Determination of the origin-destination

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1 Introduction

Let us consider a traffic network with n entrances and m exits. The number N_i of vehicles entering through each entrance i is given, as well as the number M_j of vehicles exiting through each exit j.

The question is to determine, soley from this information, the probablilities α_{ij} that a vehicle entering at entrance i chooses exit j. These probabilities should then satisfy

$$\begin{aligned} \sum_i N_i \alpha_{ij} &= M_j \qquad \forall j \\ \sum_j \alpha_{ij} &= 1 \qquad \forall i \end{aligned} \tag{1}$$

The first equation says that the number of cars which take exit j is the sum of the card from each entrance i that have selected exit j. The second equation says that all cars arriving at entrance i should select an exit.

Clearly, with n entrance and m exit, eq. (**??**) is underdetermined. Theres are $n \times m$ unknown (the α_{ij}) and only $n + m$ equations. Therefor there are a lot of possible solutions.

2 Model

To compute one solution among all the possible ones, the model we propose here is based on an **attractivity coefficient** β_j asssociated with each exit j. The idea is that the probability that a car goes to exit j depends on where this exit goes. For instance, a supermarket will be more attracive during the day than a theatre. It is likely the opposite in the evening. Also the attractivity can be assumed to be independent of the entrance. If someone goes to work, the attracitvity of its company is related to the importance of the company, not related to where the worker lives.

However, another hypothesis is added to this model. If there are two supermarkets, both with very similar properties, one can argue that people will

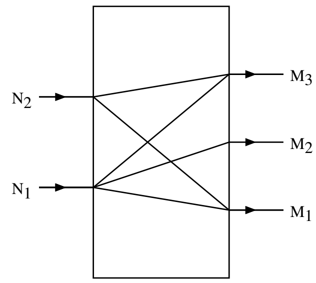

Figure 1: Example of as graph showing which network entrance is connected to which network exit. Here, there are N_1 cars arriving at entrance $i = 1$, N_2 arriving at entrance $i = 2$. Those car from entrance $i = 1$ can go to any exit, wheras cars arriving at entrance $i = 2$ can only reach exits $j = 1$ or $j = 3$.

only go the the closest one. Also if there are several access to a company (e.g. CERN) it is likely that the car will pick the entrance closest to where thex live.

Therefore our model requires that a graph of possible destinations from each entrance be specified. This graph is typically defined by someones who knows well the area and can guess which route can be excluded.

This amounts to saying that for each exit j there is a set I_j of possible entrances.

Figure 1 shows a graph for a situation of 3 entrance and 2 exits. In this example, one has

$$I_1 = \{1, 2\} \qquad I_2 = \{1\} \qquad I_3 = \{1, 2\} \tag{2}$$

We now assume that β_j is the attractivity of exit j. This amounts to saying that, for all i

$$\alpha_{ij} \propto \beta_j \tag{3}$$

Then, the reasoning goes as follows: cars from each entrance i may go to several other exits j. Therefore their fraction that goes to exit j is the ratio of the attractivity of j divided by the sum of the attractivities of the other exits possible from entrance i

$$\alpha_{ij} = \frac{\beta_j}{\sum_{k \in J_i} \beta_k} \tag{4}$$

Here, J_i is defined as the set of exits that are reachable from entrance i.

In the example of Fig. 1, one has

$$J_1 = \{1, 2, 3\} \qquad J_2 = \{1, 3\} \tag{5}$$

Note the following relation between I_j and J_i, that holds for any X_{ij}

$$\sum_{i=1}^{n} \sum_{j \in J_i} X_{ij} = \sum_{j=1}^{m} \sum_{i \in I_j} X_{ij} \tag{6}$$

For our example, the relation canbe checked easily

$$X_{11} + X_{12} + X_{13} + X_{21} + X_{23} = X_{11} + X_{21} + X_{12} + X_{13} + X_{23}$$

To demonstrate formally this relation, we first notice that the set I_j and J_i correspond to a specifying a list of neighbors in a graph. One can also use adjacency matrices to indicate which entrance is connected to which exit. For instance let us define the $n \times m$ adjacency matrix A as

$$A_{ij} = \begin{cases} 1 & \text{if entrance } i \text{ is connected to exit } j \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Similarily, one can define the $m \times n$ adjacency matrix B as

$$B_{ji} = \begin{cases} 1 & \text{if exit } j \text{ is reachable from entrance } i \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

In our example, we have

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

By their definition, it is clear that these two matrices are transposed of each other

$$A = B^T \qquad A_{ij} = B_{ji}$$

Then, for any quantity X_{ij} one has

$$\sum_{i=1}^{n} \sum_{j \in I_i} X_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} X_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} B_{ji} X_{ij} = \sum_{j=1}^{m} \sum_{i \in J_i} X_{ij} \tag{9}$$

which is the expected results.

3 The system of equations

With α_{ij} expressed by eq. (4), equations (1) for the example described in Fig. 1 read

$$\begin{aligned} \frac{\beta_1}{\beta_1 + \beta_2 + \beta_3} N_1 + \frac{\beta_1}{\beta_1 + \beta_3} N_2 &= M_1 \\ \frac{\beta_2}{\beta_1 + \beta_2 + \beta_3} N_1 &= M_2 \\ \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} N_1 + \frac{\beta_3}{\beta_1 + \beta_3} N_2 &= M_3 \end{aligned} \tag{10}$$

We can see that now we have 3 equations and 3 unknowns. The normalization of the α_{ij} is guaranteed by this construction. Indeed

$$\alpha_{11} + \alpha_{12} + \alpha_{13} = \frac{\beta_1}{\beta_1 + \beta_2 + \beta_3} + \frac{\beta_2}{\beta_1 + \beta_2 + \beta_3} + \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} = 1 \tag{11}$$

and

$$\alpha_{21} + \alpha_{23} = \frac{\beta_1}{\beta_1 + \beta_3} + \frac{\beta_3}{\beta_1 + \beta_3} = 1 \tag{12}$$

More generally, with eq. (4) and the entrance and exit sets I and J, the system of equations (1) becomes

$$\sum_{i \in I_j} N_i \frac{\beta_j}{\sum_{k \in J_i} \beta_k} = M_j \qquad j \in \{1, 2, \ldots, m\} \tag{13}$$

$$\sum_{j \in J_i} \frac{\beta_j}{\sum_{k \in J_i} \beta_k} = 1 \qquad i \in \{1, 2, \ldots, n\} \tag{14}$$

$$\tag{15}$$

We observe that here again, the second set of equations is obeyed by construction. We also observed that if β_j is a solution of (13), so is $\lambda \beta_j$, for any number $\lambda \neq 0$. However. the value of α_{ij} is the same for any λ.

Finally, note that by summing (13) over $j \in$ and using (6) we get

$$\sum_{j=1}^{m} M_j = \sum_{j=1}^{m} \sum_{i \in I_j} N_i \frac{\beta_j}{\sum_{k \in J_i} \beta_k} = \sum_{i=1}^{n} \sum_{j \in J_i} N_i \frac{\beta_j}{\sum_{k \in J_i} \beta_k} = \sum_{i=1}^{n} N_i \tag{16}$$

This is a consistency relation that must hold for the given values of N_i and M_j.

4 Iterative solution

Equation (13) is non-linear but can hopefully be solved iteratively by writing it as

$$\beta_i = \frac{M_j}{\sum_{i \in I_j} \frac{N_i}{\sum_{k \in J_i} \beta_k}} \qquad j \in \{1, 2, \ldots, m\} \tag{17}$$

One starts the iterations with a guess on the values of β_j for the right hand side, for instance $b_j = 1$, for all $j \in \{1, 2, \ldots, m\}$. If the iteration converges, one obtain a solution for the β_j and then for that α_{ij}.