Autoregressive Moving Average Process of order 1 ~~(ARMA(1,1))~~

ARMA(1, 1)

Stationary sol^n to the eq^n

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1} \quad , \quad \theta, \phi \in \mathbb{R} \; (\theta + \phi \neq 0)$$
$$\{Z_t\} \sim WN(0, \sigma^2)$$

case 1: $|\phi| < 1$

$$(1 - \phi B) X_t = (1 + \theta B) Z_t$$

$$\Rightarrow X_t = \frac{1}{1 - \phi B} (1 + \theta B) Z_t$$

$$\Rightarrow X_t = (1 + \phi B + \phi^2 B^2 + \cdots)(1 + \theta B) Z_t$$

$$= \left(1 + (\theta + \phi) \sum_{j=1}^{\infty} \phi^{j-1} B^j \right) Z_t$$

$$= Z_t + (\theta + \phi) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

case 2: $|\phi| > 1$

$$X_t = \frac{1}{1 - \phi B} (1 + \theta B) Z_t$$

$$= \frac{-\phi^{-1} B^{-1}}{1 - \phi^{-1} B^{-1}} (1 + \theta B) Z_t$$

$$= -(\phi^{-1} B^{-1} + \phi^{-2} B^{-2} + \cdots)(1 + \theta B) Z_t$$

$$= -\left[\theta \phi^{-1} Z_t + (1 + \theta \phi^{-1}) \sum_{j=1}^{\infty} \phi^{-j} B^{-j} Z_t\right]$$

$$= -\left[\theta \phi^{-1} Z_t + (1 + \theta \phi^{-1}) \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}\right]$$

case 3 : $|\phi| = 1$

We do for $\phi = 1$

$$X_t - X_{t-1} = Z_t + \theta Z_{t-1}, \quad \forall t$$

$$X_{t-1} - X_{t-2} = Z_{t-1} + \theta Z_{t-2}.$$

$$\vdots$$

$$X_{t-k} - X_{t-k-1} = Z_{t-k} + \theta Z_{t-k-1}$$

$$\therefore \quad X_t - X_{t-k-1} = Z_t + (1+\theta) \sum_{j=1}^{k} Z_{t-j} + \theta Z_{t-k-1}$$

RHS has variance $\to \infty$ as $\theta \neq -1$

LHS has variance $\leq 4\gamma(0)$ [If X_t is stationary with ACVF $\gamma(h)$]

Hence for $\phi = 1$, $\nexists$ any stationary solⁿ to ARMA(1,1)

Summary:

1) $|\phi| < 1 \rightarrow$ causal stationary solⁿ

2) $|\phi| > 1 \longrightarrow$ non-causal stationary solⁿ

3) $|\phi| = 1 \longrightarrow$ no stationary solⁿ.

Defⁿ (Invertibility): We call a process invertible if Z_t can be written linearly in terms of X_s's $(s \leq t)$.

Let's check for what values of θ, the ARMA(1,1) process is invertible.

Case 1: $|\theta|<1$,

$$Z_t = \frac{1}{1+\theta B}(1-\phi B)X_t$$

$$= (1-\theta B+\dots)(1-\phi B)X_t$$

$$= X_t - (\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^j X_{t-j}$$

Case 2: $|\theta|>1$

$$Z_t = \frac{1}{1+\theta B}(1-\phi B)X_t$$

$$= \frac{\theta^{-1}B^{-1}}{1+\theta^{-1}B^{-1}}(1-\phi B)X_t$$

$$= \theta^{-1}B^{-1}(1-\theta^{-1}B^{-1}+\theta^{-2}B^{-2}\dots)(1-\phi B)X_t$$

$$= -\phi\theta^{-1}X_t + (\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{-j-1}X_{t+j}$$

$\therefore$ The process is non-invertible.

Remark:
- For $|\theta|=1$, the process is non-invertible in the sense $\nexists$ infinite linear comb^n of $\{X_s\}_{-\infty}^{t}$ to get Z_t.
- The process is invertible in the sense that Z_t can be written as mean-square limit of variables which are finite linear comb^n of X_s's $(s\leq t)$
  L^2 convergence

Eg: $Y_t = U + Z_t$, $\quad U\sim N(0,1)$, $Z_t \overset{iid}{\sim} N(0,1)$ indep.

U can not be written as infinite linear comb^n of Y_t's

But $\bar{Y}_k \xrightarrow{L^2} U$

ACVF of ARMA(1,1)

$|\phi| < 1$,

$$X_t = Z_t + (\theta+\phi) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

$$E(X_t) = 0$$

$$Var(X_t) = \sigma^2 + (\theta+\phi)^2 \sum_{j=1}^{\infty} \phi^{2(j-1)} \sigma^2$$

$$= \sigma^2 \left[1 + (\theta+\phi)^2 \frac{1}{1-\phi^2} \right]$$

$$\gamma(1) = Cov(X_t, X_{t+1})$$

$$= Cov\left(Z_t + (\theta+\phi) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j} , Z_{t+1} + (\theta+\phi) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t+1-j} \right)$$

$$= \left((\theta+\phi) + (\theta+\phi)^2 \sum_{j=1}^{\infty} \phi^{2j-1} \right) \sigma^2$$

$$= \sigma^2 \left[(\theta+\phi) + (\theta+\phi)^2 \frac{\phi}{1-\phi^2} \right]$$

$$\gamma(h) = Cov(X_t, X_{t+h})$$

$$= E(X_t X_{t+h})$$

$$= (\theta+\phi)^2 \sum_{j=1}^{\infty}$$

$$x_t - \phi x_{t-1} = Z_t + \theta Z_{t-1}$$

$$\Rightarrow \quad x_t x_{t-h} - \phi x_{t-1} x_{t-h} = Z_t x_{t-h} + \theta Z_{t-1} x_{t-h}$$

$$\Rightarrow \quad \gamma(h) - \phi \gamma(h-1) = 0$$

$$\Rightarrow \quad \gamma(h) = \phi \gamma(h-1) = \phi^{(h-1)} \gamma(1)$$

The time series occuring in practical situation in population studies/economic/social studies are usually non-stationary. So far we have considered stationary time series.

Additive Model:

$$X_t = m_t + s_t + c_t + \varepsilon_t$$ [occuring in practical situation]

$m_t \rightarrow$ trend component

slowly changing component of the series that ~~it~~ indicates overall increase/decrease.

[population in India has increasing trend from t=1901 to t=2001]

$s_t \longrightarrow$ seasonal component

component of time series that depends on particular parts/quarter/month of a year.

[rainfall in a city has seasonal component]

$c_t \longrightarrow$ cyclical component

The component that is due to change in perspective generating the time series/long period cycles those are present in the series.

[cyclical component in stock values due to long term change in international or national market]

$\varepsilon_t \longrightarrow$ random component

component whose cause is unaccounted

Sometimes it may be more useful to consider product model.

$$X_t = m_t \times s_t \times c_t \times \varepsilon_t$$

e.g. production of wheat / size of population in a country may depend on previous year and change proportionately.

$$X_{t+1} \simeq (1+r) X_t$$

Then product model is necessary.

- We take logarithm to transform product model to additive model for easier analysis.

- If ε_t are iid noise then we estimate future values of ε_t by 0. Otherwise we ~~use~~ assume ε_t ~~follows~~ is stationary time series and using its results we predict its future values.

Estimation / Elimination of trend component

Method 1: Curve fitting

$$m_t = c_0 + c_1 t + c_2 t^2 + \dots + c_k t^k$$

Assume,

$$X_t = m_t + \varepsilon_t$$

$$\omega = \sum_{t=1}^{n} (X_t - m_t)^2$$

$$\frac{\partial \omega}{\partial c_i} = 0 \quad , \; i = 0,1,2,\dots,k$$

$$B_{(k+1)\times(k+1)} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_k \end{pmatrix}$$

$$b_{ij} = \sum_{t=1}^{n} t^{i+j}$$

$$d_i = \sum_{t=1}^{n} t^{i} X_t$$

B is vandermond matrix and it's non-singular.

$\therefore \exists$ unique $\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_k$ s.t.

$$\hat{m}_t = \sum_{i=0}^{k} \hat{c}_i t^i$$

<u>Method 2:</u> Moving Average Method

$$X_t = m_t + \varepsilon_t$$

$$\frac{1}{2q+1}\sum_{j=-q}^{q} X_{t+j} = \frac{1}{2q+1}\sum_{j=-q}^{q} m_{t+j} + \underbrace{\frac{1}{2q+1}\sum_{j=-q}^{q} \varepsilon_{t+j}}_{\approx 0}$$

Now if $m_t = a + bt$ (linear component)

$$\frac{1}{2q+1}\sum_{j=-q}^{q} m_{t+j} = m_t$$

$$\therefore \frac{1}{2q+1}\sum_{j=-q}^{q} X_{t+j} \simeq m_t$$

absorbes high freq^n components allowing low freq^n components to survive. So it is called Low-Pass Filter.

General form of Moving Average:

$$\sum_{j=-q}^{q} c_j X_{t+j}$$

It can be designed in such a way choosing c_j's s.t. it allows to pass m_t undistorted

for fixed k, m_t is a polynomial of degree $\leq k$

Also it can be designed to absorb seasonal components of fixed period.

Remark: Moving Average filter absorbs high frequency part and allows to pass low frequency part. (Low Pass filter).

Excercise:

1. $X_t - \phi X_{t-1} = Z_t + Z_{t-1}$ $[|\phi|<1]$

$Z_t \sim WN(0,\sigma^2)$

Show that Z_t cannot be expressed as

$$Z_t = \sum_{j=0}^{\infty} \psi_j X_{t-j}$$

2.

$\rightarrow$ $X_t = (1-\phi B)^{-1}(1+B) Z_t$

$= (1-\phi B)^{-1}(1+B)\alpha(B) X_t$

$= (1+B)\beta(B) X_t$

$\Rightarrow X_t = (1+B)\left(\sum_{j=0}^{\infty} \eta_j B^j\right) X_t$

$Cov(X_s, Z_t) = 0$

$s < t$

$$= \eta_0 X_t + \underbrace{\sum_{j=1}^{\infty}(\eta_j + \eta_{j-1}) X_{t-j}}_{\text{uncorrelated with } Z_t}$$

$\Rightarrow \eta_0 = 1$

$\Rightarrow \sum_{j=1}^{\infty}(\eta_j + \eta_{j-1}) X_{t-j} = 0$

$$\Rightarrow (\eta_0 + \eta_1) X_{t-1} = -\underbrace{\sum_{j=2}^{\infty}(\eta_j + \eta_{j-1}) X_{t-j}}_{\text{uncorrelated with } Z_{t-1}}$$

$\therefore \eta_1 + \eta_0 = 0$

$\Rightarrow \eta_1 = -\eta_0 = -1$

Similarly, $\eta_4 = -\eta_3 = \eta_2 = -\eta_1 = \eta_0 = 1$

$\therefore \eta_i \not\to 0$ as $i \to \infty$

$\Rightarrow \sum_{i=0}^{\infty} |\eta_i| = \infty$

$\Rightarrow \beta(B)$ is not defined at all.

2. If $X_t = Z_t - Z_{t-1}$

then, $Z_t \in$ closure $\{\text{span}\{X_j : -\infty < j \leq t\}\}$

~~then~~ [Z_t can be written as limit of finite linear comb^n of X_t's.] [Book 2 : 3.8 problem]