

$$1. a) \frac{dy(x)}{dx} + ay(x) = x^3$$

general solution: to obtain complementary function  $y = g(x)$ : , to obtain particular solution  $y = p(x)$ :

$$\frac{dy(x)}{dx} + ay(x) = 0$$

$$\frac{dy(x)}{dx} = -ay(x)$$

$$\int \frac{1}{y(x)} dy(x) = \int -a dx$$

$$\ln |y(x)| = -ax + c$$

$$\text{general solution: } y(x) = Ae^{-ax}, A = e^c$$

Let particular integral be:

$$y = C_3 x^3 + C_2 x^2 + C_1 x + C_0, \quad C_3, C_2, C_1, C_0 \text{ are constants}$$

$$\frac{dy}{dx} = 3C_3 x^2 + 2C_2 x + C_1$$

substituting,

$$3C_3 x^2 + 2C_2 x + C_1 + a(C_3 x^3 + C_2 x^2 + C_1 x + C_0) = x^3 \\ aC_3 x^3 + (3C_3 + aC_2)x^2 + (2C_2 + aC_1)x + (C_1 + aC_0) = x^3$$

Comparing coefficients,

$$aC_3 x^3 = x^3, \quad 3C_3 + aC_2 = 0, \quad 2C_2 + aC_1 = 0, \quad C_1 + aC_0 = 0$$

$$aC_3 = 1 \quad C_3 = -\frac{a}{3}C_2 \quad 2C_2 + a^2C_0 = 0 \quad C_1 = -aC_0$$

$$C_3 = \frac{1}{a} \rightarrow \frac{1}{a} = -\frac{a}{3} \left( \frac{a^2 C_0}{2} \right) \quad C_2 = \frac{a^2 C_0}{2} \quad C_1 = -a \left( -\frac{6}{a^4} \right)$$

$$\frac{1}{a} = -\frac{a^3 C_0}{6} \quad C_2 = \frac{a^2 \left( -\frac{6}{a^4} \right)}{2} \quad C_1 = \frac{6}{a^3} \\ -\frac{6}{a^4} = C_0 \quad C_2 = -\frac{3}{a^2}$$

∴ particular solution:

$$y(x) = \frac{1}{a} x^3 - \frac{3}{a^2} x^2 + \frac{6}{a^3} x - \frac{6}{a^4}$$

$$1.b) \frac{dQ(t)}{dt} - rQ(t) = te^{-at}$$

$$\text{general solution: } \frac{dQ(t)}{dt} = rQ(t)$$

$$\int \frac{1}{Q(t)} dQ(t) = \int r dt$$

$$\ln |Q(t)| = rt + c$$

$$Q(t) = Ae^{rt}, A = e^c$$

(Assuming  $r \neq -a$ ):

, particular solution:

Let particular integral be:

$$Q(t) = (C_1 t + C_0) e^{-at}$$

$$\frac{dQ(t)}{dt} = (C_1 t + C_0)(-a e^{-at}) + e^{-at}(C_1)$$

$$= -a(C_1 t + C_0)e^{-at} + C_1 e^{-at}$$

$$= -aC_1 t e^{-at} - aC_0 e^{-at} + C_1 e^{-at}$$

$$= -aC_1 t e^{-at} + (C_1 - aC_0) e^{-at}$$

$$\therefore -aC_1 t e^{-at} + (C_1 - aC_0) e^{-at} - r(C_1 t + C_0) e^{-at} = te^{-at}$$

$$-aC_1 t e^{-at} - rC_1 t e^{-at} + (C_1 - aC_0) e^{-at} - rC_0 e^{-at} = te^{-at}$$

$$(-aC_1 - rC_1) t e^{-at} + (C_1 - aC_0 - rC_0) e^{-at} = te^{-at}$$

Comparing coefficients,

$$-aC_1 - rC_1 = 1, \quad C_1 - aC_0 - rC_0 = 0$$

$$C_1 = -\frac{1}{a+r}, \quad C_1 - C_0(a+r) = 0$$

$$C_1 = C_0(a+r)$$

$$C_0 = -\frac{1}{(a+r)^2}$$

$\Rightarrow$

particular solution.

$$Q(t) = -\left(\frac{1}{a+r}t + \frac{1}{(a+r)^2}\right) e^{-at}$$

$$1. c) \frac{dx(t)}{dt} + 3x(t) = t \sin(3t)$$

To find particular solution:

Let particular integral be:

$$x(t) = (c_1 t + c_0) \cos(3t) + (d_1 t + d_0) \sin(3t)$$

$$\frac{dx(t)}{dt} = (c_1 t + c_0)(-3\sin 3t) + c_1 \cos 3t + (d_1 t + d_0)3\cos 3t + d_1 \sin 3t$$

$$\therefore (c_1 t + c_0)(-3\sin 3t) + c_1 \cos 3t + (d_1 t + d_0)3\cos 3t + d_1 \sin 3t + 3(c_1 t + c_0) \cos(3t) + 3(d_1 t + d_0) \sin(3t) = t \sin 3t$$

$$-3c_1 t \sin 3t - 3c_0 \sin 3t + c_1 \cos 3t + 3d_1 t \cos 3t + 3d_0 \cos 3t + d_1 \sin 3t + 3c_1 t \cos 3t + 3c_0 \cos 3t + 3d_1 t \sin 3t + 3d_0 \sin 3t = t \sin 3t$$

$$(-3c_1 + 3d_1) t \sin 3t + (-3c_0 + d_1 + 3d_0) \sin 3t + (3d_1 + 3c_1) t \cos 3t + (c_1 + 3d_0 + 3c_0) \cos 3t = t \sin 3t$$

Comparing coefficients,

$$-3c_1 + 3d_1 = 1, \quad 3d_1 + 3c_1 = 0, \quad -3c_0 + d_1 + 3d_0 = 0, \quad 3c_1 + 3d_0 + 3c_0 = 0$$

$$d_1 - c_1 = \frac{1}{3}, \quad c_1 + d_1 = 0$$

$$\begin{aligned} &\Downarrow \\ &2d_1 = \frac{1}{3} \\ &d_1 = \frac{1}{6} \Rightarrow c_1 = -\frac{1}{6} \end{aligned}$$

$$\begin{aligned} &d_1 + 6d_0 + 3c_1 = 0 \quad \Rightarrow \quad c_0 = -d_0 - c_1 \\ &d_0 = -\left(\frac{3c_1 + d_1}{6}\right) = -\left(\frac{3(-\frac{1}{6}) + \frac{1}{6}}{6}\right) = \frac{1}{18} \\ &= \frac{1}{18} \end{aligned}$$

$\therefore$  particular solution:

$$x(t) = \left(-\frac{1}{6}t + \frac{1}{9}\right) \cos 3t + \left(\frac{1}{6}t + \frac{1}{18}\right) \sin 3t$$

$$2. \frac{\partial f}{\partial \vec{k}} \cdot \vec{v} + \frac{1}{\tau} \frac{\partial f}{\partial \vec{k}} \vec{F} + \frac{\partial f}{\partial t} = -\frac{f - f_0}{\tau}, \quad f(\vec{k}, t), \quad f_0 = \frac{1}{\exp\left[\frac{E_k - E_F}{k_B T}\right] + 1}$$

- For  $\frac{\partial f}{\partial \vec{k}} = 0$  and  $\vec{F} = 0$  at  $t=0$ ,

the Boltzmann transport eqn reduces to :

$$\frac{\partial f}{\partial t} = -\frac{f - f_0}{\tau}$$

and we can solve it as a linear ODE via variable separation:

$$\int \frac{1}{f - f_0} df = \int -\frac{1}{\tau} dt$$

$$\ln |f - f_0| = -\frac{t}{\tau} + C$$

for the case at  $t=0$ ,  $f(\vec{k}, t=0) = f(\vec{k}, t=0)$  being the nonequilibrium distribution:

$$\ln |f(\vec{k}, t=0) - f_0| = C$$

$$\therefore \ln |f - f_0| = -\frac{t}{\tau} + \ln |f(\vec{k}, t=0) - f_0|$$

$$\ln \left| \frac{f - f_0}{f(\vec{k}, t=0) - f_0} \right| = -\frac{t}{\tau}$$

$$\frac{f(\vec{k}, t) - f_0}{f(\vec{k}, t=0) - f_0} = \exp\left[-\frac{t}{\tau}\right]$$

$$\therefore f(\vec{k}, t) = f_0 + [f(\vec{k}, t=0) - f_0] \exp\left[-\frac{t}{\tau}\right]$$

depending on the time constant  $\tau$ ,  $f(\vec{k}, t)$  will 'relax' back to the equilibrium distribution function  $f_0$  with time,

as we can see with the limit as  $t \rightarrow \infty$ :

$$\begin{aligned} \lim_{t \rightarrow \infty} f(\vec{k}, t) &= \lim_{t \rightarrow \infty} \left[ f_0 + [f(\vec{k}, t=0) - f_0] \exp\left[-\frac{t}{\tau}\right] \right] \\ &= f_0 + 0 \\ \lim_{t \rightarrow \infty} [f(\vec{k}, t)] &= \underline{\underline{f_0}} \end{aligned}$$

3. a)  $M(t) \rightarrow$  (Infants protected by maternally derived antibodies)

$S(t) \rightarrow$  (susceptible population, not yet exposed to  $x$ )

$E(t) \rightarrow$  (exposed but not yet infectious)

$I(t) \rightarrow$  (Infected and transmitting  $x$ )

$R(t) \rightarrow$  (Recovered and Immunised)

$$N(t) = M(t) + S(t) + E(t) + I(t) + R(t)$$

$$\frac{dM(t)}{dt} = b(N - S(t)) - (d + \mu) M(t)$$

$$\frac{dS(t)}{dt} = bS(t) + dM(t) - r\beta \frac{I(t)}{N(t)} S(t) - (\mu + \nu) S(t)$$

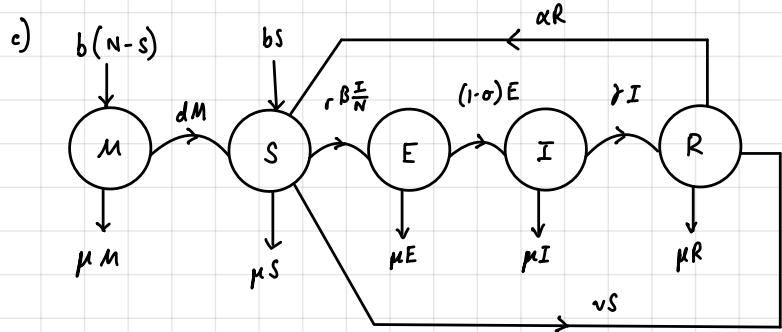
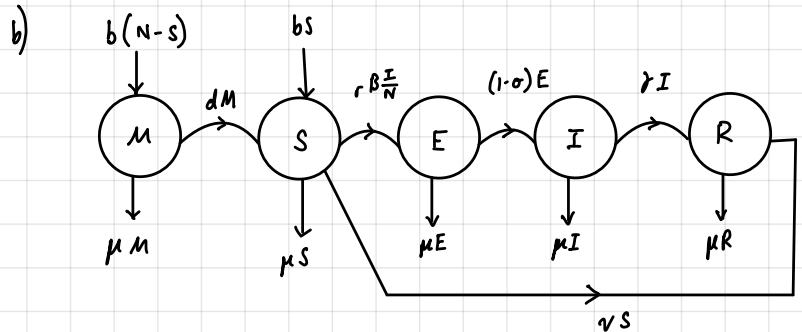
$$\frac{dE(t)}{dt} = r\beta \frac{I(t)}{N(t)} S(t) - (1 - \sigma + \mu) E(t)$$

$$\frac{dI(t)}{dt} = (1 - \sigma) E(t) - (\gamma + \mu) I(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t) + \nu S(t) - \mu R(t)$$

\* Assumptions for the MSEIR model:

1. once recovered, the individual will forever be immune.
2. once vaccinated, individuals will be immune forever.
- 3.



For waning immunity of the  $R(t)$ , the  $M(t)$ ,  $E(t)$  and  $I(t)$  models remain constant as before in the MSEIR model, but the new  $S(t)$  and  $R(t)$  are given by:

$$\frac{dS(t)}{dt} = bS(t) + dM(t) + \alpha R(t) - r\beta \frac{I(t)}{N(t)} S(t) - (\mu + \nu) S(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t) + \nu S(t) - (\mu + \alpha) R(t)$$

$$4. \frac{dT(t)}{dt} = k [T_a - T(t)]$$

$T(0) = T_0$ , solving for  $T(t)$  via variable separation:

$$\int \frac{1}{T_a - T(t)} dT(t) = \int k dt$$

$$-\ln |T_a - T(t)| = kt + C$$

$$T_a - T(t) = Ae^{-kt}, \quad A = e^{-C}$$

$$\text{at } t=0: T_a - T_0 = Ae^{-k(0)}$$

$$A = T_a - T_0$$

$$\therefore T(t) = T_a - (T_a - T_0)e^{-kt}$$

To find  $T_{\text{new}} = T(t')$ :

$$T_{\text{new}} = T_a - (T_a - T_0) e^{-kt'}$$

$$\frac{T_a - T_{\text{new}}}{T_a - T_0} = e^{-kt'}$$

$$\ln \left| \frac{T_a - T_{\text{new}}}{T_a - T_0} \right| = -kt'$$

$$t' = -\frac{1}{k} \ln \left| \frac{T_a - T_{\text{new}}}{T_a - T_0} \right|$$

$$5. \frac{ds(t)}{dt} = -r\beta s(t) \frac{I(t)}{N(t)} - \mu s(t) + bN(t)$$

$$\frac{dI(t)}{dt} = r\beta s(t) \frac{I(t)}{N(t)} - \mu I(t)$$

$$\frac{dN(t)}{dt} = bN(t) - \mu s(t) - \mu I(t)$$

$$N(t) = s(t) + I(t), \quad I(0) = I_0$$

$$\frac{dN(t)}{dt} = \frac{ds(t)}{dt} + \frac{dI(t)}{dt}$$

eliminating  $s(t)$  from  $\frac{dI(t)}{dt}$ :

$$\begin{aligned} \frac{dI(t)}{dt} &= r\beta (N(t) - I(t)) \frac{I(t)}{N(t)} - \mu I(t) \\ &= r\beta I(t) - r\beta \frac{I^2(t)}{N(t)} - \mu I(t) \\ &= (r\beta - \mu)I(t) - r\beta \frac{I^2(t)}{N(t)} \end{aligned}$$

at steady-state,  $N(t) = N$  (constant)

$$\frac{dI(t)}{dt} = (r\beta - \mu)I(t) - \frac{r\beta}{N}I^2(t)$$

$$= r\beta \left[ (1 - \frac{\mu}{r\beta})I(t) - \frac{1}{N}I^2(t) \right]$$

$$\text{Let } a = 1 - \frac{\mu}{r\beta}, \quad b = \frac{1}{N} \quad \text{and}$$

solve via variable separation method:

$$\frac{dI(t)}{dt} = r\beta \left[ aI(t) - bI^2(t) \right]$$

$\rightarrow \int \frac{1}{aI(t) - bI^2(t)} dt = \int r\beta dt, \quad \text{decompose the partial fraction:}$

$$\frac{-1}{I(bI-a)} = \frac{A}{I} + \frac{B}{bI-a}$$

$$-1 = A(bI-a) + BI$$

$$\text{Let } I=0: \quad \text{let } I = \frac{a}{b}: \quad -1 = -aA \quad -1 = B \frac{a}{b}$$

$$A = \frac{1}{a} \quad B = -\frac{b}{a}$$

$$\Rightarrow \frac{1}{I(a-bI)} = \frac{1}{aI} + \frac{b}{a(a-bI)}$$

Initial conditions  $I(0) = I_0$ :

$$\Rightarrow \frac{1}{a} \ln \left| \frac{I_0}{a-bI_0} \right| = C$$

$$\ln \left| \frac{I_0}{a-bI_0} \right| = aC$$

$$\therefore \ln \left| \frac{I(t)}{a-bI(t)} \right| = ar\beta t + ac$$

$$I(t) = (a-bI(t)) \exp [ar\beta t + ac]$$

$$I(t) + bI(t) \exp [ar\beta t + ac] = a \exp [ar\beta t + ac]$$

$$I(t) = \frac{a \exp [ar\beta t + ac]}{1 + b \exp [ar\beta t + ac]}$$

$$I(t) = \frac{a \exp [ar\beta t + ac]}{1 + b \exp [ar\beta t + ac]}$$

$$I(t) = \frac{a}{b + \exp [-ar\beta t - ac]}$$

$$I(t) = \frac{1 - \frac{\mu}{r\beta}}{\frac{1}{N} + \exp [-(r\beta - \mu)t - \ln \left| \frac{I_0}{a-bI_0} \right|]}$$

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$$S. \quad I(t) = \frac{1 - \frac{\mu}{r\beta}}{\frac{1}{N} + \frac{\exp[-(r\beta - \mu)t]}{\frac{I_0}{a - bI_0}}}$$

$$I(t) = \frac{\left(1 - \frac{\mu}{r\beta}\right)}{\frac{1}{N} + \frac{\left[\left(1 - \frac{\mu}{r\beta}\right) - \frac{1}{N}I_0\right]\exp[-(r\beta - \mu)t]}{I_0}}$$

$$I(t) = \frac{N I_0 \left(1 - \frac{\mu}{r\beta}\right)}{I_0 + \left[N \left(1 - \frac{\mu}{r\beta}\right) - I_0\right] \exp[-(r\beta - \mu)t]} \quad (\text{proved})$$

$\equiv$

$$6. a) \frac{dS(t)}{dt} = - \frac{r\beta I(t)}{N(t)} S(t)$$

$$\frac{dE(t)}{dt} = \frac{r\beta I(t)}{N(t)} S(t) - \varepsilon E(t)$$

$$\frac{dI(t)}{dt} = \varepsilon E(t) - \gamma I(t) - \delta I(t)$$

$$\frac{dR(t)}{dt} = \gamma I(t)$$

$$\frac{dD(t)}{dt} = \delta I(t)$$

b) i) From the SEIRD python simulation,

Value of peak infection: 143 people

ii) Time taken to reach peak infected population: 66 days

iii) Time taken for infected to fall below 10: 111 days

c) i) Value of peak infected: 268 people

ii) Time taken to reach peak infected: 36 days

iii) Time taken for infected to fall below 10: 69 days

