BAPC 2015

Teams UCooL and UCooLTwo

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Contents

1 Remarks

1.1 Warning!

1. Read every statement!
2. Do not copy-paste without thinking about it.
3. Be careful of overflows! Use long!
4. **Do not trust this document!**

1.2 Operations on bits

1. Check parity of n: `(n & 1) == 0`
2. 2^n: `1L << n.`
3. Test of the ith bit of n is 0: `(n & 1L << i) != 0`
4. Set the ith bit of n at 0: `n &= ~(1L << i)`
5. Set the ith bit of n at 1: `n |= (1L << i)`
6. Union: `a | b`
7. Intersection: `a & b`
8. Subtraction bits: `a & ~b`
9. Verify if n is a power of 2: `(n & (n-1) == 0)`
10. Least significant bit not null of n: `(n & (-n))`
11. Negate: `0 x7fffffff ^n`

1.3 Complexity table

n $\leq$	Maximum complexity
$[10, 11]$	$O(n!), O(n^6)$
$[15, 18]$	$O(2^n n^2)$
$[18, 22]$	$O(2^n n)$
100	$O(n^4)$
400	$O(n^3)$
$2K$	$O(n^2 \log(n))$
$10K$	$O(n^2)$
$1M$	$O(n \log(n))$
$10M$	$O(n), O(\log(n)), O(1)$

Not so obvious complexity:

$$\sum_{k=1}^{n} \frac{1}{k} = O(\log(n))$$

2 Graphs

2.1 Basics

- Adjacency matrix: $A[i][j] = 1$ if i is connected to j and 0 otherwise
- Undirected graph: $A[i][j] = A[j][i]\ \forall\ i, j\ (A = A^T)$
- Adjacency list: `LinkedList<Integer>[] g;` $g[i]$ stores all neighbors of i
- Useful alternatives:

```
HashSet<Integer>[] g; // for edge deletion
HashMap<Integer, Integer>[] g; // for weighted graph
```

- Basic classes

```
class Edge implements Comparable<Edge> {
  int o, d, w;
  public Edge(int o, int d, int w) {
    this.o = o; this.d = d; this.w = w;
  }
  public int compareTo(Edge o) {
    return w - o.w;
  }
}
```

2.2 BFS

Computes d, an array of distance from start vertex v. $d[v] = 0$, $d[u] = \infty$ if u not connected to v. If $(u, w) \in E$ and $d[u]$ known and $d[w]$ unknown, $d[w] = d[u] + 1$.

```
int[] bfsVisit(LinkedList<Integer>[] g, int v, int c[]) { //c is for connected components only
  Queue<Integer> Q = new LinkedList<Integer>();
  Q.add(v);
  int[] d = new int[g.length];
  c[v]=v; //for connected components
  Arrays.fill(d, Integer.MAX_VALUE);
  // set distance to origin to 0
  d[v] = 0;
  while(!Q.isEmpty()) {
    int cur = Q.poll();
    // go over all neighbors of cur
    for(int u : g[cur]) {
      // if u is unvisited
      if(d[u] == Integer.MAX_VALUE) { //or c[u] == -1 if we calculate connected components
        c[u] = v; //for connected components
        Q.add(u);
        // set the distance from v to u
        d[u] = d[cur] + 1;
      }
    }
  }
  return d;
}
```

2.2.1 Connected components

```
int[] bfs(LinkedList<Integer>[] g)
{
  int[] c = new int[g.length];
  Arrays.fill(c, -1);
  for(int v = 0; v < g.length; v++)
    if(c[v] == -1)
      bfsVisit(g, v, c);
  return c;
}
```

2.2.2 Girth

The girth of an undirected graph is the length of its shortest cycle (∞ if none). Complexity $O(|V||E|)$.

```java
int girth(LinkedList<Integer>[] g) {
  int girth = Integer.MAX_VALUE;
  for(int v = 0; v < g.length; v++) {
    girth = Math.min(girth, checkFromV(v, g));
  }
  return girth;
}

int checkFromV(int v, LinkedList<Integer>[] g) {
  int[] parent = new int[g.length];
  Arrays.fill(parent, -1);
  int[] d = new int[g.length];
  Arrays.fill(d, Integer.MAX_VALUE);
  Queue<Integer> Q = new LinkedList<Integer>();
  Q.add(v);
  d[v] = 0;
  while(!Q.isEmpty()) {
    int cur = Q.poll();
    for(int u : g[cur]) {
      if(u != parent[cur]) {
        if(d[u] == Integer.MAX_VALUE) {
          parent[u] = cur;
          d[u] = d[cur] + 1;
          Q.add(u);
        } else {
          return d[cur] + d[u] + 1;
        }
      }
    }
  }
  return Integer.MAX_VALUE;
}
```

2.3 DFS

Equals to BFS with *Stack* instead of *Queue* or recursive implementation. Complexity $O(|V| + |E|)$

```java
int UNVISITED = 0, OPEN = 1, CLOSED = 2;
boolean cycle; // true iff there is a cycle

void dfsVisit(LinkedList<Integer>[] g, int v, int[]
    label) {
  label[v] = OPEN;
  for(int u : g[v]) {
    if(label[u] == UNVISITED)
      dfsVisit(g, u, label);
    if(label[u] == OPEN)
      cycle = true;
  }
  label[v] = CLOSED;
}

void dfs(LinkedList<Integer>[] g) {
  int[] label = new int[g.length];
  Arrays.fill(label, UNVISITED);
  cycle = false;
  for(int v = 0; v < g.length; v++)
    if(label[v] == UNVISITED)
      dfsVisit(g, v, label);
}
```

2.3.1 Topological order

Graph must be acyclic.

```java
Stack<Integer> toposort; // add stack to global
    variables
/* ... */
void dfs(LinkedList<Integer>[] g) {
  /* ... */
  toposort = new Stack<Integer>();
  for(int v = 0; v < g.length; v++) { /* ... */ }
}

void dfsVisit(LinkedList<Integer>[] g, int v, int[]
    label) {
  /* ... */
  toposort.push(v); // push vertex when closing it
  label[v] = CLOSED;
}
```

2.3.2 Strongly connected components

Uses BFS following the topologic order.

```java
int[] scc(LinkedList<Integer>[] g) {
  // compute the reverse graph
  LinkedList<Integer>[] gt = transpose(g);
  // compute ordering
  dfs(gt);
  // !! last position will contain the number of scc
    's
  int[] scc = new int[g.length + 1];
  Arrays.fill(scc, -1);
  int nbComponents = 0;
  // simulate bfs loop but in toposort ordering
  while(!toposort.isEmpty()) {
    int v = toposort.pop();
    if(scc[v] == -1) {
      nbComponents++;
      bfsVisit(g, v, scc);
    }
  }
  scc[g.length] = nbComponents;
  return scc;
}
```

2.3.3 SCC, Bridges and Articulation Points in C

C version of SCC (shorter).

```c
void tarjanSCC(int u) {
  dfs_low[u] = dfs_num[u] = dfsNumberCounder++; //
    dfs_low[u] <= dfs_num[u]
  S.push_back(u); // stores u in a vector based on
    order of visitation
  visited[u] = 1;
  for(int j = 0; j < (int)AdjList[u].size(); j++) {
    ii v = AdjList[u][j];
    if(dfs_num[v.first] == UNVISITED)
    tarjanSCC(v.first);
    if(visited[v.first]) // condition for update
      dfs_low[u] = min(dfs_low[u], dfs_low[v.first])
    ;
  }
  if(dfs_low[u] == dfs_num[u]) { // if this is a
    root (start) of an SCC
    printf("SCC %d:", ++numSCC); // this part is
    done after recursion
    while(1) {
      int v = S.back(); S.pop_back(); visited[v] =
    0;
      printf(" %d", v);
      if(u == v) break;
    }
    printf("\n");
  }
}


int main() {
  dfs_num.assign(V, UNVISITED); dfs_low.assign(V, 0)
  ;
  visited.assign(V, 0); dfsNumberCounter = numSCC =
    0;
  for(int i = 0; i < V; i++)
    if(dfs_num[i] == UNVISITED)
      tarjanSCC(i);
}
```

Bridges are edges that, when removed, increases the number of connected components. Articulation points are the same, but for vertices.

```c
void articulationPointAndBridge(int u) {
```

```
  dfs_low[u] = dfs_num[u] = dfsNumberCounter++; //
    dfs_low[u] <= dfs_num[u]
  for(int j = 0; j < (int)AdjList[u].size(); j++) {
    ii v = AdjList[u][j];
    if(dfs_num[v.first] == UNVISITED) { // a tree
    edge
      dfs_parent[v.first] = u;
      if(u == dfsRoot) rootChildren++; // special
    case if u is a root
      articulationPointAndBridge(v.first);
      if(dfs_low[v.first] >= dfs_num[u]) // for
    articulation point
        articulation_vertex[u] = true; // store this
     information first
      if(dfs_low[v.first] > dfs_num[u]) // for
    bridge
        printf("Edge (%d %d) is a bridge\n", u, v.
    first);
      dfs_low[u] = min(dfs_low[u], dfs_low[v.first])
    ; // update dfs_low[u]
    }
    else if(v.first != dfs_parent[u]) // a back edge
     and not direct cycle
      dfs_low[u] = min(dfs_low[u], dfs_num[v.first])
    ; // update dfs_low[u]
  }
}

int main() {
  dfsNumberCounter = 0; dfs_num.assign(V, UNVISITED)
    ;
  dfs_low.assign(V, 0); dfs_parent.assign(V, 0);
    articulation_vertex.assign(V, 0);
  printf("Bridges:\n");
  for(int i = 0; i < V; i++) {
    dfsRoot = i; rootChildren = 0;
    articulationPointBridge(i);
    articulation_vertex[dfsRoot] = (rootChildren >
    1); // special case
  }
  printf("Articulation Points:\n");
  for(int i = 0; i < V; i++)
    if(articulation_vertex[i])
      printf("Vertex %d\n", i);
}
```

2.3.4 Directed Graph to toposorted DAG

In $O(n+m)$, with Tarjan SCC algo, we merge the SCCs and take the resulting DAG, (remembering their size in scc_size) which is reverse toposorted (i.e. node 0 has no outgoing edge), ready for bottom up DP (starting with node 0 ending with node N) !

```
static Integer[] dfs_num;
static int[] dfs_low, scc_id;
static BitSet visited;
static int dfsNumberCounter;
static Stack<Integer> S;
static void tarjanSCC(LinkedList<Integer>[] g, int u
    , LinkedList< LinkedList<Integer> > SCCs) {
  dfs_low[u] = dfsNumberCounter;
  dfs_num[u] = dfsNumberCounter++; // dfs_low[u] <=
    dfs_num[u]
  S.add(u); // stores u in a vector based on order
    of visitation
  visited.set(u);
  for(int v : g[u]) {
    if(dfs_num[v] == null)
      tarjanSCC(g, v, SCCs);
    if(visited.get(v)) // condition for update
      dfs_low[u] = Math.min(dfs_low[u], dfs_low[v]);
  }
  if(dfs_low[u] == dfs_num[u]) { // if this is a
    root (start) of an SCC
    LinkedList<Integer> newSCC = new LinkedList<
    Integer>();
    int id = SCCs.size();
    for (;;) {
      int v = S.pop(); visited.clear(v);
      newSCC.add(v);
      scc_id[v] = id;
      if(u == v) break;
    }
    SCCs.add(newSCC);
  }
}
static LinkedList<Integer>[] DirectedGraphToDag (
    LinkedList<Integer>[] g) {
  int n = g.length;
  dfs_num = new Integer[n];
  dfs_low = new int[n];
  scc_id = new int[n];
  visited = new BitSet(n);
  dfsNumberCounter = 0;
  S = new Stack<Integer>();
  LinkedList< LinkedList<Integer> > SCCs = new
    LinkedList< LinkedList<Integer> >();
  for(int i = 0; i < n; i++)
    if(dfs_num[i] == null)
      tarjanSCC(g, i, SCCs);
  int N = SCCs.size();
  @SuppressWarnings("unchecked")
  LinkedList<Integer>[] G = new LinkedList[N];
  scc_size = new int[N];
  int i = 0;
  for (LinkedList<Integer> SCC : SCCs) {
    G[i] = new LinkedList<Integer>();
    scc_size[i] = SCC.size();
    BitSet reachable = new BitSet(N);
    reachable.set(i);
    for (int u : SCC) {
      for (int v : g[u])
        if (!reachable.get(scc_id[v])) {
          G[i].add(scc_id[v]);
        }
    }
    i++;
  }
  return G;
}
static int[] scc_size; // bonus information
```

2.4 Minimum Spanning Tree

2.4.1 Prim

```
double prim(LinkedList<Edge>[] g) {
  boolean[] inTree = new boolean[g.length];
  PriorityQueue<Edge> PQ = new PriorityQueue<Edge>()
    ;
  // add 0 to the tree and initialize the priority
    queue
  inTree[0] = true;
  for(Edge e : g[0]) PQ.add(e);
  double weight = 0;
  int size = 1;
  while(size != g.length) {
    // poll the minimum weight edge in PQ
    Edge minE = PQ.poll();
    // if its endpoint in not in the tree, add it
    if(!inTree[minE.d]) {
      // add edge minE to the MST
      inTree[minE.d] = true;
      weight += minE.w;
      size++;
      // add edge leading to new endpoints to the PQ
      for(Edge e : g[minE.d])
        if(!inTree[e.d]) PQ.add(e);
    }
  }
  return weight;
}
```

2.4.2 Kruskal

Uses Union-Find (See section 8.4).

```
double kruskal(LinkedList<Edge> g, int n) {
  Collections.sort(g);
```

```
  UnionFind uf = new UnionFind(n);
  double w = 0;
  int c = 0;
  for(Edge e: g) {
    if(c == n-1) return w;
    if(uf.find(e.o) != uf.find(e.d)) {
      w+=e.w;
      c++;
      uf.union(e.o, e.d);
    }
  }
  return w;
}
```

2.5 Dijkstra

Shortest path from a node v to other nodes. Graph must not have any negative weighted cycle. $O((|V| + |E|) \log(|V|))$

```
double[] dijkstra(LinkedList<Edge>[] g, int v) {
  double[] d = new double[g.length];
  Arrays.fill(d, Double.POSITIVE_INFINITY);
  d[v] = 0;
  PriorityQueue<Edge> PQ = new PriorityQueue<Edge>()
    ;
  for(Edge e : g[v])
    PQ.add(e);
  while(!PQ.isEmpty()) {
    Edge minE = PQ.poll();
    if(d[minE.d] == Double.POSITIVE_INFINITY) {
      d[minE.d] = minE.w;
      for(Edge e : g[minE.dest])
        if(d[e.d] == Double.POSITIVE_INFINITY)
          PQ.add(new Edge(e.o, e.d, e.w + d[e.o]));
    }
  }
  return d;
}
```

2.6 Bellman-Ford

Shortest path from a node v to other nodes. Graph can have negative weighted cycles: Bellman-Ford won't give the correct shortest path, but will warn that a negative cycle exists. $O(|V||E|)$.

```
static double[] bellmanFord(LinkedList<Edge> gt, int
    v, int n) {
  double[] dist = new double[n];
  Arrays.fill(dist, Double.POSITIVE_INFINITY);
  dist[v] = 0;
  for(int i=0; i < n-1; i++)
    for(Edge e : gt)
      if(dist[e.o] + e.w < dist[e.d])
        dist[e.d] = dist[e.o] + e.w;
  for(Edge e : gt)
    if(dist[e.o] + e.w < dist[e.d])
      return null;
  return dist;
}
```

```
static double[] spfa (LinkedList<Edge>[] g, int s) {
  int n = g.length;
  double[] dist = new double[n];
  Arrays.fill(dist, Double.POSITIVE_INFINITY);
  Queue<Integer> q = new LinkedList<Integer>();
  BitSet inQueue = new BitSet(n);
  int[] timesIn = new int[n];
  dist[s] = 0;
  q.add(s);
  inQueue.set(s);
  timesIn[s]++;
  while (!q.isEmpty()) {
    int cur = q.poll(); inQueue.clear(cur);
    for (Edge next : g[cur]) {
      int v = next.d, w = next.w;
      if (dist[cur] + w < dist[v]) {
        dist[v] = dist[cur] + w;
        if (!inQueue.get(v)) {
          q.add(v);
          inQueue.set(v);
          timesIn[v]++;
          if (timesIn[v] >= n) {
            return null; // Infinite loop
          }
        }
      }
    }
  }
  return dist;
}
```

2.7 Floyd-Warshall

Shortest path from a node v to other nodes. Graph can have negative weighted cycles: Floyd-Warshall won't give the correct shortest path, but will warn that a negative cycle exists. Negative weighted cycles exists iif $result[v][v] < 0$. $O(|V|^3)$ in time and $O(|V|^2)$ in memory.

```
double[][] floydWarshall(double[][] A)
{
  int n = A.length;
  for(int k = 0; k < n; k++)
    for(int v = 0; v < n; v++)
      for(int u = 0; u < n; u++)
        A[v][u] = Math.min(A[v][u], A[v][k] + A[k][u
    ]);
        //or:
        A[v][u] = min(A[v][u], max(A[v][k], A[k][u])
    ); //minimax
        A[v][u] = max(A[v][u], min(A[v][k], A[k][u])
    ); //maximin
        A[v][u] = max(A[v][u], A[v][k] * A[k][u]);
    //safest path (A contains probability)
  return A;
}
```

2.8 Directed Max flow

2.8.1 Edmonds-Karps (BFS)

Path in residual graph searched via BFS. $O(|V||E|^2)$.

```
int maxflowEK(TreeMap<Integer, Integer>[] g, int
    source, int sink) {
  int flow = 0;
  int pcap;
  while((pcap = augmentBFS(g, source, sink)) != -1)
    {
    flow += pcap;
  }
  return flow;
}
```

```
int augmentBFS(TreeMap<Integer, Integer>[] g, int
    source, int sink) {
  // initialize bfs
  Queue<Integer> Q = new LinkedList<Integer>();
  Integer[] p = new Integer[g.length];
  int[] pcap = new int[g.length];
  pcap[source] = Integer.MAX_VALUE;
  p[source] = -1;
  Q.add(source);
  // compute path
  while(p[sink] == null && !Q.isEmpty()) {
    int u = Q.poll();
    for(Entry<Integer, Integer> e : g[u].entrySet())
     {
      int v = e.getKey();
      if(e.getValue() > 0 && p[v] == null) {
        p[v] = u;
```

```
            pcap[v] = Math.min(pcap[u], e.getValue());
            Q.add(v);
          }
        }
      }
      if(p[sink] == null) return -1;
      // update graph
      int cur = sink;
      while(cur != source) {
        int prev = p[cur];
        int cap = g[prev].get(cur);
        g[prev].put(cur, cap - pcap[sink]);
        Integer backcap = g[cur].get(prev);
        g[cur].put(prev, backcap == null? pcap[sink] :
        backcap + pcap[sink]);
        cur = prev;
      }
      return pcap[sink];
    }
```

2.8.2 Ford-Fulkerson

Equals to Edmonds-Karps, but with a DFS. $O(|E|f^*) = O(|V||E|^2)$ where f^* is the value of the max flow.

```
int pcap;

int maxflowFF(TreeMap<Integer, Integer>[] g, int
    source, int sink) {
  int flow = 0;
  pcap = Integer.MAX_VALUE;
  while(augmentDFS(g, source, sink, new boolean[g.
    length])) {
    flow += pcap;
    pcap = Integer.MAX_VALUE;
  }
  return flow;
}

boolean augmentDFS(TreeMap<Integer, Integer>[] g,
    int cur, int sink, boolean[] done) {
  if(cur == sink) return true;
  if(done[cur]) return false;
  done[cur] = true;
  for(Entry<Integer, Integer> e : g[cur].entrySet())
     {
    if(e.getValue() > 0) {
      int oldcap = pcap;
      pcap = Math.min(pcap, e.getValue());
      if(augmentDFS(g, e.getKey(), sink, done)) {
        g[cur].put(e.getKey(), e.getValue() - pcap);
        Integer backcap = g[e.getKey()].get(cur);
        g[e.getKey()].put(cur, backcap == null? pcap
     : backcap + pcap);
        return true;
      } else {
        pcap = oldcap;
      }
    }
  }
  return false;
}
```

2.8.3 Min cut

We search, between two nodes s and t, subsets of nodes V_1 and V_2 so as $s \in V_1$, $t \in V_2$ and $\sum_{e \in E(V_1,V_2)} w(e)$ minimum. We just have to compute the max-flow between s and t and to apply a BFS/DFS on the residual graph. All node which are visited are in V_1, others in V_2. The weight from the cut is the max-flow.

2.8.4 Maximum number of disjoint paths

For edge disjoint paths just compute the max flow with unit capacities. For vertex disjoint paths split vertices into two with unit capacity edge between them.

2.8.5 Maximum weighted bipartite matching

Assignment problem: Given a set of n persons and n jobs, and a cost matrix M, assign a job to each person such that the sum of the costs is minimized. It also works for n persons and m jobs with $n \neq m$. Just fill make a square matrix using dummy values. Can also be solve with min cost max flow but it is slower.

$O(n^3)$ solution:

```
static int[][] cost;
static int n;
static int[] lx, ly;
static int maxMatch;
static boolean[] S, T;
static int[] slack, slackx, prev, xy, yx;

static int[] minHungarian(int[][] M) {
  for(int i = 0; i < M.length; i++)
    for(int j = 0; j < M.length; j++)
      M[i][j] = -M[i][j];
  return maxHungarian(M);
}

static int[] maxHungarian(int[][] M) {
  cost = M;
  n = cost.length;
  slack = new int[n];
  slackx = new int[n];
  prev = new int[n];
  xy = new int[n];
  yx = new int[n];
  maxMatch = 0;
  for(int i = 0; i < n; i++) {
    xy[i] = -1;
    yx[i] = -1;
  }
  initLabels();
  augment();
  int ret = 0;
  int[] assignment = new int[n];
  for(int x = 0; x < n; x++) {
    ret += cost[x][xy[x]];
    assignment[x] = xy[x];
  }
  return assignment;
}

static void initLabels() {
  lx = new int[n];
  ly = new int[n];
  for(int x = 0; x < n; x++)
    for(int y = 0; y < n; y++)
      lx[x] = Math.max(lx[x], cost[x][y]);
}

static void augment() {
  if(maxMatch == n) {return;}
  int x, y, root = 0;
  int[] q = new int[n];
  int wr = 0, rd = 0;
  S = new boolean[n];
  T = new boolean[n];
  for(x = 0; x < n; x++)
    prev[x] = -1;
  for(x = 0; x < n; x++) {
    if(xy[x] == -1) {
      q[wr++] = root = x;
      prev[x] = -2;
      S[x] = true;
      break;
    }
  }
  for(y = 0; y < n; y++) {
    slack[y] = lx[root] + ly[y] - cost[root][y];
    slackx[y] = root;
  }
  while(true) {
```

```
    while(rd < wr) {
      x = q[rd++];
      for(y = 0; y < n; y++) {
        if(cost[x][y] == lx[x]+ly[y] && !T[y]) {
          if(yx[y] == -1) {break;}
          T[y] = true;
          q[wr++] = yx[y];
          addToTree(yx[y], x);
        }
      }
      if (y < n) {break;}
    }
    if (y < n) {break;}
    updateLabels();
    wr = rd = 0;
    for (y = 0; y < n; y++) {
      if (!T[y] && slack[y] == 0) {
        if(yx[y] == -1) {
          x = slackx[y];
          break;
        } else {
          T[y] = true;
          if(!S[yx[y]]) {
            q[wr++] = yx[y];
            addToTree(yx[y], slackx[y]);
          }
        }
      }
    }
    if(y < n) {break;}
  }
  if(y < n) {
    maxMatch++;
    for(int cx=x, cy=y, ty; cx!=-2; cx=prev[cx], cy=
    ty){
      ty = xy[cx];
      yx[cy] = cx;
      xy[cx] = cy;
    }
    augment();
  }
}

static void updateLabels() {
  int delta = Integer.MAX_VALUE;
  for(int y = 0; y < n; y++)
    if(!T[y])
      delta = Math.min(delta, slack[y]);
  for(int i = 0; i < n; i++) {
    if(S[i]) {lx[i] -= delta;}
    if(T[i]) {ly[i] += delta;}
    if(!T[i]) {slack[i] -= delta;}
  }
}

static void addToTree(int x, int prevx) {
  S[x] = true;
  prev[x] = prevx;
  for(int y = 0; y < n; y++) {
    if(lx[x] + ly[y] - cost[x][y] < slack[y]) {
      slack[y] = lx[x] + ly[y] - cost[x][y];
      slackx[y] = x;
    }
  }
}
```

$O(n2^n)$ solution using DP (very simple to code):

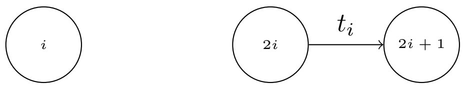
```
int n;
double[][] w;
Double[] memo;

double minCostMatching(int paired) {
  if(memo[paired] != null) return memo[paired];
  if(paired == (1 << n) - 1) return 0.0;
  double min = Double.POSITIVE_INFINITY;
  int i = 0;
  while(((paired >> i) & 1) == 1) i++;
  for(int j = i + 1; j < n; j++) {
    if(((paired >> j) & 1) == 0) {
      min = Math.min(min, w[i][j] + minCostMatching(
      paired | (1 << i) | (1 << j)));
    }
  }
  memo[paired] = min;
  return min;
}
```

2.9 Directed Min cost flow

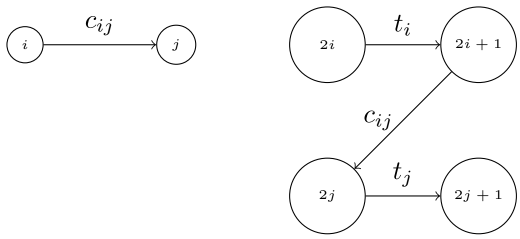
Avoiding parallel edges:

1. Split nodes



where t_i is the number of times node i can be used (usually ∞).

2. Link nodes




```
TreeMap<Integer,Edge>[] preprocess(TreeMap<Integer,
    Edge>[] g) {
  TreeMap<Integer, Edge>[] h =
  new TreeMap[2*g.length];
  for(int v = 0; v < h.length; v++)
    h[v] = new TreeMap<Integer, Edge>();
  for(int v = 0; v < g.length; v++) {
    for(Entry<Integer,Edge> entry:g[v].entrySet()) {
      int u = entry.getKey();
      Edge e = entry.getValue();
      h[2*v+1].put(2*u, e);
    }
    h[2*v].put(2*v+1, new Edge(Integer.MAX_VALUE,0))
    ;
  }
  return h;
}
```

Min cost flow analogous to max flow but using Bellman-Ford to find paths (can be made faster using Dijkstra by chaining costs).

```
int[] p;

int minCostFlow(TreeMap<Integer, Edge>[] g, int s,
    int t) {
  int mincost = 0;
  while(spfa(g, s) != null && p[t] != -1) {
    // compute path capacity
    int cur = t;
    int pcap = Integer.MAX_VALUE;
    while(cur != s) {
      int prev = p[cur];
      pcap = Math.min(pcap, g[prev].get(cur).cap);
      cur = prev;
    }
    // update graph
    cur = t;
    int pcost = 0;
    while(cur != s) {
      int prev = p[cur];
      Edge epath = g[prev].get(cur);
      pcost += epath.cost * pcap;
      // update current edge
      if(epath.cap == pcap) g[prev].remove(cur);
```

```
      else epath.cap -= pcap;
      // update reverse edge
      Edge eback = g[cur].get(prev);
      if(eback != null) eback.cap += pcap;
      else g[cur].put(prev, new Edge(pcap, -epath.
    cost))
      cur = prev;
    }
    mincost += pcost;
  }
  return mincost;
}
```

Some changes to SPFA may be necessary. Computation of global variable p containing parents is required.

2.10 Chinese Postman Problem

Given an undirected weighted graph, compute the minimum length tour that visits every edge (edges may be visited several times, unavoidable if odd degree vertices exist). The number of odd degree vertices is even. Hence we can compute the minimum weight bipartite matching between them where w_{ij} is the length of the shortest path between i and j. Then the length of the tour is given by the sum of the lengths of all edges plus the weight of the matching.

2.11 Bipartite graph

Check if bipartite

```
boolean isBipartite(LinkedList<Integer>[] g)
{
  int[] d = bfs(g);
  for(int u = 0; u < g.length; u++)
    for(Integer v: g[u])
      if((d[u]%2)!=(d[v]%2)) return false;
  return true;
}
```

2.11.1 Max Cardinality Bipartite Matching (MCBM)

Pairing of adjacent nodes. No node in two different pairs.

- Max Flow.
- Augmenting Path: path starting at non matched, ending at non-matched, even edges are matching. MCBM ssi no augmenting path. Start from non-matched, if augmenting path, augment (do not have to take all matching in the augmenting path).

MCBM : Number of matching.

2.11.2 Independent Set (or Dominating Set)

Set of vertices with no edges between them. MIS, add a vertex create an edge. In **bipartite** graph, MIS + MCBM = V.

2.11.3 Vertex Cover

Vertices such that each edge is adjacent to at least one vertex. Min Vertex Cover (MVC). In **bipartite** graph, MVC = MCBM.
In **general** graph, MVC = MIS and the MVC is the complementary of MIS.

```
static int n; // V
static int m; // vertex on the left subset of V
static LinkedList<Integer>[] g;
static int[] match;
static BitSet visited;
private static int Aug(int left) {
  if (visited.get(left)) return 0;
  visited.set(left);

  for (int right : g[left]) {
    if (match[right] == -1 || Aug(match[right]) ==
    1) {
      match[right] = left;
      return 1; // we found one matching
    }
  }

  return 0; // no matching
}

static int mcbm () {
  int MCBM = 0;
  match = new int[n];
  for (int i = 0; i < n; i++) {
    match[i] = -1;
  }
  for (int l = 0; l < m; l++) {
    visited = new BitSet(n);
    MCBM += Aug(l);
  }
  return MCBM;
}
```

3 Dynamic programming

3.1 Bottom-up

Give n objects of value $v[i]$ to 3 people such that $\max_i V_i - \min_i V_i$ is minimum (V_i is total value for person i).
$canDo[i][v_1][v_2] = 1$ if we can give the objects $0, 1, \ldots, i$ such that v_1 is going to P_1 and v_2 to P_2, 0 otherwise. v_3 is determined from the sum.

Base case $i = 0$:

- $canDo[0][0][0] = 1$
- $canDo[0][v[0]][0] = 1$
- $canDo[0][0][v[0]] = 1$

Case $i \geq 1$:
$canDo[i][v_1][v_2] =$
$canDo[i-1][v_1][v_2] \vee$
$canDo[i-1][v_1 - v[i]][v_2] \vee$
$canDo[i-1][v_1][v_2 - v[i]]$

Sol. : $\min_{v_1, v_2 : canDo[n-1][v_1][v_2]} [max(v_1, v_2, S - v_1 - v_2) - min(v_1, v_2, S - v_1 - v_2)]$

```
int solveDP() {
  boolean[][][] canDo = new boolean[v.length][sum +
    1][sum + 1];
  // initialize base cases
  canDo[0][0][0] = true;
  canDo[0][v[0]][0] = true;
  canDo[0][0][v[0]] = true;
  // compute solutions using recurrence relation
  for(int i = 1; i < v.length; i++) {
    for(int a = 0; a <= sum; a++) {
      for(int b = 0; b <= sum; b++) {
        boolean giveA = a - v[i] >= 0 && canDo[i -
    1][a - v[i]][b];
        boolean giveB = b - v[i] >= 0 && canDo[i -
    1][a][b - v[i]];
        boolean giveC = canDo[i - 1][a][b];
        canDo[i][a][b] = giveA || giveB || giveC;
      }
    }
  }
  // compute best solution
  int best = Integer.MAX_VALUE;
  for(int a = 0; a <= sum; a++) {
    for(int b = 0; b <= sum; b++) {
      if(canDo[v.length - 1][a][b]) {
        best = Math.min(best, max(a, b, sum - a - b)
    - min(a, b, sum - a - b));
      }
    }
```

```
    }
    return best;
}
```

3.2 Top-down

Same problem as bottom-up. Main idea : memoization (Remember intermediate results).

```
int solve(int i, int a, int b) {
  if(i == n) {
    memo[i][a][b] = max(a, b, sum - a - b) - min(a,
    b, sum - a - b);
    return memo[i][a][b];
  }
  if(memo[i][a][b] != null) {
    return memo[i][a][b];
  }
  int giveA = solve(i + 1, a + v[i], b);
  int giveB = solve(i + 1, a, b + v[i]);
  int giveC = solve(i + 1, a, b);
  memo[i][a][b] = min(giveA, giveB, giveC);
  return memo[i][a][b];
}
```

3.3 Knapsack problem

Given n objects of value $v[i]$ and weight$w[i]$, an integer W :

- Maximize $\sum_i x[i]v[i]$
- Such that $\sum_i x[i]w[i] \leq W$ where $x[i]=0$ (not taken) or 1 (taken)

3.3.1 No repetition

best[i][w]= best way to take objects $0, 1, \ldots, i$ in a knapsack of capacity w.

Base case:

- $best[0][w] = v[0]$ si $w[0] \leq w$
- 0 else

Other cases:
$best[i][w] =$
$\max\{best[i-1][w],$
$best[i-1][w-w[i]] + v[i]\}$

3.3.2 An object can be repeated

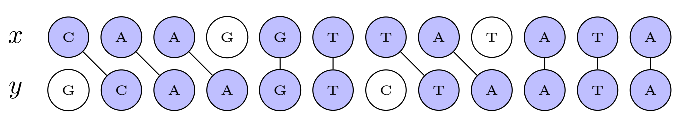
- $best[0] = 0$
- $best[w] = \max_{i:w[i]<w}\{best[w-w[i]] + v[i]\}$

3.3.3 Several knapsacks

$best[i][w_1][w_2]$ = best way to take objects $0, 1, \ldots, i$ in knapsacks of capacity w_1 and w_2.

3.4 Longest common sub-sequence (LCS)

Given two String x and y. Find the longest common subsequence between x and y.



- **Formulation:** $lcs[i][j]$ = size of
  $LCS(x[0]x[1]\cdots x[i-1], y[0]y[1]\cdots y[j-1])$
- **Base case:** $lcs[0][j] = 0$ $\quad lcs[i][0] = 0$
- **Other cases:**
  - Si $x[i-1] = y[i-1]$ alors:
    $lcs[i][j] = 1 + lcs[i-1][j-1]$
  - Si $x[i-1] \neq y[i-1]$ alors:
    $lcs[i][j] = \max\{lcs[i-1][j], lcs[i][j-1]\}$

3.5 Matrix Chain Multiplication (MCM)

Given a list of matrices, find the order minimizing the number of multiplications to compute their product.

- Number to multiply a matrix of size $n \times m$ by a matrix of size $m \times r$: $n \cdot m \cdot r$.
- Example: A : 10×30, B : 30×5 et C : 5×60.
  - For $(AB)C$: $10 \cdot 30 \cdot 5 + 10 \cdot 5 \cdot 60 = 4500$ multiplications.
  - For $A(BC)$: $30 \cdot 5 \cdot 60 + 10 \cdot 30 \cdot 60 = 27000$ multiplications.
- **Formulation :** $best[i][j]$ = min cost to multiply $A_i, \ldots, A_j$
- **Base case :** $best[i][i] = 0$
- **Other cases:**

$$best[i][j] = \min_{i \leq k < j} best[i][k] + best[k+1][j] + A_i.n_1 \times A_k.n_2 \times A_j.n_2$$

3.5.1 Generalized MCM

Given a list of objects $x[0], \ldots, x[n-1]$ and an operation $\odot$ with an associated cost, find the order in which perform the operations to minimize the total cost. The matrix product is replaced by $\odot$.

$$best[i][j] = \min_{i \leq k < j} best[i][k] + best[k+1][j] + cost(i,j,k)$$

$cost(i,j,k)$ is the cost of $(x[i] \odot \cdots \odot x[k]) \odot (x[k+1] \odot \cdots \odot x[j])$.

```
int bestParenthesize() {
  int n = x.length; // x is a global variable
  int[][] best = new int[n][n];
  for(int i = 0; i < n; i++) {
    best[i][i] = 0;
  }
  for(int l = 1; l <= n; l++) {
    for(int i = 0; i < n - l; i++) {
      int j = i + l;
      int min = Integer.MAX_VALUE;
      for(int k = i; k < j; k++) {
        min = Math.min(min, best[i][k] + best[k +
    1][j] + cost(i, j, k)); // cost is problem-
    independent
      }
      best[i][j] = min;
    }
  }
  return best[0][n - 1];
}
```

3.6 Edit distance

Given two String x and y, by performing operations on en x, compute the minimal cost to transform x into y.
We can (operation cost):

1. Remove a character (D=1)
2. Insert a character (I=1)
3. Replace a character(R=2)

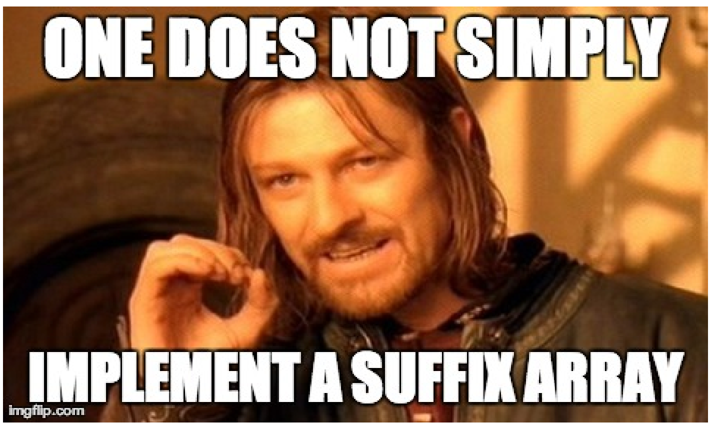
- **Formulation**:$editDist[i][j]$= min. cost to transform $x_0 \cdots x_{i-1}$ into $y_0 \cdots y_{j-1}$
- **Base case**:
$editDist[i][0] = i \cdot D \quad editDist[0][j] = j \cdot I$
- **Other cases**:

$$editDist[i][j] = \min \begin{cases} editDist[i-1][j] + D, \\ editDist[i][j-1] + I, \\ editDist[i-1][j-1] + R^* \end{cases}$$

where $R^* = R$ if $x[i-1] \neq y[j-1]$, 0 else.

```java
int editDistance(String txt1, String txt2, int I,
    int D, int R){
  int[][] d = new int[txt1.length()+1][txt2.length()
    +1];
  for(int i=0; i <= txt1.length(); i++)
    d[i][0]=i*D;
  for(int j=0; j <= txt2.length(); j++)
    d[0][j]=j*I;
  for(int i=1; i <= txt1.length(); i++){
    for(int j=1; j <= txt2.length(); j++){
      int cost;
      // Non-equality cost
      if(txt1.charAt(i-1)==txt2.charAt(j-1))
        cost = 0;
      else
        cost = R;
      // Deletion, Insertion, Replacement
      d[i][j] = Math.min(Math.min(d[i-1][j] + D, d[i
    ][j-1] + I), d[i-1][j-1] + cost);
    }
  }
  // Last computed element is the edit distance
  return d[txt1.length()][txt2.length()];
}
```

3.7 Suffix array



3.7.1 $O(n \log(n)^2)$, full matrix, need $n \leq 10K$

- Suffix array of *algorithm* = algorithm, gorithm, hm, ithm, lgorithm, m, orithm, rithm, thm
- Characterized by its starting index
Example : Suffix array of *algorithm*:

$$[0,\ 2,\ 7,\ 5,\ 1,\ 8,\ 3,\ 4,\ 6]$$

Example : Given suf_j suffix beginning at index j, and $C(i,j,k)$ comparison result of suf_j and suf_k on the 2^i first characters.

$$C(i,j,k) = \begin{cases} C(i-1,j,k) & \text{si } C(i-1,j,k) \neq 0 \\ C(i-1, j+2^{i-1}, k+2^{i-1}) & \text{else} \end{cases}$$

- Define a matrix *so* such that:

$$so[i][j] = so[i][k] \Leftrightarrow C(i,j,k) = 0$$
$$so[i][j] < so[i][k] \Leftrightarrow C(i,j,k) < 0$$
$$so[i][j] > so[i][k] \Leftrightarrow C(i,j,k) > 0$$

$so[i]$ is the order of sorted suffixes on the 2^i first characters.

- **Base case:** $so[0][j] = (int)s.charAt(i)$
Example: for $s = ccacab$ we have
$s[0] = [97, 97, 95, 97, 95, 96]$
- For every j we define a triplet (l, r, j):

$$\begin{cases} (s[i-1][j], s[i-1][j+2^{i-1}], j) & \text{si } j+2^{i-1} < n \\ (s[i-1][j], -1, j) & \text{si } j+2^{i-1} \geq n \end{cases}$$

```java
class Triple implements Comparable<Triple> {
  int l, r, index;
  public Triple(int half1, int half2, int index) {
    this.l = half1;
    this.r = half2;
    this.index = index;
  };
  public int compareTo(Triple other) {
    if(l != other.l) {
      return l - other.l;
    }
    return r - other.r;
  }
}
```

```java
int[][] suffixOrder(String s) { // O(n log^2(n))
  int n = s.length();
  int lg = (int)Math.ceil((Math.log(n) / Math.log(2)
    )) + 1;
  int[][] so = new int[lg][n];
  // initialize so[0] with character order
  for(int i = 0; i < n; i++) {
    so[0][i] = s.charAt(i);
  }
  Triple[] next = new Triple[n];
  for(int i = 1; i < lg; i++) {
    // build the next array
    for(int j = 0; j < n; j++) {
      int k = j + (1 << (i - 1));
      next[j] = new Triple(so[i - 1][j], k < n ? so[
    i - 1][k] : -1, j);
    }
    // sort next array
```

```java
    Arrays.sort(next);
    // build so[i]
    for(int j = 0; j < n; j++) {
      if(j == 0) {
        // smallest elements gets value 0
        so[i][next[j].index] = 0;
      } else if(next[j].compareTo(next[j - 1]) == 0)
      {
        // equal to previous so it gets the same value
        so[i][next[j].index] = so[i][next[j - 1].index
];
      } else {
        // largest than previous so get + 1
        so[i][next[j].index] = so[i][next[j - 1].index
] + 1;
      }
    }
  }
  return so;
}

//Calcule le Suffix Array pour un so donne:
int[] suffixArray(int[][] so) {
  int[] sa = new int[so[0].length];
  for(int j = 0; j < so[0].length; j++) {
    sa[so[so.length - 1][j]] = j;
  }
  return sa;
}

//Retourne le plus long prefixe commun de suf_j (le
    suffixe de s commencant a j = s.substr(j)) et
    suf_k pour un so donne:
int lcp(int[][] so, int j, int k) { // O(log(n)
  int lcp = 0;
  int n = so[0].length;
  for(int i = so.length - 1; i >= 0; i--) {
    if(j < n && k < n && so[i][j] == so[i][k]) {
      lcp += (1 << i);
      j += (1 << i);
      k += (1 << i);
    }
  }
  return lcp;
}

//Quelques exemples
String maxStrRepeatedKTimes(String s, int k) {
  int[][] so = suffixOrder(s);
  int[] SA = suffixArray(so);
  int n = s.length();
  int max = Integer.MIN_VALUE;
  int j = 0;
  for(int i = 0; i <= n - k; i++) {
    int lcp = lcp(so, SA[i], SA[i + k - 1]);
    if(lcp > max) {
      max = lcp;
      j = SA[i];
    }
  }
  return s.substring(j, j + max);
}

String minLexicographicRotation(String s) {
  int n = s.length();
  s += s;
  int[] SA = suffixArray(suffixOrder(s));
  int i = 0;
  while(!(0 <= SA[i] && SA[i] < n)) {
    i++;
  }
  return s.substring(SA[i], SA[i] + n);
}

class MaxLexConc implements Comparator<String> {
 public int compare(String x, String y) {
    String xy = x + y;
    String yx = y + x;
    if(xy.compareTo(yx) < 0 ||
      (xy.equals(yx) && x.length()<y.length())) {
      return 1;
    }
    return -1;
}
```

3.7.2 $O(n\log(n))$, only last line, need $n \leq 100K$

```java
static final int MAX_N = 100010;
static Integer[] tempSA, sa;
static int[] c, ra;
static int[] lcp, plcp;
static void countingSort(int n, int k) {
  int i, sum, maxi = Math.max(300, n); // up to 255
    ASCII chars or length of n
  for (i = 0; i < MAX_N; i++) c[i] = 0; // clear
    frequency table
  for (i = 0; i < n; i++) // count the frequency of
    each rank
    c[i + k < n ? ra[i + k] : 0]++;
  for (i = sum = 0; i < maxi; i++) {
    int t = c[i]; c[i] = sum; sum += t;
  }
  for (i = 0; i < n; i++)              // shuffle
    the suffix array if necessary
    tempSA[c[sa[i] + k < n ? ra[sa[i] + k] : 0]++] =
     sa[i];
  for (i = 0; i < n; i++)
    // update the suffix array SA
    sa[i] = tempSA[i];
}

static void constructSA(char[] s) { // O(n log(n))
   -> n <= 100K
  int i, k, r, n = s.length;
  tempSA = new Integer[n]; sa = new Integer[n];
  ra = new int[n]; int[] tempRA = new int[n];
  c = new int[MAX_N];
  for (i = 0; i < n; i++) ra[i] = s[i];
        // initial rankings
  for (i = 0; i < n; i++) sa[i] = i;          //
    initial SA: {0, 1, 2, ..., n-1}
  for (k = 1; k < n; k <<= 1) {             // repeat
     sorting process log n times
    countingSort(n, k);        // actually radix sort
    : sort based on the second item
    countingSort(n, 0);               // then (
    stable) sort based on the first item
    tempRA[sa[0]] = r = 0;                   // re-
    ranking; start from rank r = 0
    for (i = 1; i < n; i++)
    // compare adjacent suffices
      tempRA[sa[i]] =       // if same pair => same
    rank r; otherwise, increase r
      (ra[sa[i]] == ra[sa[i-1]] && ra[sa[i]+k] == ra
    [sa[i-1]+k]) ? r : ++r;
    for (i = 0; i < n; i++)
     // update the rank array RA
      ra[i] = tempRA[i];
  } }

static void computeLCP(char[] s) {
  int i, L, n = s.length;
  int[] phi = new int[n];
  lcp = new int[n]; plcp = new int[n];
  phi[sa[0]] = -1; // default value
  for (i = 1; i < n; i++) // compute Phi in O(n)
    phi[sa[i]] = sa[i-1];  // remember which suffix
    is behind this suffix
  for (i = L = 0; i < n; i++) { // compute Permuted
   LCP in O(n)
    if (phi[i] == -1) { plcp[i] = 0; continue; } //
    special case
    while (i + L < n && phi[i] + L < n && s[i + L]
   == s[phi[i] + L]) L++; // L will be increased
   max n times
    plcp[i] = L;
    L = Math.max(L-1, 0); // L will be decreased max
     n times
```

```
  }
  for (i = 1; i < n; i++) // compute LCP in O(n)
    lcp[i] = plcp[sa[i]]; // put the permuted LCP back to the correct position
}

static int strncmp(char[] a, int i, char[] b, int j, int n){
  for (int k=0; i+k < a.length && j+k < b.length; k++){
    if (a[i+k] != b[j+k]) return a[i+k] - b[j+k];
  }
  return 0;
}

static int[] stringMatching(char[] s, char[] p) { // string matching in O(m log n)
  int n = s.length, m = p.length;
  constructSA(s);
  int lo = 0, hi = n-1, mid = lo; // valid matching = [0 .. n-1]
  while (lo < hi) { // find lower bound
    mid = (lo + hi) / 2;
    int res = strncmp(s, sa[mid], p, 0, m); // try to find P in suffix 'mid'
    if (res >= 0) hi = mid;
    else          lo = mid + 1;
  }
  if (strncmp(s,sa[lo], p,0, m) != 0) return new int[]{-1, -1};// not found
  int[] ans = new int[]{ lo, 0} ;

  lo = 0; hi = n - 1; mid = lo;
  while (lo < hi) { // if lower bound is found, find upper bound
    mid = (lo + hi) / 2;
    int res = strncmp(s, sa[mid], p,0, m);
    if (res > 0) hi = mid;
    else         lo = mid + 1;
  }
  if (strncmp(s, sa[hi], p,0, m) != 0) hi--; // special case
  ans[1] = hi;
  return ans;
} // return lower/upper bound as the first/second item of the pair, respectively

static String LRS(char[] s) { // Longest Repeating substring
  int n = s.length;
  constructSA(s);
  computeLCP(s);
  int i, idx = 0, maxLCP = 0;

  for (i = 1; i < n; i++) // O(n)
    if (lcp[i] > maxLCP) {
      maxLCP = lcp[i];
      idx = i;
    }
  return new String(s).substring(sa[idx], sa[idx]+maxLCP);
}

static int owner(int idx,int n,int m) { return (idx < n-m-1) ? 1 : 2; }

static String LCS(String T, String P) { // Longest common substring
  int i, idx = 0;

  int m = P.length();
  char[] s = (T + "$" + P + "#").toCharArray(); // append P and '#'
  int n = s.length; // update n
  constructSA(s); // O(n log n)
  computeLCP(s); // O(n)

  int maxLCP = -1;
  for (i = 1; i < n; i++)
    if (lcp[i] > maxLCP && owner(sa[i],n,m) != owner(sa[i-1],n,m)) { // different owner
      maxLCP = lcp[i];
      idx = i;
    }

  return new String(s).substring(sa[idx], sa[idx] + maxLCP);
}
```

4 Geometry in 2D

Be careful of rounding errors. Define E in function of the problem. `Double.parseDouble` is a lot slower than `Integer.parseInt`.

```
boolean eq(double a,double b){return Math.abs(a - b) <= E;}
boolean le(double a,double b){return a < b - E;}
boolean leq(double a,double b){return a <= b + E;}
```

4.1 Vectors

4.1.1 Rotation around (0,0)

$(x, y) \leftrightarrow x + yi$
$\rho e^{i\theta} = \rho\cos(\theta) + i\rho\sin(\theta)$

(x, y) rotated by α is $(\cos(\alpha)x - \sin(\alpha)y, \sin(\alpha)x + \cos(\alpha)y)$

4.2 Points

```
class Point implements Comparable<Point>
{
  double x, y;
  public int compareTo(Point o) { //xcomp
    if(a.x < b.x) return -1;
    if(a.x > b.x) return 1;
    if(a.y < b.y) return -1;
    if(a.y > b.y) return 1;
    return 0;
  }
}

class yComp implements Comparator<Point> {
  public int compare(Point p, Point q) {
    if(p.y == q.y) {return Double.compare(p.x, q.x);}
    return Double.compare(p.y, q.y);
  }
}
```

4.2.1 Point in box

```
boolean inBox(Point p1, Point p2, Point p) {
  return Math.min(p1.x, p2.x) <= p.x && p.x <= Math.max(p1.x, p2.x) &&
         Math.min(p1.y, p2.y) <= p.y && p.y <= Math.max(p1.y, p2.y);
}
```

4.2.2 Polar sort

```
LinkedList<Point> sortPolar(Point[] P, Point o)
{
  LinkedList<Point> above = new LinkedList<Point>();
  LinkedList<Point> samePos = new LinkedList<Point>();
  LinkedList<Point> sameNeg = new LinkedList<Point>();
  LinkedList<Point> bellow = new LinkedList<Point>();
  for(Point p : P)
  {
    if(p.y > o.y)
      above.add(p);
    else if(p.y < o.y)
      bellow.add(p);
    else
    {
```

```
        if(p.x < o.x)
          sameNeg.add(p);
        else
          samePos.add(p);

      }
    }
    PolarComp comp = new PolarComp(o);
    Collections.sort(samePos, comp);
    Collections.sort(sameNeg, comp);
    Collections.sort(above, comp);
    Collections.sort(bellow, comp);
    LinkedList<Point> sorted = new LinkedList<Point>()
      ;
    for(Point p : samePos) sorted.add(p);
    for(Point p : above) sorted.add(p);
    for(Point p : sameNeg) sorted.add(p);
    for(Point p : bellow) sorted.add(p);
    return sorted;
}

class PolarCmp implements Comparator<Point> {
  static Point orig = new Point(0, 0);
  public int compare(Point p, Point q) {
    double o = orient(orig, p, q);
    if(o == 0) {
      if(p.x * p.x + p.y * p.y > q.x * q.x + q.y * q
      .y)
        return 1;
      return -1;
    }
    return -(int)Math.signum(o);
  }
}
```

4.2.3 Closest pair of points

```
double closestPair(Point[] points) {
  if(points.length == 1) {return Double.
    POSITIVE_INFINITY;}
  Arrays.sort(points, new xComp());
  double min = dist(points[0], points[1]);
  // keep track of the leftmost point
  int leftmost = 0;
  TreeSet<Point> candidates = new TreeSet<Point>(new
     yComp());
  candidates.add(points[0]);
  candidates.add(points[1]);
  for(int i = 2; i < points.length; i++) {
    Point cur = points[i];
    // eliminate points s.t cur.x - x > min
    while(cur.x - points[leftmost].x > min) {
      candidates.remove(points[leftmost]);
      leftmost++;
    }
    Point low = new Point(0, cur.y - min);
    Point high = new Point(0, cur.y + min);
    // check all points in the rectangle
    for(Point point : candidates.subSet(low, high))
      min = Math.min(min, dist(cur, point));
    candidates.add(cur);
  }
  return min;
}
```

4.2.4 Orientation

$$orient(p,q,r) = \begin{vmatrix} 1 & p_x & p_y \\ 1 & q_x & q_y \\ 1 & r_x & r_y \end{vmatrix}$$

$$orient(p,q,r) \begin{cases} = 0 & p,q,r \text{ are collinear} \\ < 0 & p \text{ -¿ } q \text{ -¿ } r \text{ is clockwise} \\ > 0 & p \text{ -¿ } q \text{ -¿ } r \text{ is counterclockwise} \end{cases}$$

$$|orient(p,q,r)| = 2 \cdot area \triangle(p,q,r)$$

```
double orient(Point p, Point q, Point r) {
  return q.x * r.y - r.x * q.y - p.x * (r.y - q.y) +
     p.y * (r.x - q.x);
}
```

4.2.5 Angle visibility

x lies strictly inside the angle formed by p,q,r iff

$$sgn(orient(p,q,x)) = sgn(orient(p,x,r))$$
$$sgn(orient(p,r,x)) = sgn(orient(p,x,q))$$

To allow it to lie on the border simply check if

$$sgn(orient(p,q,x)) = 0 \text{ or } sgn(orient(p,r,x)) = 0$$

4.2.6 Fixed radius neighbors (1D)

```
List<Double[]> findPairs1D(double[] x, double r) {
  HashMap<Integer, List<Double>> H = new HashMap<
    Integer, List<Double>>();
  // fill buckets
  for(int i = 0; i < x.length; i++) {
    int b = (int)(x[i] / r);
    if(H.containsKey(b)) {
      H.get(b).add(x[i]);
    } else {
      List<Double> L = new ArrayList<Double>();
      L.add(x[i]);
      H.put(b, L);
    }
  }
  // find pairs in consecutive buckets
  List<Double[]> pairs = new LinkedList<Double[]>();
  for(int i = 0; i < x.length; i++) {
    int b = (int)(x[i] / r);
    List<Double> bucket = H.get(b + 1);
    if(bucket != null)
      for(double y : bucket)
        if(y - x[i] <= r)
          pairs.add(new Double[] {x[i], y});
  }
  // add points in buckets
  for(List<Double> bucket : H.values())
    for(int i = 0; i < bucket.size(); i++)
      for(int j = i + 1; j < bucket.size(); j++)
        pairs.add(new Double[] {bucket.get(i),
    bucket.get(j)});
  return pairs;
}
```

4.2.7 Fixed radius neighbors (2D)

```
List<Point[]> findPairs2D(Point[] points, double r)
    {
  HashMap<Integer, List<Point>> H = new HashMap<
    Integer, List<Point>>();
  // fill buckets
  for(int i = 0; i < points.length; i++) {
    int bx = (int)(points[i].x / r);
    int by = (int)(points[i].y / r);
    int key = 33 * bx + by;
    if(H.containsKey(key)) {
      H.get(key).add(points[i]);
    } else {
      List<Point> L = new ArrayList<Point>();
      L.add(points[i]);
      H.put(key, L);
    }
  }
  // find pairs in adjacent buckets
  List<Point[]> pairs = new LinkedList<Point[]>();
  int[][] dir = new int[][] {new int[] {1,0}, new
    int[] {0,1}, new int[] {1,1}};
  for(int i = 0; i < points.length; i++) {
    int bx = (int)(points[i].x / r);
    int by = (int)(points[i].y / r);
    for(int[] d : dir) {
      List<Point> bucket = H.get(33 * (bx + d[0]) +
    (by + d[1]));
      if(bucket != null)
```

```
        for(Point y : bucket)
          if(sqDist(points[i], y) <= r * r)
            pairs.add(new Point[] {points[i], y});
    }
  }
  // add points in buckets
  for(List<Point> bucket : H.values())
    for(int i = 0; i < bucket.size(); i++)
      for(int j = i + 1; j < bucket.size(); j++)
        if(sqDist(bucket.get(i), bucket.get(j)) <= r
   * r)
          pairs.add(new Point[] {bucket.get(i),
  bucket.get(j)});
  return pairs;
}
```

4.3 Lines

General equation: $Ax + By = C$. The line through $(x_1, y_1), (x_2, y_2)$ is given by: $A = y_2 - y_1$, $B = x_1 - x_2$, $C = Ax_1 + By_1$.

4.3.1 Intersections

Intersection exists there is a solution for $A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$. This happens if and only if

$$d := \det \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \neq 0$$

Intersection is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} B_2 & -B_1 \\ -A_2 & A_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

4.3.2 Perpendicular line

The lines perpendicular to $Ax + By = C$ are

$$-Bx + Ay = D \quad \text{for } D \in \mathbb{R}$$

If we want the one that goes through (x_0, y_0) set

$$D = -Bx_0 + Ay_0$$

4.3.3 Orthogonal Symmetry

For a line, find X', the point which is the orthogonal symmetry of X on line a.
Computes the perpendicular of the given line that goes through X. Compute intersection Y. $X' = Y - (X - Y)$.

4.4 Segments

4.4.1 Intersection

- Treat segments as lines.
- If $d \neq 0$, compute line intersection (x, y).
- Segments intersect iff

$$\min(x_1, x_2) \leq x \leq \max(x_1, x_2)$$
$$\min(y_1, y_2) \leq y \leq \max(y_1, y_2)$$

```
boolean intersects(Point p1, Point p2, Point p3,
    Point p4) {
  double o1 = orient(p1, p2, p3);
  double o2 = orient(p1, p2, p4);
  double o3 = orient(p3, p4, p1);
  double o4 = orient(p3, p4, p2);
  // check first condition of the lemma
  if(o1 * o2 < 0 && o3 * o4 < 0) return true;
  // check seconds condition of the lemma
  if(o1 == 0 && inBox(p1, p2, p3)) return true;
  if(o2 == 0 && inBox(p1, p2, p4)) return true;
  if(o3 == 0 && inBox(p3, p4, p1)) return true;
  if(o4 == 0 && inBox(p3, p4, p2)) return true;
  return false;
}
```

4.4.2 Intersections problem

Given a lot of segments, return true if it exists a pair that intersects.

```
boolean segmentIntersection(Segment[] S) {
  Event[] events = new Event[2 * S.length];
  // create event points
  for(int i = 0, j = 0; i < S.length; i++) {
    events[j++] = new Event(S[i].l.x, true, S[i]);
    events[j++] = new Event(S[i].r.x, false, S[i]);
  }
  Arrays.sort(events);
  SegmentCmp cmp = new SegmentCmp();
  TreeSet&lt;Segment&gt; T = new TreeSet&lt;Segment&gt;(cmp);
  // sweep line
  for(Event event : events) {
    Segment s = event.s;
    cmp.x = event.x;
    if(event.isLeft) {
      // new segment found. check if it intersects
    one of its neighbors
      T.add(s);
      Segment above = T.higher(s);
      Segment bellow = T.lower(s);
      if((above != null && intersects(above, s)) ||
         (bellow != null && intersects(bellow, s)))
        return true;
    } else {
      // end of segment. check if its neighbors
    intersect
      Segment above = T.higher(s);
      Segment bellow = T.lower(s);
      if(above != null && bellow != null &&
    intersects(above, bellow))
        return true;
      T.remove(s);
    }
  }
  return false;
}

class Event implements Comparable<Event> {
  double x;
  boolean isLeft;
  Segment s;
  public Event(double x, boolean isLeft, Segment s)
    {
    this.x = x;
    this.isLeft = isLeft;
    this.s = s;
  }
  public int compareTo(Event other) {
    int cmp = Double.compare(x, other.x);
    // ensure that left comes before right
    if(cmp == 0) return isLeft ? -1 : 1;
    return cmp;
  }
  public String toString() {
    return x + " " + isLeft;
  }
}

class SegmentCmp implements Comparator&lt;Segment&gt; {
  double x;
  public int compare(Segment s1, Segment s2) {
    // compute A,B,C from eq Ax + by = C for each
    segment
    double A1 = s1.r.y - s1.l.y;
    double B1 = s1.l.x - s1.r.x;
    double C1 = A1 * s1.l.x + B1 * s1.l.y;
```

```
    double A2 = s2.r.y - s2.l.y;
    double B2 = s2.l.x - s2.r.x;
    double C2 = A2 * s2.l.x + B2 * s2.l.y;

    // no divisions =)
    double t1 = B2 * (C1 - A1 * x);
    double t2 = B1 * (C2 - A2 * x);
    if(t1 == t2) {
      return s1 == s2? 0 : -1;
    } else if(B1 * B2 > 0) {
      return Double.compare(t1, t2);
    } else {
      return Double.compare(t2, t1);
    }
  }
}
```

4.5 Circles

4.5.1 Circles from 3 points

- 3 non collinear points define a unique circle.
- c = intersection of bisectors of XY and YZ.

4.6 Polygon

4.6.1 Triangles

- côtés a, b, c, angles A, B, C, hauteurs h_A, h_B, h_C, $s = \frac{a+b+c}{2}$, aire S.
- Aire: $S = ah_A/2$, $S = ab\sin C/2$, $S = \sqrt{s(s-a)(s-b)(s-c)}$.
- Inradius $r = \frac{S}{s}$.
- Outradius $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- $rR = \frac{abc}{4s}$.

4.6.2 Check convexity

```
boolean isConvex(Point[] P) {
  if(P.length < 3) return false;
  double o1 = orient(P[P.length-1], P[0], P[1]);
  for (int i = 0; i < P.length; i++) {
    double o2 = orient(P[i], P[i + 1], P[i + 2]);
    if(o1 * o2 < 0) {
      return false;
    } else if (o2 != 0) {
      o1 = o2;
    }
  }
  return true;
}
```

4.6.3 Winding number

Number of times a path of points "turn around" another point. (can check if a point is inside a polygon: in this case, winding numbe ! = 0)

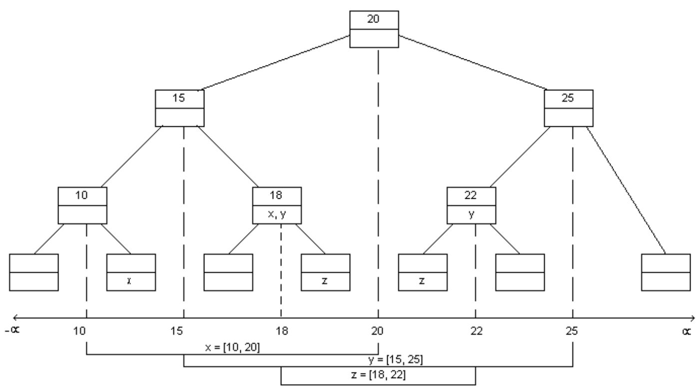
```
// assumes p is not on P
double winding(Point[] P, Point p) {
  //make a translation so p = (0, 0)
  for(Point q : P) {
    q.x -= p.x;
    q.y -= p.y;
  }
  double w = 0;
  for(int i = 0; i < P.length - 1; i++) {
    if(P[i].y * P[i + 1].y < 0) {
      // segment crosses the x-axis
      double r = (P[i].y - P[i+1].y) * P[i].x + P[i
].y * (P[i+1].x - P[i].x);
      //check for intersection with the positive x-
    axis
      if((P[i].y - P[i+1].y > 0 && r > 0) || (P[i].y
    - P[i+1].y < 0 && r < 0)) {
        // segment fully crosses the x-axis
        // - to + add 1, + to - subtract 1
        w += P[i].y < 0? 1 : -1;
      } else if(P[i].y == 0 && P[i].x > 0) {
        // the segment starts at the x-axis
        // 0 to + add 0.5, 0 to - subtract 0.5
        w += P[i+1].y > 0? 0.5 : -0.5;
      } else if(P[i+1].y == 0 && P[i+1].x > 0) {
        // the segment ends at the x-axis
        // - to 0 add 0.5, + to 0 subtract 0.5
        w += P[i].y < 0? 0.5 : -0.5;
      }
    }
  }
  return w;
}
```

4.6.4 Convex Hull

```
Point[] convexHull(Point[] points) {
  // sort points by increasing x coordinates
  Arrays.sort(points, new xComp());
  // build upper chain
  Point[] upChain = buildChain(points, 1);
  // build lower chain
  Point[] loChain = buildChain(points, -1);
  Point[] hull = new Point[upChain.length + loChain.
    length - 2];
  int i;
  // build convex hull from upper and lower chain
  for(i = 0; i < upChain.length; i++) {
    hull[i] = upChain[i];
  }
  for(int j = loChain.length - 2; j >= 1; j--) {
    hull[i] = loChain[j];
  }
  return hull;
}

Point[] buildChain(Point[] points, int sgn) {
  Point[] S = new Point[points.length];
  int k = 0;
  S[k++] = points[0]; // push points[0]
  S[k++] = points[1]; // push points[1]
  // build chain
  for(int i = 2; i < points.length; i++) {
    //double orient = orient(S[k - 2], S[k - 1],
    points[i]);
    while(k >= 2 && sgn * orient(S[k - 2], S[k - 1],
     points[i]) >= 0) {
      S[k - 1] = null; // pop
      k--;
    }
    S[k++] = points[i]; // push points[i]
  }
  return Arrays.copyOf(S, k);
}
```

4.7 Interval Tree



```
class IntervalTree {
 Node root;
 public IntervalTree(int[] x) {
   root = new Node();
   buildTree(root, 0, x.length - 1, x);
 }
 public int measure() {
   return root.measure;
 }
 public void buildTree(Node node, int i, int j, int
   [] x) {
   if(j - i == 1) {
     node.l = x[i];
     node.r = x[j];
     node.m = -1;
   } else {
     node.l = x[i];
     node.r = x[j];
     int mid = (i + j) / 2;
     Node left = new Node();
     buildTree(left, i, mid, x);
     Node right = new Node();
     buildTree(right, mid, j, x);
     node.m = x[mid];
     node.left = left;
     left.parent = node;
     node.right = right;
     right.parent = node;
   }
 }
 public void remove(int x1, int x2) {
   remove(root, x1, x2);
 }
 private void remove(Node node, int x1, int x2) {
   if(node.l == x1 && node.r == x2) {
     node.count = Math.max(0, node.count - 1);
     if(node.left == null || node.right == null) {
       node.measure = node.count == 0 ? 0 : node.
   measure;
     } else {
       node.measure = node.count == 0 ? node.left.
   measure + node.right.measure : node.measure;
     }
  } else {
     // go down the three to delete new interval
     int mid = node.m;
     if(x1 < mid && mid < x2) {
       // split
       remove(node.left, x1, mid);
       remove(node.right, mid, x2);
     } else if(node.l <= x1 && x2 <= mid) {
       // contained on left
       remove(node.left, x1, x2);
     } else {
       // contained on right
       remove(node.right, x1, x2);
     }
     // update measures when going up
     if(node.count == 0) {
       node.measure = node.left.measure + node.
   right.measure;
     }
   }
 }
 public void add(int x1, int x2) {
   add(root, x1, x2);
 }
 private void add(Node node, int x1, int x2) {
   if(node.l == x1 && node.r == x2) {
     node.measure = x2 - x1;
     node.count++;
   } else {
     // go down the three to add new interval
     int mid = node.m;
     if(x1 < mid && mid < x2) {
       // split
       add(node.left, x1, mid);
       add(node.right, mid, x2);
     } else if(node.l <= x1 && x2 <= mid) {
       // contained on left
       add(node.left, x1, x2);
     } else {
       // contained on right
       add(node.right, x1, x2);
     }
     // update measures when going up
     if(node.count == 0) {
       node.measure = node.left.measure + node.
   right.measure;
     }
   }
 }
 public class Node {
   int l, r, m;
   int count, measure;
   Node left, right, parent;
 }
}
```

4.8 Area of union of rectangles

```
long area(R[] r) {
  // sort y coordinates
  int[] y = new int[2 * r.length];
  int k = 0;
  for(R rect : r) {
    y[k++] = rect.y1;
    y[k++] = rect.y2;
  }
  Arrays.sort(y);
  // build interval tree
  IntervalTree T = new IntervalTree(y);
  // initialize event queue
  PriorityQueue<Event> Q = new PriorityQueue<Event
    >();
  for(R rectangle : r) {
    Q.add(new Event(rectangle.x1, rectangle));
    Q.add(new Event(rectangle.x2, rectangle));
  }
  long area = 0;
  Event previous = null;
  // loop over all events
  while(!Q.isEmpty()) {
    // poll next event
    Event e = Q.poll();
    if(previous == null) {
      // first vertical line
      T.add(e.r.y1, e.r.y2);
    } else {
      // found a new vertical line
      // update area by dx * tree measure
      int dx = e.x - previous.x;
      area += dx * T.measure();
      if(e.x == e.r.x1) {
        // new rectangle, add segment to T
        T.add(e.r.y1, e.r.y2);
      } else {
        // exiting rectangle, remove segment from T
        T.remove(e.r.y1, e.r.y2);
      }
    }
    // update previous
    previous = e;
  }
  return area;
}

class Event implements Comparable<Event> {
  int x;
 R r;
  public Event(int x, R r) {
    this.x = x;
    this.r = r;
  }
  public int compareTo(Event other) {
    return x - other.x;
  }
}
```

```java
class R {
  int x1, y1, x2, y2;
  public R(int x1, int y1, int x2, int y2) {
    this.x1 = x1; this.y1 = y1; this.x2 = x2; this.y2 =
     y2;
  }
}
```

5 Geometry in 3D

5.1 Cross product

With vectors $\vec{V_1} = (a_1, b_1, c_1)$ and $\vec{V_2} = (a_2, b_2, c_2)$:

$$\vec{V_1} \times \vec{V_2} = (b_1c_2 - c_1b_2, c_1a_2 - a_1c_2, a_1b_2 - b_1a_2)$$

5.2 Equation of a plane

5.2.1 with a normal vector and a point

A plane is defined by a point (x_0, y_0, z_0) and an normal vector (a, b, c).

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0 = d$$

5.2.2 with a point and two vectors in the plane

A plane is defined by a point (x_0, y_0, z_0) and two vectors $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$. We obtain the parametric equations:

$$x = x_0 + t_1\alpha_1 + t_2\alpha_2$$
$$y = y_0 + t_1\beta_1 + t_2\beta_2$$
$$z = z_0 + t_1\gamma_1 + t_2\gamma_2$$

Or we can find the normal vector of the plane by doing the vector product of the two vectors

5.2.3 with three points

Make vectors from these three points and use one of the methods above.

5.3 Equation of a line

5.3.1 With a point and a vector

A line is defined by a point (x_0, y_0, z_0) and a vector (a, b, c).

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

5.3.2 With two points

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$z = z_1 + t(z_2 - z_1)$$

5.4 Distance from a point to a line

Distance from a point $M_P = (x_p, y_p, z_p)$ to a line defined with a point $M_L = (x_l, y_l, z_l)$ and a vector $\vec{V} = (a, b, c)$ equals to

$$\frac{||\vec{M_LM_P} \times \vec{V}||}{||\vec{V}||}$$

5.5 Distance from a point to a plane

The distance to a plane is 0 if a point is in the plane.

$$\frac{|ax_p + by_p + cz_p - d|}{\sqrt{a^2 + b^2 + c^2}}$$

5.6 Orthogonal projection of a point on a line

If p_p is the point, s the direction vector of the line and p_l the base point for the vector, the projection is

$$\frac{(p_p - p_l) \cdot s}{s \cdot s} s + p_l$$

5.7 Orthogonal projection of a point on a plane

$$P_p = (x + \lambda a, y + \lambda b, z + \lambda c)$$

$$\lambda = -\frac{ax_p + by_p + cz_p - d}{a^2 + b^2 + c^2}$$

5.8 Orthogonal projection of a line on a plane

Take two points of the line, project them on the plane, recreate the line from the two new points.

5.9 Finding if a point is in a 3D polygon

Take any ray in the plane of the polygon, starting from the point you want to check (simply fix one of the coordinate of the point to find the ray); if it intersects an even number number of time with the sides of the polygon, the point is inside it.

5.10 Intersection of a line and a plane

Given a plane $ax + by + cz = d$ and a line with parametric equations:

$$x = x_0 + \alpha t$$

$$y = y_0 + \beta t$$

$$z = z_0 + \gamma t$$

The value of t associated with the intersection is

$$t = \frac{d - ax_0 - by_0 - cz_0}{a\alpha + b\beta + c\gamma}$$

6 Math

6.1 Permutations, Combinations, Arrangements... *untested*

```
void nextPerm(int[] p) {
  int n = p.length;
  int k = n - 2;
  while(k >= 0 && p[k] >= p[k + 1]) {k--;}
  int l = n - 1;
  while(p[k] >= p[l]) {l--;}
  swap(p, k, l);
  reverse(p, k + 1, n);
}

LinkedList<Integer> getIPermutation(int n, int index
    ) {
  LeftRightArray lr = new LeftRightArray(n);
  lr.freeAll();
  LinkedList<Integer> perm = new
  LinkedList<Integer>();
  getPermutation(lr, index, fact(n), perm);
  return perm;
}

void getPermutation(LeftRightArray lr, int i, long
    fact, LinkedList<Integer> perm) {
  int n = lr.size();
  if(n == 1) {
    perm.add(lr.freeIndex(0, false));
  } else {
    fact /= n;
    int j = (int)(i / fact);
    perm.add(lr.freeIndex(j, true));
    i -= j * fact;
    getPermutation(lr, i, fact, perm);
  }
}

int[] getICombinadic(int n, int k, long i) {
  int[] comb = new int[k];
  int j = 0;
  for(int z = 1; z <= n; z++) {
    if ( k == 0 ) {
      break;
    }
    long threshold = C(n - z, k - 1);
    if (i < threshold) {
      comb[j] = z - 1;
      j++;
      k = k - 1;
    } else if (i >= threshold) {
      i = i - threshold;
    }
  }
  return comb;
}

void combinations(int n, int k) {
  combinations(n, 0, new int[k], 0);
}

void combinations(int n, int j, int[] comb, int k) {
  if(k == comb.length) {
    System.out.println(Arrays.toString(comb));
  } else {
    for(int i = j; i < n; i++) {
      comb[k] = i;
      combinations(n, i + 1, comb, k + 1);
    }
  }
}
void subsets(int[] set) {
  int n = (1 << set.length);
  for(int i = 0; i < n; i++) {
    int[] sub = new int[Integer.bitCount(i)];
    int k = 0, j = 0;
    while((1 << j) <= i) {
      if((i & (1 << j)) == (1 << j)) {
        sub[k++] = set[j];
      }
      j++;
    }
    System.out.println(Arrays.toString(sub));
  }
}
```

6.2 Decomposition in unit fractions *untested*

Write $0 < \frac{p}{q} < 1$ as a sum of $\frac{1}{k}$

```
void expandUnitFrac(long p, long q) {
  if(p != 0) {
    long i = q % p == 0 ? q/p : q/p + 1;
    System.out.println("1/" + i);
    expandUnitFrac(p*i-q, q*i);
  }
}
```

6.3 Combination

Number of combinations of k elements within n ones (C_n^k)
Special case : $C_n^k \; mod \; 2 = n \oplus m$

```
long C(int n, int k) {
  double r = 1;
  k = Math.min(k, n - k);
  for(int i = 1; i <= k; i++)
  r /= i;
  for(int i = n; i >= n - k + 1; i--)
  r *= i;
  return Math.round(r);
}
```

6.3.1 Catalan numbers

$\text{cat}(n) = \frac{C_n^{2n}}{n+1}$ $\text{cat}(n+1) = \frac{(2n+2)(2n+1)}{(n+2)(n+1)}\text{cat}(n)$

- distinct binary trees with n vertices.
- expressions containing n pairs of parentheses correctly matched (e.g. $n = 3$ $()()(), ()(()), (())(), ((())), (()())$).
- parenthesize $n+1$ factors (e.g. $n = 3$ $(ab)(cd), a(b(cd)), ((ab)c)(d), (a(bc))(d), a((bc)d)$).
- triangulate a convex polygon of $n+2$ sides.
- number of monotonic paths along the edge of a $n \times n$ grid which do not pass above de diagonal.

Compute all Catalan number $\leq n$

```
long[] allCatalan(int n) {
  long[] catalanNumbers = new long[n];
  catalanNumbers[0] = 1;
  for(int i = 1; i < n; i++) {
    int j = i - 1;
    long b = j * j;
    long a = 4 * b + 6 * j + 2;
    b += 3 * j + 2;
    catalanNumbers[i] = catalanNumbers[j] * a/b;
  }
  return catalanNumbers;
}
```

6.4 Fibonacci series

$f(0) = 0$, $f(1) = 1$ et $f(n) = f(n-1) + f(n-2)$.
The following relation enables us to compute every number of the series in $O(log(n))$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

6.5 Cycle finding

```
int[] floydCycleFinding (int x0) {
  int tortoise = f(x0), hare = f(f(x0));
  while (tortoise != hare) {
    tortoise = f(tortoise);
    hare = f(f(hare)); }
  int mu = 0; hare = x0; // first
  while (tortoise != hare) {
    tortoise = f(tortoise); hare = f(hare); mu++; }
  int lambda = 1; hare = f(tortoise); // length
  while (tortoise != hare) {
    hare = f(hare); lambda++; }
  return new int[] {mu, lambda};
}
```

6.6 Number theory

6.6.1 Misc

$$ax \le b \Leftrightarrow x \le \left\lfloor \frac{b}{a} \right\rfloor \quad ax \ge b \Leftrightarrow x \le \left\lceil \frac{b}{a} \right\rceil \quad \left\lceil \frac{a}{b} \right\rceil = \left\lfloor \frac{a+b-1}{b} \right\rfloor$$

```
long gcd (long a, long b) {
  return (b == 0) ? a : gcd(b, a % b);
}
long lcm (long a, long b) {
  return a * (b / gcd(a,b));
}
long modInverse (long a, long b) {
  return big(a).modInverse(big(b)).longValue();
}
long modInverse (long a, long b) {
  extendedEuclid(a, b);
  return x;
}
```

In prime factorization of n, the power of p is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

```
int factopower (int n, int p) {
  int pow = 0;
  while (n > 0) {
    pow += n / p;
    n /= p;
  }
  return pow;
}
```

6.6.2 Équations diophantiennes

$ax + by = c$. $d = \gcd(a, b)$, no sol si d divise pas c sinon $(a, b) = (x(n/d) + (b/d)n, y(n/d) + (a/d)n)$ où $ax + by = d$ $n \in \mathbb{Z}$.

```
static int x, y;
static int extendedEuclid(int a, int b) {
  if (b == 0) { x = 1; y = 0; return a; }
  int d = extendedEuclid(b, a % b);
  int x1 = y;
  int y1 = x - (a / b) * y;
  x = x1;
  y = y1;
  return d;
}
```

6.6.3 Chinese remainder theorem

```
static long[] chinese (long[] b, long[] m) {
  long x = b[0], l = m[0];
  for (int i = 1; i < m.length; i++) {
    long m1 = m[i], b1 = b[i];
    long d = gcd(l, m1);
    if ((x - b1) % d != 0) return null;
    long lcm = l * (m1 / d);
    long t1 = ((((x - b1) / d) % lcm) * (modInverse(
    m1/d, l/d) % lcm)) % lcm;
    x = (b1 + ((t1 * m1) % lcm)) % lcm;
    l = lcm;
  }
  return new long[] {x, l};
}
```

6.6.4 Euler phi

$\phi(N) = N \times \prod_{p|N}(1 - \frac{1}{p}) = \#\{k < N | \gcd(k, N) = 1\}$

```
long phi(long n, int primes[]) {
  long ans = n; // Method 1
  for (int i = 0; i < primes.length && primes[i] *
    primes[i] <= n; i++) {
    int p = primes[i];
    if (n % p == 0) ans -= ans / p;
    while (n % p == 0) ans /= p;
  }
  if (n != 1) ans -= ans / n;
  return ans;
}
for (int i = 1; i <= 1000000; i++) phi[i] = i;
for (int i = 2; i <= 1000000; i++) // Method 2
  if (phi[i] == i) // i is prime
    for (int j = i; j <= 1000000; j += i)
      phi[j] = (phi[j] / i) * (i - 1);
```

- If $\phi(1) = 1$, $n = \sum_{d|n} \phi(d)$.
- p prime iff there exists a number relatively prime with p of order $p - 1$ (primitive root of p).
- There is $\phi(d)$ number of orders d modulo p.
- If g is order d mod p, $\{g^k | k = 1, \dots, d-1 : (k, d) = 1\}$ are the $\phi(d)$ numbers of order d mod p.

Let $\phi_S(n) = \sum_{i=1}^{n} \phi(i)$.

$$\phi_S(n) = \frac{n^2 + n}{2} - \sum_{d=2}^{n} \phi_S\left(\left\lfloor \frac{n}{d} \right\rfloor\right).$$

Discrete log

$$\begin{aligned} a^x \equiv a^y \pmod{n} &\Leftrightarrow x \equiv y \pmod{O_n(a)} \\ &\Leftarrow x \equiv y \pmod{\phi(n)} \end{aligned}$$

and in particular, if g is a primitive root of p,

$$g^x \equiv g^y \pmod{p} \Leftrightarrow x \equiv y \pmod{p-1}$$

so for an equation $(p \not| a, b)$

$$a^{k_1} \equiv b^{k_2} \pmod{p}$$

we take ℓ_1 and ℓ_2 such that $a = g^{\ell_1}$ and $b = g^{\ell_2}$ and it becomes

$$k_1\ell_1 \equiv k_2\ell_2 \pmod{p-1}$$

6.6.5 Quadratic residue (QR)

p *odd* prime. Let g primitive root mod p. $\forall n$, g^{2n} is QR mod p and g^{2n+1} is not. There is $\frac{p-1}{2}$ QR and $\frac{p-1}{2}$ not QR.

$$\begin{aligned} \left(\frac{a}{p}\right) &\equiv a^{\frac{p-1}{2}} \pmod{m} \\ &= \prod_{r=1}^{\frac{p-1}{2}} \varepsilon(ar) \end{aligned}$$

where $\varepsilon(x) = 1$ if $x \equiv 1, \dots, \frac{p-1}{2} \pmod{p}$ and -1 otherwise. b odd ($\left(\frac{a}{b}\right) = 1$ does not mean a QR mod b !!!)

$$\left(\frac{a}{b}\right) \triangleq \prod \left(\frac{a}{p_i}\right)^{e_i}$$

- $\left(\frac{-1}{b}\right) = 1$ iff $b \equiv 1 \pmod 4$.
- $\left(\frac{2}{b}\right) = 1$ iff $b \equiv \pm 1 \pmod 8$.

b odd

$$\left(\frac{ac}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{c}{b}\right)$$

a, b odd

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2}\frac{b-1}{2}}.$$

```
static long modpow (long a, long n, long m) {
  if (n == 0) {
    return 1 % m;
  }
  if (n % 2 == 0) {
    long demi = modpow(a, n/2, m);
    return (demi * demi) % m;
  } else {
    return (modpow(a, n-1, m) * a) % m;
  }
}
static long modular_sqrt(long a, long p) {
  /*
     Solve the congruence of the form:
     x^2 = a (mod p)
     And returns x. Note that p - x is also a root.

     0 is returned is no square root exists for
     these a and p.
     */
  /*
     The Tonelli-Shanks algorithm is used (except
     for some simple cases in which the solution
     is known from an identity). This algorithm
     runs in polynomial time (unless the
     generalized Riemann hypothesis is false).
     */
  // Simple cases
  if (legendre_symbol(a, p) != 1) {
    return 0;
  } else if (a == 0) {
    return 0;
  } else if (p == 2) {
    return a;
  } else if (p % 4 == 3) {
    return modpow(a, (p + 1) / 4, p);
  }

  /* Partition p-1 to s * 2^e for an odd s (i.e.
     reduce all the powers of 2 from p-1)
     */
  long s = p - 1;
  long e = 0;
  while (s % 2 == 0) {
    s /= 2;
    e += 1;
  }

  /* Find some 'n' with a legendre symbol n|p = -1.
     Shouldn't take long.*/
  long n = 2;
  while (legendre_symbol(n, p) != -1) {
    n += 1;
  }

  /* x is a guess of the square root that gets
    better
   * with each iteration.
   * b is the "fudge factor" - by how much we're off
   * with the guess. The invariant x^2 = ab (mod p)
   * is maintained throughout the loop.
   * g is used for successive powers of n to update
   * both a and b
   * r is the exponent - decreases with each update
   */
  long x = modpow(a, (s + 1) / 2, p);
  long b = modpow(a, s, p);
  long g = modpow(n, s, p);
  long r = e;

  for (;;) {
    long t = b;
    long m = 0;
    for (m = 0; m < r; m++) {
      if (t == 1) {
        break;
      }
      t = (t * t) % p;
    }

    if (m == 0) {
      return x;
    }

    long pow2 = 1;
    for (int i = 0; i < r-m-1; i++) { pow2 *= 2; }
    long gs = modpow(g, pow2, p);
    g = (gs * gs) % p;
    x = (x * gs) % p;
    b = (b * g) % p;
    r = m;
  }
}


static long legendre_symbol1(long a, long p) {
  // p is prime and a is rel. prime to b
  long ls = modpow(a, (p - 1) / 2, p);
  return ls == p - 1 ? -1 : ls;
}
static long legendre_symbol(long a, long b) {
  // b is odd and rel. prime to a
  a %= b;
  if (a == 0) {
    return 0;
  }
  int exp2 = 0;
  while (a % 2 == 0) {
    a /= 2;
    exp2++;
  }
  int cur = 1;
  if (exp2 % 2 == 1 && (b % 8 == 3 || b % 8 == 5)) {
    cur *= -1;
  }
  if (a < 0) {
    if (b % 4 == 3) {
      cur *= -1;
    }
    a *= -1;
  }
  if (a == 1) {
    return cur;
  }
  if (a % 4 == 3 && b % 4 == 3) {
    cur *= -1;
  }
  return cur * legendre_symbol(b, a);
}
```

6.7 Linear equations

Solve $Ax = b$.

```
double[] gaussElim(double[][] A, double[] b) {
  int N  = b.length;
  for(int p = 0; p < N; p++) {
    int max = p;
    for(int i = p + 1; i < N; i++) {
      if(Math.abs(A[i][p])>Math.abs(A[max][p])) {
        max = i;
      }
    }
    swap(A, p, max);
```

```java
      swap(b, p, max);
      // singular or nearly singular
      if(Math.abs(A[p][p]) <= E) {
        return null;
      }
      // pivot within A and b
      for(int i = p + 1; i < N; i++) {
        double alpha = A[i][p] / A[p][p];
        b[i] -= alpha * b[p];
        for(int j = p; j < N; j++) {
          A[i][j] -= alpha * A[p][j];
        }
      }
    }
    // back substitution
    double[] x = new double[N];
    for(int i = N - 1; i >= 0; i--) {
      double sum = 0.0;
      for(int j = i + 1; j < N; j++) {
        sum += A[i][j] * x[j];
      }
      x[i] = (b[i] - sum) / A[i][i];
    }
    return x;
}
```

6.8 Ternary Search

Find minimum of unimodal function.

```java
double ternarySearch(double left, double right) {
  if(right - left < E) {
    return (right + left) / 2;
  }
  double leftThird = (left * 2 + right) / 3;
  double rightThird = (left + right * 2) / 3;
  //minimize >, maximize <
  if(f(leftThird) > f(rightThird)) {
    return ternarySearch(leftThird, right);
  }
  return ternarySearch(left, rightThird);
}
```

6.9 Integration

Compute integral.

```java
double integral(double a, double b) {
  double h = b - a;
  double c = (a + b) / 2.0;
  double d = (a + c) / 2.0;
  double e = (b + c) / 2.0;
  double Q1 = h/6  * (f(a) + 4*f(c) + f(b));
  double Q2 = h/12 * (f(a)+4*f(d)+2*f(c)+4*f(e)
                   +f(b));
  if (Math.abs(Q2 - Q1) <= E) {
    return Q2 + (Q2 - Q1) / 15;
  } else {
    return integral(a, c) + integral(c, b);
  }
}
```

7 Strings *untested*

Reverse a String

```java
new StringBuilder(line).reverse().toString()
```

7.1 Longest palindrome

```java
int[] calculateAtCenters(String s) {
  int n = s.length();
  int[] L = new int[2 * n + 1];
  int i = 0, palLen = 0, k = 0;
  while(i < n) {
    if((i > palLen) &&
        (s.charAt(i - palLen - 1)==s.charAt(i))) {
      palLen += 2;
      i += 1;
      continue;
    }
    L[k++] = palLen;
    int e = k - 2 - palLen;
    boolean found = false;
    for(int j = k - 2; j > e; j--) {
      if(L[j] == j - e - 1) {
        palLen = j - e - 1;
        found = true;
        break;
      }
      L[k++] = Math.min(j - e - 1, L[j]);
    }
    if(!found) {
      i += 1;
      palLen = 1;
    }
  }
  L[k++] = palLen;
  int e = 2 * (k - n) - 3;
  for(i = k - 2; i > e; i--) {
    int d = i - e - 1;
    L[k++] = Math.min(d, L[i]);
  }
  return L;
}

String getPalindrome(String s, int[] L) {
  int max = L[0];
  int maxI = 0;
  for(int i = 1; i < L.length; i++) {
    if(L[i] > max) {
      max = L[i];
      maxI = i;
    }
  }
  int b = 0, e = 0;
  b = maxI / 2 - L[maxI] / 2;
  e = maxI / 2 + L[maxI] / 2;
  e += maxI % 2 == 0 ? 0 : 1;
  return s.substring(b, e);
}

String getPalindrome(String s)
{
  return getPalindrome(s, calculateAtCenters(s));
}
```

7.2 Occurences in a string

KMP(s,p) returns occurences index of p in s.

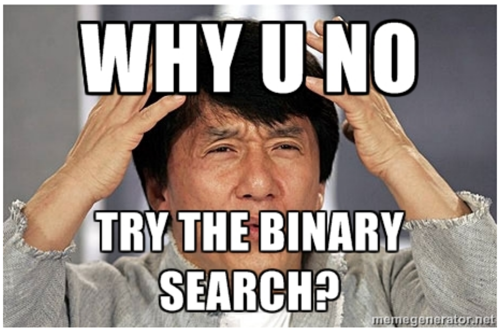
```java
int[] kmpPreprocess(char[] p) {
  int m = p.length;
  int[] b = new int[m+1];
  int i = 0, j = -1; b[0] = -1; // starting values
  while (i < m) { // pre-process the pattern string
    p
    while (j >= 0 && p[i] != p[j]) j = b[j]; // if
    different, reset j using b
    i++; j++; // if same, advance both pointers
    b[i] = j;
  }
  return b; }

LinkedList<Integer> kmpSearchAll(char[] s, char[] p)
     { // text, pattern
  int[] b = kmpPreprocess(p); // back table
  int n = s.length, m = p.length;
  LinkedList<Integer> found = new LinkedList<Integer
    >();
  int i = 0, j = 0; // starting values
  while (i < n) { // search through string s
    while (j >= 0 && s[i] != p[j]) j = b[j]; // if
    different, reset j using b
    i++; j++; // if same, advance both pointers
    if (j == m) { // a match found when j == m
      found.add(i-j);
```

```
      j = b[j]; // prepare j for the next possible
    match
    } }
  return found; }

int kmpSearchFirst(char[] s, char[] p) { // text,
    pattern
  int[] b = kmpPreprocess(p); // back table
  int n = s.length, m = p.length;
  int i = 0, j = 0; // starting values
  while (i < n) { // search through string s
    while (j >= 0 && s[i] != p[j]) j = b[j]; // if
    different, reset j using b
    i++; j++; // if same, advance both pointers
    if (j == m) { // a match found when j == m
      return i - j;
    } }
  return n - j; }
```

8 Miscellaneous



8.1 The answer

```
int reponse() { return 42; }
```

8.2 Sort algorithms *untested*

```
int findKth(int[] A, int k, int n) {
  if(n <= 10) {
    Arrays.sort(A, 0, n);
    return A[k];
  }
  int nG = (int)Math.ceil(n / 5.0);
  int[][] group = new int[nG][];
  int[] kth = new int[nG];
  for(int i = 0; i < nG; i++) {
    if(i == nG - 1 && n % 5 != 0) {
      group[i] = Arrays.copyOfRange(A, (n/5)*5, n);
      kth[i] = findKth(group[i], group[i].length /
    2,
                       group[i].length);
    } else {
      group[i] = Arrays.copyOfRange(A, i*5, (i+1)*5)
    ;
      kth[i] = findKth(group[i], 2, group[i].length)
    ;
    }
  }
  int M = findKth(kth, nG / 2, nG);
  int[] S = new int[n];
  int[] E = new int[n];
  int[] B = new int[n];
  int s = 0, e = 0, b = 0;
  for(int i = 0; i < n; i++) {
    if(A[i] < M) {
      S[s++] = A[i];
    } else if(A[i] > M) {
      B[b++] = A[i];
    } else {E[e++] = A[i];}
  }
  if(k < s) {
    return findKth(S, k, s);
  } else if(k >= s + e) {
    return findKth(B, k - s - e, b);
  }
  return M;
}

int[] countSort(int[] A, int k) { // O(n + k)
  int[] C = new int[k];
  for(int j = 0; j < A.length; j++) {
    C[A[j]]++;
  }
  for(int j = 1; j < k; j++) {
    C[j] += C[j - 1];
  }
  int[] B = new int[A.length];
  for(int j = A.length - 1; j >= 0; j--) {
    B[C[A[j]] - 1] = A[j];
    C[A[j]]--;
  }
  return B;
}

int[][] radixSort(int[][] nums, int k) { // O(d*(n+k
    ))
  int n = nums.length;
  int m = nums[0].length;
  int[][] B = null;
  for(int i = m - 1; i >= 0; i--) {
    int[] C = new int[k];
    for(int j = 0; j < n; j++) {
      C[nums[j][i]]++;
    }
    for(int j = 1; j < k; j++) {
      C[j] += C[j - 1];
    }
    B = new int[n][];
    for(int j = n - 1; j >= 0; j--) {
      B[C[nums[j][i]] - 1] = nums[j];
      C[nums[j][i]] = C[nums[j][i]] - 1;
    }
    nums = B;
  }
  return nums;
}

int mergeSort(int[] a) {
  int n = a.length;
  if(n == 1) {return 0;}
  int m = n / 2;
  int[] left = Arrays.copyOfRange(a, 0, m);
  int[] right = Arrays.copyOfRange(a, m, n);
  int inv = mergeSort(left);
  inv += mergeSort(right);
  inv += merge(left, right, a);
  return inv;
}

int merge(int[] left, int[] right, int[] a) {
  int i = 0, l = 0, r = 0, inv = 0;
  while(l < left.length && r < right.length) {
    if(left[l] <= right[r]) {
      a[i++] = left[l++];
    } else {
      inv += left.length - l;
      a[i++] = right[r++];
    }
  }
  for(int j = l; j < left.length; j++) {
    a[i++] = left[j];
  }
  for(int j = r; j < right.length; j++) {
    a[i++] = right[j];
  }
  return inv;
}

int countMinSwapsToSort(int[] a) {
  int[] b = a.clone();
  Arrays.sort(b);
  int nSwaps = 0;
  for(int i = 0; i < a.length; i++) {
```

```
    // cuidado com elementos repetidos!
    int j = Arrays.binarySearch(b, a[i]);
    if(b[i] == a[j] && i != j) {
      nSwaps++;
      swap(a, i, j);
    }
  }
  for(int i = 0; i < a.length; i++) {
    if(a[i] != b[i]) {
      nSwaps++;
    }
  }
  return nSwaps;
}

//Count (i, j):h[i] <= h[k] <= h[j], k = i+1,...,j
   -1.
int countVisiblePairs(int[] h) { // O(n)
  int n = h.length;
  int[] p = new int[n];
  int[] r = new int[n];
  Stack<Integer> S = new Stack<Integer>();
  for(int i = 0; i < n; i++) {
    int c = 0;
    if(S.isEmpty()) {
      S.push(h[i]);
      p[i] = 0;
    } else {
      if(S.peek() == h[i]) {
        p[i] = p[i - 1] + 1 - r[i - 1];
      } else {
        while(!S.isEmpty() && S.peek() < h[i]) {
     S.pop();
     c++;
   }
   p[i] = c;
   r[i] = c;
   if(!S.isEmpty()) {
     p[i]++;
   }
      }
    S.push(h[i]);
    }
  }
  return sum(p);
}

void shuffle(Object[] a)
{
  int N = a.length;
  for (int i = 0; i < N; i++) {
    int r = i + (int) (Math.random() * (N-i));
    swap(a, i, r);
  }
}
```

8.3 Huffman (compression)

Usually used for characters, but usable with everything in which we can count occurrences.
Make a prefix tree we use to decode and we unstack to encode.

```
class HuffmanNode implements Comparable<HuffmanNode>
{
  public boolean isLeaf;
  public int occurences;
  public int charIndex;
  public HuffmanNode left,right;
  public HuffmanNode(HuffmanNode left, HuffmanNode
    right)
  {
    this.occurences = left.occurences+right.
    occurences;
    this.left = left;
    this.right = right;
    isLeaf = false;
  }
  public HuffmanNode(int charIndex, int occurences)
  {
    this.charIndex = charIndex;
    this.occurences = occurences;
    isLeaf = true;
  }
  @Override
  public int compareTo(HuffmanNode o) {
    return occurences-o.occurences;
  }
}
HuffmanNode getHuffmanTree(int[] occurences) {
  PriorityQueue<HuffmanNode> q = new PriorityQueue<
    HuffmanNode>();
  for(int i = 0; i < occurences.length; i++)
    q.add(new HuffmanNode(i, occurences[i]));
  while(q.size() != 1) {
    HuffmanNode right = q.poll();
    HuffmanNode left = q.poll();
    q.add(new HuffmanNode(left, right));
  }
  return q.poll();
}
void getHuffmanTable(HuffmanNode tree, BitSet[]
    result, BitSet current, int pos){
  if(tree.isLeaf) {
    BitSet finalBitSet = new BitSet();
    for(int i = 0; i < pos; i++)
      finalBitSet.set(i, current.get(pos-i-1));
    result[tree.charIndex] = finalBitSet;
  } else {
    BitSet leftBitSet = new BitSet();
    leftBitSet.or(current);
    leftBitSet.set(pos, false);
    getHuffmanTable(tree.left, result, leftBitSet,
    pos+1);

    BitSet rightBitSet = new BitSet();
    rightBitSet.or(current);
    rightBitSet.set(pos, true);
    getHuffmanTable(tree.right, result, rightBitSet,
     pos+1);
  }
}

//n=occurences.length
static BitSet[] getHuffmanTable(int n, HuffmanNode
    tree) {
  BitSet[] result = new BitSet[n];
  getHuffmanTable(tree, result, new BitSet(), 0);
  return result;
}
```

8.4 Union Find

```
static class UnionFind {
  int[] depth; int[] leader; int[] size;
  public UnionFind(int n) {
    depth = new int[n]; leader = new int[n]; size =
    new int[n];
    Arrays.fill(depth, 1); Arrays.fill(size, 1);
    for(int i = 0; i < n; i++) leader[i] = i;
  }
  public int find(int a) {
    if(a != leader[a])
      leader[a] = find(leader[a]);
    return leader[a];
  }
  public void union(int a, int b) {
    int leaderA = find(a);
    int leaderB = find(b);
    if(leaderA == leaderB) return;
    if(size[leaderA] > size[leaderB]) {
      union(leaderB, leaderA); return;
    }
    leader[leaderA] = leaderB;
    depth[leaderB] = Math.max(depth[leaderA]+1,
    depth[leaderB]);
    size[leaderB] += size[leaderA];
  }
}
```

8.5 Fenwick Tree (RSQ solver)

```
static class FenwickTree {
  private int[] ft;
  private int LSOne(int S) { return (S & (-S)); }
  public FenwickTree(int n) { // ignore index 0
    ft = new int[n+1];
    for (int i = 0; i <= n; i++) ft[n] = 0;
  }
  public int rsq(int b) { // returns RSQ(1, b)
      PRE 1 <= b <= n
    int sum = 0; for (; b > 0; b -= LSOne(b)) sum +=
     ft[b];
    return sum;
  }
  public int rsq(int a, int b) { // returns RSQ(a, b
    ) PRE 1 <= a,b <= n
    return rsq(b) - (a == 1 ? 0 : rsq(a - 1));
  }
  void adjust(int k, int v) { // n = ft.size() - 1
      PRE 1 <= k <= n
    for (; k < ft.length; k += LSOne(k)) ft[k] += v;
  }
}
```

9 Being first?