>}I_1(ar)+bc^{out}_{2,</>}K_1(ar), \end{aligned} \tag{3.30}$$

with $b=\frac{a}{M+\mu}=\sqrt{\frac{M-\mu}{M+\mu}}$. Where we have used $ar\geq 0$ to rewrite the superposition of the Bessel functions of first kind with imaginary argument, for which then $Y_1(-iar)=-\frac{2i}{\pi}K_1(ar)+I_1(ar)$. The relations $J_0(ix)=I_0(x)$, $J_1(ix)=iI_1(x)$ and $Y_0(-ix)=-\frac{2}{\pi}K_0(x)-iI_0(x)$ are valid for all $x\in\mathbb{R}$. The functions $I_n(x)$ and $K_n(x)$ for $n=0,1$ are the modified Bessel functions of first and second kind respectively (A.2).

For the region (II) the solutions have to be convergent for $r\to\infty$. Because $I_0(ar)$ and $I_1(ar)$ go to infinity while $K_0(ar)$ and $K_1(ar)$ go to zero for $r\to\infty$, the only valid solutions are given for the constant $c^{out}_{1,>}=0$. For the region (I) we have to have finite solutions for $r\to 0$. Because in this case $K_0(ar)$ is divergent while $I_0(ar)$ goes to one and $I_1(ar)$ goes to zero for $r\to 0$, the only valid solution in this region is given for the constant $c^{out}_{2,<}=0$. Further for *high* values of $a=\sqrt{M^2-\mu^2}$, in region (I) for $r<R_{in}$ the Bessel function $I_0(ar)$ has a *fast* decay to 1 and the Bessel function $I_1(ar)$ has a *fast* decay to zero. In region (II) the Bessel functions $K_0(ar)$ and $K_1(ar)$ have both for $r>R_{out}$ a *fast* decay to zero. In the end we get for the distinct regions the following solutions:

$$\begin{aligned} &\text{for region (I):} && f^{out}_{<}(r)=c^{out}_{<}I_0(ar), \\ & && g^{out}_{<}(r)=-bc^{out}_{<}I_1(ar), \\ &\text{for region (II):} && f^{out}_{>}(r)=c^{out}_{>}K_0(ar), \\ & && g^{out}_{>}(r)=bc^{out}_{>}K_1(ar), \\ &\text{for region (III):} && f^{in}(r)=c^{in}_1J_0(\mu r)+c^{in}_2Y_0(\mu r), \\ & && g^{in}(r)=c^{in}_1J_1(\mu r)+c^{in}_2Y_1(\mu r). \end{aligned} \tag{3.31}$$

We assume that the solutions for the inner and outer boundaries are independent from each other and have a negligible overlap or, idealized, no overlap in the inner superconducting region. For

this we have to search independent solutions at the inner and outer boundaries for which we label the different solutions of the inner region (III) also by $<$ for the solution at the inner boundary and $>$ for the solution at the outer boundary. The solutions of the inner and outer regions have to be continuous at the boundaries such the constants are determined by

$$\begin{aligned} &\text{(1) inner boundary:} && f_<^{out}(R_{in}) = f_<^{in}(R_{in}), \\ & && g_<^{out}(R_{in}) = g_<^{in}(R_{in}), \\ &\text{(2) outer boundary:} && f_>^{out}(R_{out}) = f_>^{in}(R_{out}), \\ & && g_>^{out}(R_{out}) = g_>^{in}(R_{out}). \end{aligned} \tag{3.32}$$

Which gives us the relations

$$\begin{aligned} &\text{(1) inner boundary:} && I_0(aR_{in}) = c_1^< J_0(\mu R_{in}) + c_2^< Y_0(\mu R_{in}), \\ & && -bI_1(aR_{in}) = c_1^< J_1(\mu R_{in}) + c_2^< Y_1(\mu R_{in}), \\ &\text{(2) outer boundary:} && K_0(aR_{out}) = c_1^> J_0(\mu R_{out}) + c_2^> Y_0(\mu R_{out}), \\ & && bK_1(aR_{out}) = c_1^> J_1(\mu R_{out}) + c_2^> Y_1(\mu R_{out}), \end{aligned} \tag{3.33}$$

Both systems of equations can be written as a Matrix equation

$$\begin{aligned} &\text{(1) inner boundary:} && \begin{pmatrix} I_0(aR_{in}) \\ -bI_1(aR_{in}) \end{pmatrix} = M_{in} \cdot \begin{pmatrix} c_1^< \\ c_2^< \end{pmatrix}, \\ &\text{(2) outer boundary:} && \begin{pmatrix} K_0(aR_{out}) \\ bK_1(aR_{out}) \end{pmatrix} = M_{out} \cdot \begin{pmatrix} c_1^> \\ c_2^> \end{pmatrix}, \end{aligned} \tag{3.34}$$

where

$$M_{in/out} = \begin{pmatrix} J_0(\mu R_{in/out}) & Y_0(\mu R_{in/out}) \\ J_1(\mu R_{in/out}) & Y_1(\mu R_{in/out}) \end{pmatrix}, \tag{3.35}$$

which for our case ($\mu r > 0$) are real 2×2 matrices. And as long as $det\left(M_{in/out}\right) \neq 0$ we find solutions for both boundaries. The full functions for the inner boundary then read

$$f_<(r) = \begin{cases} I_0(ar)\,, & \text{for } r \leq R_{in} \\ c_1^< J_0(\mu r) + c_2^< Y_0(\mu r)\,, & \text{for } R_{in} \leq r < R_{out} \end{cases} \tag{3.36}$$

$$g_<(r) = \begin{cases} -bI_1(ar)\,, & \text{for } r \leq R_{in} \\ c_1^< J_1(\mu r) + c_2^< Y_1(\mu r)\,, & \text{for } R_{in} \leq r < R_{out} \end{cases} \tag{3.37}$$

and for the outer boundary

$$f_{>}(r) = \begin{cases} c_1^{>} J_0(\mu r) + c_2^{>} Y_0(\mu r) \ , & \text{for } R_{in} < r \leq R_{out} \\ K_0(ar) \ , & \text{for } R_{out} \leq r \end{cases} \tag{3.38}$$

$$g_{>}(r) = \begin{cases} c_1^{>} J_1(\mu r) + c_2^{>} Y_1(\mu r) \ , & \text{for } R_{in} < r \leq R_{out} \\ bK_1(ar) \ , & \text{for } R_{out} \leq r \end{cases} \tag{3.39}$$

and as our result the wave function of zero energy state existing at the inner boundary reads

$$\Psi_0^{R_{in}}(r,\theta) = e^{-V(r)} \begin{pmatrix} f_{<}(r) \\ g_{<}(r)e^{i\theta} \\ g_{<}(r)e^{-i\theta} \\ -f_{<}(r) \end{pmatrix} \tag{3.40}$$

and at the outer boundary it reads

$$\Psi_0^{R_{out}}(r,\theta) = e^{-V(r)} \begin{pmatrix} f_{>}(r) \\ g_{>}(r)e^{i\theta} \\ g_{>}(r)e^{-i\theta} \\ -f_{>}(r) \end{pmatrix} \tag{3.41}$$

for which we can say, that for a suitable given fixed superconducting region size and a high enough superconducting gap in the superconducting region and a high enough magnetic gap in the magnetic regions one has highly localized states at these inner and outer boundaries between these two regions. For this functions of the inner and outer boundary solutions we want to have a closer look on the dependence on the distance r of the four components of each edge state. On the one hand considering the outer regions for high enough (compared to the chemical potential) effective magnetic field strength M_0, how we discussed above, the Bessel functions have a fast decay into the outer regions and the functions $f_{</>}(r)$ and $g_{</>}(r)$ have a fast decay into the magnetic region (I) and (II) respectively. To be more descriptive, the dependence of the four components of each boundary solution on the magnetic field strength M_0 are shown in Figure 3.2 as functions of the radius r for a fixed angle of $\theta = 0$ (larger figures are given in the appendix).

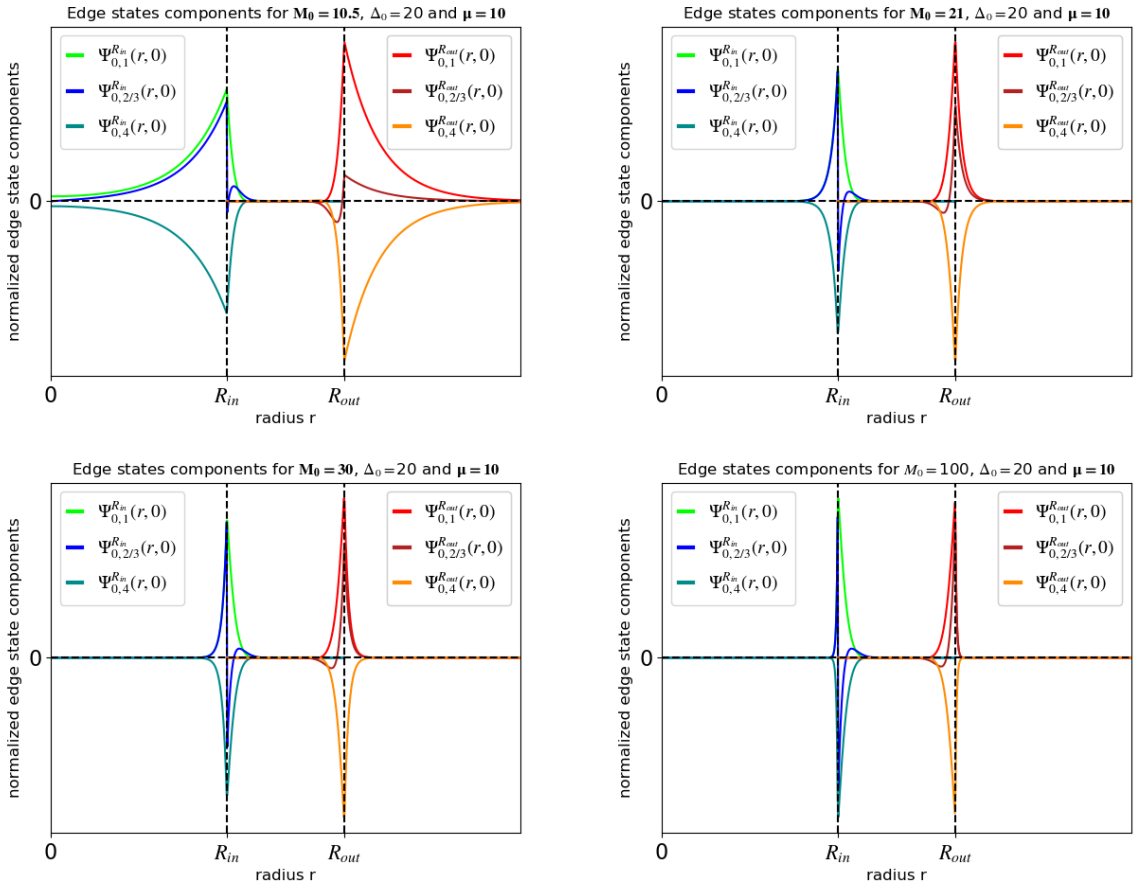


Figure 3.2.: The wave functions components of the edge state solutions (at a fixed angle of $\theta = 0$) are shown for different values of the magnetic field strengths M_0 to demonstrate its r-dependency for different (relative to μ) magnetic field strengths M_0. Both magnetic regions are treated equal, although for our intended intentions it will be the magnetic field strength $M_0 \gg \mu$ fixed in the magnetic region for $r < R_{in}$ [28].

We see that for $M_0 \gg \mu$ the components of the wave functions at the boundaries have a vanishing penetration into the outer magnetic regions (compare with the figure in the right bottom of Figure 3.2), while for $M_0 \to \mu$ they can extend over the outer regions (compare with the figure in the left top of Figure 3.2). On the other hand considering the behaviour into the superconducting region we have to have a closer look on the exponentially dependence on $-V(r)$. We can see that the edge states described by the wave functions of eq. (3.40) and eq. (3.41) are localized around the inner and outer edges. To be more precise let us consider the function $V(r)$

at the two boundaries and their behaviour into the inner region

$$\begin{aligned} \text{(outer boundary)}\ V(r) &= \int_{r}^{R_{out}} \mathrm{d}r' \vartheta(r'), \\ \text{(inner boundary)}\ V(r) &= \int_{R_{in}}^{r} \mathrm{d}r' \vartheta(r'), \end{aligned} \tag{3.42}$$

where in the two outer regions it is $\vartheta = 0$, which leads to $V = 0$ for both outer regions. But in the inner region (III) we have $\vartheta \geq 0$ for $R_{in} \leq r \leq R_{out}$, so that $V(R_{in}) = 0$ and $V(R_{out}) = 0$ but other wise $V(r) > 0$ for $R_{in} < r < R_{out}$ and we have an exponentially decay into the superconducting region (III) from the boundaries. We see that the bigger the superconducting gap Δ_0, the faster the decay of our solutions into the superconducting region. Assuming a constant superconducting gap in the inner region (III) $\vartheta(r) = \Delta_0$ for $R_{in} < r < R_{out}$ and reinsert the Fermi velocity v_F, which we had set $v_F = 1$ in the beginning, we can write

$$\begin{aligned} \text{(outer boundary)}\ V(R_{in}) &= \frac{(R_{out} - R_{in})}{v_F}\Delta_0, \\ \text{(inner boundary)}\ V(R_{out}) &= \frac{(R_{out} - R_{in})}{v_F}\Delta_0, \end{aligned} \tag{3.43}$$

where $\xi = \frac{v_F}{\Delta_0}$ is the effective superconducting coherence length and we can write the factor

$$\tau = e^{-\frac{(R_{out} - R_{in})}{\xi}} \tag{3.44}$$

each for the overlap at the opposite boundary. For this we can see that for a finite distance of the two boundaries the superconducting gap Δ_0 has to be high enough to suppress an substantive overlap of the two solutions. Again this behavior of all components of each solution is shown for different values of the strength of the superconducting gap in Figure 3.3.

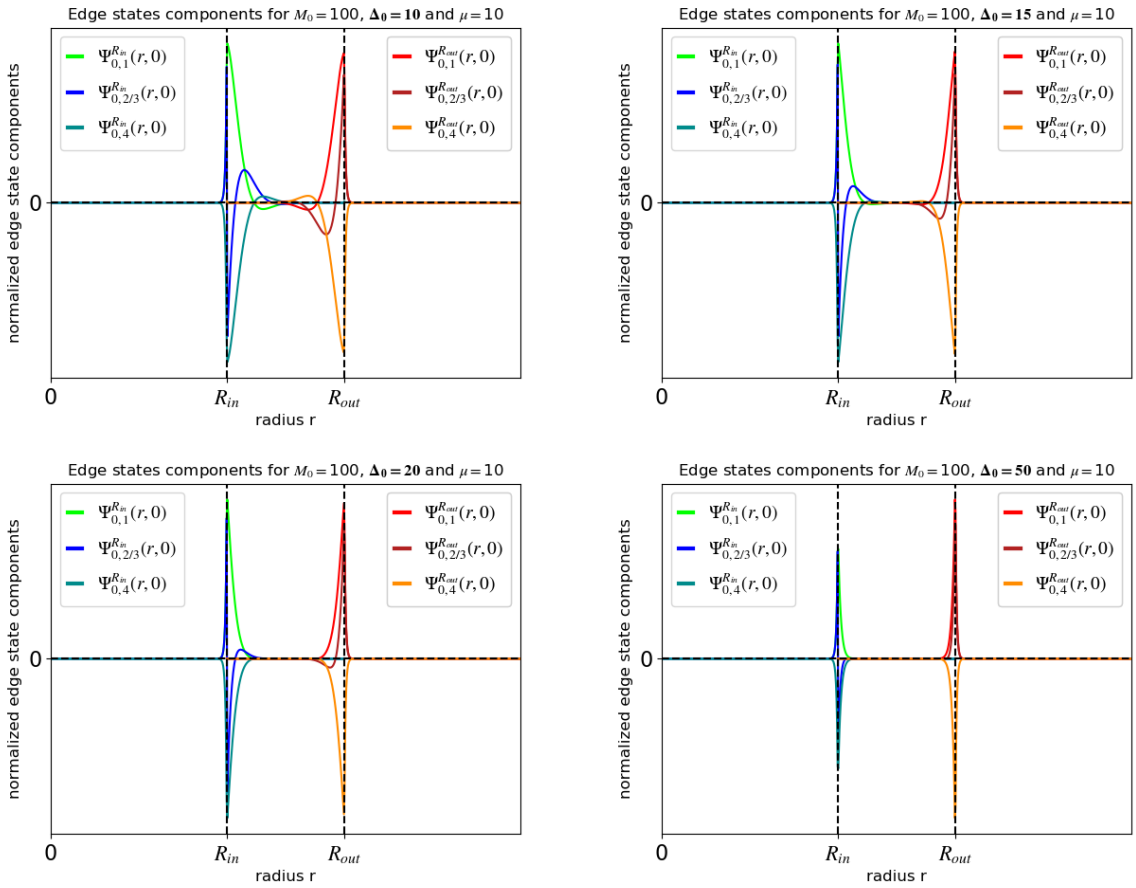


Figure 3.3.: The wave functions components of the edge state solutions (at a fixed angle of $\theta = 0$) are shown in its r-dependency for different values of the strength of the superconducting gap Δ_0 [28].

It should be mentioned that in the superconducting region the chemical potential has no influence on the strength of the overlap and it only provides for higher oscillations of the individual wave function components (see Figure B.11 for visualization). So for a suitable fixed distance $R_{out} - R_{in}$ we can see that the wave functions have no overlap in the inner region and a fast decay for each of the two edges each in both radial directions away from the respective edge as long as the magnetic field M_0 and the superconducting gap Δ_0 are strong enough (compare the plot for $M_0 = 100$ and $\Delta_0 = 50$ in Figure 3.3). So for both we can speak about strong localized edge state solutions of zero energy.

For simplicity we have ignored the London penetration depth of the magnetic field at the boundaries, which in fact can broaden the region of our edge states. And it should be clear that in reality due to the finite distance $R_{out} - R_{in}$ between the inner and outer boundaries our wave function will have an overlap, which splits up the zero energy state into two very small energy values distinct from zero.

To see that these edge modes are Majoranas, we can define the edge mode operators $\gamma_0^{R_{in}\ (R_{out})}$

and $(\gamma_0^{R_{in}\ (R_{out})})^\dagger$ destroy and create a particle in state $\Psi_0^{R_{in}\ (R_{out})}$ with zero energy as

$$\begin{aligned}\gamma_0^\alpha &= \int \mathrm{d}\boldsymbol{r}(\Psi_0^\alpha(\boldsymbol{r}))^\dagger\Psi(\boldsymbol{r})\ ,\\ (\gamma_0^\alpha)^\dagger &= \int \mathrm{d}\boldsymbol{r}\Psi^\dagger(\boldsymbol{r})\Psi_0^\alpha(\boldsymbol{r})\ ,\end{aligned} \tag{3.45}$$

where we have used the label α for the labels for the inner (outer) boundary $R_{in},(R_{out})$ labels. We know that our eigenstates have the form $\Psi_0^\alpha = (a_\alpha, b_\alpha, b_\alpha^*, -a_\alpha^*)^T$ (for which the functions $f(r),\ g(r)$ in fact have to be real) and therefore we have in fact

$$\begin{aligned}(\gamma_0^\alpha)^\dagger &= \int \mathrm{d}\boldsymbol{r}\left(a_\alpha\psi_\uparrow^\dagger + b_\alpha\psi_\downarrow^\dagger + b_\alpha^*\psi_\downarrow + a_\alpha^*\psi_\uparrow\right)\\ &= \int \mathrm{d}\boldsymbol{r}\left[f_\alpha(r)\left(\psi_\uparrow^\dagger+\psi_\uparrow\right) + g_\alpha(r)\left(e^{-i\theta}\psi_\downarrow^\dagger + e^{i\theta}\psi_\downarrow\right)\right]e^{-V(r)}\\ &= \gamma_0^\alpha\ ,\end{aligned} \tag{3.46}$$

only due to the particle-hole symmetry. Due to our zero mode operators are fermionic we have zero energy Majorana operators. All in all the solutions $\Psi_0^{R_{in}\ (R_{out})}$ are Majorana bound states of zero energy for a suitable chosen set of system parameters. That is why our edge states operators are indeed operators of Majorana edge modes with

$$\gamma_0^{R_{in}} = \left(\gamma_0^{R_{in}}\right)^\dagger \text{ and } \gamma_0^{R_{out}} = \left(\gamma_0^{R_{out}}\right)^\dagger\ . \tag{3.47}$$

Further one could assume that their existing low energy exited states at the boundaries $\Psi_{\pm k}^{R_{in}}$ and $\Psi_{\pm k}^{R_{out}}$, which are eigenstates with energy $\pm E_k$ for which one can define the creation and annihilation operators

$$\begin{aligned}\gamma_k^\alpha &= \int \mathrm{d}\boldsymbol{r}(\Psi_k^\alpha(\boldsymbol{r}))^\dagger\Psi(\boldsymbol{r})\ ,\\ (\gamma_k^\alpha)^\dagger &= \int \mathrm{d}\boldsymbol{r}\Psi^\dagger(\boldsymbol{r})\Psi_k^\alpha(\boldsymbol{r})\ ,\end{aligned} \tag{3.48}$$

for which it is $(\gamma_k^\alpha)^\dagger = \gamma_{-k}^\alpha$. If we treat the low energy excitations perturbatively and consider the system Hamiltonian the perturbation part of the Hamiltonian depends only on θ and therefore

we can define the operators γ_k^α in real-space as

$$\begin{aligned}\gamma^\alpha(\theta) &= C\int \mathrm{d}k e^{ik\theta}\gamma_k^\alpha \\ &= \left(C\int \mathrm{d}k e^{-ik\theta}(\gamma_k^\alpha)^\dagger\right)^\dagger \\ &= \left(C\int \mathrm{d}k e^{-ik\theta}\gamma_{-k}^\alpha\right)^\dagger \\ &= \left(C\int \mathrm{d}k e^{ik\theta}\gamma_k^\alpha\right)^\dagger \\ &= (\gamma^\alpha(\theta)),^\dagger\end{aligned} \tag{3.49}$$

where C is some real constant. And also furthermore the excited states have to be localized in r-direction. So for low energy excitation states we get localized Majorana states. L. Fu and C. L. Kane have shown [27] that the low energy excitations have the energies are given by $E(k) \propto v\cdot k$ or in real space $E(r) \propto \frac{v}{r}$, where v is some velocity depending on the system parameters M_0, μ and Δ_0. And at the boundaries we have $E^{in} \propto \frac{v}{R_{in}}$ and $E^{out} \propto \frac{v}{R_{out}}$ for which one can see, that small enough size of the superconducting ring prevents the zero eigenstates for higher excitation states. So with (3.40) and (3.41) we have found one dimensional Majorana bound states of zero energy, which are localized at the edges of the superconducting region.

If we consider the conditions for the appearance of these Majorana zero modes, we see that as long as local perturbations are sufficiently small the zero energy modes remain localized at the boundaries of the ring. This means the above found states of zero energy are stable against local perturbations like an electromagnetic potential $\boldsymbol{A}'$ which is smaller than one, which creates a magnetic flux quantum, because only for such high vector potential the phase of the effective superconducting pairing gap will change by an integer number. Further if we consider local perturbations of a Zeemann splitting field $\boldsymbol{M}'$ it has no influences on our solutions as long as it is not higher as the upper critical field H_{c2} of the superconductor or it breaks down the magnetic field on the surface $\boldsymbol{M}$ to values $M < \mu$ for which we will have no longer a topological phase transition. But this kind of perturbation would not be called local. Due to considering surface states of a three dimensional topological insulator local spin-orbit interactions, which are sufficiently small enough that they will not destroy the surface states of the given topological insulator, can not destroy the localized Majorana zero modes. Because the only condition that the superconducting region is in a topological non-trivial phase is that Δ_0 never becomes zero in this region, the edge modes have a stability against local perturbations of external potentials deforming the superconducting pairing gap Δ_0 as long it is Δ_0 stays high enough so that for a given finite ring size the overlap of the wave functions at the inner and outer boundary remains negligible.

So we explicit showed, that it is possibly by choosing appropriately the system parameters R_{in}, R_{out}, M_0, Δ_0, μ at the surface of the quantum anomalous Hall insulator to create a pair of zero energy Majorana bound states, which is one of three basic operation steps make a quantum computer, enabling the initialization of the N qubit system in the ground state $|\text{g.s.}\rangle$. The second of the three basic operation steps making a quantum computer is the adiabatically rearranging of the Majorana edge modes, which modifies the phase of the systems state and performs a basic quantum computation (what is called *braiding*). It is our next task to investigate the possibility of *braiding* in the present architecture.

3.2. Braiding of Majorana zero modes

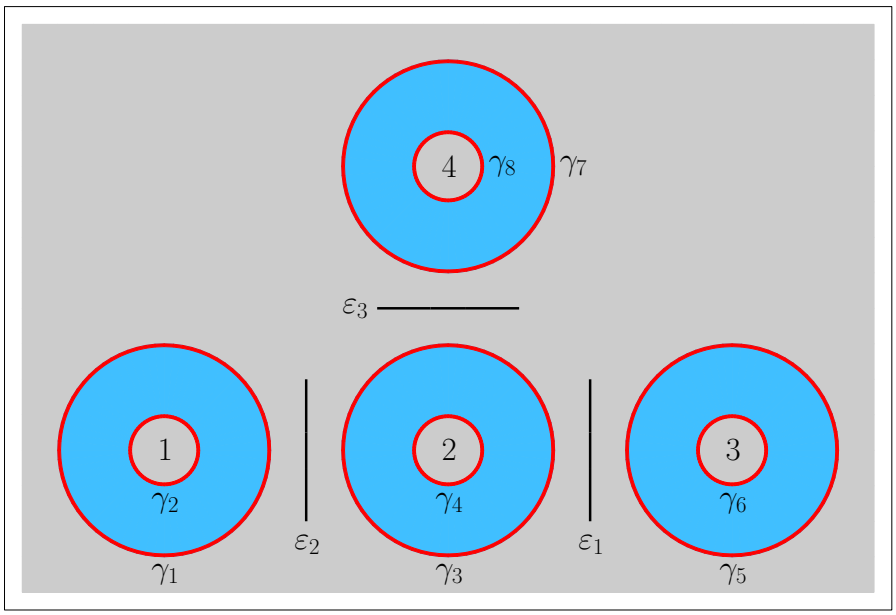


Figure 3.4.: A schematic view of our basis setup for the exchange process of the two Majorana zero modes γ_1 and γ_5 localized at the outer boundaries of the two outer rings one and three. The grey coloured background represents the surface of the thin ferromagnetic three dimensional topological insulator. The number in the center of each ring labels the superconducting rings (blue), so that the grey areas represent the magnetic and the blue ones the superconducting regions. The red inner and outer circles symbolically represent the localized Majorana zero modes labelled and numbered by γ_j. The thin lines labelled by ε_1, ε_2 and ε_3 represent our gates. If one of these lines is continuous or has an opening in the middle it represents a closed ($\varepsilon_i = 0$) or an open ($\varepsilon_i = 1$) gate. In this picture above all gates are closed [22].

In this part we want to determine the quantum statistic of a certain exchange process concerning two of the outer Majorana bound states belonging to two different rings. The space in which the exchange of the Majorana zero modes takes place is two dimensional, because it is assumed to be done only at the surface of the three dimensional topological insulator. Due to Majorana fermions have no charge ($\gamma_i = \gamma_i^\dagger$) or other inherit characteristics which make them distinguishable the only thing which gives them an identity is their position in space. That means, that if we exchange the positions of two of these Majorana fermions, we have to end up in the same system after the exchange process as before, which gives us the condition for our Hamiltonian, which describes the system, to be unchanged after the exchange process. That means for a given state of our quantum system it can only has a change in its phase still solving the Schrödinger equation. This change in the phase of the quantum system state gives us rise to the available quantum statistics.

Now we want to consider the full architecture presented in the previous chapter, with four rings arranged in a T-shape. A schematic view of our used architecture is given by Figure 3.4. We assume that the chemical potential at the surface in between some of the rings is manipulable by applied gates in the region, where we have no superconducting ring (see previous chapter for detailed description). In the previous part of this chapter we have derived the Majorana wave functions at the inner and outer boundary for such a ring. Here we look first and foremost on the Majorana wave function at the outer boundary. We have seen, that for a suitable size of the rings and a suitable distance between the different rings as for a suitable chosen system parameter set, that the overlap between the boundary Majorana wave functions is negligible and we can assume that all Majorana zero modes are well separated from each other. In addition we want to assume, that the system is brought initially into a topological regime, whereby the eight Majorana zero modes at the boundaries are created out of the vacuum[2].

Comparing with the results for these wave functions, one can see that for a suitable distance between two rings and by shifting (via gating) the chemical potential closer to the magnitude of the magnetic field at the surface M_0 the overlap between the outer Majorana edge wave functions of two neighbouring rings becomes more an more relevant. For a high enough overlap of two outer Majorana wave functions we get one fused wave function extending over the region of both of these neighbouring rings and the space along the gate in between them. This fused wave function represents a normal Fermi wave function.

We want to call this process of shifting the chemical potential between two rings at the surface via gating *opening the gate*. The reverse process, where we decrease exact this chemical potential, we want to call *closing the gate*.

By *closing the gate* between such two rings with overlapping outer wave functions, we separate them by decreasing their overlap up to a negligible value, which again leads to two well separated outer Majorana zero modes. But all this holds only if we assume that the changing of the chemical potential is done *adiabatically* in a quantum mechanical sense, i.e. it fulfils the adiabatic theorem, what means, that the systems remains all the time in its instantaneous time dependent ground state.

So the speed of the process is limited by the size of the energy gap between our zero modes and the energy of the next excited states. For the system of eight Majorana zero modes γ_i with $i = 1,2,3,4,5,6,7,8$ we have four corresponding fermions of zero energy state constructed out

[2]This can be done for example by switching on the external magnetic field creating exact one flux quantum and adjusting the chemical potential to make the outer region topological trivial, which both together enable the existence of Majorana zero modes at the ring boundaries.

of pairs of the eight Majorana zero modes

$$f_j = \frac{1}{2}\left(\gamma_{2j-1} + i\gamma_{2j}\right) \ , \quad j = 1,2,3,4. \tag{3.50}$$

This means creation or destruction of one of these four fermions costs no energy and the ground states with and without these fermions have the same energy. So for four of these zero energy fermions we have $2^4 = 16$ degenerated ground states for which we can construct a basis consisting of 16 basis states.

Our goal now is to work out how the system state behaves under exchanging adiabatically two of the Majorana zero modes. Due to these present highly localized quasiparticles should have all the time zero energy for which the system stays for all the time in its degenerated ground state and these Majorana zero modes are non-distinguishable, if we look at them individually, we expect that an exchange in the positions of two of them leads to non-abelian exchange statistics and braiding of them is possible. The adiabatic exchange will be carried out by a gating process based on the described gating mechanism above. Therefore we first explain the steps of the process and give a mathematically description for it. Then we will work out the Berry phase as an outcome of this adiabatic exchange process. First consider the system of eight Majorana zero modes at the eight boundaries, which are assumed to be created out of the vacuum. Let us describe the three gates between the rings with functions of time t as $\varepsilon_i(t) \in [0,1]$ for $i = 1,2,3$, where we use the convention

$$\begin{aligned} \varepsilon_i(t) &= 1 \quad \text{, for gate } i \text{ is switched on,} \\ \varepsilon_i(t) &= 0 \quad \text{, for gate } i \text{ is switched off.} \end{aligned} \tag{3.51}$$

When a gate function between to neighboured rings is zero, the Majorana zero modes localized there will (idealized) have no overlap. When we open a gate adiabatically the gate function becomes 1 and the overlap of our previous boundary modes become maximal giving a common wave function of an ordinary fermion. The system state can be written as a superposition of the basis states lying in the degenerated ground state given by all 8 Majorana zero modes. So that the motion of the Majorana zero modes through moving their positions of localization determines the time evolution of the systems state, which can be totally described by the time evolution of all three gates together. We have a freedom in the choice which pair of Majorana zero modes constructs a fermion[3] and for simpler calculations we choose the following basis

$$\begin{aligned} f_1 &= \frac{1}{2}(\gamma_1 + i\gamma_5), \quad f_2 &&= \frac{1}{2}(\gamma_3 + i\gamma_7), \\ f_3 &= \frac{1}{2}(\gamma_2 + i\gamma_6), \quad f_4 &&= \frac{1}{2}(\gamma_4 + i\gamma_8), \end{aligned} \tag{3.52}$$

[3] It is only a basis change to transform between the different constructions.

for which we have the relations

$$\begin{aligned} \gamma_1 &= f_1^\dagger + f_1, \quad \gamma_5 &&= i(f_1^\dagger - f_1), \\ \gamma_3 &= f_2^\dagger + f_2, \quad \gamma_7 &&= i\left(f_2^\dagger - f_2\right) \end{aligned} \tag{3.53}$$

and for γ_2, γ_4, γ_6 and γ_8 analogue. Our chosen basis consisting of 16 states can be written as

$$\begin{aligned} |0000\rangle, \qquad & |1111\rangle = f_1^\dagger f_2^\dagger f_3^\dagger f_4^\dagger |0000\rangle, \\ |1000\rangle = f_1^\dagger |0000\rangle, \qquad & |0100\rangle = f_2^\dagger |0000\rangle, \\ |0010\rangle = f_3^\dagger |0000\rangle, \qquad & |0001\rangle = f_4^\dagger |0000\rangle, \end{aligned} \tag{3.54}$$

and the other ten basis states are written analogue. This is a complete basis for the Hilbert space of the set of the 8 Majorana modes. Beside that, this basis states are all eigenstates of the operators $P_l = (1 - 2f_l^\dagger f_l) = i\gamma_k\gamma_m$, where $f_l = \frac{1}{2}(\gamma_m + i\gamma_k)$. This operator is what is called a fermion parity operator and one can proof that it is $P_l(f_l^\dagger |...,n_{l-1},0,n_{l+1},...\rangle) = -|...,n_{l-1},1,n_{l+1},...\rangle$ and $P_l(f_l |...,n_{l-1},1,n_{l+1},...\rangle) = +|...,n_{l-1},0,n_{l+1},...\rangle$.
The total fermion parity operator then reads $P = \prod_{l=1}^4 P_l$ and for the eigenstates of the parity operator it is $P|n\rangle = \pm|n\rangle$, such that linear combinations of states of different fermion parity are forbidden. In isolated closed systems the total fermion parity has to be a conserved quantity. In the following we consider our system of four rings as an isolated system[4]. The effective Hamiltonian of the interactions by gating the outer Majorana zero modes can be described as

$$\begin{aligned} H_{eff} &= i\left(n_1\varepsilon_1(t)\gamma_3\gamma_5 + n_2\varepsilon_2(t)\gamma_1\gamma_3 + n_3\varepsilon_3(t)\gamma_7\gamma_3\right) \\ &= \alpha(t)(f_2 f_1 + f_2^\dagger f_1) - \beta(t)(f_2^\dagger f_1^\dagger + f_2 f_1^\dagger) + \varepsilon_3(t)(1 - 2f_2^\dagger f_2), \end{aligned} \tag{3.55}$$

where $\alpha(t) = \varepsilon_1(t) - i\varepsilon_2(t)$ and $\beta(t) = \varepsilon_1(t) + i\varepsilon_2(t)$ and while all $n_i = \pm 1$ for $i = 1,2,3$ are definite up to a minus sign here we choose $n_i = 1$ for all $i = 1,2,3$. One easily check, that all the inner Majorana zero modes commute with H_{eff} for all times t and that is why the 16×16 Hamiltonian matrix reduces to a 4×4 matrix in the basis given by

$$|00\rangle,\ |11\rangle = f_1^\dagger f_2^\dagger |00\rangle,\ |10\rangle = f_1^\dagger |00\rangle,\ |01\rangle = f_2^\dagger |00\rangle\ . \tag{3.56}$$

[4] In a network of more than four rings this subsystem of four rings could be in a linear superposition of different parities.

In this basis our effective Hamiltonian can be written as the following matrix

$$H_{eff} = \begin{pmatrix} \varepsilon_3 & \varepsilon_1 - i\varepsilon_2 & 0 & 0 \\ \varepsilon_1 + i\varepsilon_2 & -\varepsilon_3 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & \varepsilon_1 + i\varepsilon_2 \\ 0 & 0 & \varepsilon_1 - i\varepsilon_2 & -\varepsilon_3 \end{pmatrix} = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix}, \tag{3.57}$$

which consists of the two 2×2 submatrices H_{even} in the subspace of even parity $\{|00\rangle, |11\rangle\}$ and H_{odd} in the subspace of odd parity $\{|10\rangle, |01\rangle\}$. Further with (3.53) one can easily check that the Majorana zero mode operators γ_i for $i \in 1,3,5,7$ map the the states of the even subspace into the states of the odd subspace and vice versa such that $\gamma_i |even(odd)\rangle \propto |odd(even)\rangle$ with $|even\rangle \in \{|00\rangle, |11\rangle\}$ and $|odd\rangle \in \{|10\rangle, |01\rangle\}$.

Due to the fermion number parity conservation we can solve the problem for even and odd subspace separately. We rewrite the Hamiltonian matrices of the even and odd parity subspace by expressing them in terms of the Pauli matrices σ_1, σ_2 and σ_3 and write them in terms of two vectors

$$\begin{aligned} H_{even} &= \varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3 \quad &= \boldsymbol{\varepsilon}_{even}\cdot\boldsymbol{\sigma} \\ H_{odd} &= \varepsilon_1\sigma_1 - \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3 \quad &= \boldsymbol{\varepsilon}_{odd}\cdot\boldsymbol{\sigma} \end{aligned}, \tag{3.58}$$

where $\boldsymbol{\varepsilon}_{even} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$ and $\boldsymbol{\varepsilon}_{odd} = (\varepsilon_1, -\varepsilon_2, \varepsilon_3)^T$ are the vectors of the gate functions for the even and odd subspace respectively and $\boldsymbol{\sigma}$ is the vector of the Pauli matrices (A.1). It is $H_{odd} = H^*_{even}$ and H_{even} resembles the Hamiltonian of a spin-$\frac{1}{2}$ particle in a magnetic field, where $\boldsymbol{\varepsilon}$ corresponds to the magnetic field $\boldsymbol{B}$. For each of the vectors $\boldsymbol{\varepsilon}$ its length $|\boldsymbol{\varepsilon}| \equiv \varepsilon$ corresponds to the positive ones of the eigenvalues $\lambda_\pm = \pm\sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}$ of the submatrix Hamiltonian to which it belongs to. Let us first consider the subspace of even parity. The gate functions are the set of system parameters, which will change in time. We map our process of gating on to a sphere by defining the general vector of gate functions in spherical coordinates as $\boldsymbol{\varepsilon} = (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t))^T = \varepsilon(\sin\theta(t)\cos\phi(t),\ \sin\theta(t)\sin\phi(t),\ \cos\theta(t))^T$, where we map our gate functions on the set of parameters $(\phi(t), \theta(t))$, which are spherical angles depending on time. Here for our choice describing the present system this general gate vector is equal to $\boldsymbol{\varepsilon}_{even}$, while $\boldsymbol{\varepsilon}_{odd} = \varepsilon(\sin\theta(t)\cos\phi(t),\ -\sin\theta(t)\sin\phi(t),\ \cos\theta(t))^T$, but this is not valid for every choice of a basis and the components $\boldsymbol{\varepsilon}_{even}$ can differ in a minus sign too. In spherical coordinates our Hamiltonian matrix reads

$$H_{even} = \begin{pmatrix} \varepsilon\ \cos\theta & \varepsilon\ \sin\theta e^{-i\phi} \\ \varepsilon\ \sin\theta e^{i\phi} & -\varepsilon\ \cos\theta \end{pmatrix}. \tag{3.59}$$

In the same way the Hamiltonian matrix of the odd parity subspace reads

$$H_{odd} = \begin{pmatrix} \varepsilon \cos\theta & \varepsilon \sin\theta e^{i\phi} \\ \varepsilon \sin\theta e^{-i\phi} & -\varepsilon \cos\theta \end{pmatrix}. \tag{3.60}$$

Before we continue we want to give a description how the exchange process works with the time dependent gate functions. First consider the picture when all gates are closed as shown in Figure 3.4. Before we start the exchange process we initialize a state of our system, where the outer boundary Majorana modes of the two rings (ring one and ring four) in the middle are combined to an ordinary fermion by opening slowly enough the first gate till it is $\varepsilon_3(t_0) = 1$ and let the other two gates at $t = t_0$ be zero. Let us call this system state of our process the initial state, which is shown in picture (a) of Figure 3.5. For this our initial Hamiltonian is $H_{eff}(t_0) = i\gamma_7\gamma_3$ and $\boldsymbol{\varepsilon}(t_0) = (0,0,1)^T$.

step 1: Slowly open the gate one described by $\varepsilon_1(t_0) = 0 \rightarrow \varepsilon_1(t_1) = 1$, while the other gates remain in the state of t_0, which gives us $H_{eff}(t_1) = i(\gamma_7\gamma_3 + \gamma_3\gamma_5)$ and $\boldsymbol{\varepsilon}(t_1) = (1,0,1)^T$ at time $t = t_1$. This will extend the Majorana zero mode, which was initially localized at the outer boundary of ring three, over a region going through gate one and three over the boundaries of ring two, three and four (Figure 3.5b).

step 2: Now adiabatically closing the gate three by $\varepsilon_3(t_1) = 1 \rightarrow \varepsilon_3(t_2) = 0$, while again the other ones stay equal, we get $H_{eff}(t_2) = i\gamma_3\gamma_5$ and $\boldsymbol{\varepsilon}(t_2) = (1,0,0)^T$. Now ring one and ring four, each have an outer Majorana zero mode , while the other ones exhibit only the inner ones. One could say, that the initially at the outer boundary of ring three positioned Majorana zero mode γ_5 is localized at the outer boundary of ring four (Figure 3.5c).

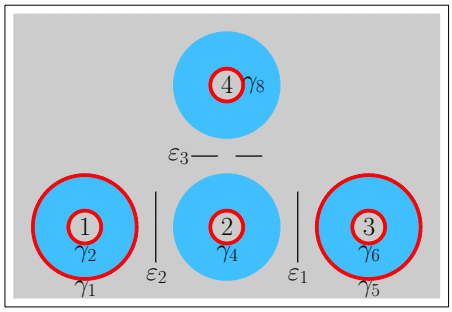
step 3: Let us open gate two by $\varepsilon_2(t_2) = 0 \rightarrow \varepsilon_2(t_3) = 1$, while keeping the other gates unchanged. This leads to the extension of the outer Majorana zero mode of ring one over the boundaries of the three rings one, two and three. And our Hamiltonian reads $H_{eff}(t_3) = i(\gamma_1\gamma_3 + \gamma_3\gamma_5)$ and it is $\boldsymbol{\varepsilon}(t_3) = (1,1,0)^T$ (Figure 3.5d).

step 4: By closing gate one and sending $\varepsilon_1(t_3) = 1 \rightarrow \varepsilon_1(t_4) = 0$ and let the other ones unaffected, we get $H_{eff}(t_4) = i\gamma_1\gamma_3$and $\boldsymbol{\varepsilon}(t_4) = (0,1,0)^T$. Two outer Majorana zero modes localized at the outer boundaries of ring three and four respectively (Figure 3.5e).

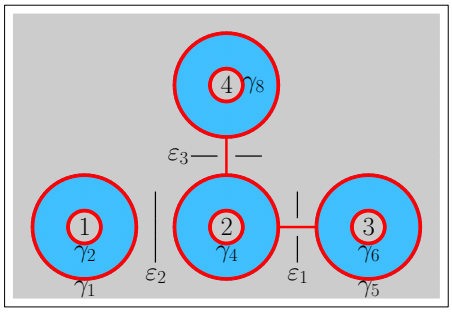
step 5: We open again gate three represented by $\varepsilon_3(t_4) = 0 \rightarrow \varepsilon_3(t_5) = 1$, which results in $H_{eff}(t_5) = i(\gamma_1\gamma_3 + \gamma_7\gamma_3)$ and $\boldsymbol{\varepsilon}(t_5) = (0,1,1)^T$. Now the Majorana zero mode, which at time t_4 was still localized at the outer boundary of ring four, is extended over the outer boundaries of ring one, two and four (Figure 3.5f).

step 6: At last we close again gate two with $\varepsilon_2(t_4) = 1 \rightarrow \varepsilon_2(t_5) = 0$ and come back to the initial Hamiltonian $H_{eff}(t_6) = i\gamma_7\gamma_3 = H_{eff}(t_0)$ and and $\boldsymbol{\varepsilon}(t_5) = (0,0,1)^T = \boldsymbol{\varepsilon}(t_0)$. The system in which we end up now looks like our initial one at time $t = t_0$, but only with γ_1 at the original position of γ_5 and γ_5 at the original position of γ_1 (Figure 3.5g).

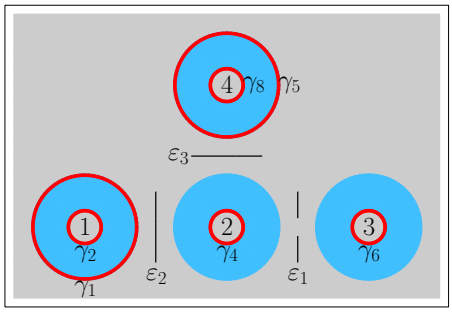
Step 1 up to step 6 exchange the Majorana zero modes γ_1 and γ_5 in their positions and due to the Majorana zero modes are non-distinguishable except in their positions our system state and therefore our Hamiltonian is left unchanged compared to the initial one after the exchange process. Reiterate once more step 1 up to step 6 will bring us back to the original system, so that after twelve steps all Majorana zero modes should be at their initial positions.
Now we want to calculate the Berry phase, which the wave function lying in one of the degenerated ground states of even or odd parity gets by such an exchange process. We choose the subspace of even parity for detailed descriptions, but for the case of odd parity it is all analogue.



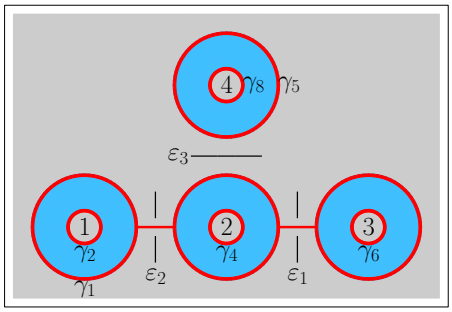
(a) Initial state at time $t = t_0$



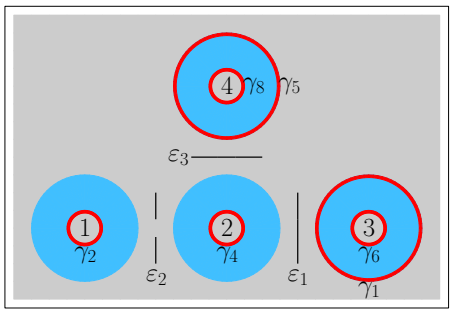
(b) After first step at time $t = t_1$



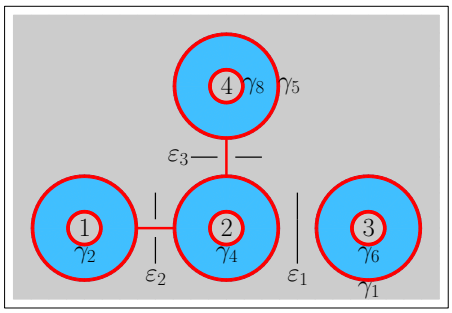
(c) After second step at time $t = t_2$



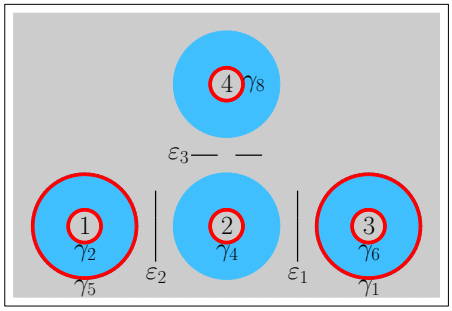
(d) After third step at time $t = t_3$



(e) After fourth step at time $t = t_4$



(f) After fifth step at time $t = t_5$



(g) After sixth step at time $t = t_6$

Figure 3.5.: The *braid protocol* describing the process adiabatically exchanging the Majorana zero modes γ_1 and γ_5 in their positions by gating [22].

First we determined the eigenvectors of our Hamiltonian matrix (3.59) to the eigenvalues $\lambda_\pm = \pm|\boldsymbol{\varepsilon}|$ in spherical coordinates, which can be determined as normalized vectors due to our vector of gate functions $\boldsymbol{\varepsilon}$ and therefore the eigenvalues never become zero (never close all gates at the same time). The eigenvectors are given by

$$|e_+\rangle_{even} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}, \; |e_-\rangle_{even} = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \tag{3.61}$$

and analog we get for the eigenvectors of the odd parity subspace

$$|e_+\rangle_{odd} = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}, \; |e_-\rangle_{odd} = \begin{pmatrix} \sin\frac{\theta}{2}e^{i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}. \tag{3.62}$$

For even and odd parity the lowest eigenvalue is $\lambda_- = -\varepsilon$, so that the ground state can be determined as $|e_-\rangle_{even} \equiv |\text{g.s.}\rangle_{even}$ and $|e_-\rangle_{odd} \equiv |\text{g.s.}\rangle_{odd}$. To determine the Berry phase $e^{i\varphi}$ we calculate the Berry curvature $\mathscr{F}$. For both we use

$$\begin{aligned} \mathscr{A}_\phi = -i\langle\text{g.s.}|\,\partial_\phi\,|\text{g.s.}\rangle, \;\; \mathscr{A}_\theta = -i\langle\text{g.s.}|\,\partial_\theta\,|\text{g.s.}\rangle, \\ \mathscr{F}_{\theta\phi} = \partial_\theta\mathscr{A}_\phi - \partial_\phi\mathscr{A}_\theta, \end{aligned} \tag{3.63}$$

where $\mathscr{A}_\phi$ and $\mathscr{A}_\theta$ is the ϕ-component and the θ-component of the Berry connection respectively and we have used the short hand notation for the partial differentiation $\partial_\alpha \equiv \frac{\partial}{\partial\alpha}$. For the even parity subspace we calculate

$$\mathscr{A}_\phi = -\sin^2\frac{\theta}{2}, \quad \mathscr{A}_\theta = 0, \quad \mathscr{F}_{\theta\phi} = -\frac{1}{2}\sin\theta \tag{3.64}$$

and one gets

$$i\varphi_{even} = -i\int_0^\theta\int_0^\phi \mathrm{d}\theta'\mathrm{d}\phi'\mathscr{F}_{\theta'\phi'} = \frac{i}{2}\int_0^\Omega \mathrm{d}\Omega' = \frac{i}{2}\Omega\,, \tag{3.65}$$

where $\mathrm{d}\Omega' = \mathrm{d}\theta'\mathrm{d}\phi'\sin\theta'$ is the differential of the solid angle. And this Berry phase gives for the even parity case:

$$e^{i\varphi_{even}} = e^{i\frac{\Omega}{2}}. \tag{3.66}$$

For the odd parity subspace we get analogue

$$\begin{aligned} &\mathscr{A}_\phi = \sin^2\frac{\theta}{2}, \quad \mathscr{A}_\theta = 0, \qquad \mathscr{F}_{\theta\phi} = \frac{1}{2}\sin\theta \\ &\text{and} \qquad\qquad i\varphi_{odd} = -\frac{i}{2}\Omega \end{aligned} \tag{3.67}$$

and with this Berry phase we have

$$e^{i\varphi_{odd}} = e^{-i\frac{\Omega}{2}}. \tag{3.68}$$

For both one can see that we have a 4π periodicity in Ω. By the way for even and odd case the Berry curvature fulfils

$$\int \mathcal{F}_{\theta\phi} \mathrm{d}\theta \wedge \mathrm{d}\phi = 2\pi C, \quad C \in \mathbb{Z}\,, \tag{3.69}$$

where C is called the Chern number.

Now we want to determine the solid angle Ω for our special process described above.

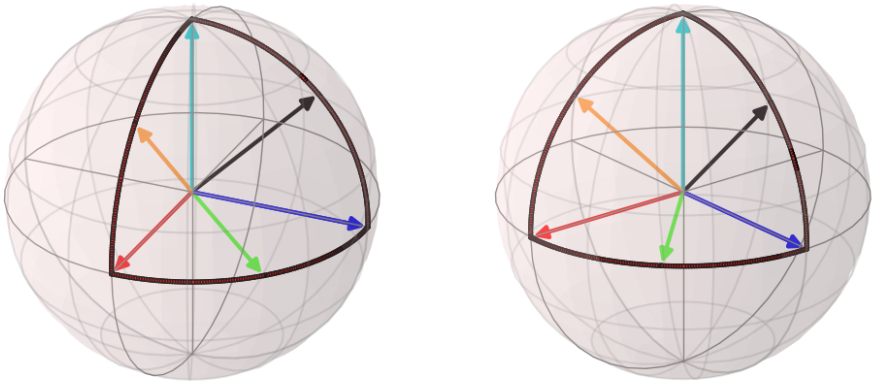


Figure 3.6.: The figure above shows all normalized gate vectors $\hat{\boldsymbol{\varepsilon}}(t)$ of the time evolution steps at times t_0 up to t_6. The shown arrows are $\hat{\boldsymbol{\varepsilon}}(t_0) = (0,0,1)^T$ (cyan), $\hat{\boldsymbol{\varepsilon}}(t_1)$ (orange), $\hat{\boldsymbol{\varepsilon}}(t_2)$ (red), $\hat{\boldsymbol{\varepsilon}}(t_3)$ (green), $\hat{\boldsymbol{\varepsilon}}(t_4)$ (blue), $\hat{\boldsymbol{\varepsilon}}(t_5)$ (black) and again $\hat{\boldsymbol{\varepsilon}}(t_6)$ (cyan). At time t_6 the curve connecting all arrowheads is closed and the solid angle, which is covered by the motion of $\hat{\boldsymbol{\varepsilon}}(t)$ during this time is $\Omega = \frac{\pi}{2}$, an eighth solid angle of the total sphere [28].

Representing the three gates together into the normalized gate vector $\hat{\boldsymbol{\varepsilon}}(t) = \frac{\boldsymbol{\varepsilon}(t)}{|\boldsymbol{\varepsilon}(t)|}$ we can map the time evolution of our system state onto a curve at the surface of a sphere. For the solid angle Ω we are only interested in the time dependence of the direction of the vector $\hat{\boldsymbol{\varepsilon}}(t)$. Because the time evolution of the gate functions is the same for the even and odd parity subspace the solid angle of this time evolution is for both the solid angle covered by the time evolution of the general unit gate vector $\hat{\boldsymbol{\varepsilon}}(t)$. Now it should be clear, that the unit vector $\hat{\boldsymbol{\varepsilon}}(t)$ maps the time dependence of the subsystems states lying always in the ground state onto a closed curve on the surface of an unit sphere. Consider Figure 3.6 where the mapping of the time evolution of the general vector of our gate function onto the time evolution of the two angles $(\phi(t), \theta(t))$ is shown. At time t_0 it is $\hat{\boldsymbol{\varepsilon}}(t_0) = (0,0,1)^T$ (the cyan colored vector), $\hat{\boldsymbol{\varepsilon}}(t_1) = \frac{1}{\sqrt{2}}(1,0,1)^T$ (orange), $\hat{\boldsymbol{\varepsilon}}(t_2) = (1,0,0)^T$ (red), $\hat{\boldsymbol{\varepsilon}}(t_3) = \frac{1}{\sqrt{2}}(1,1,0)^T$ (green), $\hat{\boldsymbol{\varepsilon}}(t_4) = (0,1,0)^T$ (blue), $\hat{\boldsymbol{\varepsilon}}(t_5) = \frac{1}{\sqrt{2}}(0,1,1)^T$(black) and again $\hat{\boldsymbol{\varepsilon}}(t_6) = (0,0,1)^T$ (cyan). While from time t_0 up to time t_6 the angle ϕ and the angle θ each goes from zero to $\frac{\pi}{2}$. So that after step 6 our curve is closed and the unit vector has covered an solid angle of $\Omega = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta' \mathrm{d}\phi' \sin(\theta') = \frac{\pi}{2}$.

So the final result for the Berry phase in even and odd subspace is

$$\begin{aligned} e^{i\varphi_{even}} &= e^{i\frac{\pi}{4}}, \\ e^{i\varphi_{odd}} &= e^{-i\frac{\pi}{4}}. \end{aligned} \tag{3.70}$$

At this point it should be mentioned, that the sign in the Berry phase depends on how we have chosen the n_i in the beginning, but the even and odd case will always have opposite sign. While above we have considered a counter-clockwise exchange of the Majorana modes γ_1 and γ_5, we could also consider a clockwise exchange by opening in the first step adiabatically the gate ε_2 and so on, which will give us the time evolution of $\hat{\boldsymbol{\varepsilon}}(t_0) = (0,0,1)^T$, $\hat{\boldsymbol{\varepsilon}}(t_1) = \frac{1}{\sqrt{2}}(0,1,1)^T$, $\hat{\boldsymbol{\varepsilon}}(t_2) = (0,1,0)^T$, $\hat{\boldsymbol{\varepsilon}}(t_3) = \frac{1}{\sqrt{2}}(1,1,0)^T$, $\hat{\boldsymbol{\varepsilon}}(t_4) = (1,0,0)^T$, $\hat{\boldsymbol{\varepsilon}}(t_5) = \frac{1}{\sqrt{2}}(1,0,1)^T$ and again $\hat{\boldsymbol{\varepsilon}}(t_6) = (0,0,1)^T$. We see that for the case of clockwise exchange the vector $\hat{\boldsymbol{\varepsilon}}(t)$ moves along the same curve but in counter direction, which gives us a solid angle $\Omega = -\frac{\pi}{2}$, being of opposite sign to the solid angle of the previous counter-clockwise process (integration of ϕ goes from $\frac{\pi}{2}$ to zero). So for both the sign of the Berry phase depends on the direction of exchanging the Majorana modes, i.e. if it is clockwise or counter-clockwise. So for a clockwise exchange we will get

$$\begin{aligned} e^{i\varphi_{even}} &= e^{-i\frac{\pi}{4}}, \\ e^{i\varphi_{odd}} &= e^{i\frac{\pi}{4}}. \end{aligned} \tag{3.71}$$

So for this resulting Berry phase of the above described process we have demonstrated non-abelian statistics. In the following we want to give a detailed description of the meaning of this determined Berry phase for our system. Therefore we assume the counter-clockwise exchange and we want to describe the exchange by an unitary matrix acting on the system state $|\Psi\rangle$. For the above choice of pairing the Majorana modes to ordinary fermion operators we can write $|\Psi\rangle = |m_1, m_2, m_3, m_4\rangle$, where $f_i^\dagger f_i |m_1, m_2, m_3, m_4\rangle = m_i |\Psi\rangle$ with $m_i \in 0,1$ is the occupation number for a fermion in state created by $f_i^\dagger$ for $i = 1,2,3,4$. In our given choice it is $f_1 = \frac{1}{2}(\gamma_1 + i\gamma_5)$, so that the operator $\gamma_1\gamma_5$ will commute with all fermion operators except with f_1 for which we can write

$$\begin{aligned} \gamma_1\gamma_5 f_1^\dagger &= -i f_1^\dagger, \\ \gamma_1\gamma_5 f_1 &= i f_1 \ . \end{aligned} \tag{3.72}$$

So that we have $\gamma_1\gamma_5 |\Psi\rangle = -i|\Psi\rangle$ if $n_1 = 1$ (fermion state is occupied) and $\gamma_1\gamma_5 |\Psi\rangle = i|\Psi\rangle$ if $n_1 = 0$ (unoccupied). This is why we can write the unitary matrix describing the time evolution by acting on the state of the system as

$$U = e^{\frac{\pi}{4}\gamma_1\gamma_5} \ , \tag{3.73}$$

for which we get

$$U|\Psi\rangle = e^{\pm i\frac{\pi}{4}}|\Psi\rangle \ , \tag{3.74}$$

where the plus sign results for an even parity space and the minus sign for an odd parity. This operator is known as the generator of the braid group of the braid interchange of the two Majorana fermions γ_1 and γ_5 in the ground state of degeneracy four [3]. One can easily check that it is $(\gamma_1\gamma_5)\gamma_1 = -\gamma_5\gamma_1\gamma_1 = -\gamma_5$ and $(\gamma_1\gamma_5)\gamma_5 = \gamma_1$. Therefore the braid operator $U = e^{\frac{\pi}{4}\gamma_1\gamma_5}$ can be written as the braid matrix

$$U_{15} = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \tag{3.75}$$

in the subspace of the outer boundary Majorana zero modes given by equation (3.56). Also we can write the Majorana operators as matrices in this subspace as

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \gamma_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{3.76}$$

for which one see, that we have $U_{15}U_{15}\gamma_1 = -\gamma_5$ and $U_{15}U_{15}\gamma_5 = \gamma_1$ and $(U_{15})^4\gamma_1 = -\gamma_1$ and $(U_{15})^4\gamma_5 = -\gamma_5$. Or we can also write $U_{15}\gamma_1 U_{15}^\dagger = -\gamma_5$ and $U_{15}\gamma_5 U_{15}^\dagger = \gamma_1$. Here $(U_{15})^2$ represents a motion of γ_1 and γ_5 by 2π, which means that we perform the process described in step 1 up to step 6 twice in the same direction (both times counter-clockwise or clockwise) and we see that this braids the Majorana zero modes γ_1 and γ_5. Performing this process four times leads to the mapping $\gamma_i \to -\gamma_i$ of the two Majorana zero modes. At this point it is important to remember, that we have chosen our basis arbitrarily and that due to the invariance of a basis transformation into an other ground state basis the result for the Berry phase above is valid for all possible chosen pairings of the eight Majorana zero modes laying in the ground state degeneracy. Above we have chosen our basis in such way that our calculation is simpler and clearer, but later we are more interested in braiding two Majorana zero modes which belong to different fermion states like the exchange of γ_1 with γ_5 (if one is interested, other examples of basis choice are given in the appendix (C)). For the following chosen basis

$$\begin{aligned} f_1 &= \frac{1}{2}(\gamma_1 + i\gamma_3), \quad f_2 &&= \frac{1}{2}(\gamma_7 + i\gamma_5), \\ f_3 &= \frac{1}{2}(\gamma_2 + i\gamma_6), \quad f_4 &&= \frac{1}{2}(\gamma_4 + i\gamma_8), \end{aligned} \tag{3.77}$$

or even in the basis for

$$\begin{aligned} f_1 &= \frac{1}{2}(\gamma_1 + i\gamma_2), \quad f_2 &= \frac{1}{2}(\gamma_6 + i\gamma_5), \\ f_3 &= \frac{1}{2}(\gamma_3 + i\gamma_4), \quad f_4 &= \frac{1}{2}(\gamma_8 + i\gamma_7), \end{aligned} \tag{3.78}$$

for which each pairing of Majoranas to one fermion corresponds to one of the rings. If we consider one case, in which γ_1 and γ_5 belong to different fermions, then we can always find

$$\gamma_1\gamma_5 = \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \pm i & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp i \\ 0 & 0 & \mp i & 0 \end{pmatrix} = \begin{pmatrix} \pm\sigma_1 & 0 \\ 0 & \mp\sigma_1 \end{pmatrix}, \tag{3.79}$$

so that for this case in general the braid matrix in the even or odd parity subspace is given by $U_{15} = e^{\pm\frac{\pi}{4}\sigma_1}$, where σ_1 is the Pauli matrix (A.1). We can see, that if we braid them twice in the same direction and end up in the original state, where γ_1 is localized at the outer boundary of ring one and γ_5 is localized at the outer boundary of ring three, the phase of our wave function changes in total by $\frac{\pi}{2}$, which gives us an pre-factor of i, so that repeating step 1 up to step 6 in total four times in the same direction, we get a pre-factor of -1 for both, even and odd parity subspace. To get back the originally wave function (pre-factor of 1) we have to do it eighth times. So we have shown, that the exchange of the two outer Majorana zero modes by the process described above corresponds to the braid interchange or what is known as braiding non-abelian statistics [3] of two Majorana zero modes in the four times degenerated ground state. So braiding Majorana zero modes in the the given architecture will be possible. In the previous part of this chapter we have discussed, that our Majorana zero modes have to be stable against local perturbations as long as these are sufficiently small enough. So this is why braiding these Majorana zero modes has to be stable for the same local perturbations and as long as the speed of braiding them is adiabatically, so that the Majorana zero modes are prevented from excitations to energetically higher states. So we have shown that an adiabatically exchange of two outer Majorana zero modes gives a braiding of them.

3.3. Fusion and read out method

For the read out of the state of one present qubit it is proposed to use single electron transistors to make use of the proposed *parity-to-charge conversion* [6]. Therefore each of the two outer ones of the superconducting islands should be capacitively coupled to one single electron transistor. In fact we assume that the given quantum bit system is useable in a Coulomb blockade regime, where each superconducting ring island represents a possible Coulomb island. These assumptions can be justified, due to the temperature has to be very low anyway and the superconducting islands are of mesoscopic size (a few micro meter). The single electron transistor consists basically of metallic quantum dot or the so called single electron transistor island in the Coulomb blockade regime. This island is on the one hand through a metallic wire capacitively coupled to the superconducting ring island and on the other hand it is coupled to two leads by tunnelling junctions and is tunable by a capacitively coupled gate voltage V_g (for more details to a single electron transistor we refer to the introduction (1.2.1.3)). In addition each superconducting island is also coupled to an extra side gate $V_{i,g}^{\mathrm{sc}}$, which should enable to control the Coulomb or *charging* energy E_C of each superconducting island. This should ensure the controlling of the fermion parity in this island and effectively setting the boundary modes occupied or unoccupied. So the extra side gate $V_{i,g}^{\mathrm{sc}}$ of the ith ring enables a fine-tuning into the topological ground state degeneracy point away from this point, so that we are able to create controlled a pair of Majorana zero modes out of the vacuum at the boundaries of each ring. In the topological degeneracy point the ground states of even and odd parity becomes degenerated, i.e. adding or subtracting an extra electron in the ground state does not change the energy of the system. In this topological degeneracy one can describe the systems state by pairing the Majorana zero modes to ordinary fermion operators f_i and use the fermion occupation number operators $\hat{n}_i = f_i^\dagger f_i$ which can be either zero or one. This is the way how we have described the system state in the previous part. While in the topological degeneracy the charge on the ring island can not be a suitable quantity describing the system state, away from this point the energy for even and odd states will split up in different energies and the charge becomes a quantum number which describes the system state by the charge state $|Q_i\rangle$ described by the charge Q_i of each island [6]. Of cause we assume that the hole of superconducting ring island is further threaded by exact one flux quantum Φ_0 and that the chemical potential in adjust such that high localized Majorana zero modes occur in the topological degeneracy point at the inner and outer boundaries of the superconducting rings.

3.3.1. Fusion

To fuse two Majorana states into an ordinary fermion state, that either can be occupied or unoccupied, means in general to find a way to split up the ground state degeneracy, so that there is an energy shift between the occupied and unoccupied fermionic state. To fuse two Majorana

edge states of the outer boundaries of neighbouring rings, a gate in between them is used to hybridize their wave functions to one Dirac fermion. One can describe this into an abstract way saying fusion of two anyons σ of the kind of the given Majorana zero modes is described by $\sigma \times \sigma = \mathrm{I} + \Psi$, where I represents the vacuum channel and Ψ the particle channel. This means fusion of two Majorana fermions only has the two possibilities to end up either in an unoccupied fermion state (the vacuum I) or in an occupied fermion state Ψ. In general one could detect fusion in a system consisting of just two neighboured rings. Therefore one would first initialize both rings in a well defined charge state, so that the system state is given by the charges Q_1 and Q_2 of both rings as $|Q_1, Q_2\rangle$. For this one has to use the side gates of each ring. Let us call this the initial system state. From this state we are able to create Majorana zero modes out of the vacuum in two different ways. Close the gate between the two rings and hold it closed while generating a pair of Majorana zero modes at the boundaries of each ring, which will composed to the fermion states $f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ for the one ring and $f_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$ for the other ring. So the state of the system is described by the basis states $|00\rangle_f$, $|10\rangle_f = f_1^\dagger |00\rangle_f$, $|01\rangle_f = f_2^\dagger |00\rangle_f$ and $|11\rangle_f = f_1^\dagger f_2^\dagger |00\rangle_f$. Now, if we again shift the charging energy away from the degeneracy point to fuse the inner and the outer modes at each ring with each other, we have to obtain the vacuum fusion channel, because we have fused the same pair of Majoranas into the vacuum, which we have had created out of the vacuum without any manipulations done in the time between creation and fusion. This represents a trivial fusion process, which could be used to control if fusion would work correctly. By charge sensing via a single electron transistor one would detect the same signal for the fusion outcome as for the initial charge state which are both $|Q_1, Q_2\rangle$. Further due to in the charging energy regime the to rings can be considered as two decoupled Coulomb-islands, we can assume that we are able using the single electron transistor and the side gates at the rings to initialize the charge state, such that the single electron transport through the single electron transistor will be zero. So for this trivial fusion, the current through the single electron transistor remains zero. In the other case we let the gate between the two rings open and fine tune first the system into the degenerated parity ground state regime, so that we create out of the vacuum two Majorana zero modes at the inner boundaries of each ring pairing to the fermion state $f_1 = \frac{1}{2}(\gamma_2 + i\gamma_4)$. Then we adiabatically close the gate between the two rings and create two more Majorana zero modes at the outer boundaries of the rings out of the vacuum, such that they combine to a second fermion $f_2 = \frac{1}{2}(\gamma_3 + i\gamma_1)$. Again we read out the charging states of each ring separately, in which we would read out a state of an other pairing of Majoranas as we have had created out of the vacuum, namely $c_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $c_2 = \frac{1}{2}(\gamma_3 + i\gamma_4)$, for which we can transform the basis given by the f_i into the new basis given by the c_i in the even parity space by

$$\begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix} = F \begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} . \tag{3.80}$$

This second possibility of fusion with two rings would end up into a probabilistic charge read out, in which we get with equal probability an occupied and an unoccupied state, i.e. sensing charge or no charge. This two fusion possibilities in a setup consisting of two superconducting rings could be seen as the analog *fusion protocol* to this, which was given for the nanowirebased fusion and braiding processes by David Aasen *et al.* [6].

Here we want to read out the information stored in the quantum bit by previous braiding. Concrete this means, that we assume to create our Majorana zero mode pairs out of the vacuum at the boundaries of each ring as described in the previews chapter. To do this, we could adjust at each ring first the charging energy, such that we have no extra charge on it as we have described above for the two ring islands, and then we want to shift them into the degeneracy point, for which we get the Majorana zero modes at the boundaries of each ring. Then we carry out the above described braiding process and braid the Majoranas γ_1 and γ_5 once. This yields us a phase manipulated system state. After this braiding we want to read out the information about the manipulated system state (we expect with equal probability an occupied or an unoccupied fermion state for each of the two rings). So we are interested in the information stored in each of the two rings separately after manipulation by braiding. Due to the information, which we want to read out, is stored non-local in the localized spatial separated Majorana zero modes, we first have to fuse them to an original fermion. But we have to ensure that the only operation, which has changed the fusion outcome from the trivial one, was the braiding process. If we would create the Majorana zero modes out of the vacuum and fuse them in the same way without any operation in the time between creation and fusion, we end up in the trivial fusion outcome (the vacuum) as described above for a system of two rings. So this is, why fusion of the outer modes does not work in this case, because it creates itself just a probabilistic fusion outcome (as described for the system of two rings above). But here we can not hybridize our Majorana modes over the superconducting region by gating, used for braiding outer modes.While we can extend the outer Majorana modes in the magnetic region by shifting the chemical potential near to the value of the strength of the magnetic field at the surface, we have no inherent adjustable system parameter in the superconducting region, because the superconducting pairing gap is fixed as a material constant, and the chemical potential in this region only has the influence on the oscillation frequency, but not on the penetration depth into the superconducting region. So here, for the read out process it is necessary to fuse only the inner and outer Majorana bound states of ring one and ring three by *parity-to-charge conversion* [6]. Therefore consider the initial state described in the previous part and the state after one braiding process of the system shown in Figure 3.5a and in Figure 3.5g. There are existing two Majorana zero modes localized at the outer boundaries of ring one and three, and there are four modes each localized at the inner boundary in each ring, while gate ε_3 is open, so that no Majorana zero modes are localized at

the outer boundaries of the two middle rings (ring two and ring four). This is the situation in which our system is given, when we start the read out process. Let be the system at time t_0 in the chosen ground state basis in which it is for example either $f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $f_2 = \frac{1}{2}(\gamma_5 + i\gamma_6)$ or $f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $f_2 = \frac{1}{2}(\gamma_6 + i\gamma_5)$ so that γ_1 and γ_5 belong to different fermions, because we have had created them out of the vacuum at each of the rings independently from each other. We assume that the initial charge state before we fine tune the system into the degenerated ground state manifold is given by $|Q_1, Q_3\rangle$. Imagine that the present system is in the state after one braid process as shown in Figure 3.5g (at time t_6). Then we can describe the system in the new basis for the pairs of Majorana fermions at the inner and outer boundaries of ring one and three. Considering the first case of chosen initial basis in terms of fermion creation and annihilation operators, we can describe this new pairing as

$$c_1 = \frac{1}{2}(\gamma_5 + i\gamma_2)\ ,\ c_2 = \frac{1}{2}(\gamma_1 + i\gamma_6)\ , \tag{3.81}$$

so that $n_1 = c_1^\dagger c_1$ and $n_2 = c_2^\dagger c_2$ will give us the occupation number of ring one and ring two respectively. In this basis we have the new basis states $|00\rangle_c$ and $|11\rangle_c = c_1^\dagger c_2^\dagger |00\rangle_c$ for the even and $|01\rangle_c = c_2^\dagger |00\rangle_c$ and $|10\rangle_c = c_1^\dagger |00\rangle_c$ for the odd parity subspace. We label our original basis, which we used to describe the system before braiding as $|n_1, n_2\rangle_f$. Then we can determine that our new basis states can be written terms of the old basis states as

$$|00\rangle_c = \frac{1}{\sqrt{2}}(|00\rangle_f - |11\rangle_f) \text{ and } |11\rangle_c = \frac{1}{\sqrt{2}}(|00\rangle_f + |11\rangle_f) \tag{3.82}$$

and that is what interests us in the read out of the information stored in the pairing of inner and outer modes of each of the outer rings. We can describe the basis change by a unitary operation

$$\begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix} = F \begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix}\ , \tag{3.83}$$

where the matrix F is known as the non trivial fusion matrix[5] for Ising anyons [2]. When we use for this fusion outcome (after braiding once) the *parity-to-charge* conversion analogue to [6] we expect a new charge state given by $\frac{1}{\sqrt{2}}(|Q_1, Q_3\rangle \pm e^{i\alpha}|Q_1 - 1, Q_3 + 1\rangle)$, where the non universal phase factor $e^{i\alpha}$ could be produced by the charge noise as in [6] explained. This new charge state leads to a a probabilistically read out for which we assume to measure with equal probability a single electron transport can occur or not occur in the single electron transistor. The $\pm$ sign depends on the chosen basis state by pairing the Majorana zero modes suitable in its context. With the above chosen basis we get the minus sign. Further we have assumed that during the

[5] A trivial fusion matrix would be a diagonal matrix, which for example is given if we will braid two Majoranas which belong to the same fermion state and fuse them [2].

total braid process no charges can exit or enter the system, why the total system charge has to be conserved. And we assume that no charge can tunnel from ring one or from ring three onto one of the other ring islands. So, as a fusion outcome, this kind of charge state is controlled by adjusting the ring island side gates in a suitable way.
For the second case of mentioned initial basis we get analog as the new basis after braiding

$$c_1 = \frac{1}{2}(\gamma_5 + i\gamma_2)\ ,\ c_2 = \frac{1}{2}(\gamma_6 + i\gamma_1) \tag{3.84}$$

again, so that $n_1 = c_1^\dagger c_1$ and $n_2 = c_2^\dagger c_2$ will give us the occupation number of ring one and ring two respectively. Then we can determine that our new basis states can be written terms of the old basis states as

$$|00\rangle_c = \frac{1}{\sqrt{2}}(|00\rangle_f - i|11\rangle_f) \text{ and } |11\rangle_c = \frac{1}{\sqrt{2}}(|11\rangle_f - i|00\rangle_f). \tag{3.85}$$

We can write the unitary fusion matrix in this case as

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_0 - i\sigma_1), \tag{3.86}$$

where σ_0 is the 2×2 unit matrix and σ_2 is the second Pauli matrix. So finally we can summarize the most important facts for a controlled fusion. First, with extra applied gate voltages at each ring it should be possible to initialize deterministically and reproducibly the system in one of the degenerated parity ground state manifolds. Second, with this gate voltages at each ring we should be able to control smoothly enough the charging energy at each island, such that we lift the degeneracy of the ground state and get the charge on a superconducting ring island as the new quantity characterizing the system state, for which we speak about the charge state $|Q_i\rangle$ of an island. And in the regime of a degenerated ground state manifold we have states described by the occupation numbers n_i. Here we will assume the idealized case, where we have either an charge state of exact one ($n_i = 1$) or no electron charge ($n_i = 0$)[6]. This controlled charging of the superconducting ring islands should be possible due to the small size of only a few micrometer of each of the rings, which gives us (at the prevailing very low temperatures ($\sim T < 1K$)) a Coulomb-blockade regime, for which such a controlled charging of a so called Coulomb island is possible [7]. After we have fused the Majorana fermions to an original fermion we can read out the information by using a single electron transistor, which will detect the presence of either no or one single electron on the superconducting ring. To get a mathematical description of the read out we consider the superconducting ring as an charged quantum island, which has the charge ne, where n is either $n = 0$ (no electron) or $n = 1$ (one electron exist on the island).

[6]By shifting the zero charge state artificially this should be possible always.

We want to leave it in this demonstrated parallelism to the fusion protocol for a system of nano wires [6] and in the following we concentrate on the more detailed view on the read out method by the use of single electron transistors. How fusion looks like in detail has to be discussed in a different frame.

3.3.2. Read out

As mentioned above, for the read out we want to use single electron transistors each coupled capacitively to one of the outer rings, which are interesting for charge sensing. Here we are interested in the charge state of ring one and ring three after we have exchanged their outer Majorana zero modes.

3.3.2.1. Read out circuits

In Figure 3.7 it is shown one of the two Coulomb blockade circuits used for the read out, $i = 1,2$ two read out circuits. A single electron transistor consists of one Coulomb island, which is capacitively coupled to a gate voltage $V_{i,g}$ controlling the charge occupation of this island and which is connected via tunnel junctions to two applied voltages, which we call the left $V_L^{(i)}$ and the right $V_R^{(i)}$ voltage. In the following we call this Coulomb island of the single electron transistor the SET-island. This SET-island is the component of the single electron transistor which is capacitively coupled to one superconducting island of the qubit.
We can adjust the gate voltage $V_{0,g}^{(i)}$ of the SET-island in such a way, that the island is occupied by exact one charged particle $n_{i,0} = 1$ or with no particle $n_{i,0} = 0$ [7]. We call $\left|n_{i,0}\right\rangle$ the charge state of the SET-island.

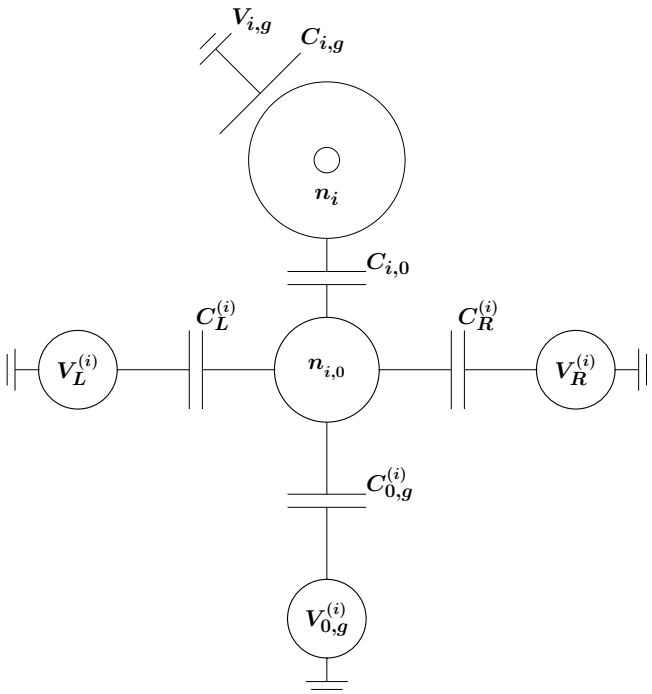


Figure 3.7.: Here it is shown schematically one of the circuits for the read out. The index i labels the respective circuit belonging to one of the two superconducting ring islands, which we want to read out with a single electron transistor. In our case we have such a circuit for read out either for ring one (n_1) and for ring three (n_2). Each of the two required single electron transistors ($n_{i,0}$) couple capacitively with $C_{i,0}$ to the respective ring island n_i. The charging energy of the superconducting ring island is controlled by the capacitively ($C_{i,g}$) coupled side gate ($V_{i,g}$). The gate voltage of the island of the single electron transistor is given by $V_{0,g}^{(i)}$ and couples via $C_{0,g}^{(i)}$. The left and right transport electrodes with applied voltages $V_L^{(i)}$ and $V_R^{(i)}$ of the single electron transistor are coupled with $C_L^{(i)}$ and $C_R^{(i)}$. The single electron transport in each of the single electron transistors appears by single electron tunnelling from or to the left or right onto or from the island of the single electron transistor [22].

Remember how we derived the effective Hamiltonian (1.22) for a general Coulomb blockade circuit determining the charging energy (1.23) of the Coulomb islands in this circuit. Here for the Coulomb circuit shown in Figure 3.7 using (1.20) we have the capacitance matrix with its

four elements

$$\begin{aligned}\hat{C}_i &= \begin{pmatrix} C_{00}^{(i)} & C_{01}^{(i)} \\ C_{10}^{(i)} & C_{11}^{(i)} \end{pmatrix} , \\ C_{00}^{(i)} &= C_{i,0} + C_{i,g} , \\ C_{11}^{(i)} &= C_{i,0} + C_L^{(i)} + C_R^{(i)} + C_{0,g}^{(i)} , \\ C_{01}^{(i)} &= C_{10}^{(i)} = -C_{i,0} ,\end{aligned} \tag{3.87}$$

and its inverse is given by

$$\begin{aligned}\hat{C}_i^{-1} &= \frac{1}{\det \hat{C}^{(i)}} \begin{pmatrix} C_{11}^{(i)} & -C_{01}^{(i)} \\ -C_{01}^{(i)} & C_{00}^{(i)} \end{pmatrix} , \\ \det \hat{C}_i &= C_{00}^{(i)} C_{11}^{(i)} - (C_{01}^{(i)})^2 ,\end{aligned} \tag{3.88}$$

where the index 0 labels the SET-island and the index i the ring island i. Plug this into (1.23) we get directly the charging energy matrix for our given circuit. For the charges induced by the voltages applied at each of the both islands we have for the SET-island $q_0^{(i)}$ and for the ring island of the circuit i q_i with

$$\begin{aligned} q_0^{(i)} &= -(C_L^{(i)} V_L^{(i)} + C_R^{(i)} V_R^{(i)} + C_{0,g}^{(i)} V_{0,g}^{(i)}), \\ q_i &= -C_{i,g} V_{i,g} . \end{aligned} \tag{3.89}$$

For the time of the pure read out process we want to assume, that $q_i = 0$ and we set $n_{0,g}^{(i)} = q_0^{(i)}/e$ which describes the number of shifted charges of the SET-island and is determined by the gate voltage $V_{0,g}^{(i)}$ of the single electron transistor. Further the effective Hamiltonian (1.22) for this circuit reads

$$H_{i,S} = E_{00}^{(C,i)} (n_{i,0} - n_{0,g}^{(i)})^2 + E_{11}^{(C,i)} n_i^2 + 2E_{10}^{(C,i)} (n_{i,0} - n_{0,g}^{(i)}) n_i , \tag{3.90}$$

where we have omitted the term, which does not depend on any island charge numbers, and we have used that the off-diagonal elements of the charging matrix are equal. This effective Hamiltonian describes the pure system out of the both Coulomb islands in the circuit and is called the system Hamiltonian which will be the main interesting part, because it includes both island charge states, which are describable by the quantum numbers $n_{i,0}$ and n_i. Later we will

use these charge numbers as charge number operators written in operator notation

$$\begin{aligned} \hat{n}_i &= c_i^\dagger c_i \ , \\ \hat{n}_{i,0} &= -i\partial_{\varphi_i} \ , \end{aligned} \tag{3.91}$$

where the operators $c_i^\dagger$ and c_i create and annihilate an electron on the ring island and the charge number operator of the SET-island is written in a phase representation with $\varphi_i = e\int_{-\infty}^{t} \mathrm{d}t' V_{i,0}(t')$, where $V_{i,0}$ is the voltage of the SET-island. The phase operator $\hat{\varphi}_i$ obeys the commutation relation $[\hat{\varphi}_i, \partial_{\varphi_i}] = i$, so that the eigenvalues of $\hat{n}_{0,g}^{(i)}$ are quantized to integers. So the phase operator $e^{i\varphi_i}$ corresponds to a momentum shift operator with $e^{\pm i\varphi_i} n_{i,0} e^{\mp i\varphi_i} = n_{i,0} \pm 1$ [9]. In the following we will drop the index i for the circuit, because in the following all will be equivalent for both circuits and we will speak in general about one of these circuits.

Now we want to calculate the energy differences between the different charge number states of the the SET-island and the ring island. Therefore we first assume that both charge numbers are restricted to be either zero or one, i.e. $n_0, n \in \{0,1\}$. Further we plug in the results for the charging energy matrix elements and we first do not use any assumptions for the induced charges q_0 and q. As the energy depending on the charge numbers n_0 and n we get

$$E_{el}(n,n_0) = \frac{e^2 C_{01}}{2(C_{11}C_{00} - C_{01}^2)} \left[(n - q/e)^2 \frac{C_{00}}{C_{01}} + (n_0 - q_0/e)^2 \frac{C_{11}}{C_{01}} + 2(n - q/e)(n_0 - q_0/e) \right], \tag{3.92}$$

where - for better readability - we have not plugged in the expressions for the elements of the capacitance matrix (3.87). For fixed values of the charge number of the ring island n we can calculate the energy differences by $\Delta E_{el}^n(n_0) = E_{el}^n(n_0 \pm 1) - E_{el}^n(n_0)$ and with the restriction, that n_0 has to be either zero or one, we get

<u>for $n = 0$:</u>

$$\begin{aligned} \Delta E^0(0) &= \frac{e^2 C_{01}}{2(C_{11}C_{00} - C_{01}^2)} \left[(1 - 2q_0/e)\frac{C_{11}}{C_{01}} - 2q/e \right], \\ \Delta E^0(1) &= -\Delta E^0(0), \end{aligned} \tag{3.93}$$

<u>for $n = 1$:</u>

$$\begin{aligned} \Delta E^1(0) &= \frac{e^2 C_{01}}{2(C_{11}C_{00} - C_{01}^2)} \left[(1 - 2q_0/e)\frac{C_{11}}{C_{01}} + 2(1 - q/e) \right], \\ \Delta E^1(1) &= -\Delta E^1(0). \end{aligned} \tag{3.94}$$

In the following we want to determine the time evolution of the charge states of the SET-islands $n_{i,0}$ depending on the charge states of the ring islands n_i. This time evolution will tell us how we

can use a single electron transistor to read out the charge state of a given ring island. Here it is necessary to consider the quantum mechanical analogue of the master equation solving the full problem, because we have a circuit in which superpositions of charge states are possible in contrast to a classical consideration, where a state is either present with a probability of one or zero and not something in between. Nevertheless we start with a brief introduction into the description of classical master equation for a circuit with pure classical states, where we have no superpositions of charge states. Therefore we will determine the classical rates for changing from one into the other state. Then we consider a quantum mechanical analogue of the master equation to determine the rate matrix and verify the classical limit correspondence.

3.3.2.2. Classical description and the master equation

First we want to describe the time evolution of the SET-island charge state in a classical approach. Therefore we consider the pure electrostatic energy difference and, how we described in the introduction, we take into account the applied voltages of the electrodes, as far as $V_L \neq V_R$, and write the energy differences for extraction (label *ex*) and addition (label *ad*) of an electron from or to the island as

$$\begin{aligned} \Delta E^n_{ad,R(L)}(n_0) &= \Delta E^n_{el}(n_0) - eV_{R(L)} \ , \\ \Delta E^n_{ex,R(L)}(n_0) &= \Delta E^n_{el}(n_0) + eV_{R(L)} \ . \end{aligned} \tag{3.95}$$

In the following we skip the index n for the fixed ring island charge state. The master equation is described by

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{n_0}(t) = -\sum_{n_0'} \Gamma^{out}_{n_0 \to n_0'} p_{n_0}(t) + \sum_{n_0'} \Gamma^{in}_{n_0' \to n_0} p_{n_0'}(t) \ , \tag{3.96}$$

where $\Gamma^{in(out)}_{n_0 \to n_0'}$ is the in-rate (out-rate) at charge state n_0 to charge state n_0'. *Incoming* means that the charge state changes from an other charge state to the state of consideration and *out going* means, that the state of consideration changes into an other state. Here we have assumed that only $n_0 = 0, 1$ are valid charge numbers, which means that all probabilities $p_{n_0} = 0$ for $n_0 \neq 0$ and $n_0 \neq 1$. So we can write two equations for the time evolution of the state probability

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} p_0(t) &= -\Gamma^{out}_{0\to 1} p_0(t) + \Gamma^{in}_{1\to 0} p_1(t) \ , \\ \frac{\mathrm{d}}{\mathrm{d}t} p_1(t) &= -\Gamma^{out}_{1\to 0} p_1(t) + \Gamma^{in}_{1\to 0} p_0(t) \ , \end{aligned} \tag{3.97}$$

for which we can see that we can write the system of differential equation in matrix representation by

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{p}(t) = \begin{pmatrix} -\Gamma^{out}_{0\to 1} & \Gamma^{in}_{1\to 0} \\ \Gamma^{in}_{1\to 0} & -\Gamma^{out}_{1\to 0} \end{pmatrix} \boldsymbol{p}(t) = \boldsymbol{\Gamma} \boldsymbol{p}(t) \ , \tag{3.98}$$

where $\mathbf{\Gamma}$ is called the rate matrix. So for a detailed description of the single-electron transport in the Coulomb circuit we have to determine the elements of the rate matrix.
First let us describe the rate from the left (right) lead to the island as $\vec{\Gamma}_{L(R)}$ and from the left (right) to the island as $\overleftarrow{\Gamma}_{L(R)}$ Y. V. Nazarov shows [7] how one can find argumentatively from a classical approximation, that the rate at finite temperature can be calculated by

$$\overleftrightarrow{\Gamma}_{L(R)}(n_0) = -\frac{\sigma_{L(R)}}{e^2}\frac{\Delta E(n_0)}{\exp\left(-\Delta E(n_0)/k_B T\right)-1}\ , \tag{3.99}$$

where $\Delta E(n_0)$ is one of the corresponding energy differences $\Delta E^n_{ad/ex,R(L)}(n_0)$ given by (3.95). In the limit of zero temperature $T \to 0$ the rate is given by

$$\lim_{T\to 0}\overleftrightarrow{\Gamma}_{L(R)}(n_0) = -\frac{\sigma_{L(R)}}{e^2}\Delta E(n_0)\Theta(-\Delta E(n_0))\ . \tag{3.100}$$

So that the rates are only non vanishing for positive energy differences $\Delta E(n_0) < 0$ or more precise, the single-electron transfer from or to the island of one of the two leads is blocked if the corresponding energy difference is $\Delta E^n_{ad/ex,R(L)}(n_0) > 0$.

3.3.2.3. Quantum mechanical solution of the master equation

Our goal is to solve the equations of motion for our system. We want to describe the time evolution of the state of the given system during the read out process. Instead of the direct time evolution of the wave function of the system it is practicable to determine the time evolution of the density matrix of the given quantum system. Because the coupling of each SET-island to the 'bath' electrons is assumed to be weak we can use a weak-coupling approximation to determine the time evolution of the systems density matrix. Our starting point in the following is based on the references about special tunnelling in Coulomb blockade regimes [29, 30]. First let us consider the effective Hamiltonian describing our system, which essentially consists of two Coulomb blockade circuits of the kind described above, i.e. two totally decoupled and separated Coulomb superconducting ring islands with each coupled capacitively to its own single electron transistor. Let use call the circuit with the superconducting ring one *circuit* $i = 1$ and that with ring three *circuit* $i = 2$. The Hamiltonian for each of these subsystems can be written as

$$H_i = H_{i,S} + H_{i,B} + H_{i,I}\ , \tag{3.101}$$

where i labels the circuits and S labels the system Hamiltonian, which only consists of the quantized charges of the ring and the SET-island. The label B represents the bath Hamiltonian which consists of the Hamiltonians of the both leads of the transport electrodes at the SET-

island $H_{i,L}$ and $H_{i,R}$ and the electrons in the SET-island itself $H_{\text{i,island}}$, so that it is

$$\begin{aligned} H_{i,B} &= H_{i,L} + H_{i,R} + H_{\text{i,island}} \ , \\ H_{i,L} &= \sum_k c^\dagger_{i,L,k} c_{i,L,k} \varepsilon_{i,L,k} + \mu_{i,L} \sum_k c^\dagger_{i,L,k} c_{i,L,k} \ , \\ H_{i,R} &= \sum_k c^\dagger_{i,R,k} c_{i,R,k} \varepsilon_{i,R,k} + \mu_{i,R} \sum_k c^\dagger_{i,R,k} c_{i,R,k} \ , \\ H_{\text{i,island}} &= \sum_k d^\dagger_{i,k} d_{i,k} \varepsilon_{i,k} \ , \end{aligned} \tag{3.102}$$

where $\mu_{i,L} = eV_L^{(i)}$ and $\mu_{i,R} = eV_R^{(i)}$ are the chemical potentials of the left and right lead of each subsystem respectively and the indices k encapsulate both electron degrees of freedom the spin σ and momentum $\boldsymbol{k}$ with $k = (\sigma, \boldsymbol{k})$. The operators $c^\dagger_{i,R(L),k}$ ($c_{i,R(L),k}$) create (annihilate) electrons with the degrees of freedom k in the right (left) lead of the transport electrodes in the single electron transistor and the operators $d^\dagger_{i,k}$ ($d_{i,k}$) create (annihilate) electrons with k in the metal of the SET-island. All this together forms a 'bath' of electrons in which the system states of interest are embedded.

The effective system Hamiltonian $H_{i,S}$ is described by (3.90) and under the use of the charge number operators described above in (3.91) it can be written as

$$H_{i,S} = E_{00}^{(C,i)} (\hat{n}_{i,0} - n_{0,g}^{(i)})^2 + E_{11}^{(C,i)} (c_i^\dagger c_i)^2 + 2E_{10}^{(C,i)} (\hat{n}_{i,0} - n_{0,g}^{(i)}) c_i^\dagger c_i \ . \tag{3.103}$$

In the following we will use the shift operator $e^{-i\varphi_i}$ which corresponds to the phase operator of the charge number operator $n_{i,0}$ (we refer to the description of equation (3.91)). And the interaction Hamiltonian $H_{i,I}$ represents the interaction of the SET-island with the coupled left and right leads and can be written as

$$H_{i,I} = \sum_{\sigma=L,R} \left[\sum_{q,k} \tilde{t}^{i,\sigma}_{q,k} d^\dagger_{i,q} c_{i,\sigma,k} e^{i\varphi_i} + \sum_{q,k} (\tilde{t}^{i,\sigma}_{k,q})^* c^\dagger_{i,\sigma,k} d_{i,q} e^{-i\varphi_i} \right] , \tag{3.104}$$

where $\tilde{t}^{i,\sigma}_{q,k}$ are the tunnel parameters between the left (right) ($\sigma = L(R)$) lead and the SET-island. So the total effective Hamiltonian, which consists of two of such independent Coulomb circuits as shown in Figure 3.7 can be written as the sum of the Hamiltonians for each of the circuits

$$H = \sum_{i=1}^{2} H_i \ . \tag{3.105}$$

Because we assume that the two systems are complete independent from each other, first we consider in the following only one of the two circuit systems and skip the index i.

If one considers the above given description for the total effective Hamiltonian, it is noticeable

that the dependence of the ring island charge state, in which we are interested to measure, is only given in the system Hamiltonian $H_{i,S}$. First we determine the Bloch-equation for the given Hamiltonian H_i (3.101). Because we assume that the system state at time $t = 0$, where the read out process starts is well defined, we assume a weak-interacting SET-island with the surrounding 'bath', we can rewrite the density operator in the *interaction representation*

$$\rho(t) = e^{-iH_I t}\rho'(0)e^{iH_I t} \tag{3.106}$$

and we get the reduced equation of motion (1.35) with

$$H_I(t) = e^{iH_0 t}H_I e^{-iH_0 t} \ . \tag{3.107}$$

Under the assumption that the initial state at time $t = 0$ is given, such that the interaction does not generate any first-order dynamics, such that one has $tr_B([H_I(t), \rho(0)])$ and by assuming that the weak interaction is so weak that the influence on the bath is small, which makes the *Born approximation* applicable. Treating the 'bath' approximately constant in time and that the Markov approximation is a good approximation in this present regime, we can use the *Markovian master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\int_0^\infty \mathrm{d}s \ tr_B\left([H_I(t), [H_I(t-s), \rho_S(t)\otimes\rho_B]]\right) \ , \tag{3.108}$$

where for a more detailed derivation of the Markovian master equation we refer to the introduction (1.2.2.2).

Due to the Markovian approximation means locality in time it is remarkable that here an important quantity for which the Markov approximation is a good approximation is on the one hand the charging energy and on the other hand the transport voltage V for which the Markov approximation is good if it is $|s| \leq \min(1/eV, 1/E^{(C)})$.

Now we want to solve this Markovian master equation for one of the Coulomb circuits. Therefore let us introduce a description of the state of the present system. Due to the parity conservation of the qubit system, we can assume that the qubit state is fully described by the charge numbers of the both ring islands n_1 and n_2 (which we want to read out) laying in one of the parity subspaces (even and odd parity subspace). Here we want to assume that the qubit states lie in the even parity subspace and we can write the qubit states labeled by Greek letters as $|\alpha\rangle = |n_1 n_2\rangle \in \{|00\rangle, |11\rangle\} = \{|0_{qb}\rangle, |1_{qb}\rangle\}$. We write the states of the both SET-islands directly by the charge numbers $n_{1,0}$ and $n_{2,0}$ with $|n_{i,0}\rangle \in \{|0\rangle, |1\rangle\} = \{|0_i\rangle, |1_i\rangle\}$. So the total state of both read out circuits can be written as $|a\rangle = |\alpha\, n_{1,0}\, n_{2,0}\rangle = |\alpha\rangle\,|n_{1,0}\rangle\,|n_{2,0}\rangle$ which we label by the Latin

letters $a, b, ...$ So we have all in all 8 system states

$$|a\rangle, |b\rangle, |c\rangle, |d\rangle \in \Big\{ \big|0_{qb}0_10_2\big\rangle, \big|0_{qb}0_11_2\big\rangle, \big|0_{qb}1_10_2\big\rangle, \big|0_{qb}1_11_2\big\rangle, \big|1_{qb}0_10_2\big\rangle, \big|1_{qb}0_11_2\big\rangle, \big|1_{qb}1_10_2\big\rangle, \big|1_{qb}1_11_2\big\rangle \Big\}, \tag{3.109}$$

so that the density matrix of the total system can be written as a 8×8 matrix with 64 entries $\rho_S^{ab} \equiv \rho^{\alpha,\beta}_{n_1^0\, n_2^0, m_1^0\, m_2^0}$ (we used $n_{i,0} \equiv n_i^0$ for a better readability) for which the density operator can be written as

$$\rho_S(t) = \sum_{c,d} |c\rangle \, \rho_S^{cd}(t) \, \langle d| \, . \tag{3.110}$$

The operators in the Hamiltonian acting on the system states as

$$\begin{aligned} c_i^\dagger c_i |a\rangle &= n_i |a\rangle \ , \\ \hat{n}_{i,0} |a\rangle &= n_{i,0} |a\rangle \ , \\ e^{\pm i\hat{\varphi}_1} |a\rangle &= \big|\alpha \ (n_{1,0} \pm 1) \ n_{2,0}\big\rangle \ , \\ e^{\pm i\hat{\varphi}_2} |a\rangle &= \big|\alpha \ n_{1,0} (n_{2,0} \pm 1)\big\rangle \ , \end{aligned} \tag{3.111}$$

where $|a\rangle = \big|\alpha \ n_{1,0} \ n_{2,0}\big\rangle$. The Hamiltonian $H_{i,S}$ has the eigenvalues $E_{el}(n_i, n_{i,0})$ and acts on the system states as $H_{i,S} |a\rangle = E_{el}(n_i, n_{i,0}) |a\rangle$.

The calculations in the following are done all for the circuit $i = 1$ and we will skip again the index i for the both circuits and write them only if it is necessary, but all is analogue for both circuits and they can be consider individually, since we assumed, that they are decoupled from each other. Now we use the Markovian master equation (3.108) and we start by resolving the encapsulated commutators on its right side. We use

$$\langle a| \frac{\mathrm{d}}{\mathrm{d}t} \rho_S(t) |b\rangle = \frac{\mathrm{d}}{\mathrm{d}t} \rho_S^{ab}(t) \tag{3.112}$$

and applying this to all terms of the right side of (3.108), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho_S^{ab}(t) = -\sum_d \Big(\Gamma_{\mathrm{I}}^{ab,db} \rho_S^{db}(t) + \Gamma_{\mathrm{II}}^{ab,ad} \rho_S^{ad}(t) \Big) + \sum_{c,d} (\Gamma_{\mathrm{III}}^{ac,db} + \Gamma_{\mathrm{IV}}^{ac,db}) \rho_S^{cd}(t) \tag{3.113}$$

with

$$\begin{aligned}
\text{(I)}\ \Gamma_{\mathrm{I}}^{ab,db} &= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a| H_I(t)H_I(t-s)(\rho_S(t)\otimes\rho_B)\,|b\rangle\right] \\
&= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a| H_I(t)H_I(t-s)\,|d\rangle\,\rho_B\right] \\
\text{(II)}\ \Gamma_{\mathrm{II}}^{ab,ad} &= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a|\,(\rho_S(t)\otimes\rho_B)H_I(t-s)H_I(t)\,|b\rangle\right] \\
&= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\rho_B\,\langle d| H_I(t-s)H_I(t)\,|b\rangle\right] \\
\text{(III)}\ \Gamma_{\mathrm{III}}^{ac,db} &= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a| H_I(t)(\rho_S(t)\otimes\rho_B)H_I(t-s)\,|b\rangle\right] \\
&= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a| H_I(t)\,|c\rangle\,\rho_B\,\langle d| H_I(t-s)\,|b\rangle\right] \\
\text{(IV)}\ \Gamma_{\mathrm{IV}}^{ac,db} &= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a| H_I(t-s)(\rho_S(t)\otimes\rho_B)H_I(t)\,|b\rangle\right] \\
&= \int_0^{+\infty} \mathrm{d}s\ \mathrm{tr}_B\left[\langle a| H_I(t-s)\,|c\rangle\,\rho_B\,\langle d| H_I(t)\,|b\rangle\right].
\end{aligned} \tag{3.114}$$

Comparing this with the classical master equation (3.96), the functions in (3.114) can be identified as the tunnelling rates and the density matrix elements as the probabilities. In the following we determine these rate functions first for one of the two Coulomb circuits. Therefore we note that $H_S \equiv H_S(\hat{n},\hat{n_0}) \equiv H_S(c^\dagger c,\hat{\varphi})$, $H_B \equiv H_B(c_{L,q}^\dagger c_{L,q}, c_{R,q}^\dagger c_{R,q}, d_k^\dagger d_k)$ with $H_\sigma \equiv H_\sigma(c_{\sigma,q}^\dagger c_{\sigma,q})$ for $\sigma = L,R$ and $H_{\mathrm{island}} \equiv H_{\mathrm{island}}(d_k^\dagger d_k)$ for which the following commutation relations are valid

$$\begin{aligned}
[H_B,\hat{\varphi}] &= 0, \qquad [H_B,H_S] = 0, \\
[H_\sigma,H_{\sigma'}] &= 0, \quad [H_\sigma,H_{\mathrm{island}}] = 0, \ \text{ for } \sigma,\sigma' = R,L
\end{aligned} \tag{3.115}$$

and $[H_0,H_I] \neq 0$. Using the commutation relations we can use that it is

$$e^{iH_0 t} = e^{iH_S t}\, e^{iH_L t}\, e^{iH_R t}\, e^{iH_{\mathrm{island}} t}\ . \tag{3.116}$$

We can write the time depending interaction Hamiltonian as

$$H_I(t) = \sum_{\sigma,q,k} \left(\tilde{t}^{\sigma}_{k,q} d^{\dagger}_k(t) c_{\sigma q}(t) e^{iH_S t} e^{i\hat{\varphi}} e^{-iH_S t} + (\tilde{t}^{\sigma}_{k,q})^* c^{\dagger}_{\sigma q}(t) d_k(t) e^{iH_S t} e^{-i\hat{\varphi}} e^{-iH_S t} \right), \tag{3.117}$$

where the creation and annihilation operators are represented in the Heisenberg picture

$$\begin{aligned} d_k(t) &= e^{iH_B t} d_k e^{-iH_B t}, & d^{\dagger}_k(t) &= e^{iH_B t} d^{\dagger}_k e^{-iH_B t}, \\ c_{\sigma q}(t) &= e^{iH_B t} c_{\sigma q} e^{-iH_B t}, & c^{\dagger}_{\sigma q}(t) &= e^{iH_B t} c^{\dagger}_{\sigma q} e^{-iH_B t}. \end{aligned} \tag{3.118}$$

Further we use the commutation relations $[H_S, d_k(t)] = 0$ and $[H_S, c_{\sigma q}(t)] = 0$. Note that here $|\alpha\ n_0 \pm 2\rangle$ are assumed to be forbidden states. We get

$$\begin{aligned} \langle a| H_I(t) H_I(t-s) |d\rangle = \sum_{\substack{\sigma,\ q,\ k \\ \sigma',q',k'}} & (\tilde{t}^{\sigma}_{k,q} (\tilde{t}^{\sigma'}_{q',k'})^* d^{\dagger}_k(t) c_{\sigma q}(t) c^{\dagger}_{\sigma' q'}(t-s) d_{k'}(t-s) \boldsymbol{\delta}_{a,d} e^{i\Delta E^-_a s} \\ + & (\tilde{t}^{\sigma}_{q,k})^* \tilde{t}^{\sigma'}_{q',k'} c^{\dagger}_{\sigma q}(t) d_k(t) d^{\dagger}_{k'}(t-s) c_{\sigma',q'}(t-s) \boldsymbol{\delta}_{a,d} e^{i\Delta E^+_a s}), \end{aligned} \tag{3.119}$$

$$\begin{aligned} \langle d| H_I(t-s) H_I(t) |b\rangle = \sum_{\substack{\sigma,\ q,\ k \\ \sigma',q',k'}} & (\tilde{t}^{\sigma}_{k,q} (\tilde{t}^{\sigma'}_{q',k'})^* d^{\dagger}_k(t-s) c_{\sigma q}(t-s) c^{\dagger}_{\sigma' q'}(t) d_{k'}(t) \boldsymbol{\delta}_{d,b} e^{-i\Delta E^-_d s} \\ + & (\tilde{t}^{\sigma}_{q,k})^* \tilde{t}^{\sigma'}_{q',k'} c^{\dagger}_{\sigma q}(t-s) d_k(t-s) d^{\dagger}_{k'}(t) c_{\sigma',q'}(t) \boldsymbol{\delta}_{d,b} e^{-i\Delta E^+_d s}), \end{aligned} \tag{3.120}$$

$$\begin{aligned} \langle a| H_I(t) |c\rangle = \sum_{\sigma,\ q,\ k} & (\tilde{t}^{\sigma}_{k,q} d^{\dagger}_k(t) c_{\sigma q}(t) \boldsymbol{\delta}_{\alpha,\gamma} \boldsymbol{\delta}_{n^0_1, l^0_1+1} \boldsymbol{\delta}_{n^0_2, l^0_2} e^{i\Delta E^-_a t} \\ + & (\tilde{t}^{\sigma}_{q,k})^* c^{\dagger}_{\sigma q}(t) d_k(t) \boldsymbol{\delta}_{\alpha,\gamma} \boldsymbol{\delta}_{n^0_1, l^0_1-1} \boldsymbol{\delta}_{n^0_2, l^0_2} e^{i\Delta E^+_a t}), \end{aligned} \tag{3.121}$$

$$\begin{aligned} \langle d| H_I(t-s) |b\rangle = \sum_{\sigma,\ q,\ k} & (\tilde{t}^{\sigma}_{k,q} d^{\dagger}_k(t-s) c_{\sigma q}(t-s) \boldsymbol{\delta}_{\delta,\beta} \boldsymbol{\delta}_{j^0_1, m^0_1+1} \boldsymbol{\delta}_{j^0_2, m^0_2} e^{-i\Delta E^+_d (t-s)} \\ + & (\tilde{t}^{\sigma}_{q,k})^* c^{\dagger}_{\sigma q}(t-s) d_k(t-s) \boldsymbol{\delta}_{\delta,\beta} \boldsymbol{\delta}_{j^0_1, m^0_1-1} \boldsymbol{\delta}_{j^0_2, m^0_2} e^{-i\Delta E^-_d (t-s)}) \end{aligned} \tag{3.122}$$

and $\langle a| H_I(t-s) |c\rangle$ and $\langle d| H_I(t) |b\rangle$ are analogue to equation (3.122) and (3.121) respectively. Here it is $\boldsymbol{\delta}_{d,b} \neq 0$ only, if $|d\rangle = |b\rangle$, other wise it will be zero. In the following we will use a short hand notation for expressions like

$$\boldsymbol{\delta}_{d,b\pm1} = \boldsymbol{\delta}_{\delta,\beta} \boldsymbol{\delta}_{j^0_1, m^0_1 \pm 1} \boldsymbol{\delta}_{j^0_2, m^0_2} \tag{3.123}$$

and we denote $|a\rangle = |\alpha\rangle \left|n^0_1\right\rangle \left|n^0_2\right\rangle$, $|b\rangle = |\beta\rangle \left|m^0_1\right\rangle \left|m^0_2\right\rangle$, $|c\rangle = |\gamma\rangle \left|l^0_1\right\rangle \left|l^0_2\right\rangle$ and $|d\rangle = |\delta\rangle \left|j^0_1\right\rangle \left|j^0_2\right\rangle$. With $\Delta E^{\pm}$ we represent the energy differences of the energies of different states, which are given

as the eigenvalues of $H_{i,S}$ (3.92), so that it is

$$\Delta E_a^{\pm} = E_{el}(\alpha, n_1^0, n_2^0) - E_{el}(\alpha, n_1^0 \pm 1, n_2^0) \; . \tag{3.124}$$

Now we plug in these expressions into the equations (I) - (IV) of (3.114). For further calculations we use the fact that for the density operator of the canonical ensemble in thermal equilibrium is $\hat{\rho}_B = Z_B^{-1} e^{-\beta_{\text{th}} H_B}$, where $Z_B = \text{tr}_B[e^{-\beta_{\text{th}} H_B}]$ denotes the partition function and $\beta_{\text{th}} = 1/k_B T$ is the reciprocal of the thermodynamic temperature with k_B is the Boltzmann constant and T is the temperature. The expectation value of an operator $A(t,s)$ with respect to the reservoir is

$$\text{tr}_B[\hat{\rho}_B A(t,s)] = Z_B^{-1} \text{tr}_B[e^{-\beta_{\text{th}} H_B} A(t,s)] = \langle A(t,s) \rangle_B \; . \tag{3.125}$$

Here we have for example expectation values for terms consisting of annihilation and creation operators of the electrons in the reservoir like

$$\langle c_{\sigma q}^{\dagger}(s) d_k(s) d_{k'}^{\dagger}(0) c_{\sigma',q'}(0) \rangle_B = e^{-i\varepsilon_k s} e^{i\varepsilon_{\sigma q} s} e^{i\mu_\sigma s} \delta_{\sigma\sigma'} \delta_{qq'} \delta_{kk'} f(\varepsilon_{\sigma q})(1 - f(\varepsilon_k)) \; , \tag{3.126}$$

where $\mu_\sigma = eV_\sigma$ is the chemical potential of the lead $\sigma = R, L$ in the single electron transistor (here of the Coulomb circuit $i = 1$) and the ε are the band energies given in (3.102). The function $f(\varepsilon) = \left(1 + e^{\beta_{\text{th}} \varepsilon}\right)^{-1}$ is the Fermi-Dirac distribution function, here of the reservoir electrons with energy ε.
Being careful with all commutation relations, one gets the results for the traces in (3.114).
Now for simplicity we assume, that approximatively tunnelling for electrons belonging to the reservoir is independent of the corresponding band energy. Therefore we have tunnelling parameters, which are constant in q and k, i.e. $\tilde{t}_{k,q}^{\sigma} = \tilde{t}^{\sigma} \equiv \text{const.}$ for all q, k. With this assumption the results of the traces are given by

$$\begin{aligned} \text{tr}_B\left[\langle a| H_I(t) H_I(t-s) |d\rangle \rho_B\right] = \boldsymbol{\delta}_{a,d} \Bigg\{ & e^{i\Delta E_a^- s} \sum_\sigma |\tilde{t}^\sigma|^2 e^{-i\mu_\sigma s} \sum_q e^{-i\varepsilon_{\sigma q} s} (1 - f(\varepsilon_{\sigma q})) \sum_k e^{i\varepsilon_k s} f(\varepsilon_k) \\ & + e^{i\Delta E_a^+ s} \sum_\sigma |\tilde{t}^\sigma|^2 e^{i\mu_\sigma s} \sum_q e^{i\varepsilon_{\sigma q} s} f(\varepsilon_{\sigma q}) \sum_k e^{-i\varepsilon_k s} (1 - f(\varepsilon_k)) \Bigg\} \end{aligned} \tag{3.127}$$

and for the second term

$$\begin{aligned} \text{tr}_B\left[\rho_B \langle d| H_I(t-s) H_I(t) |b\rangle\right] = \boldsymbol{\delta}_{d,b} \Bigg\{ & e^{-i\Delta E_d^- s} \sum_\sigma |\tilde{t}^\sigma|^2 e^{i\mu_\sigma s} \sum_q e^{i\varepsilon_{\sigma q} s} (1 - f(\varepsilon_{\sigma q})) \sum_k e^{-i\varepsilon_k s} f(\varepsilon_k) \\ & + e^{-i\Delta E_d^+ s} \sum_\sigma |\tilde{t}^\sigma|^2 e^{-i\mu_\sigma s} \sum_q e^{-i\varepsilon_{\sigma q} s} f(\varepsilon_{\sigma q}) \sum_k e^{i\varepsilon_k s} (1 - f(\varepsilon_k)) \Bigg\} \end{aligned} \tag{3.128}$$

and for the third term

$$\begin{aligned}
&\mathrm{tr}_B\left[\langle a|H_I(t)|c\rangle\,\rho_B\,\langle d|H_I(t-s)|b\rangle\right] \\
&= \Bigg\{ \boldsymbol{\delta}_{a,c+1}\boldsymbol{\delta}_{d,b-1} e^{i\Delta E_a^- t} e^{-i\Delta E_b^-(t-s)} \sum_\sigma |\tilde{t}^\sigma|^2 e^{-i\mu_\sigma s} \sum_q e^{-i\varepsilon_{\sigma q} s} f(\varepsilon_{\sigma q}) \sum_k e^{i\varepsilon_k s}\left(1-f(\varepsilon_k)\right) \\
&+ \boldsymbol{\delta}_{a,c-1}\boldsymbol{\delta}_{d,b+1} e^{i\Delta E_a^+ t} e^{-i\Delta E_b^+(t-s)} \sum_\sigma |\tilde{t}^\sigma|^2 e^{i\mu_\sigma s} \sum_q e^{i\varepsilon_{\sigma q} s}\left(1-f(\varepsilon_{\sigma q})\right) \sum_k e^{-i\varepsilon_k s} f(\varepsilon_k) \Bigg\}
\end{aligned} \tag{3.129}$$

and the last term

$$\begin{aligned}
&\mathrm{tr}_B\left[\langle a|H_I(t-s)|c\rangle\,\rho_B\,\langle d|H_I(t)|b\rangle\right] \\
&= \Bigg\{ \boldsymbol{\delta}_{a,c+1}\boldsymbol{\delta}_{d,b-1} e^{i\Delta E_a^-(t-s)} e^{-i\Delta E_b^- t} \sum_\sigma |\tilde{t}^\sigma|^2 e^{i\mu_\sigma s} \sum_q e^{i\varepsilon_{\sigma q} s} f(\varepsilon_{\sigma q}) \sum_k e^{-i\varepsilon_k s}\left(1-f(\varepsilon_k)\right) \\
&+ \boldsymbol{\delta}_{a,c-1}\boldsymbol{\delta}_{d,b+1} e^{i\Delta E_a^+(t-s)} e^{-i\Delta E_b^+ t} \sum_\sigma |\tilde{t}^\sigma|^2 e^{-i\mu_\sigma s} \sum_q e^{-i\varepsilon_{\sigma q} s}\left(1-f(\varepsilon_{\sigma q})\right) \sum_k e^{i\varepsilon_k s} f(\varepsilon_k) \Bigg\} .
\end{aligned} \tag{3.130}$$

The electrons of the 'bath' (island, left and right lead reservoirs) can approximately described as a nearly free electron system with density of states $\nu^\sigma(\varepsilon)$. Then we can write

$$\sum_q e^{\pm i\varepsilon_{\sigma q} s} f(\varepsilon_{\sigma q}) = \int_{-\infty}^{+\infty} \mathrm{d}\varepsilon\; \nu^\sigma(\varepsilon)\, e^{\pm\varepsilon s} f(\varepsilon) \,. \tag{3.131}$$

From here we can either integrate first over energy and then over time or first over time and then over energy. For the integration over energy we use the assumption, that the interval of the electron energies next to each other $\Delta_\varepsilon = \varepsilon_{q+1} - \varepsilon_q$ is *very small* relative to the electronic energy scale of the system [7], and therefore the density of states can be considered as approximatively constant $\nu^\sigma(\varepsilon) \simeq \nu_0^\sigma = \text{const.}$. So that the integral over the energy becomes

$$\int_{-\infty}^{+\infty} \mathrm{d}\varepsilon\; \nu^\sigma(\varepsilon)\, e^{\pm\varepsilon s} f(\varepsilon) \simeq \nu_0^\sigma \int_{-\infty}^{+\infty} \mathrm{d}\varepsilon\; e^{\pm\varepsilon s} f(\varepsilon) \,. \tag{3.132}$$

We calculate the total rates here by integrating out first over time and then over energy. The results of integrating first over energy can be viewed in the appendix ((D.24) and (D.25)). The result after inserting our previous results into the total Markovian master equation can be viewed in the appendix given equation (D.22). Then first integrating out over time gives us a real and an imaginary part, which can be viewed in the appendix in equation (D.23). Second, by integration over energy we calculate the real and imaginary parts of the time integration separately. The

[7] We have already used this assumption to introduce the concept of the classical approximation of the charging energy in the present regime.

real parts can be calculated exactly, while the imaginary parts can only calculated with two approximations. The first approximation we have to use, is to assume that the temperature goes to zero ($\beta_{\text{th}} \to \infty$), which is a good approximation in this regime, due to we have to assume anyway very low temperatures for the system exhibiting Majorana zero mode and for the system parts of read out circuits being in a Coulomb regime for the relative high ring sizes. The second approximation is, that we have to introduce a *upper energy cut-off* $\Lambda \gg \left|\Delta^{\pm}_{k,\sigma}\right|$ with

$$\Delta^{\pm}_{k,\sigma} = \Delta E^{\pm}_{k} \pm \mu_{\sigma}, \quad \text{for } k = a,b \ , \ \sigma = R,L \ , \tag{3.133}$$

which are the energy differences of different occupation states of the SET-island (between n^0_1 and $n^0_1 \pm 1$) given by equation (3.124) and the chemical potential $\mu_\sigma = e \cdot V_\sigma$ for $\sigma = R,L$, the right and the left lead. At this point remember to the equation (3.95) representing the energy differences derived out of classical considerations. By comparing, we see that $\Delta^{\pm}_{k,\sigma}$ corresponds to $\Delta E^{n}_{ad/ex,R(L)}(n_0)$, the energy difference necessary to add or extract an electron from the left or right lead from or to the SET-island and it is important to keep in mind, that we can add only an electron to this island when this is in charge state $n^0_1 = 0$ and we can extract only an electron when this is in charge state $n^0_1 = 1$. This means, that we have the energy difference $\Delta^{+}_{k,\sigma}$ when we can add an electron and the difference $\Delta^{-}_{k,\sigma}$ when we can extract an electron from the SET-island. Reminding that the upper energy cut-off represents the upper boundary by integrating over the reservoir energies, it is visible, that the physics given by our results are only valid for reservoir energies, which are lower as this upper energy cut-off.
As a final result we get the following expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho^{ab}_S(t) = -\Gamma^{a,b}_{\text{out}}\,\rho^{ab}_S(t) + \Gamma^{a-1,b-1}_{\text{in,-}}\rho^{a-1,b-1}_S(t) + \Gamma^{a+1,b+1}_{\text{in,+}}\rho^{a+1,b+1}_S(t) \ , \tag{3.134}$$

where we have identified the *total out-rate* with

$$\underline{\textit{out-rate}}\text{:} \ \ \Gamma^{a,b}_{\text{out}} = \Gamma^{a,b}_{\text{out,+}} + \Gamma^{a,b}_{\text{out,-}}$$

$$\begin{aligned}\Gamma^{a,b}_{\text{out},\pm} = \sum_{\sigma} \alpha^{\sigma}_0 \quad & \left\{ \pi \cdot \left[\frac{\Delta^{\pm}_{a,\sigma}}{1 - e^{-\beta_{\text{th}}\Delta^{\pm}_{a,\sigma}}} + \frac{\Delta^{\pm}_{b,\sigma}}{1 - e^{-\beta_{\text{th}}\Delta^{\pm}_{b,\sigma}}} \right] \right. \\ & \left. + \frac{i}{2}\left[\Delta^{\pm}_{a,\sigma} \ln\left(\frac{\Lambda}{\left|\Delta^{\pm}_{a,\sigma}\right|} \right) - \Delta^{\pm}_{b,\sigma} \ln\left(\frac{\Lambda}{\left|\Delta^{\pm}_{b,\sigma}\right|} \right) \right] \right\}\end{aligned} \tag{3.135}$$

and the *total in-rate* with

$$\underline{\textit{in-rate}}\text{:} \ \ \Gamma^{a,b}_{\text{in}} = \Gamma^{a+1,b+1}_{\text{in,+}} + \Gamma^{a-1,b-1}_{\text{in,-}} \tag{3.136}$$

$$\Gamma_{\text{in},\pm}^{a\pm1,b\pm1} = e^{i(\Delta E_a^{\pm} - \Delta E_b^{\pm})t} \sum_{\sigma} \alpha_0^{\sigma} \left\{ \pi \cdot \left[\frac{\Delta_{a,\sigma}^{\pm}}{e^{\beta_{\text{th}}\Delta_{a,\sigma}^{\pm}} - 1} + \frac{\Delta_{b,\sigma}^{\pm}}{e^{\beta_{\text{th}}\Delta_{b,\sigma}^{\pm}} - 1} \right] - \frac{i}{2} \left[\Delta_{a,\sigma}^{\pm} \ln\left(\frac{\Lambda}{\left|\Delta_{a,\sigma}^{\pm}\right|} \right) - \Delta_{b,\sigma}^{\pm} \ln\left(\frac{\Lambda}{\left|\Delta_{b,\sigma}^{\pm}\right|} \right) \right] \right\} . \tag{3.137}$$

We used that at low voltages V_σ[8] the tunnelling conductance for a tunnelling junction with tunnelling parameter $\tilde{t}^\sigma$ between lead and Coulomb island of a single electron transistor can be described by $\frac{4\pi^2 e^2}{h} |\tilde{t}^\sigma|^2 \cdot \nu_0^\sigma \nu_0^I$ [31] (and compare with [29, 30]). We use ν_0^σ for the density of states of the leads and ν_0^I for the density of states of the reservoir electrons in the SET-island. So we have identified tunnel conductance between the SET-island and the lead labeled by σ as

$$\alpha_0^\sigma = |\tilde{t}^\sigma|^2 \nu_0^\sigma \nu_0^I = \frac{h}{4\pi^2 e^2 R_{\tilde{t}}^\sigma} > 0 \tag{3.138}$$

and it is

$$\sum_\sigma \alpha_0^\sigma = \frac{h}{4\pi^2 e^2} \left(\frac{R_{\tilde{t}}^R + R_{\tilde{t}}^L}{R_{\tilde{t}}^R R_{\tilde{t}}^L} \right) , \tag{3.139}$$

where $R_{\tilde{t}}^{L(R)}$ is the tunnelling resistance for the left (right) lead and $R_K = \frac{h}{e^2}$ is the von Kitzling constant or rather $\sigma_0 = 2\frac{e^2}{h}$ is the conductance quantum value (h denotes the Planck constant). Comparing the real part of the out-rates in equation (3.135) with the equation (3.99), we see that they correspond to the classical rate result for $T > 0$. Considering only the parts in the curly braces of both, in and out rates, we see, that for the case of negative energy differences $\Delta_{k,\sigma}^{\pm} < 0$ (which means that $\Delta E_k^{\pm} < \mp\mu_\sigma$) it is $\Delta_{k,\sigma}^{\pm}(1 - \exp(-\beta_{\text{th}}\Delta_{k,\sigma}^{\pm}))^{-1} = \left|\Delta_{k,\sigma}^{\pm}\right|(\exp(\beta_{\text{th}}\left|\Delta_{k,\sigma}^{\pm}\right|) - 1)^{-1}$ and vice versa $\Delta_{k,\sigma}^{\pm}(\exp(\beta_{\text{th}}\Delta_{k,\sigma}^{\pm}) - 1)^{-1} = \left|\Delta_{k,\sigma}^{\pm}\right|(1 - \exp(-\beta_{\text{th}}\left|\Delta_{k,\sigma}^{\pm}\right|))^{-1}$, so that the out-rates for $\Delta_{k,\sigma}^{\pm} < 0$ are equal to the curly braces of the in-rates for $\Delta_{k,\sigma}^{\pm} > 0$ and the curly braces of the in-rates for $\Delta_{k,\sigma}^{\pm} < 0$ are equal to these of the out-rates for $\Delta_{k,\sigma}^{\pm} > 0$. This means, by switching the sign of the applied voltages the tunnelling process will change its direction. So choosing the voltages at the left and right lead in the way that $V_L = -V_R = V/2$, this will mean, that by switching to $V_L = -V_R = -V/2$, the tunnelling rates are the same, but in an other direction. By the way the simplest way to understand these tunnelling processes would be to assume that the voltages of the left and right lead are of opposite sign, but they have equal absolute values. And for the case that $\Delta_{k,\sigma}^{\pm} = 0$ the divergent parts of the rates cancel each other, so that all rates are zero for $\left|\Delta E_k^{\pm}\right| = \left|eV/2\right|$, which agrees with the results for a single electron transistor on its own [9]. An important fact is, that the out-rates have no time dependence, while the in-rates have a time dependent phase factor of $e^{i(\Delta E_a^{\pm} - \Delta E_b^{\pm})t}$. In the following we

[8] What needs to be guaranteed to be in a Coulomb blockade regime, for which the SET-island has only the above described charge state possibilities (eV_σ is not allowed to exceed the energy, which is needed to add (extract) one electron if the SET-island is in the charge state of $n_0 = 0$ ($n_0 = 1$).

want to consider the impact of the time dependence on the elements of the density matrix (in particular the diagonal entries). From equation (3.136) and with the condition, that the SET-island charge state can only be zero or one, we see that, if both, the charge state $|a\rangle = \left|\alpha\ n_1^0\ n_2^0\right\rangle$ and $|b\rangle = \left|\beta\ m_1^0\ m_2^0\right\rangle$ have non equal SET-island charge states $n_1^0 \neq m_1^0$, the both in-rates are zero, and therefore the total in-rate vanishes. Calculating in the same way the tunnelling rates for the circuit $i = 2$ leads to the same results as for these of circuit $i = 1$ done above [9], so that therefore all in rates are zero for $n_2^0 \neq m_2^0$ as well. So in the case of $n_1^0 \neq m_1^0$ or $n_2^0 \neq m_2^0$ we have no time dependent rate terms. In the following we consider only the parts with $n_1^0 = m_1^0$ and $n_2^0 = m_2^0$.

Let $t = 0$ be the time, at which the read out process starts. We want to determine the time evolution of a given initial density matrix $\hat{\rho}(t = 0)$. In general the equation of motion given by equation (3.134) provides the combination of the two system states $|a\rangle = \left|\alpha\ n_1^0\ n_2^0\right\rangle$ and $|b\rangle = \left|\beta\ m_1^0\ m_2^0\right\rangle$. Here we want to consider the case for $n_1^0 = m_1^0$ and $n_2^0 = m_2^0$. The both decoupled subsystems can be treated again independently and it is sufficient to solve the problem only for one of the read out circuits. Here we choose again subsystem $i = 1$. So, for a fixed SET-island charge state n_1^0, the equation for the 8×8 density matrix with elements $\rho_S^{ab} \equiv \rho^{\alpha,\beta}_{n_1^0\ n_2^0, m_1^0\ m_2^0}$ reduces to one of the 2×2 submatrices with elements $\rho^{\alpha,\beta}_{n_1^0}(t)$ [10]. So we have to solve the equation of motion given by (3.134) for an initial density matrix $\hat{\rho}_{n_1^0}(t = 0)$, which is defined in the following. As a result of braiding, the system being determined by the both superconducting ring island states is assumed to be in a pure state $|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$ at time $t = 0$. The related density operator can be written as

$$\hat{\rho}(0) = |\Psi_{in}\rangle \langle\Psi_{in}| . \tag{3.140}$$

We assume that the SET-island is adjusted to be in the charge state $n_1^0 = 0$ at initial time $t = 0$. Then the initial density matrix can be written in the *state-space* as

$$\hat{\rho}_{n_1^0}(0) = \left\{\rho^{\alpha\beta}_{n_1^0}(0)\right\} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \delta_{n_1^0,0} , \tag{3.141}$$

where the Greek letters still label the state of the superconducting ring islands in the even

[9] Before tracing over the 'bath' we get mixed terms, but under the trace over the bath states, these mixed terms are zero, so that we get no interaction terms for the rates.

[10] Remember that $|\alpha\rangle, |\beta\rangle \in \left\{\left|0_{qb}\right\rangle, \left|1_{qb}\right\rangle\right\}$ are the possible quantum bit states of even parity affected by the read out process.

subspace. In *Liouville space* [32, 33] one can write this matrix as a vector

$$\hat{\boldsymbol{\rho}}_{n_1^0}(0) = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -i \\ i \end{pmatrix} \delta_{n_1^0,0} \tag{3.142}$$

and we can represent the time evolution of the density matrix $\hat{\rho}_{n_1^0}$ in matrix representation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{\rho}}_{n_1^0}(t) = \hat{\Gamma}\hat{\boldsymbol{\rho}}_{n_1^0} \,. \tag{3.143}$$

If now we consider the equation of motion (3.134) for this special case, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{n_1^0}^{\alpha,\beta}(t) = -\Gamma_{\mathrm{out}}^{\alpha,\beta,n_1^0}\rho_{n_1^0}^{\alpha,\beta}(t) + \Gamma_{\mathrm{in,-}}^{\alpha,\beta,n_1^0-1}\rho_{n_1^0-1}^{\alpha,\beta}(t) + \Gamma_{\mathrm{in,+}}^{\alpha,\beta,n_1^0+1}\rho_{n_1^0+1}^{\alpha,\beta}(t) \tag{3.144}$$

and for each pair (α,β) we get the two coupled differential equations

$$\begin{aligned} &\text{(i)} \quad \frac{\mathrm{d}}{\mathrm{d}t}\rho_0^{\alpha,\beta}(t) = -\Gamma_{\mathrm{out}}^{\alpha,\beta,0}\rho_0^{\alpha,\beta}(t) \quad + \Gamma_{\mathrm{in,+}}^{\alpha,\beta,1}\rho_1^{\alpha,\beta}(t) \,, \quad (n_1^0 = 0) \\ &\text{(ii)} \quad \frac{\mathrm{d}}{\mathrm{d}t}\rho_1^{\alpha,\beta}(t) = -\Gamma_{\mathrm{out}}^{\alpha,\beta,1}\rho_1^{\alpha,\beta}(t) \quad + \Gamma_{\mathrm{in,-}}^{\alpha,\beta,0}\rho_0^{\alpha,\beta}(t) \,, \quad (n_1^0 = 1) \,, \end{aligned} \tag{3.145}$$

which gives us for each pair (α,β) a linear system of two differential equations. This means that for $t > 0$ the SET-island charge states $n_1^0 = 0$ and $n_1^0 = 1$ will be coupled.
To solve this system of differential equations we introduce the vector notation for the (α,β)-component of the time dependent density operator

$$\boldsymbol{\rho}^{\alpha,\beta}(t) = \begin{pmatrix} \rho_0^{\alpha,\beta}(t) \\ \rho_1^{\alpha,\beta}(t) \end{pmatrix} \tag{3.146}$$

and the matrix $\hat{M}^{\alpha,\beta} \in \mathbb{C}^{2\times 2}$ as

$$\hat{M}^{\alpha,\beta} = \begin{pmatrix} M_{00}^{\alpha,\beta} & M_{01}^{\alpha,\beta} \\ M_{10}^{\alpha,\beta} & M_{11}^{\alpha,\beta} \end{pmatrix} = \begin{pmatrix} -\Gamma_{\mathrm{out}}^{\alpha,\beta,0} & \Gamma_{\mathrm{in,+}}^{\alpha,\beta,1} \\ \Gamma_{\mathrm{in,-}}^{\alpha,\beta,0} & -\Gamma_{\mathrm{out}}^{\alpha,\beta,1} \end{pmatrix} \,, \tag{3.147}$$

so that we can write the system of differential equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho}^{\alpha,\beta}(t) = \hat{M}^{\alpha,\beta} \cdot \boldsymbol{\rho}^{\alpha,\beta}(t) \,. \tag{3.148}$$

Using the rate equations (3.135) and (3.137) we see, that the diagonal matrix elements of $\hat{M}^{\alpha,\beta}$ are time independent while the off-diagonal elements are time dependent in that way, that we can

write them in a product of a time dependent phase factor and a time independent multiplicand

$$\begin{aligned} M_{01}^{\alpha,\beta} &= e^{i\Delta_0 t} \quad \left|M_{01}^{\alpha,\beta}\right| , \\ M_{10}^{\alpha,\beta} &= e^{-i\Delta_0 t} \quad \left|M_{10}^{\alpha,\beta}\right| , \end{aligned} \tag{3.149}$$

where $\Delta_0 = E_{el}(\alpha,0,n_2^0) - E_{el}(\alpha,1,n_2^0) - E_{el}(\beta,0,n_2^0) + E_{el}(\beta,1,n_2^0)$ is the difference of the electric energy of the SET-island states. Further we define the energy difference $\Delta_0^\sigma = \Delta_0 + 2\mu_\sigma$ and the rate difference $\Sigma = \pi \sum_{\sigma=L,R} \alpha_0^\sigma \Delta_0^\sigma$, we introduce the new designations $\Gamma_1^{\alpha,\beta} \equiv \left|M_{01}^{\alpha,\beta}\right|$ and $\Gamma_2^{\alpha,\beta} \equiv \left|M_{10}^{\alpha,\beta}\right|$. Then the matrix reads

$$\hat{M}^{\alpha,\beta} = \begin{pmatrix} -\Gamma_1^{\alpha,\beta} + \Sigma & e^{i\Delta_0 t}\,\Gamma_1^{\alpha,\beta} \\ e^{-i\Delta_0 t}\,\Gamma_2^{\alpha,\beta} & -\Gamma_2^{\alpha,\beta} - \Sigma \end{pmatrix} . \tag{3.150}$$

Now we use the gauge transformation

$$\begin{aligned} \tilde{\rho}_0^{\alpha,\beta}(t) &= e^{-i\Delta_0 t} \quad \rho_0^{\alpha,\beta}(t), \\ \tilde{\rho}_1^{\alpha,\beta}(t) &= e^{i\Delta_0 t} \quad \rho_1^{\alpha,\beta}(t) \end{aligned} \tag{3.151}$$

and we set $\xi = \Sigma - \frac{i}{2}\Delta_0$. So we get the time independent evolution matrix

$$\hat{\mathscr{M}}^{\alpha,\beta} = \begin{pmatrix} -\Gamma_1^{\alpha,\beta} + \xi & \Gamma_1^{\alpha,\beta} \\ \Gamma_2^{\alpha,\beta} & -\Gamma_2^{\alpha,\beta} - \xi \end{pmatrix} , \tag{3.152}$$

for which its determinant $\det\hat{\mathscr{M}}^{\alpha,\beta} = -\xi(-\Gamma_2^{\alpha,\beta} + \Gamma_1^{\alpha,\beta} - \xi)$ is non zero as long as

$$\begin{aligned} &\text{(i)} \quad \xi \neq 0 \\ &\text{(ii)} \quad \xi \neq \Gamma_1 - \Gamma_2 \end{aligned} \tag{3.153}$$

are both fulfilled. Then we can determine the eigenvalues $\lambda_+^{\alpha,\beta}$ and $\lambda_-^{\alpha,\beta}$ by diagonalizing $\hat{\tilde{M}}^{\alpha,\beta}$ with

$$D^{\alpha,\beta} = \mathrm{diag}\left(\lambda_+^{\alpha,\beta}, \lambda_-^{\alpha,\beta}\right) = {}^{\mathcal{D}}\mathcal{U}^{-1}\hat{\mathscr{M}}^{\alpha,\beta}{}^{\mathcal{D}}\mathcal{U} , \tag{3.154}$$

where the matrix ${}^{\mathcal{D}}\mathcal{U}$ is time independent. Therefore we have

$$\boldsymbol{p}^{\alpha,\beta}(t) = {}^{\mathcal{D}}\mathcal{U}^{-1}\tilde{\boldsymbol{\rho}}^{\alpha,\beta}(t){}^{\mathcal{D}}\mathcal{U} \iff \tilde{\boldsymbol{\rho}}^{\alpha,\beta}(t) = {}^{\mathcal{D}}\mathcal{U}\,\boldsymbol{p}^{\alpha,\beta}(t){}^{\mathcal{D}}\mathcal{U}^{-1} , \tag{3.155}$$

so that we get a differential equation equivalent to equation (3.148)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}^{\alpha,\beta}(t) = D^{\alpha,\beta}\boldsymbol{p}^{\alpha,\beta}(t) = \begin{pmatrix} \lambda_{+}^{\alpha,\beta} p_0^{\alpha,\beta}(t) \\ \lambda_{-}^{\alpha,\beta} p_1^{\alpha,\beta}(t) \end{pmatrix} . \tag{3.156}$$

Using the fact that $\boldsymbol{p}^{\alpha,\beta}(t)$ is a linear superposition of the two density components given in equation (3.151) and using the initial conditions for the density matrix components (3.142), we can solve this equation for $\boldsymbol{p}^{\alpha,\beta}(t)$ and we get

$$\begin{aligned} p_0^{\alpha,\beta}(t) &= e^{\lambda_{+}^{\alpha,\beta} t} \quad p_0^{\alpha,\beta}(0) \\ p_1^{\alpha,\beta}(t) &= e^{\lambda_{-}^{\alpha,\beta} t} \quad p_1^{\alpha,\beta}(0) \, , \end{aligned} \tag{3.157}$$

where the eigenvalues are given by

$$\lambda_{\pm}^{\alpha,\beta} = \frac{1}{2}\left[-\left(\Gamma_1^{\alpha,\beta} + \Gamma_2^{\alpha,\beta}\right) \pm \sqrt{\left(\Gamma_1^{\alpha,\beta} + \Gamma_2^{\alpha,\beta}\right)^2 + 4\xi\left(\Gamma_1^{\alpha,\beta} - \Gamma_2^{\alpha,\beta} + \xi\right)} \right] . \tag{3.158}$$

Finally we can use the inverse transformation of (3.155) to determine the time dependent density matrix components. Due to here we are only interested in the time dependence of these components, it is enough, that we know, that our solutions have the following structure

$$\rho_{n_1^0}^{\alpha,\beta}(t) = c_{n_1^0,\eta}^{\alpha,\beta} \eta_{n_1^0}^{\alpha,\beta} e^{\lambda_{+}^{\alpha,\beta} t} + c_{n_1^0,\zeta}^{\alpha,\beta} \zeta_{n_1^0}^{\alpha,\beta} e^{\lambda_{-}^{\alpha,\beta} t} \, , \tag{3.159}$$

where $c_{n_1^0,\zeta}^{\alpha,\beta}$, $c_{n_1^0,\eta}^{\alpha,\beta} \in \left\{-\frac{i}{2}, \frac{i}{2}\right\}$ and $c_{n_1^0,\zeta}^{\alpha,\alpha}$, $c_{n_1^0,\eta}^{\alpha,\alpha} = \frac{1}{2}$. $\eta_{n_1^0}^{\alpha,\beta}$ and $\zeta_{n_1^0}^{\alpha,\beta}$ are constants depending only on the time independent parameter in the components of the matrix in equation (3.152) and they fulfil the initial condition, that for $t = 0$ the components of the density matrix correspond to these ones of equation (3.141), so that it is $\eta_{n_1^0}^{\alpha,\beta} + \zeta_{n_1^0}^{\alpha,\beta} = \delta_{n_1^0,0}$ for all $\alpha, \beta \in \{0_{qb}, 1_{qb}\}$ and all $n_1^0 \in \{0,1\}$.

We assume, that during the time of the read out process the charge states of the both ring islands stay unchanged. We are interested in the time evolution of the diagonal elements of the density matrix ($\alpha = \beta$). Therefore we only need to determine the sign of the eigenvalues $\lambda_{\pm}^{\alpha} \equiv \lambda_{\pm}^{\alpha,\alpha}$. Plug in the results for the tunnelling rates into the equation for the eigenvalues (3.158) we get

$$\begin{aligned} \left(\Gamma_1^{\alpha} + \Gamma_2^{\alpha}\right) &= \pi \sum_{\sigma=R,L} \alpha_0^{\sigma} \frac{\Delta_{a,\sigma} \sinh\left(\beta_{\mathrm{th}} \Delta_{a,\sigma}\right)}{\sinh\left(\beta_{\mathrm{th}} \Delta_{a,\sigma}/2\right)} \\ \left(\Gamma_1^{\alpha} - \Gamma_2^{\alpha}\right) &= -\xi = -2\pi \sum_{\sigma=R,L} \alpha_0^{\sigma} \Delta_{a,\sigma} \, , \end{aligned} \tag{3.160}$$

where we have omitted one of the ring island charge states label, because they are equal anyway.

We get for the eigenvalues

$$\lambda_{\pm}^{\alpha} = -\left|(\Gamma_1^{\alpha}+\Gamma_2^{\alpha})\right| \pm \left|(\Gamma_1^{\alpha}+\Gamma_2^{\alpha})\right| = \begin{cases} 0\,, & \text{for } \lambda_{+}^{\alpha} \\ -2\left|\left(\Gamma_1^{\alpha}+\Gamma_2^{\alpha}\right)\right|\,, & \text{for } \lambda_{-}^{\alpha} \end{cases} \tag{3.161}$$

where we have used that it is

$$\frac{\Delta_{a,\sigma}\sinh\left(\beta_{\text{th}}\Delta_{a,\sigma}\right)}{\sinh\left(\beta_{\text{th}}\Delta_{a,\sigma}/2\right)} > 0, \quad \text{for } \forall \Delta_{a,\sigma} \in \mathbb{R}\,, \tag{3.162}$$

so that is $\left(\Gamma_1^{\alpha}+\Gamma_2^{\alpha}\right) = \left|\left(\Gamma_1^{\alpha}+\Gamma_2^{\alpha}\right)\right|$ [11].
The eigenvalues $\lambda_{+}^{\alpha} = 0$ and $\lambda_{-}^{\alpha} = -2\left|\left(\Gamma_1^{\alpha}+\Gamma_2^{\alpha}\right)\right|$ correspond to the static solution. Because we get as a final result the long time asymptotic limit

$$\lim_{t\to\infty} \rho_{n_1^0}^{\alpha}(t) = \frac{1}{2}\eta_{n_1^0}^{\alpha} \equiv \left(\rho_{n_1^0}^{\alpha}\right)_{\infty} \tag{3.163}$$

for which we have convergence for the diagonal elements of the density matrix to a fixed constant value which depends on both, the SET-island charge state and the ring island charge state, so that we have for these the stationary solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{n_1^0}^{\alpha}\right)_{\infty} = 0\,. \tag{3.164}$$

If we consider the problem in general for the total rate matrix we can perform the gauge done above to get a total rate matrix which does not depend on time. This is because the only time dependence of the rate matrix is given by time dependent phase factor. After the gauge transformation the general rate matrix has the same rate elements as before, but only to its diagonal elements are added the energy differences which originally were in the exponent of the time dependent exponential functions. By assuming that any rate existing in the rate matrix is much smaller than any energy difference of the system $\Gamma^{\alpha,\beta}_{n_1^0,m_1^0,n_2^0,m_2^0} \ll \left|\Delta\varepsilon_{\eta}\right|$, we can diagonalize the rate matrix approximately, because approximatively all off-diagonal elements of the gauged rate matrix are much smaller then the diagonal elements, which are approximatively equal to the corresponding energy difference of the exponential time function. So in the end we get for the total rate matrix in this approximation the stationary solution. The assumption that any rate is much smaller than any energy difference in the system $\Gamma^{\alpha,\beta}_{n_1^0,m_1^0,n_2^0,m_2^0} \ll \left|\Delta\varepsilon_{\eta}\right|$ is justified by the fact, that it is a requirement, that our system state is not influenced by the read out method with

[11] The function $(\Gamma_1^{\alpha}+\Gamma_2^{\alpha})$ is a symmetric function of the energy differences and in peculiar it is $(\Gamma_1^{\alpha}+\Gamma_2^{\alpha}) = 4\pi \sum_{\sigma=R,L} \alpha_0^{\sigma} = \frac{h}{\pi e^2}\left(\frac{R_i^R+R_i^L}{R_i^R R_i^L}\right)$ for the case of zero energy difference $\Delta_{a,\sigma}^{\pm} = \Delta E_a^{\pm} \pm \mu_{\sigma} = 0$, where the absolute value of the energy of the applied voltage eV_{σ} of the corresponding lead is equal to the electrostatic energy difference between two different charge states of the SET-island .

the single electron transistor. So in principle a read out of the the quantum bit state should be possible with two of such Coulomb circuits as shown in Figure 3.7, by measuring long enough. If we consider the results for the tunnelling rates equation (3.137) and (3.137), we see that the component, which depends on the state of the quantum bit is the energy difference (3.133). Therefore the state of the quantum bit will have a noticeable influence on the time independent and the time dependent parts of the rate functions.

4. Conclusion and outlook

Now we come back from a journey through the physics of a new proposed quantum bit. After diving into the propose of a new quantum bit architecture, we started with the question *Is it possible, that the proposed architecture harbour topological protected Majorana zero modes?* The answer now is yes, under some strict conditions we should be able to create a topological phase transition between these geometrical boundaries. The architecture offers this simply by shifting the chemical potential of the surface states of the three dimensional topological insulator by apply a voltage at its bulk. The conditions we have to comply with are for example very low temperatures, a superconducting material with a high enough upper critical magnetic field and one has to take a good choice of the kind of magnetic doping to get a clean superconducting gap. Our model architecture described above is very idealized and it does not take into account all what could prevent the appearance of zero energy excitations at the geometric boundaries. If one could manage to construct such a perfect device, which will require a lot of technical skill, it promises high localized and topological protected Majorana zero modes and therefore a stable quantum bit state, due to the way of localizing the zero modes by adjusting the chemical potential in a suitable way. Theoretically it should be possible overcoming all hurdles to enable the appearance of these Majorana zero modes.
Since the appearance of zero modes is not enough to make it useable for quantum computation, our journey has had to go on. We checked, if new kind of architecture supports braiding.
We found out, that this architecture provides a new interesting way of braiding. Even if with its *T-shaped* arrangement of rings, braiding looks similar to the famous well known nanowire architecture [6], it provides an other way of exchanging the Majorana zero modes in their positions. While the T-shaped nanowire quantum bits are moved continuously from one site to another, without substantially changing their extension, the movement of the Majorana zero modes in the present architecture works in another way. Here we move the Majorana zero modes by gating, too, but their extension in space is not constant in time. We have seen, that for braiding we have to enlarge and reduce the boundaries, at which they are localized (remember the change in extension over one ring boundary to the extension over three ring boundaries and vice versa). We have not checked, if and how it could possible to gate the Majorana zero modes from the boundary of one ring over the boundaries of three rings. It could be very difficult in the area between the rings, that the Majorana zero modes do not suffer any unwanted excitations by

interacting with the environment or that they simply smear out. This is an outstanding task. Further the initialization of the quantum bit and also the adiabatically gating of the outer Majorana zero modes requires a very sensitive adjusting of gate voltages.
An other outstanding task is to check how one can use this new quantum bit architecture as basic building block of networks. The stability and the controlled initializing, braiding, fusion and read out and the possibility to simply combine more of these ring constructions on a surface up to networks of ring architectures, makes them interesting for the future of building quantum computer devices. But the technical advantages (very low temperatures, high external magnetic fields, clean material doping,...) still pose a major obstacle.
If the construction of such a device becomes reality, it could offer new possibilities to create Majorana surface code.

Appendix

A. Quantities and definitions

- Quantum of charge: $e = 1.6021766208 \times 10^{-19} C$
- Planck constant: $h = 6.626070040(81) \times 10^{-34} Js = 4.135667662(25) \times 10^{-15} eVs$
- Magnetic flux quantum: $\Phi_0 = \frac{h}{2e} = 2.067833831(13) \times 10^{-15} Wb$
- Boltzmann constant: $k_B = 1.38064852(79) \times 10^{-23} JK^{-1} = 8.6173303(50) \times 10^{-5} eVK^{-1}$
- Electric constant: $\varepsilon_0 \simeq 8.85418781710^{-12} Fm^{-1}$
- Quantum of conductance: $\sigma_0 = \frac{2e^2}{h} = 7.7480917310(18) \times 10^{-5} \frac{e}{s}$
- von Kitzling constant: $R_K = \frac{h}{e^2} = 25812.8074555(59)\Omega$

(The values are looked up from [34].)

A.1. Spin Pauli matrices

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

A.2. Bessel functions

Bessel functions of first and second kind are solutions of the Bessel differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - n^2) y = 0 \,. \tag{A.2}$$

Modified Bessel functions of first and second kind are solutions of the modified Bessel

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} - (x^2 - n^2) y = 0 \,. \tag{A.3}$$

Here we are only interested in the case of n is a nonnegative integer $n = 0, 1$ and $x \in \mathbb{R}$. The general solution for equation (A.2) case reads

$$y(x) = c_1 J_n(x) + c_2 Y_n(x) \ , \tag{A.4}$$

where $J_n(x)$ are called the *Bessel functions of first kind* and $Y_n(x)$ are called the *Bessel functions of second kind* [35]. The both are related by

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} \ . \tag{A.5}$$

The Bessel functions of first kind can be expressed as series

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{n+2k}}{k!(n+k)!} \ . \tag{A.6}$$

For the Bessel functions of first and second kind it is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} J_0(x) &= -J_1(x), & \frac{\mathrm{d}}{\mathrm{d}x} J_1(x) &= \frac{1}{2}\left(J_0(x) - J_2(x)\right), \\ \frac{\mathrm{d}}{\mathrm{d}x} Y_0(x) &= -Y_1(x), & \frac{\mathrm{d}}{\mathrm{d}x} Y_1(x) &= \frac{1}{2}\left(Y_0(x) - Y_2(x)\right), \\ \lim_{x\to\pm\infty} J_n(x) &= 0, & \lim_{x\to\pm\infty} J_n(x) &= 0. \end{aligned} \tag{A.7}$$

The *modified Bessel functions of first and second kind* are The general solution for equation (A.3) reads

$$y(x) = c_1 I_n(x) + c_2 K_n(x) \ , \tag{A.8}$$

where $I_n(x)$ ar called the *modified Bessel functions of first kind* and $K_n(x)$ are called the *modified Bessel functions of second kind* [35].
The both are related by

$$K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)} \ . \tag{A.9}$$

The modified Bessel functions of first kind can be expressed as series

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}x)^{n+2k}}{k!(n+k)!} \ . \tag{A.10}$$

For the modified Bessel functions of first and second kind it is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} I_0(x) &= I_1(x), & \frac{\mathrm{d}}{\mathrm{d}x} I_1(x) &= \frac{1}{2}\left(I_0(x) + I_2(x)\right), \\
\frac{\mathrm{d}}{\mathrm{d}x} K_0(x) &= -K_1(x), & \frac{\mathrm{d}}{\mathrm{d}x} K_1(x) &= \frac{1}{2}\left(-K_0(x) - K_2(x)\right), \\
\lim_{x\to\pm 0} I_0(x) &= 1, & \lim_{x\to\pm\infty} K_0(x) &= 0, \\
\lim_{x\to\pm 0} I_1(x) &= 0, & \lim_{x\to\pm\infty} K_1(x) &= 0.
\end{aligned} \tag{A.11}$$

More Informations about Bessel function are written in [35].

B. Boundary solutions

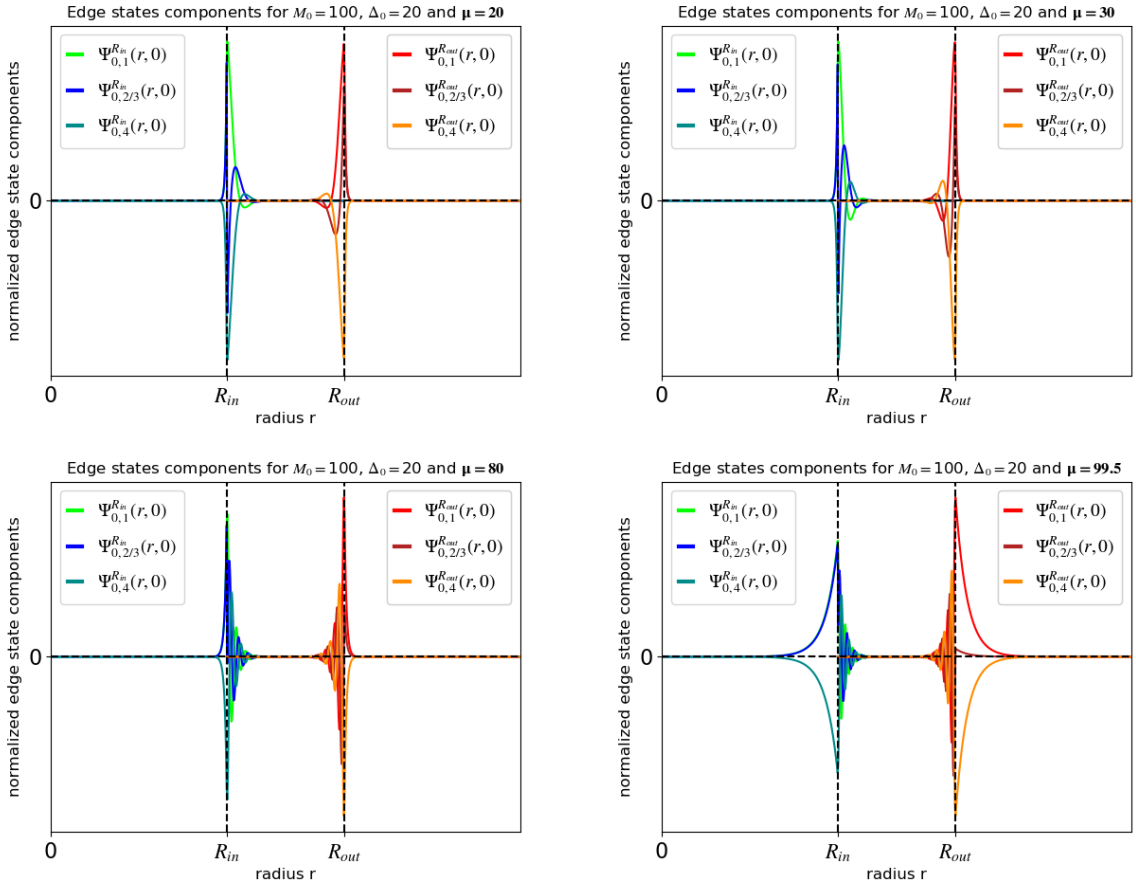


Figure B.11.: The wave functions components of the edge state solutions (at a fixed angle of $\theta = 0$) are shown for different values of the chemical potential μ to demonstrate its influence in the superconducting region. Here one can see, that it does not change the extension of the boundary solutions into the superconducting region, but it leads to higher oscillations. The higher extension into the outer magnetic regions come from the fact, that we left the magnetic field constant and therefore for higher μ the difference between them becomes smaller.

C. braiding in different basis

For the following chosen basis

$$\begin{aligned} f_1 &= \frac{1}{2}(\gamma_1 + i\gamma_3), \quad f_2 &= \frac{1}{2}(\gamma_5 + i\gamma_7), \\ f_3 &= \frac{1}{2}(\gamma_2 + i\gamma_6), \quad f_4 &= \frac{1}{2}(\gamma_4 + i\gamma_8), \end{aligned} \tag{C.12}$$

one will get the effective Hamiltonian

$$\begin{aligned} H_{eff} &= i(\varepsilon_3(t)\gamma_3\gamma_1 + \varepsilon_1(t)\gamma_5\gamma_3 + \varepsilon_2(t)\gamma_7\gamma_3) \\ &= \alpha(t)(f_2 f_1 + f_1^\dagger f_2) - \beta(t)(f_1^\dagger f_2^\dagger + f_2^\dagger f_1) + \varepsilon_3(t)(1 - 2f_1^\dagger f_1). \end{aligned} \tag{C.13}$$

Here gate ε_3 is chosen to be between ring two and one, gate ε_2 is chosen to be between ring two and four and the gate ε_1 is chosen to be between ring two and three. To exchange gate three and two does not change the braiding direction, so we get the same solid angle $\Omega = \frac{\pi}{2}$.

$$\begin{aligned} H_{even} &= \varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3 \quad &= \boldsymbol{\varepsilon}_{even}\cdot\boldsymbol{\sigma}, \\ H_{odd} &= \varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 - \varepsilon_3\sigma_3 \quad &= \boldsymbol{\varepsilon}_{odd}\cdot\boldsymbol{\sigma}, \end{aligned} \tag{C.14}$$

$$H_{even} = \begin{pmatrix} \varepsilon\,\cos\theta & \varepsilon\,\sin\theta e^{-i\phi} \\ \varepsilon\,\sin\theta e^{i\phi} & -\varepsilon\,\cos\theta \end{pmatrix}. \tag{C.15}$$

In the same way the Hamiltonian matrix of the odd parity subspace reads

$$H_{odd} = \begin{pmatrix} -\varepsilon\,\cos\theta & \varepsilon\,\sin\theta e^{-i\phi} \\ \varepsilon\,\sin\theta e^{i\phi} & \varepsilon\,\cos\theta \end{pmatrix}. \tag{C.16}$$

The eigenvectors corresponding to the ground state are given by

$$|e_-\rangle_{even} = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \tag{C.17}$$

yielding to

$$\mathcal{F}_{\theta\phi} = -\frac{1}{2}\sin(\theta) \tag{C.18}$$

and analog we get for the eigenvectors of the odd parity subspace

$$|e_-\rangle_{odd} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} \\ -\sin\frac{\theta}{2} \end{pmatrix}. \tag{C.19}$$

yielding to

$$\mathcal{F}_{\theta\phi} = \frac{1}{2}\sin(\theta). \tag{C.20}$$

So we get the same final result for the Berry-Phase in even and odd subspace is

$$\begin{aligned} e^{i\varphi_{even}} &= e^{i\frac{\pi}{4}}, \\ e^{i\varphi_{odd}} &= e^{-i\frac{\pi}{4}}. \end{aligned} \tag{C.21}$$

D. Details of determining the tunnelling rates

D.1. Integration over time

Plug in the approximation of equation (3.132) into the bath averages of the equations (3.127), (3.128), (3.129) and (3.130) one gets the following expression for the Makovian master equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = ... \\ &... = -\rho_S^{a,b}(t)\sum_\sigma \alpha_0^\sigma \int_{-\infty}^{+\infty}\mathrm{d}\varepsilon f(\varepsilon)\int_{-\infty}^{+\infty}\mathrm{d}\varepsilon'(1-f(\varepsilon'))\int_0^{+\infty}\mathrm{d}s\left[\left(e^{i\Delta^-_{a,\sigma}s}+e^{i\Delta^+_{a,\sigma}s}\right)e^{i(\varepsilon-\varepsilon')s}+\left(e^{-i\Delta^-_{b,\sigma}s}+e^{-i\Delta^+_{b,\sigma}s}\right)e^{-i(\varepsilon-\varepsilon')s}\right] \\ &+\rho_S^{a-1,b-1}(t)e^{i(\Delta E^-_a-\Delta E^-_b)t}\sum_\sigma \alpha_0^\sigma \int_{-\infty}^{+\infty}\mathrm{d}\varepsilon f(\varepsilon)\int_{-\infty}^{+\infty}\mathrm{d}\varepsilon'(1-f(\varepsilon'))\int_0^{+\infty}\mathrm{d}s\left[e^{-i\Delta^-_{a,\sigma}s}e^{i(\varepsilon-\varepsilon')s}+e^{i\Delta^-_{b,\sigma}s}e^{-i(\varepsilon-\varepsilon')s}\right] \\ &+\rho_S^{a+1,b+1}(t)e^{i(\Delta E^+_a-\Delta E^+_b)t}\sum_\sigma \alpha_0^\sigma \int_{-\infty}^{+\infty}\mathrm{d}\varepsilon f(\varepsilon)\int_{-\infty}^{+\infty}\mathrm{d}\varepsilon'(1-f(\varepsilon'))\int_0^{+\infty}\mathrm{d}s\left[e^{-i\Delta^+_{a,\sigma}s}e^{i(\varepsilon-\varepsilon')s}+e^{i\Delta^+_{b,\sigma}s}e^{-i(\varepsilon-\varepsilon')s}\right], \end{aligned} \tag{D.22}$$

where the energy difference $\Delta^\pm_{k,\sigma}$ is given in equation (3.133), the energy difference $\Delta E^\pm_k$ is given in equation (3.124) and the lead depending tunnelling constant α_0^σ is given by equation (3.138) in the main text. Here we first integrate the terms of equation (D.22) over time. For all term we have the following forms of equations:

$$\int_0^{+\infty}\mathrm{d}s\exp\left(p_1(\Delta^{p_2}_{k,\sigma}+p_3(\varepsilon-\varepsilon'))s\right) = p_1\cdot i(\Delta^{p_2}_{k,\sigma}+p_3(\varepsilon-\varepsilon'))^{-1}+\pi\delta(\Delta^{p_2}_{k,\sigma}+p_3(\varepsilon-\varepsilon')) \tag{D.23}$$

for $p_i \in \{+,-\}$ and $i = 1,2,3$ are different combinations of signs.

D.2. Rates in energy integral representation

By integrating only over time and using the residue theorem one gets for the *out-rates*

$$\begin{aligned}\Gamma^{a,b}_{\text{out},+} &= -T^2\pi^2\sum_\sigma \alpha_0^\sigma \int_0^{+\infty} \mathrm{d}s\, \frac{e^{i(\Delta^-_{a,\sigma})s}+e^{i(\Delta^+_{a,\sigma})s}}{\sinh^2(T\pi(s+i0))}\,,\\ \Gamma^{a,b}_{\text{out},-} &= -T^2\pi^2\sum_\sigma \alpha_0^\sigma \int_0^{+\infty} \mathrm{d}s\, \frac{e^{-i(\Delta^-_{b,\sigma})s}+e^{-i(\Delta^+_{b,\sigma})s}}{\sinh^2(T\pi(s-i0))}\end{aligned} \tag{D.24}$$

and for the *in-rates*

$$\begin{aligned}\Gamma^{a+1,b+1}_{\text{in},+} &= -e^{i(\Delta E^+_a-\Delta E^+_b)t}T^2\pi^2\sum_\sigma \alpha_0^\sigma \int_0^{+\infty} \mathrm{d}s \left\{\frac{e^{i(\Delta^+_{b,\sigma})s}}{\sinh^2(T\pi(s-i0))}+\frac{e^{-i(\Delta^+_{a,\sigma})s}}{\sinh^2(T\pi(s+i0))}\right\},\\ \Gamma^{a-1,b-1}_{\text{in},-} &= -e^{i(\Delta E^-_a-\Delta E^-_b)t}T^2\pi^2\sum_\sigma \alpha_0^\sigma \int_0^{+\infty} \mathrm{d}s \left\{\frac{e^{-i(\Delta^-_{b,\sigma})s}}{\sinh^2(T\pi(s-i0))}+\frac{e^{-i(\Delta^-_{a,\sigma})s}}{\sinh^2(T\pi(s+i0))}\right\},\end{aligned} \tag{D.25}$$

where the α_0^σ given in (3.138) and $\Delta^\pm_{k,\sigma}$ are given in (3.133). The final Makovian master equation is again given by (3.134).

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Köln, den 17. April 2018

(Maike Schön)