Orthogonal Latin Squares With Holes

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Orthogonal Latin Squares

Two latin squares of order n, L and L' are said to be *orthogonal* if upon superimposition, the resulting square contains the n^2 pairs of $[n] \times [n]$ as its entries.

$$\begin{array}{|ccccc|}\hline 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|ccccc|}\hline 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \\ \hline \end{array}$$

$$\rightarrow \begin{array}{|ccccc|}\hline (1,1) & (2,2) & (3,3) & (4,4) & (5,5) \\ (2,3) & (3,4) & (4,5) & (5,1) & (1,2) \\ (3,5) & (4,1) & (5,2) & (1,3) & (2,4) \\ (4,2) & (5,3) & (1,4) & (2,5) & (3,1) \\ (5,4) & (1,5) & (2,1) & (3,2) & (4,3) \\ \hline \end{array}$$

$N(n)$

$N(n)$ denotes the maximum number of *mutually orthogonal latin squares* (*MOLS*) of order n.

- $N(5) = 4$:

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

Holey Latin Squares

A holey latin square is just a latin square, along with hole sets $H_1, ..., H_l$ such that the entries indexed by $H_i \times H_i$ are empty. Additionally, if $x \in H_k$, then the rows and columns indexed by H_k contain the elements of $[n] \setminus H_k$.

5	6	3	4	1	2
2	1	6	5	3	4
6	5	1	2	4	3
4	3	5	6	2	1
1	4	2	3		
3	2	4	1		

1	2	5	6	3	4
6	5	1	2	4	3
4	3	6	5	1	2
5	6	4	3	2	1
2	4	3	1		
3	1	2	4		

A holey latin square has type $h_1^{n_1}...h_l^{n_l}$ if there are exactly n_i holes of size h_i.

$N(h^n)$

We are concerned with the case when the holes have the same size, and they partition $[n]$. $N(h^n)$ denotes the maximum number of mutually orthogonal squares of type h^n.

- $N(1^4) = 2$:

$$\begin{array}{|cccc|}\hline & 3 & 4 & 2 \\ 4 & & 1 & 3 \\ 2 & 4 & & 1 \\ 3 & 1 & 2 & \\ \hline\end{array} \qquad \begin{array}{|cccc|}\hline & 4 & 2 & 3 \\ 3 & & 4 & 1 \\ 4 & 1 & & 2 \\ 2 & 3 & 1 & \\ \hline\end{array}$$

- $N(2^4) = 2$:

$$\{(1,2,7,0),(2,7,1,0),(7,1,2,0),(3,6,5,0),(6,5,3,0),(5,3,6,0)\}$$

6-$HMOLS(2^q)$

Take $q \equiv 1 \pmod 4$ a prime power, and ω primitive in $\mathbb{F}_q$. Define $C_0 = \{\omega^0, \omega^4, \omega^8, ..., \omega^{(q-1)-4}\}$, and for $1 \leq i \leq 3$ define $C_i = \omega^i C_0$. Let

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and let v_i equal the ith row of V, $1 \leq i \leq 8$. We will use V to construct blocks defining a set of 6-HMOLS(2^q) on $\mathbb{F}_q \times \mathbb{Z}_2$.

6-HMOLS(2^q)

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Build vectors $u = (u_i : 1 \le i \le 8)$ and $u' = (u'_i : 1 \le i \le 8)$ so that $B = \{xu \circ v_1,\ \omega xu \circ v_2,\ \omega^2 xu \circ v_3,\ \omega^3 xu \circ v_4,\ xu' \circ v_5,\ \omega xu' \circ v_6,\ \omega^2 xu' \circ v_7,\ \omega^3 xu' \circ v_8 : x \in C_0\}$ has distinct i, j differences, with no difference equal to 0_0 or 0_1.

6-HMOLS(2^q)

We get the following table of allowed cosets for the quotients $(u_j - u_i)/(u'_j - u'_i)$

$i \backslash j$	2	3	4	5	6	7	8
1	0	0, 2	0	2	1, 3	2	0, 1, 2, 3
2		2	1, 3	0, 1, 2, 3	0	0, 2	2
3			2	0	0, 1, 2, 3	0	1, 3
4				0, 2	0	0, 1, 2, 3	2
5					2	1, 3	0
6						2	0, 2
7							0

6-HMOLS(2^q)

In addition to below $q = 81, 121, 125, 169$ were also done.

$q = 101$: $[1, 87, 86, 31, 24, 64, 59, 0]$
$[1, 45, 22, 77, 88, 25, 83, 0]$

$q = 109$: $[1, 3, 10, 103, 23, 42, 101, 0]$
$[1, 31, 74, 101, 85, 13, 23, 0]$

$q = 113$: $[1, 52, 53, 6, 86, 101, 3, 0]$
$[1, 53, 27, 75, 70, 39, 89, 0]$

$q = 137$: $[1, 30, 110, 51, 74, 107, 25, 0]$
$[1, 34, 74, 66, 99, 122, 128, 0]$

$q = 149$: $[1, 56, 91, 108, 104, 141, 81, 0]$
$[1, 3, 39, 38, 31, 66, 145, 0]$

$q = 157$: $[1, 121, 37, 51, 4, 137, 31, 0]$
$[1, 58, 123, 126, 110, 45, 57, 0]$

Allowed Cosets for Order 12 Hadamard Matrix

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}$$

Allowed Cosets for Order 12 Hadamard Matrix

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$0, 3$	3	2	$\emptyset$	4
1		$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	0	$\emptyset$	2	1	$1, 4$	$\emptyset$
2			$\emptyset$	$\emptyset$	$\emptyset$	2	4	$\emptyset$	0	$\emptyset$	$2, 5$
3				$\emptyset$	$\emptyset$	4	3	$0, 3$	$\emptyset$	2	$\emptyset$
4					$\emptyset$	$\emptyset$	2	$\emptyset$	$1, 4$	1	0
5						$2, 5$	$\emptyset$	4	$\emptyset$	0	5
6							$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7								$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8									$\emptyset$	$\emptyset$	$\emptyset$
9										$\emptyset$	$\emptyset$
10											$\emptyset$

K-HMOLS(2^q)

The order 16 Hadamard matrix admits allowed cosets for every i, j pair.

9-HMOLS(2^{617}): $[1, 506, 250, 28, 11, 375, 294, 380, 0, 393, 155]$
$[1, 474, 298, 318, 112, 480, 139, 108, 0, 343, 127]$

10-HMOLS(2^{1009}):
$[1, 510, 81, 865, 744, 652, 17, 765, 0, 237, 669, 91]$
$[1, 687, 337, 10, 963, 575, 472, 593, 0, 179, 794, 68]$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}
$$

$i \backslash j$	1	2	3	4	5	6	7	8
0	4	4	4	2, 6	1, 3, 5, 7	2, 6	4	0, 1, 2, 3, 4, 5
1		2, 6	1, 3, 5, 7	4	4	4	2, 6	0
2			2, 6	4	4	4	1, 3, 5, 7	0
3				4	4	4	2, 6	0
4					2, 6	1, 3, 5, 7	4	0, 4
5						2, 6	4	0, 2, 4, 6
6							4	0, 4
7								0
8								
9								
10								
11								
12								
13								
14								