Mathematics for A.I.

Linear Programming in Resource Allocation

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1. Introduction

1.1 Resource Allocation Background

The division of resources among alternative options with the goal of maximizing total return or minimizing total expenses is known as an allocation issue. A collection of resources in predetermined quantities, a series of tasks to be completed, each requiring a certain quantity of resources, and a set of expenses or returns for each task and resource are the components of such issues. The challenge is figuring out how much of each resource to use for each task [1].

1.2 Linear Programming Background

A mathematical model whose needs and aim are represented by linear relationships can be optimized using a technique known as linear programming (LP), also referred to as linear optimization.

To facilitate deeper comprehension, suppose a postman must deliver 6 letters in a day from the post office (located at point A) to different houses (P, Q, R, S, T). The distance between the houses is indicated on the lines as given in 'Figure 1 – Distance between '. If the postman wants to find the shortest route that will enable him to deliver the letters as well as save on fuel, then it becomes a linear programming problem. Thus, LP will be used to get the optimal solution which will be the shortest route in this example [2].

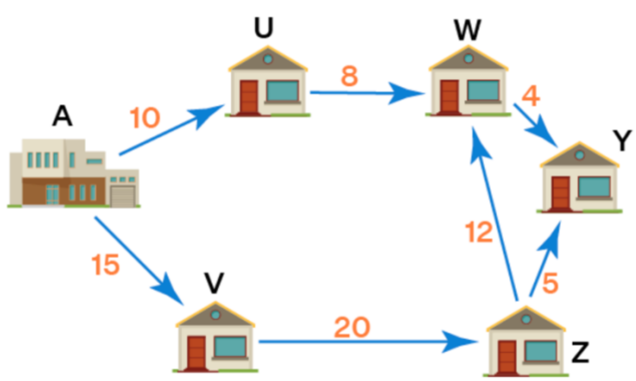

Figure 1 – Distance between houses [2].

Organizations face the problem of properly allocating resources to achieve great results while adhering to limits in many real-world circumstances. Such optimization challenges may be systematically addressed with linear programming which creates an objective function and limitations before using mathematical methods of optimization to find the best possible solutions. In the realm of Artificial intelligence, statistics and probability is a key mathematical model which is where linear programming lie which are widely used in machine learning and data analysis.

2. Numerical Methods for Linear Programming

2.1 Linear Programming formula

Firstly, the basic structure of LP problem is being depicted in terms of a canonical form when optimizing objective functions under constraints, this is shown as [3].

$$\mathrm{Max}f = c_1x_1 + c_2x_2 + \cdots + c_px_p \quad \text{s.t.}$$

$$A_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p \leq b_1$$

$$A_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p \leq b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mp}x_p \leq b_m$$

$$X_1, x_2, \ldots\ldots, x_p \geq 0. \tag{1}$$

Equation 1 – canonical form of linear programming problem

Where:

f – objective function (a hyperplane)

m – the first m inequalities are the structural constraints

p – the last p inequalities are the nonnegativity conditions

The solution to equation 1 is found within the p-dimensional feasible region k, formed by the constraint system and nonnegativity conditions [3].

A linear programming problem involves decision variables (x, y), an objective function (Z), constraints, and non-negative restrictions. The objective function maximizes or minimizes the solution, while constraints limit the variables' value, ensuring they always have non-negative values [2].

Linear Programming general formula can be expressed as.

Objective Function:

$$Z = ax + by. \qquad (2)$$

Constraints:

$$cx + dy \le e, \quad fx + gy \le h. \qquad (3)$$

The inequalities can also be "$\geq$ "

Non-negative restrictions: $x \geq 0, y \geq 0$

Equation 2 – General formula of linear programming problem

2.2 Simplex Method

The simplex method is a fast and valuable tool for solving complex linear programming problems. However, its slowness in certain problems has puzzled theoreticians. The ellipsoid method, a new algorithm, has shown little prospect of outperforming the simplex method in practice [4]. It involves repeatedly shifting from one variable to another along the borders of the feasible region.

2.3 Interior point Method

Interior Point Methods are optimization algorithms used to find the optimal solution of mathematical problems by moving from one point on the objective function to another in the feasible region. They are commonly used for linear and nonlinear programming problems. The primal-dual interior-point method is the most theoretically elegant and successful computationally, developed by researchers like Kojima, Monteiro, McShane, Choi, and Lustig [5].

Interior point methods and, simplex method are two examples of optimization techniques that are made possible by linear programming, which implies linear objective function and constraints. Larger issues are difficult for the simplex approach to solve because of the exponential rise of possible solutions. However, it is effective for small to moderate difficulties. Although they can handle non-linear constraints more effectively and are appropriate for bigger problems, interior point approaches may demand more time and processing resources [6].

3. Simplex method workings

Assuming there is a company with two products A and B, it has limited resources in terms of labor materials. Each unit of A requires an hour to execute. The profit per units is 4 for A and 6 for B. It has a total of 11hrs of labor and 27units of material available for production. Lastly it needs 2 units and 5 units of resources respectively. Simplex method is being used below to show how the company can maximize its profits given the source constraints.

Figure 2 shows the graphical representation of the solution, where it resides in vertices of the graph as linear programming can also be solved graphically [7].
Linear programming has solutions at the vertex points, with the maximum being the point furthers away from the origin and the minimum being the closest to the origin

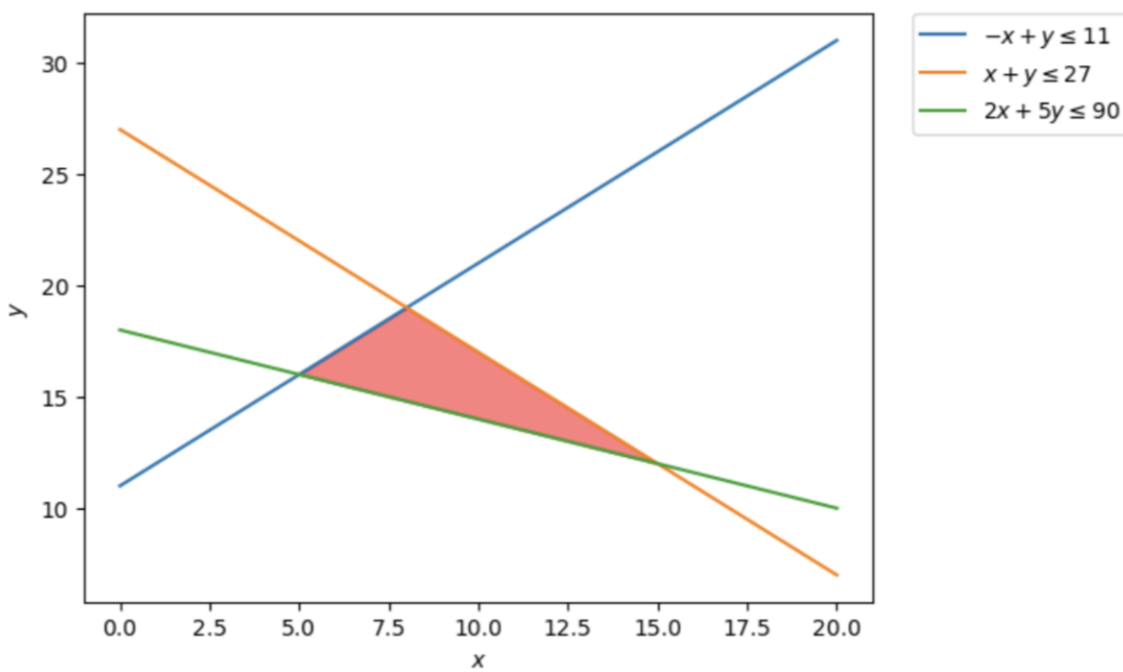

Figure 2 -Graphical representation of Linear programming *[7]*.

Proceeding with simplex method;

a) Identify the decision variables.

The decision variables are the quantities of products A and B as indicated in the scenario.

$$x_1: Number\ of\ units\ of\ A$$
$$x_2: Number\ of\ units\ of\ B$$

b) Write up the goal function.

The goal is to maximize the profit, and the profit function is:

$$Profit = 4x_1 + \ 6x_2 \qquad (4)$$

c) Record the constraints.

$$Labor\ constraint: -x_1 + \ x_2 \leq 11\ (total\ available\ labor\ hours)$$
$$Material\ constraint: x_1 + \ x_2 \leq 27 (total\ available\ material\ units)$$
$$Resource\ constraint: 2x_1 + \ 5x_2 \leq 90 (total\ available\ material\ units)$$
$$Non-negativity\ constraints: x_1, x_2 \geq 0\ (total\ available\ labor\ hours) \qquad (5)$$

d) Non-detrimental constraint.

This step requires ensuring that the variables are greater than or equal 0. Which is regarded fundamental within the constraints section. With regards to this, the non-negativity constraint has already been included in the previous step, ensuring that the decision variables are greater than or equal to 0.

e) Solve the LP problem using the preferred method.

Using simplex method, we proceed by setting the initial tableau.

Initial simplex matrix

$$-x_1 + \ x_2 + \ s_1 = 11$$
$$x_1 + \ x_2 + \ s_2 = 27$$
$$2x_1 + \ 5x_2 + \ s_3 = 90 \qquad (6)$$

$$\begin{array}{lcccccc} coefficients & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ s_1 & -1 & 1 & 1 & 0 & 0 & 11 \\ s_2 & 1 & 1 & 0 & 1 & 0 & 27 \\ s_3 & 2 & 5 & 0 & 0 & 1 & 90 \\ \square & -4 & -6 & 0 & 0 & 0 & 0 \end{array} \tag{7}$$

Where:

Row 1 represents the labor constraint with slack variable S1.

Row 2 represents the material constraint with slack variable S2.

Row 3 represents the resources constraint with slack variable S3.

Row 4 represents the coefficients of the objective function (unit profit).

<u>Determine the entering variable and exiting variable.</u>

Identify the column with the highest negative entry. This is called the pivot column (entering variable). As seen product A is has the highest negative entry of -5. And the lowest ratio is the labor constraint which is the exiting variable.

$$Entering\ value : maximum\ negative\ value = -6$$

$$Exiting\ value : \frac{RHS}{Coresponding\ coefficient}\ in\ entering\ variable$$

$$Exiting\ variable\ is\ s_1\ \ and\ \ x_2\ is\ the\ entering\ variable$$

$$\begin{array}{lcccccc} coefficients & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ s_1 & -1 & 1 & 1 & 0 & 0 & 11 = \frac{11}{1} \\ s_2 & 1 & 1 & 0 & 1 & 0 & 27 = \frac{27}{1} \\ s_3 & 2 & 5 & 0 & 0 & 1 & 90 = \frac{90}{5} \\ \square & -4 & -6 & 0 & 0 & 0 & 0 \end{array} \tag{8}$$

Next, Gaussian elimination is executed to assign the value 1 to the pivot element and 0 to the remaining elements in the pivot column, this step is repeated until there till the bottom-most row has no negative entries in the coefficient x1 and x2 [7].

$$Guassian\ elimination;$$

$$R_2 = R_2 - R_1$$

$$R_3 = R_3 - 5R_1$$

$$R_4 = R_4 - 6R_1 \qquad (9)$$

$$Revised\ tableau;$$

$$\begin{array}{ccccccc} New\ BFS & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ x_2 & -1 & 1 & 1 & 0 & 0 & 11 \\ s_2 & 2 & 0 & -1 & 1 & 0 & 16 \\ s_3 & 7 & 0 & -5 & 0 & 1 & 35 \\ \square & -10 & 0 & 6 & 0 & 0 & 66 \end{array} \qquad (10)$$

As 6 is the only positive coefficient in the objective function row, s3 is our new pivot column and recalculate the ratios of the corresponding RHS to the corresponding pivot column entry:

$$Entering\ value : maximum\ negative\ value = -10$$

$$Guassian\ elimination;$$

$$R_1 = R_1 + \frac{2}{7}R_3$$

$$R_2 = R_2 - \frac{2}{7}R_3$$

$$R_3 = \frac{R_3}{7}$$

$$R_4 = R_4 + \frac{10}{7}R_3 \qquad (11)$$

$$Revised\ tableau;$$

$$\begin{array}{ccccccc} New\ BFS & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ x_2 & 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ s_2 & 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ x_1 & 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ ⬚ & 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{array} \quad (12)$$

There is still a negative entering $-\frac{8}{7}$ in the bottom row, hence further iteration.

$$Entering\ value : maximum\ negative\ value = -\frac{8}{7}$$

$$Guassian\ elimination;$$

$$R_1 = R_1 + \frac{2}{7}R_2$$

$$R_2 = R_2 - \frac{R_3}{0.43}$$

$$R_3 = R_3 + \frac{5}{7}R_2$$

$$R_4 = R_4 + \frac{8}{7}R_2 \quad (13)$$

$$Revised\ tableau;$$

$$\begin{array}{ccccccc} New\ BFS & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ x_2 & 0 & 1 & 0 & -\frac{2}{7} & \frac{11}{49} & 12 \\ s_1 & 0 & 0 & 1 & \frac{7}{3} & -\frac{2}{3} & 14 \\ x_1 & 1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 15 \\ ⬚ & 0 & 0 & 0 & \frac{8}{3} & \frac{54}{49} & 132 \end{array} \quad (14)$$

Therefore, the optimal values are;

$$x_1 = 15\ units\ of\ product\ A$$

$$x_2 = 12\ units\ of\ product\ B$$

$$Profit = 4(15) + \ 6(12)$$

$$Maximum\ profit = 132$$

By following these iterative steps and updating the tableau accordingly, we have determined the optimal production quantities of products A and B that maximize the company's profits within the given resource constraints.

4. Discussion and conclusion

Linear Programming has several advantages in with regards to problem optimizations as it enables effective resolution of issues in which the design variables' goal function is linear producing fast calculation times and unique solutions. It's necessity for the objective function and constraints to be linear that may not adequately capture real world problems is its limitation [8].

The simplicity and efficiency of the simplex method makes it a popular solution for small to moderate sized linear programming problems. However, it struggles with large scale problems due to its exponential increase in possible solutions and its tendency to get stuck at local optima. This emphasizes the need for an alternative optimization algorithm such as interior point, which uses a primal dual approach to navigate complex solution spaces efficiently, improving convergence properties and scalability for larger problem instances [9].

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