1. a) $\frac{dy(x)}{dx} + ay(x) = x^3$

general solution: to obtain complementary function $y = g(x)$:

$$\frac{dy(x)}{dx} + ay(x) = 0$$

$$\frac{dy(x)}{dx} = -ay(x)$$

$$\int \frac{1}{y(x)}\, dy(x) = \int -a\, dx$$

$$\ln|y(x)| = -ax + c$$

general solution: $y(x) = Ae^{-ax}$, $A = e^c$

, to obtain particular solution $y = p(x)$:

Let particular integral be:

$y = C_3x^3 + C_2x^2 + C_1x + C_0$, C_3, C_2, C_1, C_0 are constants

$$\frac{dy}{dx} = 3C_3x^2 + 2C_2x + C_1$$

Substituting,

$$3C_3x^2 + 2C_2x + C_1 + a(C_3x^3 + C_2x^2 + C_1x + C_0) = x^3$$

$$aC_3x^3 + (3C_3 + aC_2)x^2 + (2C_2 + aC_1)x + (C_1 + aC_0) = x^3$$

Comparing coefficients,

$aC_3x^3 = x^3$, $3C_3 + aC_2 = 0$, $2C_2 + aC_1 = 0$, $C_1 + aC_0 = 0$

$aC_3 = 1$ $\qquad C_3 = -\frac{a}{3}C_2$ $\qquad 2C_2 - a^2C_0 = 0$ $\qquad C_1 = -aC_0$

$C_3 = \frac{1}{a}$ $\longrightarrow$ $\frac{1}{a} = -\frac{a}{3}\left(\frac{a^2C_0}{2}\right)$ $\qquad C_2 = \frac{a^2C_0}{2}$ $\qquad C_1 = -a\left(-\frac{6}{a^4}\right)$

$\frac{1}{a} = -\frac{a^3C_0}{6}$ $\qquad C_2 = \frac{a^2\left(-\frac{6}{a^4}\right)}{2}$ $\qquad C_1 = \frac{6}{a^3}$

$-\frac{6}{a^4} = C_0$ $\qquad C_2 = -\frac{3}{a^2}$

$\therefore$ particular solution:

$$y(x) = \frac{1}{a}x^3 - \frac{3}{a^2}x^2 + \frac{6}{a^3}x - \frac{6}{a^4}$$

1.b) $\frac{dQ(t)}{dt} - rQ(t) = te^{-\alpha t}$

general solution: $\frac{dQ(t)}{dt} = rQ(t)$

$$\int \frac{1}{Q(t)}\, dQ(t) = \int r\, dt$$

$$\ln|Q(t)| = rt + c$$

$$Q(t) = Ae^{rt}, \quad A = e^c$$

(Assuming $r \neq -\alpha$):

, particular solution:

Let particular integral be:

$$Q(t) = (c_1t + c_0)e^{-\alpha t}$$

$$\frac{dQ(t)}{dt} = (c_1t + c_0)(-\alpha e^{-\alpha t}) + e^{-\alpha t}(c_1)$$

$$= -\alpha(c_1t + c_0)e^{-\alpha t} + c_1e^{-\alpha t}$$

$$= -\alpha c_1te^{-\alpha t} - \alpha c_0e^{-\alpha t} + c_1e^{-\alpha t}$$

$$= -\alpha c_1te^{-\alpha t} + (c_1 - \alpha c_0)e^{-\alpha t}$$

$$\therefore -\alpha c_1te^{-\alpha t} + (c_1 - \alpha c_0)e^{-\alpha t} - r(c_1t + c_0)e^{-\alpha t} = te^{-\alpha t}$$

$$-\alpha c_1te^{-\alpha t} - rc_1te^{-\alpha t} + (c_1 - \alpha c_0)e^{-\alpha t} - rc_0e^{-\alpha t} = te^{-\alpha t}$$

$$(-\alpha c_1 - rc_1)te^{-\alpha t} + (c_1 - \alpha c_0 - rc_0)e^{-\alpha t} = te^{-\alpha t}$$

Comparing coefficients,

$-\alpha c_1 - rc_1 = 1$, $\qquad c_1 - \alpha c_0 - rc_0 = 0$

$c_1 = -\frac{1}{\alpha + r}$, $\qquad c_1 - c_0(\alpha + r) = 0$

$c_1 = c_0(\alpha + r)$

$c_0 = -\frac{1}{(\alpha + r)^2}$

$\Rightarrow$ particular solution:

$$Q(t) = -\left(\frac{1}{\alpha + r}t + \frac{1}{(\alpha + r)^2}\right)e^{-\alpha t}$$

1. c) $\frac{dx(t)}{dt} + 3x(t) = t\sin(3t)$

To find particular solution,

Let particular integral be:

$$x(t) = (c_1 t + c_0)\sin(3t) + (d_1 t + d_0)\cos(3t)$$

$$\frac{dx(t)}{dt} = (c_1 t + c_0)(3\cos 3t) + c_1 \sin(3t) + (d_1 t + d_0)(-3\sin 3t) + d_1 \cos 3t$$

$$\therefore (c_1 t + c_0)(3\cos 3t) + c_1\sin(3t) + (d_1 t + d_0)(-3\sin 3t) + d_1\cos 3t + 3(c_1 t + c_0)\sin(3t) + 3(d_1 t + d_0)\cos(3t) = t\sin 3t$$

$$3c_1 t\cos 3t + 3c_0\cos 3t + c_1\sin(3t) - 3d_1 t\sin 3t - 3d_0\sin 3t + d_1\cos 3t + 3c_1 t\sin 3t + 3c_0\sin 3t + 3d_1 t\cos 3t + 3d_0\cos 3t = t\sin 3t$$

comparing coefficients,

$$3c_1 + 3d_1 = 0 \quad , \quad 3c_1 - 3d_1 = 1 \quad , \quad 3c_0 + d_1 + 3d_0 = 0 \, , \quad c_1 - 3d_0 + 3c_0 = 0$$

$$\Rightarrow 6d_1 = -1$$

$$d_1 = -\frac{1}{6}$$

$$\therefore 3c_1 - 3\left(-\frac{1}{6}\right) = 1$$

$$c_1 = \frac{1}{6}$$

$$\Rightarrow 3c_0 + d_1 + c_1 + 3c_0 = 0$$

$$6c_0 + d_1 + c_1 = 0$$

$$c_0 = \frac{-\left(-\frac{1}{6}\right) - \frac{1}{6}}{6}$$

$$= 0$$

$$\therefore c_1 - 3d_0 = 0$$

$$d_0 = \frac{\frac{1}{6}}{3}$$

$$d_0 = \frac{1}{18}$$

$\therefore$ particular solution: $x(t) = \frac{1}{6}t\sin 3t - \frac{1}{6}t\cos 3t + \frac{1}{18}\cos 3t$

2. $\frac{\partial f}{\partial \vec{r}} \cdot \vec{v} + \frac{1}{\hbar}\frac{\partial f}{\partial \vec{k}}\vec{F} + \frac{\partial f}{\partial t} = -\frac{f - f_0}{\tau}$, $f(\vec{k}, t)$, $f_0 = \frac{1}{\exp\left[\frac{\epsilon_k - \epsilon_F}{k_B T}\right] + 1}$

- For $\frac{\partial f}{\partial \vec{r}} = 0$ and $\vec{F} = 0$ at $t = 0$,

the Boltzmann transport eqn reduces to:

$$\frac{\partial f}{\partial t} = -\frac{f - f_0}{\tau}$$

and we can solve it as a linear ODE via variable separation:

$$\int \frac{1}{f - f_0}\, df = \int -\frac{1}{\tau}\, dt$$

$$\ln|f - f_0| = -\frac{t}{\tau} + c$$

for the case at $t = 0$, $f(\vec{k}, t) = f(\vec{k}, t=0)$ being the nonequilibrium distribution:

$$\ln|f(\vec{k}, t=0) - f_0| = c$$

$$\therefore \ln|f - f_0| = -\frac{t}{\tau} + \ln|f(\vec{k}, t=0) - f_0|$$

$$\ln\left|\frac{f - f_0}{f(\vec{k}, t=0) - f_0}\right| = -\frac{t}{\tau}$$

$$\frac{f - f_0}{f(\vec{k}, t=0) - f_0} = \exp\left[-\frac{t}{\tau}\right]$$

$$\therefore f(\vec{k}, t) = f_0 + \left[f(\vec{k}, t=0) - f_0\right]\exp\left[-\frac{t}{\tau}\right]$$

depending on the time constant τ, $f(\vec{k},t)$ will 'relax' back to the equilibrium distribution function f_0 with time,

as we can see with the limit as $t \to \infty$:

$$\lim_{t\to\infty} f(\vec{k},t) = \lim_{t\to\infty}\left[f_0 + \left(f(\vec{k},t=0) - f_0\right)\exp\left[-\frac{t}{\tau}\right]\right]$$

$$= f_0 + 0$$

$$\lim_{t\to\infty}\left[f(\vec{k},t)\right] = \underline{\underline{f_0}}$$

3. a) $M(t) \rightarrow$ (Infants protected by maternally derived antibodies)

$S(t) \rightarrow$ (susceptible population, not yet exposed to x)

$E(t) \rightarrow$ (exposed but not yet infectious)

$I(t) \rightarrow$ (Infected and transmitting x)

$R(t) \rightarrow$ (Recovered and Immunised)

* Assumptions for the MSEIR model:

1. once recovered, the individual will forever be immune.
2. once vaccinated, individuals will be immune forever.
3.

$N(t) = M(t) + S(t) + E(t) + I(t) + R(t)$

$$\frac{dM(t)}{dt} = b\left(N - S(t)\right) - (d+\mu)M(t)$$

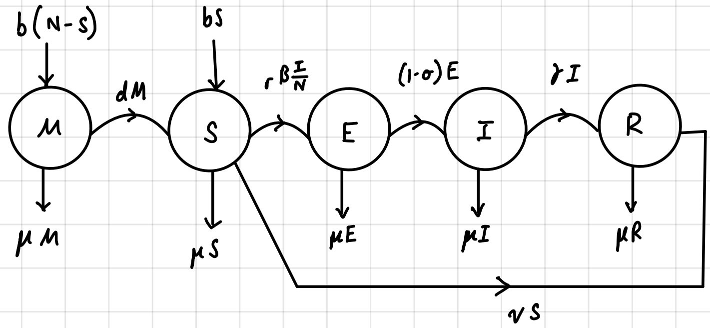
$$\frac{dS(t)}{dt} = bS(t) + dM(t) - r\beta S(t)\frac{I(t)}{N(t)} - (\mu+\nu)S(t)$$

$$\frac{dE(t)}{dt} = r\beta S(t)\frac{I(t)}{N(t)} - (1-\sigma+\mu)E(t)$$

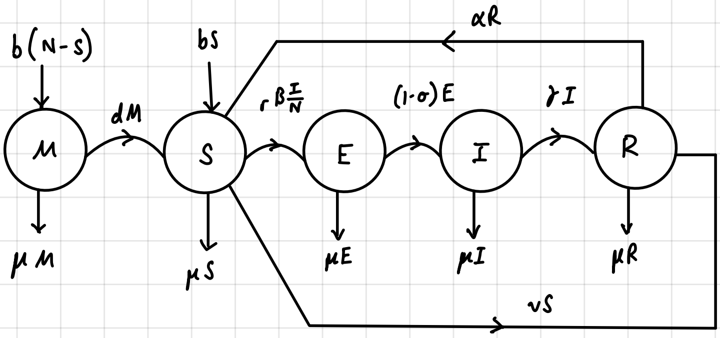
$$\frac{dI(t)}{dt} = (1-\sigma)E(t) - (\gamma+\mu)R(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t) + \nu S(t) - \mu R(t)$$

b)



c)



For waning immunity of the $R(t)$, the $M(t)$, $E(t)$ and $I(t)$ models remain constant as before in the MSEIR model, but the new $S(t)$ and $R(t)$ are given by:

$$\frac{dS(t)}{dt} = bS(t) + dM(t) + \alpha R(t) - r\beta S(t)\frac{I(t)}{N(t)} - (\mu+\nu)S(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t) + \nu S(t) - (\mu+\alpha)R(t)$$

4. $\frac{dT(t)}{dt} = k\left[T_a - T(t)\right]$

$T(0) = T_0$, solving for $T(t)$ via variable separation:

$$\int \frac{1}{T_a - T(t)}\, dT(t) = \int k\, dt$$

$$-\ln\left|T_a - T(t)\right| = kt + C$$

$$T_a - T(t) = Ae^{-kt}, \quad A = e^{-C}$$

$$\text{at } t=0: \quad T_a - T_0 = Ae^{-k(0)}$$

$$A = T_a - T_0$$

$$\therefore T(t) = T_a - (T_a - T_0)e^{-kt}$$

To find $T_{new} = T(t')$:

$$T_{new} = T_a - (T_a - T_0)e^{-kt'}$$

$$\frac{T_a - T_{new}}{T_a - T_0} = e^{-kt'}$$

$$\ln\left|\frac{T_a - T_{new}}{T_a - T_0}\right| = -kt'$$

$$t' = -\frac{1}{k}\ln\left|\frac{T_a - T_{new}}{T_a - T_0}\right|$$

5. $\frac{dS(t)}{dt} = -r\beta S(t)\frac{I(t)}{N(t)} - \mu S(t) + bN(t)$

$$\frac{dI(t)}{dt} = r\beta S(t)\frac{I(t)}{N(t)} - \mu I(t)$$

$$\frac{dN(t)}{dt} = bN(t) - \mu S(t) - \mu I(t)$$

$N(t) = S(t) + I(t)$, $I(0) = I_0$

$$\frac{dN(t)}{dt} = \frac{dS(t)}{dt} + \frac{dI(t)}{dt}$$

eliminating $S(t)$ from $\frac{dI(t)}{dt}$:

$$\frac{dI(t)}{dt} = r\beta\left(N(t) - I(t)\right)\frac{I(t)}{N(t)} - \mu I(t)$$

$$= r\beta I(t) - r\beta\frac{I^2(t)}{N(t)} - \mu I(t)$$

$$= (r\beta - \mu)I(t) - r\beta\frac{I^2(t)}{N(t)}$$

at steady-state, $N(t) = N$ (constant)

$$\frac{dI(t)}{dt} = (r\beta - \mu)I(t) - \frac{r\beta}{N}I^2(t)$$

$$= r\beta\left[\left(1 - \frac{\mu}{r\beta}\right)I(t) - \frac{1}{N}I^2(t)\right]$$

Let $a = 1 - \frac{\mu}{r\beta}$, $b = \frac{1}{N}$ and

solve via variable separation method:

$$\frac{dI(t)}{dt} = r\beta\left[aI(t) - bI^2(t)\right] \rightarrow$$

$$\int \frac{1}{aI(t) - bI^2(t)}\, dt = \int r\beta\, dt,$$

decompose the partial fraction:

$$\frac{-1}{I(bI-a)} = \frac{A}{I} + \frac{B}{bI-A}$$

$$-1 = A(bI-a) + BI$$

Let $I=0$: , let $I = \frac{a}{b}$:

$-1 = -aA$ $-1 = B\frac{a}{b}$

$A = \frac{1}{a}$ $B = -\frac{b}{a}$

$$\Rightarrow \frac{1}{I(a-bI)} = \frac{1}{aI} + \frac{b}{a(a-bI)}$$

$$\int \frac{1}{I(t)(a-bI(t))}\, dt = \int r\beta\, dt$$

$$\int\left[\frac{1}{aI(t)} + \frac{b}{a(a-bI(t))}\right]dt = r\beta t + C$$

$$\frac{1}{a}\ln|I(t)| + \frac{b}{a}\ln|a - bI(t)|\left(-\frac{1}{b}\right) = r\beta t + C$$

$$\frac{1}{a}\left[\ln\left|\frac{I(t)}{a - bI(t)}\right|\right] = r\beta t + C$$

Initial conditions $I(0) = I_0$:

$$\Rightarrow \frac{1}{a}\ln\left|\frac{I_0}{a - bI_0}\right| = C$$

$$\ln\left|\frac{I_0}{a - bI_0}\right| = aC$$

$$\therefore \ln\left|\frac{I(t)}{a - bI(t)}\right| = ar\beta t + aC$$

$$I(t) = (a - bI(t))\exp\left[ar\beta t + aC\right]$$

$$I(t) + bI(t)\exp\left[ar\beta t + aC\right] = a\exp\left[ar\beta t + c\right]$$

$$I(t) = \frac{a\exp\left[ar\beta t + c\right]}{1 + b\exp\left[ar\beta t + ac\right]}$$

$$I(t) = \frac{a\exp\left[ar\beta t + c\right]}{1 + b\exp\left[ar\beta t + ac\right]}$$

$$I(t) = \frac{a}{b + \exp\left[-ar\beta t - ac\right]}$$

$$I(t) = \frac{1 - \frac{\mu}{r\beta}}{\frac{1}{N} + \exp\left[-(r\beta - \mu)t - \ln\left|\frac{I_0}{a - bI_0}\right|\right]}$$

(cont. next page)

5. $$I(t) = \frac{1 - \frac{\mu}{r\beta}}{\frac{1}{N} + \frac{\exp[-(r\beta - \mu)t]}{\frac{I_0}{a - bI_0}}}$$

$$I(t) = \frac{\left(1 - \frac{\mu}{r\beta}\right)}{\frac{1}{N} + \frac{\left[\left(1 - \frac{\mu}{r\beta}\right) - \frac{1}{N} I_0\right] \exp[-(r\beta - \mu)t]}{I_0}}$$

$$I(t) = \frac{N I_0 \left(1 - \frac{\mu}{r\beta}\right)}{I_0 + \left[N\left(1 - \frac{\mu}{r\beta}\right) - I_0\right] \exp[-(r\beta - \mu)t]}$$ (proved)

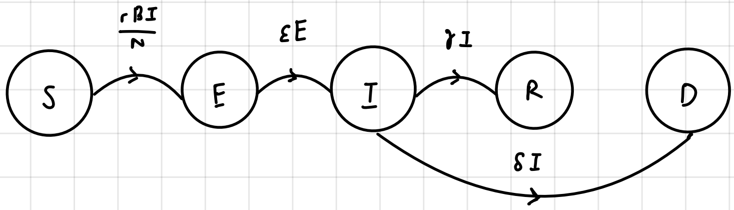
6. a) $$\frac{dS(t)}{dt} = -\frac{r\beta I(t)}{N(t)} S(t)$$

$$\frac{dE(t)}{dt} = \frac{r\beta I(t)}{N(t)} S(t) - \varepsilon E(t)$$

$$\frac{dI(t)}{dt} = \varepsilon E(t) - \gamma I(t) - \delta I(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t)$$

$$\frac{dD(t)}{dt} = \delta I(t)$$



b) i) From the SEIRD python simulation,

Value of peak infection: 143 people

ii) Time taken to reach peak infected population: 66 days

iii) Time taken for infected to fall below 10: 111 days

c) i) Value of peak infected: 268 people

ii) Time taken to reach peak infected: 36 days

iii) Time taken for infected to fall below 10: 69 days