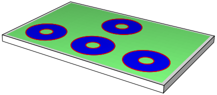
Physical properties of entangled Majorana fermion states on textured surfaces of topological insulators

Colloquium
Maike Schön

February 19, 2018



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Content

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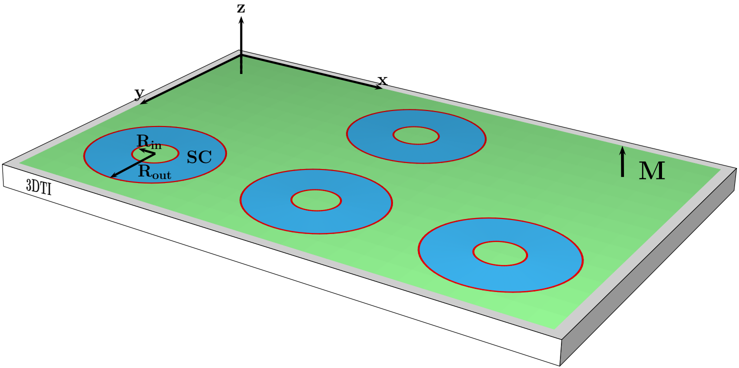
3. ***Fusion***

Main characteristics of a quantum bit architecture

1. ***Existence* of topological protected Majorana zero modes located at the boundaries**
2. ***Controlled* quantum state *manipulations***
   (**braiding** of Majorana bound states)
3. ***Fusion***
4. ***Read out***
   (method for initialization and read out of the quantum bit states)

The architecture



SC: superconducting ring (blue)
3DTI: three dimensional topological insulator thin film (white)
M: ferromagnetic dopant (green)

Question:

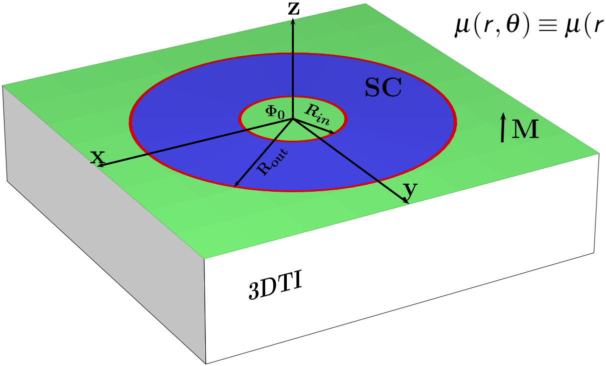
Is that possibly a
new potential quantum bit architecture?

Subsystem of consideration effective (2D) surface Hamiltonian

$\mathscr{H}_{\text{surface}} = \frac{i}{2}\left[\sigma\mathbf{p}, \sigma\mathbf{n}\right] = \frac{i}{2}\mathbf{pn} + \frac{1}{2}\left(\mathbf{n}\left(\mathbf{p}\times\sigma\right) + \left(\mathbf{p}\times\sigma\right)\mathbf{n}\right)$
(regime of low-energy excitations)

electrochemical potential:

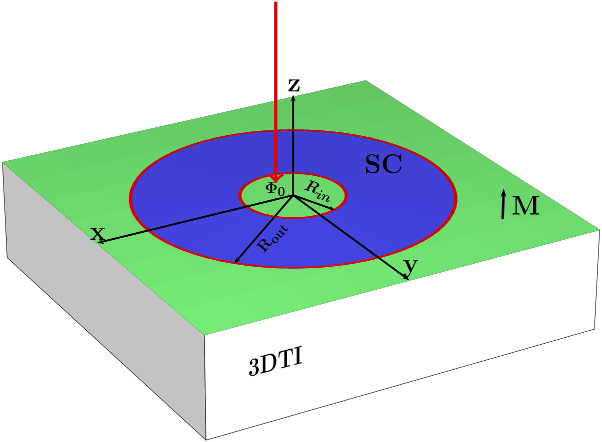
$\mu(r,\theta) \equiv \mu(r)$



effective surface Hamiltonian of the 3DTI

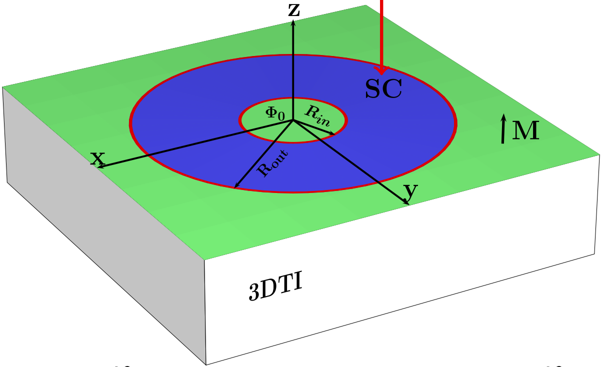
$$\mathscr{H}_{\text{3DTI}} = v_F\left(\boldsymbol{\sigma}\times\boldsymbol{p}\right)\hat{\boldsymbol{e}}_z\tau_z - \mu\sigma_0\tau_z = v_F\left(\sigma_x p_y - \sigma_y p_x\right)\tau_z - \mu\sigma_0\tau_z$$

Subsystem of consideration magnetic flux quantum $\Phi_0 = \frac{h}{2e}$



$\Phi_0 = \frac{h}{2e}$ created by external magnetic field $\boldsymbol{B} = |B|\,\hat{\boldsymbol{e}}_z$ (set $\hbar = 1$)

Subsystem of consideration   proximitized super conductor $\Delta(r,\theta)$

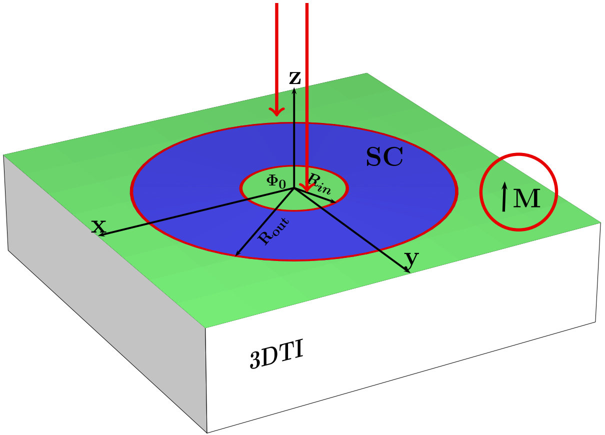


$$\Delta(r,\theta) = \Delta_0 e^{i\theta}\Theta(R_{out}-r)\Theta(r-R_{in}) = \vartheta(r)e^{i\theta} \text{ (blue)}$$

effective proximity induced Hamiltonian:

$$\mathscr{H}_\Delta = \frac{1}{2}\left[\Delta\left(\tau_x + i\tau_y\right) + \Delta^*\left(\tau_x - i\tau_y\right)\right]$$

Subsystem of consideration   doped magnetic field $M(r)$

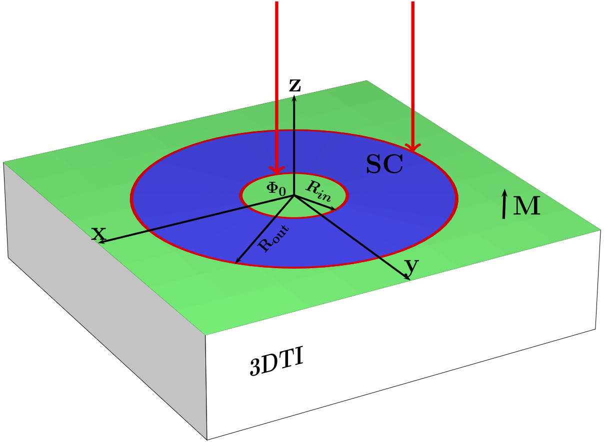


$$M(r,\theta) \equiv M(r) = M_0[\Theta(r - R_{out}) + \Theta(R_{in} - r)] \text{ (green)}$$

effective magnetic Hamiltonian: $\mathscr{H}_{\mathrm{M}} = \boldsymbol{M}(r)\boldsymbol{\sigma}\tau_0 = M(r)\sigma_z\tau_0$

Subsystem of consideration

geometric boundaries R_{in} and R_{out}



Goal: find states of zero energy at the boundaries R_{in} and R_{out} (red)

The full Hamiltonian matrix for the low energy surface states:

$$\mathscr{H} = \mathscr{H}_{\mathrm{3DTI}} + \mathscr{H}_{\Delta} + \mathscr{H}_{\mathrm{M}} = \begin{pmatrix} -\mu + M & v_F p_+ & \Delta & 0 \\ v_F p_- & -\mu - M & 0 & \Delta \\ \Delta^* & 0 & \mu + M & -v_F p_+ \\ 0 & \Delta^* & -v_F p_- & \mu - M \end{pmatrix}$$

The total Hamiltonian: $H = {}^1\!/_2 \Psi^\dagger \mathscr{H} \Psi$

(in polar coordinates)
$p_+ = e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right)$, $p_- = -e^{+i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right)$, $p_+ = p_-^*$

$\psi = \left(\psi_\uparrow, \psi_\downarrow\right)^T$ (spin space)
$\Psi = ((\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger))^T$ (Nambu spinor space)

Important property of the Hamiltonian matrix: $\mathscr{H}$

particle-hole symmetry: $\Xi\mathscr{H}\Xi = -\mathscr{H}$,
with particle-hole symmetry operator $\Xi = \sigma_x \tau_y \hat{C}$

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$\mathscr{H}\Psi_E = E\Psi_E$ and $E = 0$

$\Rightarrow \Xi\Psi_0 = \Psi_0$

$\Rightarrow \Psi_0 = (a, b, b^*, -a^*)^T = \Psi_0(r, \theta)$

with $a(r,\theta) \rightarrow \psi_\uparrow$ and $b(r,\theta) \rightarrow \psi_\downarrow$

different topological phases in different geometrical regions

magnetic region

$$\varepsilon_M(\boldsymbol{k}) = -\mu \pm \sqrt{v^2k^2 + M_0^2} \Rightarrow \varepsilon_M(\boldsymbol{k}=0) = -\mu \pm |M_0|$$

$$\begin{array}{lcccl} \text{(i)} & |M_0| & < & |\mu| & \text{(topological region)} \\ \text{(ii)} & |M_0| & > & |\mu| & \text{(topological trivial region)} \end{array}$$

superconducting region:

$$\varepsilon(\boldsymbol{k}) = \sqrt{(\mu \pm v_F\,|\boldsymbol{k}|)^2 + |\Delta_0|^2}$$

$$|\Delta_0| \neq 0 \quad \text{(topological region)}$$

assume: $\mu(r) \equiv \mu \equiv const.$ **after adjustment**

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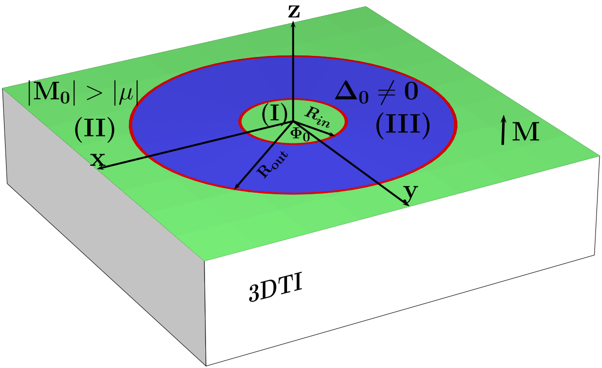
chemical potential

assume: $\mu(r) \equiv \mu \equiv const.$ **after adjustment**

solve for each region separately

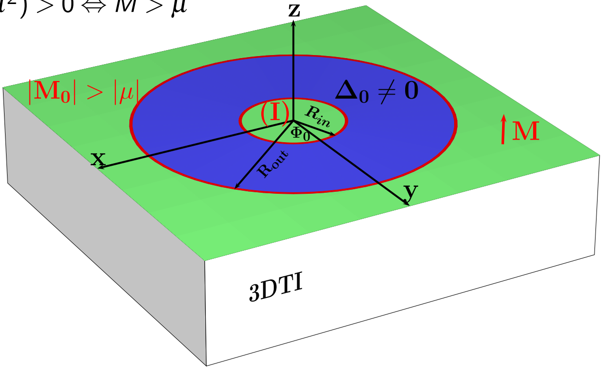


for each region reduce to a set of two **Bessel differential equations**

in the following $f(r) \to \psi_\uparrow$ and $g(r) \to \psi_\downarrow$

solutions for different regions

$a = \sqrt{(M^2 - \mu^2)} > 0 \Leftrightarrow M > \mu$



for region (I):

$$f_<^{out}(r) = c_<^{out} I_0(ar)$$
$$g_<^{out}(r) = -bc_<^{out} I_1(ar)$$

solutions for different regions

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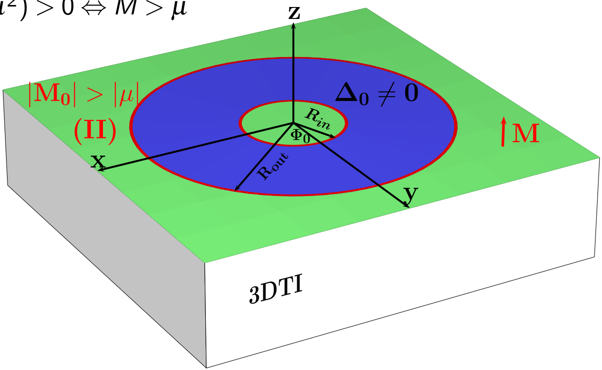
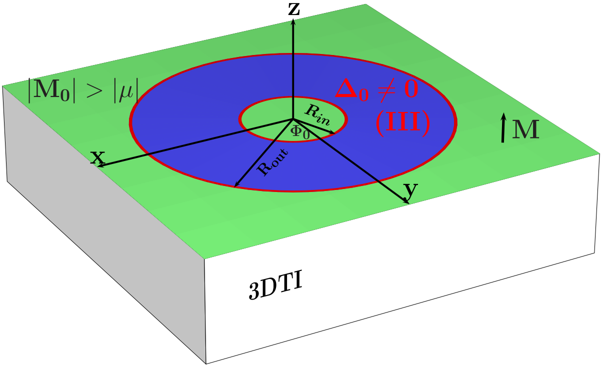


for region (II):

$$f_{>}^{out}(r) = c_{>}^{out} K_0(ar)$$
$$g_{>}^{out}(r) = bc_{>}^{out} K_1(ar)$$

solutions for different regions



for region (III):
$$f^{in}(r) = c_1^{in} J_0(\mu r) + c_2^{in} Y_0(\mu r)$$
$$g^{in}(r) = c_1^{in} J_1(\mu r) + c_2^{in} Y_1(\mu r)$$

Wave functions of zero energy at the boundaries

$$\Psi^{R_{in}}{}_0(r,\theta) = e^{-V(r)} \begin{pmatrix} f_<(r) \\ g_<(r)e^{i\theta} \\ g_<(r)e^{-i\theta} \\ -f_<(r) \end{pmatrix}$$

$$\Psi^{R_{out}}{}_0(r,\theta) = e^{-V(r)} \begin{pmatrix} f_>(r) \\ g_>(r)e^{i\theta} \\ g_>(r)e^{-i\theta} \\ -f_>(r) \end{pmatrix}$$

outer boundary: $V(r) = \int\limits_r^{R_{out}} \mathrm{d}r' \vartheta(r'),$

inner boundary: $V(r) = \int\limits_{R_{in}}^{r} \mathrm{d}r' \vartheta(r'),$

with $\vartheta(r) = \Delta_0 \Theta(R_{out} - r)\Theta(r - R_{in})$

When they are localized at the boundaries?

$$\Psi^{R_{in}}{}_0(r,\theta) = e^{-V(r)} \begin{pmatrix} f_<(r) \\ g_<(r)e^{i\theta} \\ g_<(r)e^{-i\theta} \\ -f_<(r) \end{pmatrix}$$

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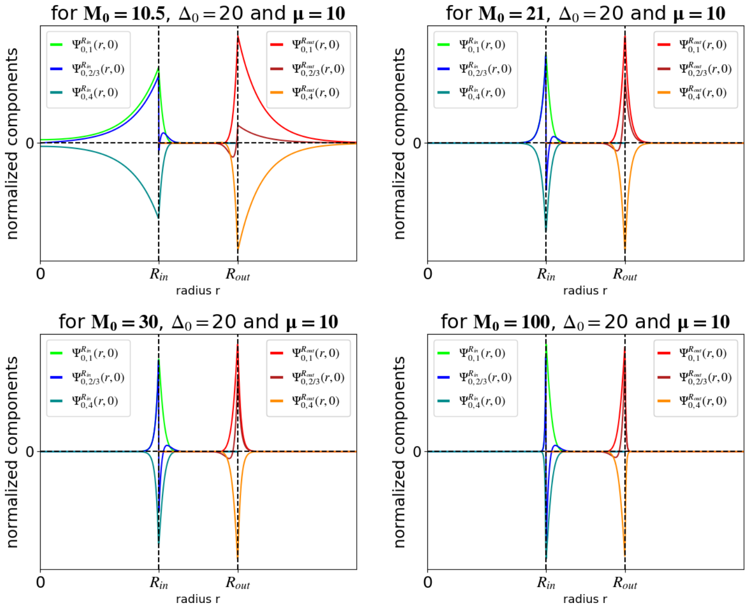
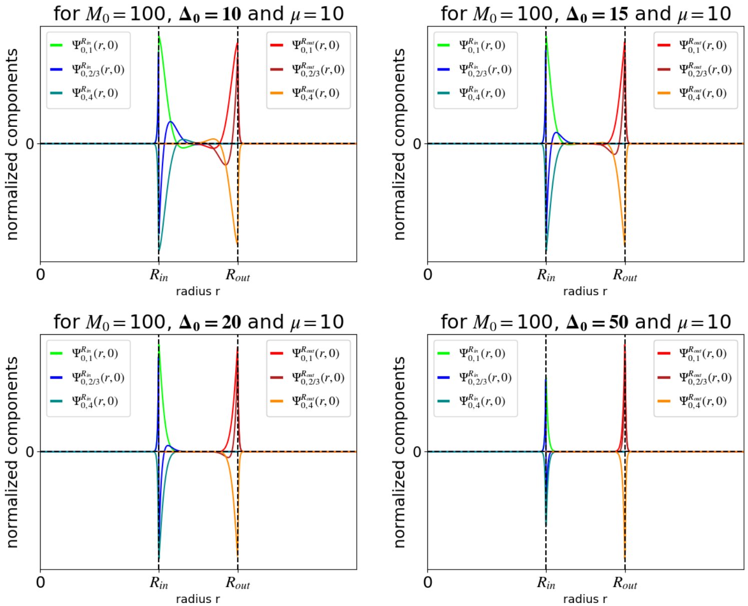
Edge state components for different magnetic fields relative to μ



Edge state components for different values for gap strength Δ_0 (SC)



Majorana zero mode operators

$\Psi = ((\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger))^T$

$$\gamma_0^\alpha = \int \mathrm{d}\boldsymbol{r} (\Psi_0^\alpha(\boldsymbol{r}))^\dagger \Psi(\boldsymbol{r}) \ ,$$
$$(\gamma_0^\alpha)^\dagger = \int \mathrm{d}\boldsymbol{r} \Psi^\dagger(\boldsymbol{r}) \Psi_0^\alpha(\boldsymbol{r}) \ ,$$

for real functions $f(r)$, $g(r)$ it is

$$\gamma_0^{R_{in}} = \left(\gamma_0^{R_{in}}\right)^\dagger \quad \text{and} \quad \gamma_0^{R_{out}} = \left(\gamma_0^{R_{out}}\right)^\dagger$$

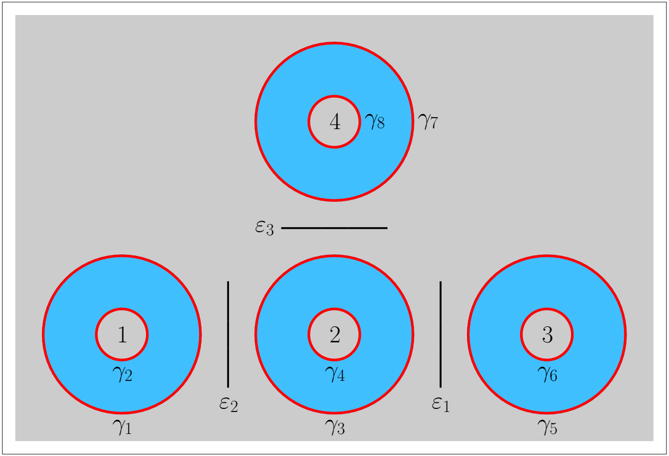
$$\Psi_0^{R_{in}}(r,\theta) = e^{-V(r)} \begin{pmatrix} f_<(r) \\ g_<(r)e^{i\theta} \\ g_<(r)e^{-i\theta} \\ -f_<(r) \end{pmatrix}, \quad \Psi_0^{R_{out}}(r,\theta) = e^{-V(r)} \begin{pmatrix} f_>(r) \\ g_>(r)e^{i\theta} \\ g_>(r)e^{-i\theta} \\ -f_>(r) \end{pmatrix}$$

Questions

How to provide ***Controlled*** **quantum state** ***manipulations***?

What are the ***exchange statistics***
of the present Majorana bound states?

schematic view: basis setup for an exchange process of two Majorana zero modes



$$\varepsilon_i(t) = 1 \quad \text{, for gate } i \text{ is switched on,}$$
$$\varepsilon_i(t) = 0 \quad \text{, for gate } i \text{ is switched off.}$$

adiabatically exchange of two Majorana zero modes

Throughout the exchange process:

stay in the degenerated ground state manifold!

Avoid ground state excitations!

The speed of the exchange process is limited by the energy gap.

adiabatically exchange of two Majorana zero modes

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adiabatic time evolution process from and back to an initial systems parameter set

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Throughout the exchange process:

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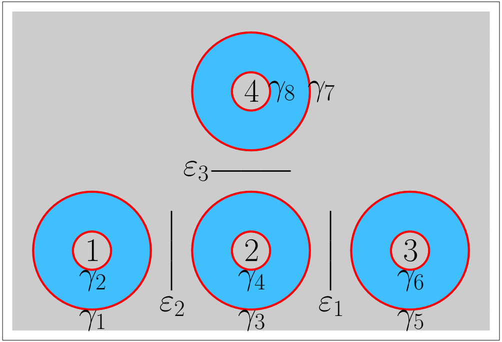
adiabatic time evolution process from and back to an initial systems parameter set
$\Rightarrow$ **unitary evolution of the systems ground state**

$\Rightarrow$ **braiding**

$\rightarrow$ use the concept of **Berry phase**

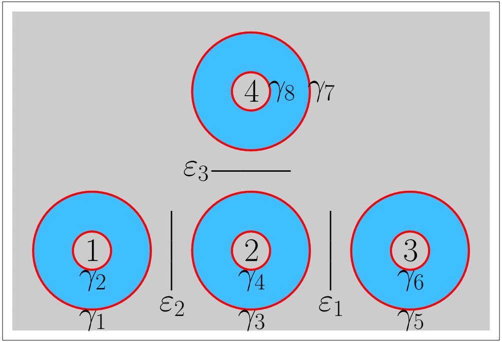
effective system Hamiltonian



$$
\begin{aligned}
H_{eff} &= i\left(n_1\varepsilon_1(t)\gamma_3\gamma_5 + n_2\varepsilon_2(t)\gamma_1\gamma_3 + n_3\varepsilon_3(t)\gamma_7\gamma_3\right) \\
&= \alpha(t)(f_2 f_1 + f_2^\dagger f_1) - \beta(t)(f_2^\dagger f_1^\dagger + f_2 f_1^\dagger) + \varepsilon_3(t)(1 - 2f_2^\dagger f_2),
\end{aligned}
$$

$\alpha(t) = \varepsilon_1(t) - i\varepsilon_2(t)$, $\beta(t) = \varepsilon_1(t) + i\varepsilon_2(t)$ and we choose $n_i = 1$ for all $i = 1, 2, 3$

$2^4 = 16$ fold degenerated ground state manifold



$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_5), \quad f_2 = \frac{1}{2}(\gamma_3 + i\gamma_7),$$
$$f_3 = \frac{1}{2}(\gamma_2 + i\gamma_6), \quad f_4 = \frac{1}{2}(\gamma_4 + i\gamma_8),$$

effective system Hamiltonian

description of
the time evolution of the system state
in the ground state manifold

$$
\begin{aligned}
H_{eff} &= i\left(n_1\varepsilon_1(t)\gamma_3\gamma_5 + n_2\varepsilon_2(t)\gamma_1\gamma_3 + n_3\varepsilon_3(t)\gamma_7\gamma_3\right) \\
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effective system Hamiltonian

the time evolution of the system state

$$\begin{aligned} H_{eff} &= i\left(n_1\varepsilon_1(t)\gamma_3\gamma_5 + n_2\varepsilon_2(t)\gamma_1\gamma_3 + n_3\varepsilon_3(t)\gamma_7\gamma_3\right) \\ &= \alpha(t)(f_2 f_1 + f_2^\dagger f_1) - \beta(t)(f_2^\dagger f_1^\dagger + f_2 f_1^\dagger) + \varepsilon_3(t)(1 - 2f_2^\dagger f_2), \end{aligned}$$

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steps of reduction

effective system Hamiltonian

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steps of reduction

- matrix representation (16×16) matrix

effective system Hamiltonian

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steps of reduction

- matrix representation (16×16) matrix
- all fermion operators consisting of inner boundary modes **commute** with H_{eff}!
  $\Rightarrow$ reduction to a (4×4) matrix
  $|00\rangle\,,\ |11\rangle = f_1^\dagger f_2^\dagger |00\rangle\,,\ |10\rangle = f_1^\dagger |00\rangle\,,\ |01\rangle = f_2^\dagger |00\rangle$

effective system Hamiltonian

$$H_{eff} = \begin{array}{c} \begin{array}{cccc} |00\rangle & |11\rangle & |10\rangle & |01\rangle \end{array} \\ \begin{pmatrix} \varepsilon_3 & \varepsilon_1 - i\varepsilon_2 & 0 & 0 \\ \varepsilon_1 + i\varepsilon_2 & -\varepsilon_3 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & \varepsilon_1 + i\varepsilon_2 \\ 0 & 0 & \varepsilon_1 - i\varepsilon_2 & -\varepsilon_3 \end{pmatrix} \end{array} = \begin{pmatrix} H_{even} & 0 \\ 0 & H_{odd} \end{pmatrix}$$

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- **parity conservation**
  $\Rightarrow$ either *even* or *odd* parity subspace $\rightarrow (2 \times 2)$ matrix

effective system Hamiltonian

$$H_{even} = \varepsilon_1 \sigma_1 + \varepsilon_2 \sigma_2 + \varepsilon_3 \sigma_3 = \boldsymbol{\varepsilon}_{even} \cdot \boldsymbol{\sigma},$$
$$H_{odd} = \varepsilon_1 \sigma_1 - \varepsilon_2 \sigma_2 + \varepsilon_3 \sigma_3 = \boldsymbol{\varepsilon}_{odd} \cdot \boldsymbol{\sigma}$$

steps of reduction

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  $|00\rangle\,,\ |11\rangle = f_1^\dagger f_2^\dagger\,|00\rangle\,,\ |10\rangle = f_1^\dagger\,|00\rangle\,,\ |01\rangle = f_2^\dagger\,|00\rangle$
- **parity conservation**
  $\Rightarrow$ either *even* or *odd* parity subspace $\rightarrow (2 \times 2)$ matrix

mapping on a sphere

reduced **time evolution of the system state**

$$H_{even} = \begin{pmatrix} \varepsilon \ \cos\theta & \varepsilon \ \sin\theta\, e^{-i\phi} \\ \varepsilon \ \sin\theta\, e^{i\phi} & -\varepsilon \ \cos\theta \end{pmatrix}$$

$\varepsilon = |\boldsymbol{\varepsilon}_{even}|$

eigenvalues: $\lambda_{\pm} = \pm\sqrt{\varepsilon_1(t)^2 + \varepsilon_2(t)^2 + \varepsilon_3(t)^2}$

mapping of system state time evolution onto the
time evolution of the gate vector

normalized gate vector: $\hat{\boldsymbol{\varepsilon}}(t) = \frac{\boldsymbol{\varepsilon}(t)}{|\boldsymbol{\varepsilon}(t)|}$, for all time: $\boldsymbol{\varepsilon}(t) \neq 0$

$\rightarrow$ **unit vector moving on a sphere**

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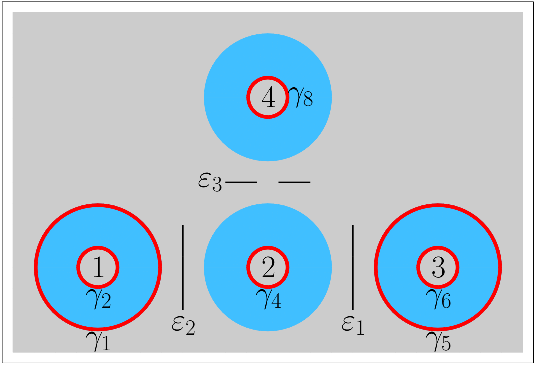
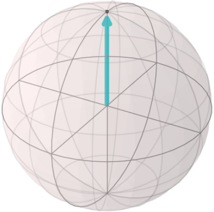
Consider now the time evolution...

$t = t_0$



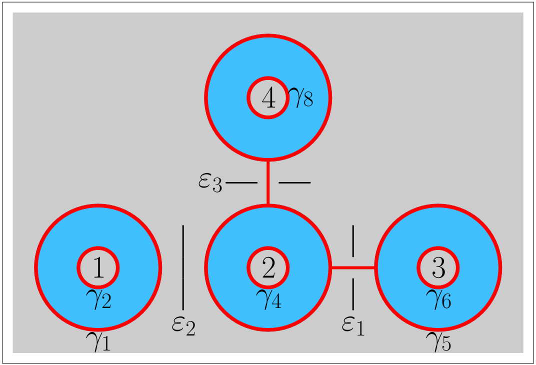


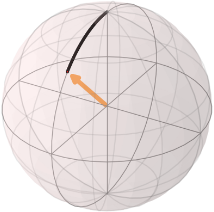
$$\hat{\boldsymbol{\varepsilon}}(t_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

initial Hamiltonian

$$H_{eff}(t_0) = i\gamma_7\gamma_3$$

$t = t_1$

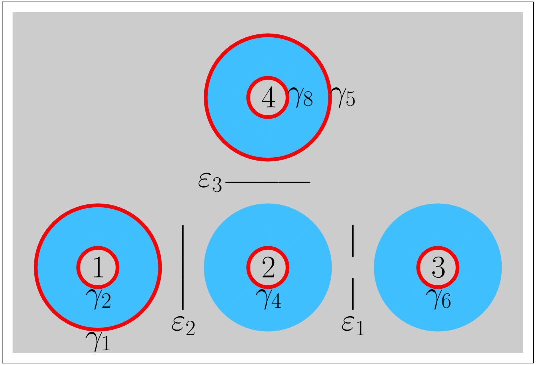


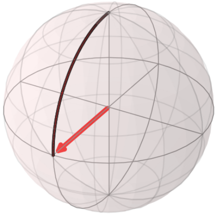


$$\hat{\boldsymbol{\varepsilon}}(t_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$H_{eff}(t_1) = i(\gamma_7\gamma_3 + \gamma_3\gamma_5)$$

$t = t_2$

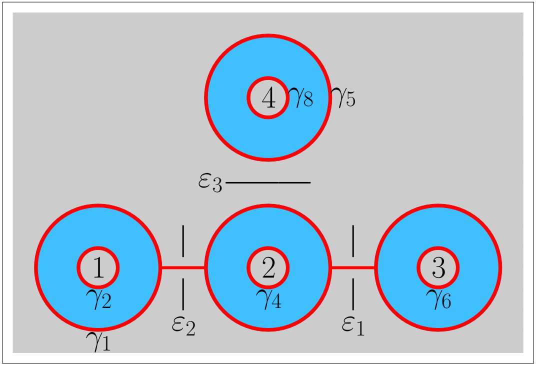


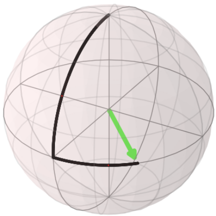


$$\hat{\boldsymbol{\varepsilon}}(t_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$H_{eff}(t_2) = i\gamma_3\gamma_5$$

$t = t_3$

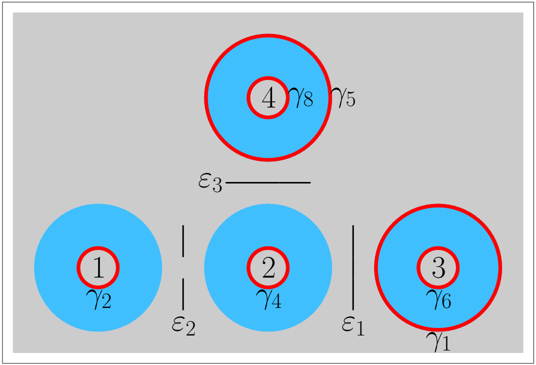


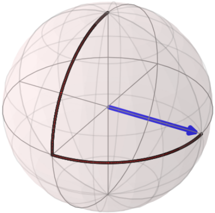


$$\hat{\boldsymbol{\varepsilon}}(t_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$H_{eff}(t_3) = i(\gamma_1\gamma_3 + \gamma_3\gamma_5)$$

$t = t_4$

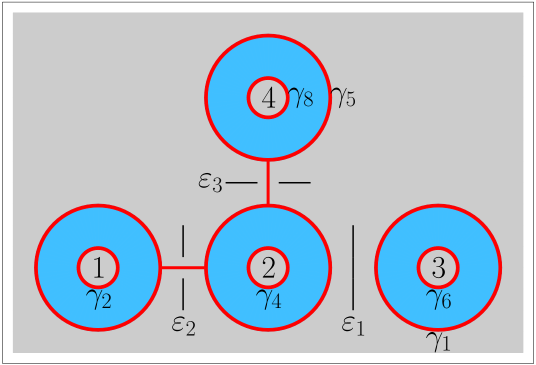


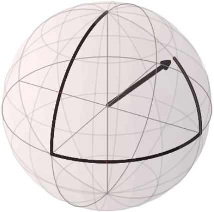


$$\hat{\boldsymbol{\varepsilon}}(t_4) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$H_{eff}(t_4) = i\gamma_1\gamma_3$$

$t = t_5$

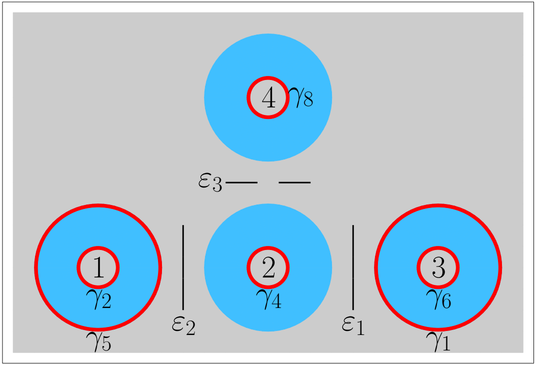


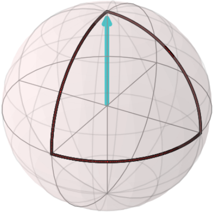


$$\hat{\boldsymbol{\varepsilon}}(t_5) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$H_{eff}(t_5) = i(\gamma_1\gamma_3 + \gamma_7\gamma_3)$$

$t = t_6$



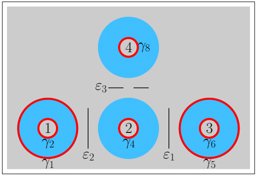


$$\hat{\boldsymbol{\varepsilon}}(t_6) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

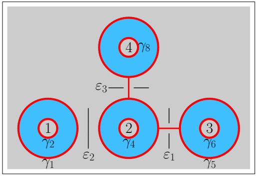
final Hamiltonian

$$H_{eff}(t_6) = i\gamma_7\gamma_3 = H_{eff}(t_0)$$

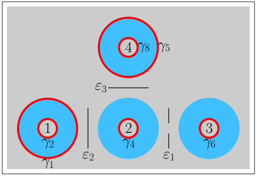
sketch of the braid process



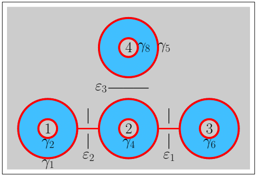
$t = t_0$



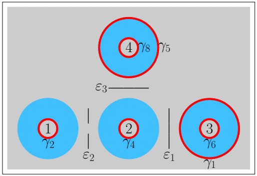
$t = t_1$



$t = t_2$



$t = t_3$



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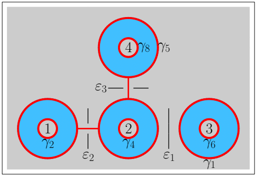

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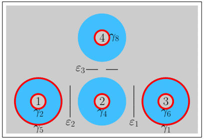
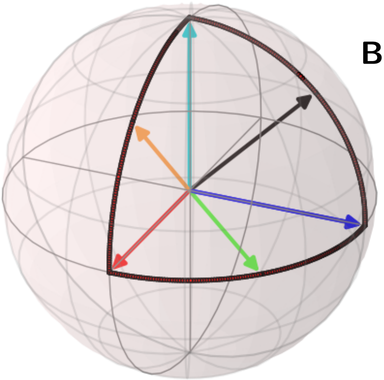


$t = t_6$

Berry phase



Berry curvature: $\mathscr{F}_{\theta\phi} = \mp\frac{1}{2}\sin\theta$

Berry phase:

$$i\varphi_{even} = -i\int_0^\theta\int_0^\phi \mathrm{d}\theta'\mathrm{d}\phi'\mathscr{F}_{\theta'\phi'} = \pm\frac{i}{2}\Omega$$

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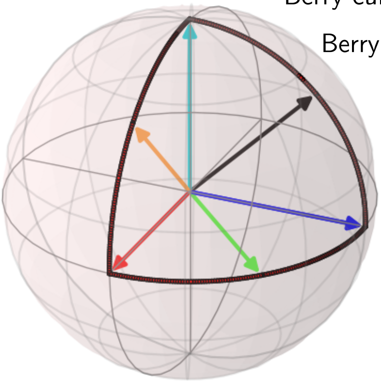

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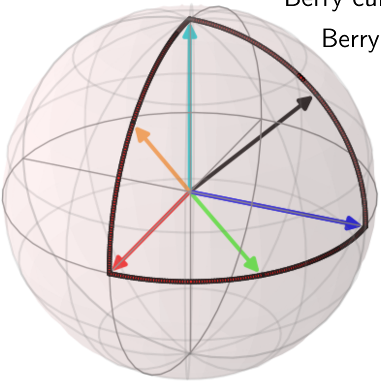

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results

for the even and odd parity space

$$\boldsymbol{e^{i\varphi_{even}} = e^{i\frac{\pi}{4}}}$$

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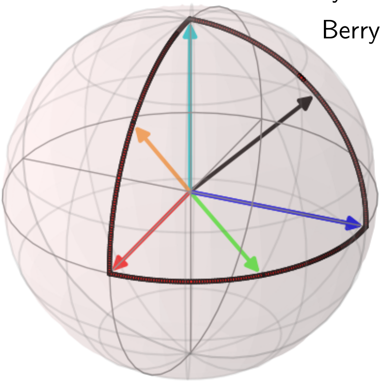
Berry phase



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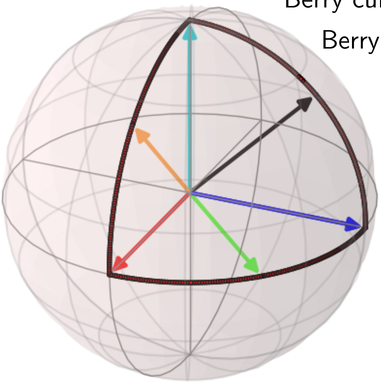

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acting on system states: $\boldsymbol{U\,|\Psi\rangle = e^{\pm i\frac{\pi}{4}}\,|\Psi\rangle}$

with $\boldsymbol{U_{15}\gamma_1 U_{15}^{\dagger} = -\gamma_5}$ and $\boldsymbol{U_{15}\gamma_5 U_{15}^{\dagger} = \gamma_1}$

fusion

anyon types of mzm

$$\sigma \times \sigma = \mathrm{I} + \Psi$$

vacuum channel particle channel

split up the ground state degeneracy

- **gating (adjust μ) $\rightarrow$ overlap of 2 outer Majorana wave functions**

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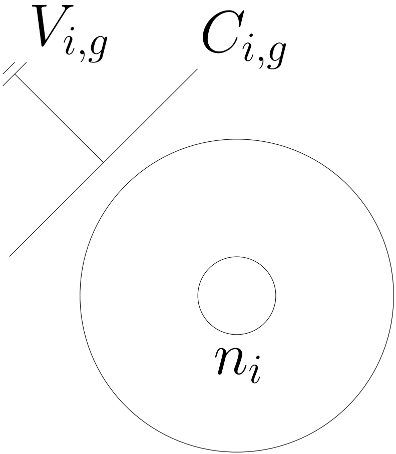
- **parity-to-charge conversion** $\rightarrow$ inner and outer Majoranas to charge state

parity-to-charge conversion

Fuse inner and outer Majoranas of one ring



Goal: read out the information stored in a quantum bit after processing manipulations.

- **initialize the quantum bit in a well defined state**

- create pairs of Majorana zero mode out of the vacuum (degenerated ground state manifold)

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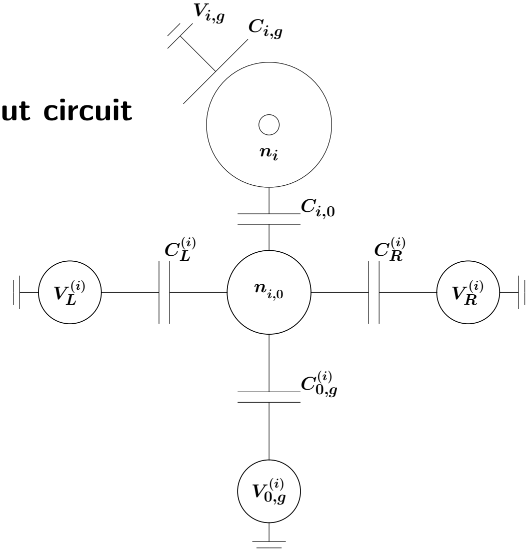
- **read out** the final system state

initialization: *parity-to-charge conversion* (side gate at each ring)

One read out circuit



$E_{el}(n_i, n_{i,0})$

read out: *charge sensing* (**S**ingle **E**lectron **T**ransistor)

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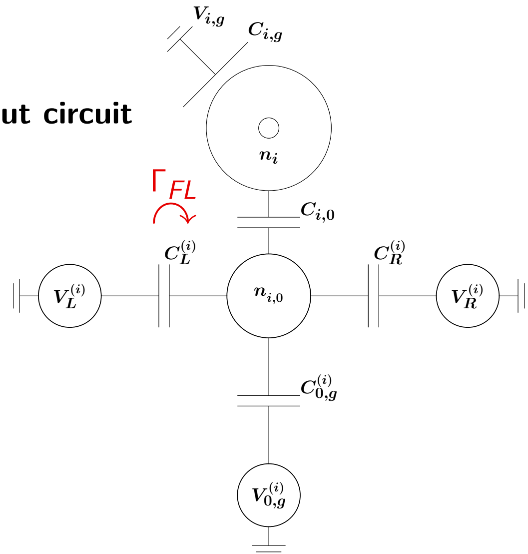

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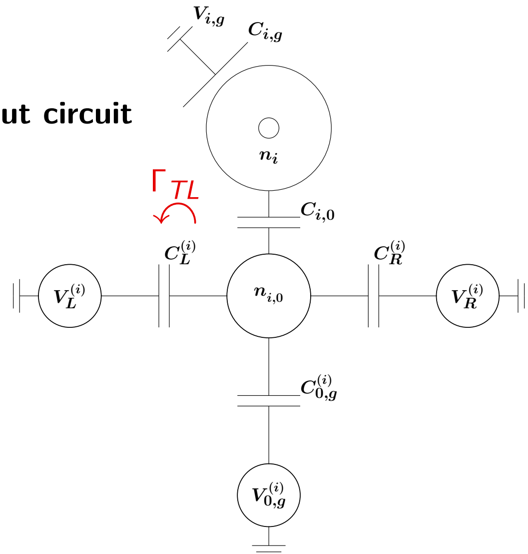

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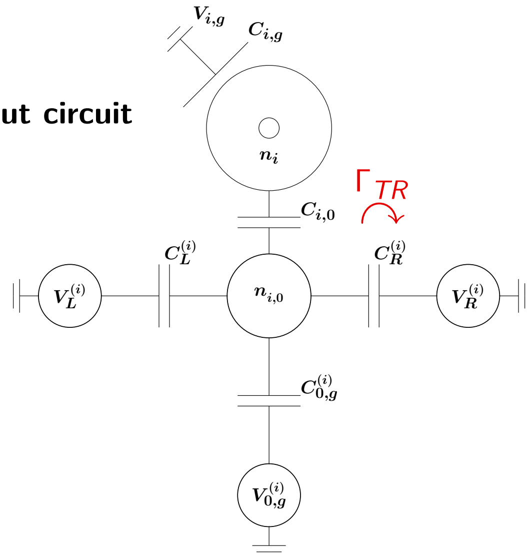

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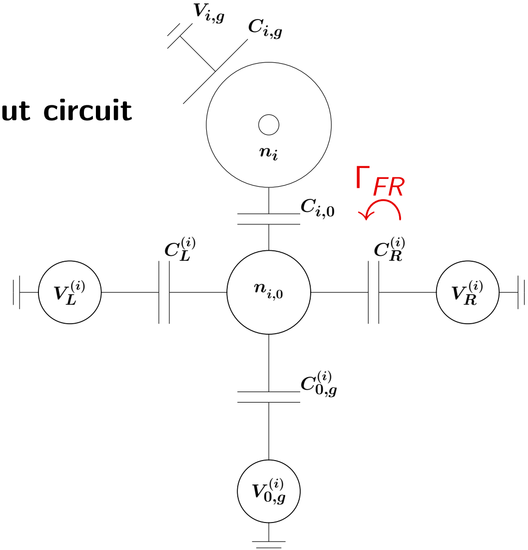

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Quantum mechanical solution of the master equation

System of two read out circuits:

$$|a\rangle, |b\rangle, |c\rangle, |d\rangle \in \Big\{ \left|0_{qb}0_1 0_2\right\rangle, \left|0_{qb}0_1 1_2\right\rangle, \left|0_{qb}1_1 0_2\right\rangle, \left|0_{qb}1_1 1_2\right\rangle, \left|1_{qb}0_1 0_2\right\rangle, \left|1_{qb}0_1 1_2\right\rangle, \left|1_{qb}1_1 0_2\right\rangle, \left|1_{qb}1_1 1_2\right\rangle \Big\}$$

general notation: $|a\rangle = |\alpha \ n_{1,0} \ n_{2,0}\rangle$, $|b\rangle = |\beta \ m_{1,0} \ m_{2,0}\rangle$,...

Goal: determine the tunneling rates Γ in the SET and its dependence on the quantum bit state

The final Makovian master equation

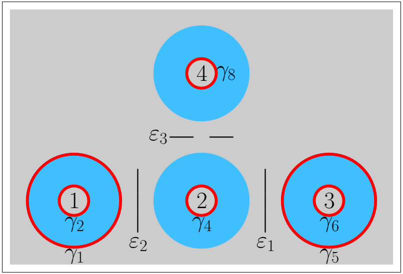
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S^{ab}(t) = -\Gamma_{\text{out}}^{a,b}\,\rho_S^{ab}(t) + \Gamma_{\text{in,-}}^{a-1,b-1}\rho_S^{a-1,b-1}(t) + \Gamma_{\text{in},+}^{a+1,b+1}\rho_S^{a+1,b+1}(t)$$

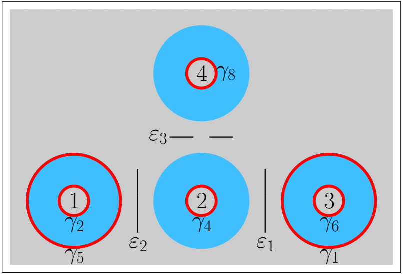
notation: $a \pm 1 = |\alpha \;\; n_{1,0} \pm 1 \;\; n_{2,0}\rangle$, $b \pm 1 = |\beta \;\; m_{1,0} \pm 1 \;\; m_{2,0}\rangle$

$\Gamma_{\text{in},\pm}^{a\pm1,b\pm1} \equiv \Gamma_{\text{in},\pm}(\alpha, \beta, n_{1,0}, m_{1,0}, t)$, $\Gamma_{\text{out}}^{a,b} \equiv \Gamma_{\text{out}}(\alpha, \beta, n_{1,0}, m_{1,0})$

$\rightarrow$ **current** through the SET depends on the fixed qubit charge state

Fusion inner and outer Majoranas after braiding





$$t_0:\ f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2),\quad f_2 = \frac{1}{2}(\gamma_6 + i\gamma_5)$$

$$t_6:\ c_1 = \frac{1}{2}(\gamma_5 + i\gamma_2)\quad c_2 = \frac{1}{2}(\gamma_6 + i\gamma_1)$$

$$\begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix}$$

Reduced density matrix in the long time limit

$$\begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix}$$

initial state after one braiding:

$|00\rangle \rightarrow |\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$

$$\hat{\rho}_{in}(0) = |\Psi_{in}\rangle \langle\Psi_{in}| = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \xrightarrow{t\rightarrow+\infty} \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\rho}_{\infty}$$

$\rightarrow$ measuring **long** enough:
get **both** possible qubit state **outcomes with equal probability**

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get **both** possible qubit state **outcomes with equal probability**

Reduced density matrix in the long time limit

$$\begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix}$$

initial state after one braiding:

$$|00\rangle \rightarrow |\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$$

$$\hat{\rho}_{in}(0) = |\Psi_{in}\rangle\,\langle\Psi_{in}| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \xrightarrow{t\rightarrow+\infty} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\rho}_\infty$$

$\rightarrow$ measuring **long** enough:
get **both** possible qubit state **outcomes with equal probability**

Measure the braid outcome

$$\begin{pmatrix} |00\rangle_f \\ |11\rangle_f \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} |00\rangle_c \\ |11\rangle_c \end{pmatrix}$$

initial state after one braiding:

$$|00\rangle \rightarrow |\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$$

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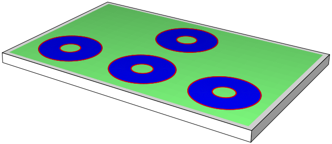
$\rightarrow$ measuring **long** enough:
get **both** possible qubit state **outcomes with equal probability**

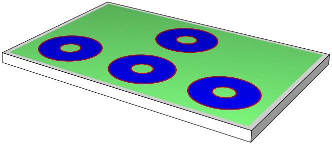
Conclusion

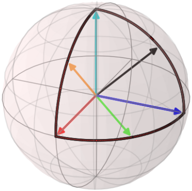
- **Existence of localized Majorana zero modes**



Conclusion

- Existence of **localized Majorana zero modes**
- **Braiding** of Majorana zero modes is possible





Conclusion

- Existence of **localized Majorana zero modes**
- **Braiding** of Majorana zero modes is possible
- **Measuring** of the outcome of braiding is realizable