$$\leq P\left(\frac{Z_n}{2n-1} \geq \frac{1}{2}\right).$$

$$\leq P\left(\left|\frac{Z_n}{2n-1} - p\right| \geq \frac{1}{2} - p\right)$$

Chebyshev.

$$\leq \frac{1}{\left(\frac{1}{2}-p\right)^2} \operatorname{Var}\left(\frac{Z_n}{2n-1}\right).$$

$$= \frac{1}{\left(\frac{1}{2}-p\right)^2} \cdot \frac{p(1-p)}{(2n-1)} \longrightarrow 0 \text{ as } n \to \infty.$$

Example:- Suppose that X follows Double Exponential ~~that~~ that is, its density is.

$$f(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$

Calculate the CHF of X.

$$\int_{-\infty}^{\infty} \cos tx \, \frac{1}{2} \cdot \cancel{e^{-|x|}} \, dx.$$

$$= \int_{0}^{\infty} \cos tx \, e^{-x} \, dx.$$

$$\int_{-\infty}^{\infty} e^{itx} f(x)\, dx.$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{(it+1)x} dx + \int_{0}^{\infty} e^{(it-1)x} dx\right]$$

$$= \frac{1}{2}\left(\frac{1}{1+it} + \frac{1}{1-it}\right).$$

$$= \frac{1}{1+t^2}.$$

Inversion Theorem:-

Suppose that the CHF ϕ of X is integrable on $\mathbb{R}$, that is,

$$\int_{-\infty}^{\infty} |\phi(t)|\,dt < \infty$$

Then, for $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$$ is density of X.

Pf:- Since,

$$\int_{-\infty}^{\infty} |\phi(t)|\,dt < \infty$$

it holds that for every x, the integral

$$\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$$ exists.

If $a < b$ are such that $P(X \in \{a,b\}) = 0$, then uniqueness theorem tells us that

$$P(a \le X \le b) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\,dt$$

$$\int_{-\infty}^{\infty}\left|\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\right|dt \le (b-a)\int_{-\infty}^{\infty}|\phi(t)|\,dt < \infty.$$

Hence,

$$P(a \le X \le b) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\,dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{a}^{b} e^{-itx}\,dx\,\phi(t)\,dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{a}^{b} e^{-itx}\phi(t)\,dx\,dt.$$

(Fubini) $= \frac{1}{2\pi} \int_a^b \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt\, dx.$

$$= \int_a^b \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt\, dx.$$

$$= \int_a^b f(x)\, dx.$$

The above holds for a, b s.t. $P(X \in \{a, b\}) = 0$.

In order to complete the proof, it suffies to show that X is continuous, Fix $x \in \mathbb{R}$.

$\exists$ a_n, b_n such that $a_n \uparrow x$, $b_n \downarrow x$ and $P(X \in \{a_1, b_1, a_2, b_2, \dots\}) = 0$.

$$P(X = x) \leq P(a_n \leq X \leq b_n) = \int_{a_n}^{b_n} f(t)\, dt.$$

Letting $n \to \infty$. we get.

$$P(X = x) = 0.$$

Example:- Suppose $X \sim$ Double exp.

$$\int_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|} dx = \frac{1}{1+t^2}, \quad t \in \mathbb{R}.$$

Since, $\int_{-\infty}^{\infty} \frac{dt}{1+t^2} < \infty$, inversion thm applies

this gives us,

$$f(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx}}{1+t^2}\, dt \qquad \text{is density of } X.$$

Note:- If a R.V. has two continuous densities, then they are identical

$$g(x) = 2f(-x) = E\left[e^{ixZ}\right] \text{ where } Z \sim \text{cauchy}.$$

Hence f is ~~cauchy~~, continuous. & density of X.

Thus, $f(x) = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$.

i.e. $$\int_{-\infty}^{\infty} \frac{e^{-itx}}{\pi(1+t^2)}\, dt = e^{-|x|}, \quad x \in \mathbb{R}.$$

Replacing $(-x)$ by y we get,

$$E\left[e^{iyZ}\right] = e^{-|y|}, \quad y \in \mathbb{R},$$

$Z \sim$ Cauchy.

Suppose, Z_1 & Z_2 are i.i.d. from Cauchy.

Define $Y = \frac{Z_1+Z_2}{2}$

~~$E(e^{iyZ_1}$~~

$$\phi_Y(t) = E\left(e^{itY}\right) = E\left[e^{\frac{1}{2}} \cdot e^{itZ_1} \cdot e^{itZ_2}\right].$$

$$= e^{\frac{1}{2}} E\left[e^{itZ_1}\right] E\left[e^{itZ_2}\right]$$

$$= e^{\frac{1}{2}}\, e^{-|t|}\, e^{-|t|}.$$

$$= e^{-|t|}.$$

Invoking uniqueness theorem $\frac{Z_1+Z_2}{2}$ also follows Cauchy.

Theorem:-

If X & Y are such that

$$E\left[e^{tX}\right] = E\left[e^{tY}\right] < \infty \quad \text{for } |t| < \varepsilon.$$

for some $\varepsilon > 0$, then $X \overset{d}{=} Y$.

Pf:- Let, ϕ_X & ϕ_Y denote the respective CHF of X & Y. In view of the uniqueness theorem it suffices to show that

$$\phi_X(t) = \phi_Y(t) \quad \text{for every } t \in \mathbb{R}.$$

the hypothesis imply that all the moments of X & Y exists and are finite and match.

Further, Since $E\left[e^{\pm \varepsilon X}\right] < \infty$, we get,

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E(X^n), \quad |t| \le \varepsilon,$$

and likewise for Y.

Hence, $\phi_X(t) = \phi_Y(t)$, $|t| \le \varepsilon$.

Our next claim is that for any $t_0 \in \mathbb{R}$ and $t \in \mathbb{R}$ such that $|t - t_0| \le \varepsilon$, it holds that,

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \phi_X^{(n)}(t_0), \quad (*)$$

and likewise for Y.

~~The RHS of (*)~~

for $n \ge 0$, $\phi^{(n)}(t_0) = E\left[(iX)^n e^{it_0 X}\right]$.

Define for $n \geq 0$,

$$Z_n = \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} (iX)^k e^{it_0X}.$$

as $n \to \infty$; $Z_n \to e^{itX}$ a.s.

$$|Z_n| \leq \sum_{k=0}^{n} \frac{|t-t_0|^k}{k!} |X|^k . \leq e^{|(t-t_0)X|} \leq e^{\varepsilon|X|}.$$

Since, $E(e^{\varepsilon|X|}) < \infty$, DCT implies that

$$E[e^{itX}] = \lim_{n\to\infty} E(Z_n).$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} E[(iX)^k e^{it_0X}]$$

$$= \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} E[(iX)^k e^{it_0X}].$$

Since, ϕ_X and ϕ_Y matchs in $[-\varepsilon, \varepsilon]$, it follows that

$$\phi_X^{(n)}(\varepsilon) = \phi_Y^{(n)}(\varepsilon), \quad n \geq 1.$$

Applying $(*)$ with $t_0 = \varepsilon$, we get that.

$$\phi_X(t) = \phi_Y(t) \quad \forall\, t \in [\varepsilon, 2\varepsilon].$$

Proceeding inductively, it can be shown that

$$\phi_X(t) = \phi_Y(t) \quad ; t \geq 0.$$

similar arguement works for $t < 0$ and ~~hence~~ $\blacksquare$

hence, ~~$\phi_X(t) =$~~

Example:-

Suppose that X has density

$$f(x) = \frac{1}{2^{3/2}\sqrt{\pi}\,|x|} \exp\left(-\frac{1}{2}(\ln|x|)^2\right) 1(x \neq 0), x \in \mathbb{R}.$$

~~Let $g(x) =$~~

$\longrightarrow$

$$g(x) = \begin{cases} \left(1 + \sin(2\pi \ln x)\right) f(x) &, \ x > 0, \\ f(x) &, \ x \le 0. \end{cases}$$

Fix $n \ge 0$,

$$\int_0^\infty x^n \sin(2\pi \ln x) f(x)\, dx.$$

$$= c \int_0^\infty x^n \sin(2\pi \ln x) \cdot \frac{1}{x} e^{-\frac{1}{2}(\ln x)^2}\, dx.$$

$$\cancel{c \int_0^\infty e^{ny} \sin(2\pi y)\, e^{-\frac{y^2}{2}}\, dy}$$

$$y = \ln x - n.$$

$$dy = \frac{dx}{x}$$

$$= c \int_{-\infty}^{\infty} e^{n(n+y)}\, e^{-(y+n)^2/2} \sin(2\pi(n+y))\, dy.$$

$$= c \int_{-\infty}^{\infty} \cancel{e^{(n+y)}}\, e^{(y+n)\left(n - \frac{y+n}{2}\right)} \sin(2\pi(n+y))\, dy.$$

$$= c \int_{-\infty}^{\infty} e^{\frac{n^2 - y^2}{2}} \sin(2\pi y)\, dy.$$

$$= c \cdot e^{n^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \sin(2\pi y)\, dy = 0. \left[\because \text{odd function}\right]$$

i.e.
$$\int_{-\infty}^{\infty} x^n g(x)\,dx = \int_{-\infty}^{\infty} x^n f(x)\,dx, \quad n \geq 0.$$

for $n=0$, the above implies g is a density. If Y is a random variable with density g then all moments of X & Y are finite and match, even though their distribution are not same.

Conditional Distribution & Expectation:-

Definition:- Suppose that f is the joint density of (X,Y) and
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad x \in \mathbb{R}.$$
is the marginal density of X. The conditional density of Y given $X=x$, for any x such that $f_X(x) > 0$, is defined by,
$$f_{Y|x}(y) = \frac{f(x,y)}{f_X(x)}, \quad y \in \mathbb{R}.$$

In this chapter unless mentioned otherwise, X and Y are R.V.s. with joint density f & $f_X = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad x \in \mathbb{R}.$

Thm:- If $E|Y| < \infty$ then for almost all x such that $f_X(x) > 0$, it holds that
$$\int_{-\infty}^{\infty} |y|\, f_{Y|x}(y)\,dy < \infty.$$

Pf: Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \int_{-\infty}^{\infty} |y| \, f_{Y|x}(y) \, dy, & \text{if } f_X(x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

and $h: \mathbb{R}^2 \to \mathbb{R}$ by:

$$h(x,y) = \begin{cases} f_{Y|x}(y), & \text{if } f_X(x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$E(|Y|) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y| \, f(x,y) \, dx \, dy.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y| \, h(x,y) \, f_X(x) \, dx \, dy.$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{\infty} |y| \, h(x,y) \, dy \, dx.$$

$$= \int_{-\infty}^{\infty} f_X(x) \, g(x) \, dx.$$

Since, the above integral is finite, it follows that $\{x \in \mathbb{R} : f_X(x) \, g(x) = \infty\}$ is a set of measure zero.

$\{x \in \mathbb{R} : f_X(x) > 0 \text{ and } g(x) = \infty\}$
$\subseteq \{x \in \mathbb{R} : f_X(x)\, g(x) = \infty\}$

Definition:- Suppose that $E(|Y|) < \infty$.
The conditional expectation of Y given $X = x$, for any x s.t. $f_X(x) > 0$ is defined by

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y\, f_{Y|x}(y)\, dy =: g(x).$$

The following NOTATION is commonly used:

$$E(Y \mid X) = g(X).$$

Example:-
Suppose that (X, Y) ~~follows~~ $\sim BVN$ with $E(X) = E(Y) = 0$, $Var(X) = Var(Y) = 1$ and ~~correlation~~ $corr(X, Y) = \rho \in (-1, 1)$.

Recall that; the joint density of (X, Y) is.

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right), \quad x, y \in \mathbb{R}.$$

and the marginal density of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

$\therefore$ the conditional density of Y given $X = x$ is.

$$f_{Y|x}(y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot e^{-\frac{1}{2(1-\rho^2)}(y - \rho x)^2}, \quad y \in \mathbb{R}.$$

i.e. given $X = x$ the conditional distribution of Y is $N(\rho x, 1 - \rho^2)$.

$X \sim N(0,1)$.
$Z \sim N(0,1)$.
indep.
$Y = \rho X + \sqrt{1-\rho^2}\, Z$.
given $X = x$.
$Y = \rho x + \sqrt{1-\rho^2}\, Z$
$Z \sim N(0,1)$.
$Y \sim N(\rho x, 1-\rho^2)$

$$E(Y \mid X = x) = \rho x.$$

i.e. $E(Y \mid X) = \rho X$.

Theorem, If $E|Y| < \infty$, then.

$E(E(Y|X)) = E(Y)$ [formula of double expectation]

Pf: For $x \in \mathbb{R}$, define.

$$g(x) = \begin{cases} \int_{-\infty}^{\infty} y\, f_{Y|x}(y)\, dy, & \text{if } f_X(x) > 0 \text{ and the integral exist} \\ 0, & \text{o.w.} \end{cases}$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{Y|x}(y)\, f_X(x)\, dx\, dy.$$

[Tonelli] $$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{\infty} y\, f_{Y|x}(y)\, dy\, dx.$$

$$= \int_{-\infty}^{\infty} f_X(x)\, g(x)\, dx.$$

$$= E[g(X)].$$

$$= E(E(Y|X)).$$

Exc: Suppose that X and Y are as in the previous exc. Calculate.

$E(X^8 Y^2)$.

FACT:- If Y and Z have finite expectation then
$$E(Y+Z\mid X) = E[Y\mid X] + E[Z\mid X].$$

FACT:- If $E|Y| < \infty$ and $E|f(X)Y| < \infty$, where $f:\mathbb{R}\to\mathbb{R}$ is such that
$$E(f(X)Y\mid X) = f(X)\,E(Y\mid X).$$

Corollary: $E|Y|<\infty$ and $E|Z|<\infty$, for $\alpha,\beta\in\mathbb{R}$.
$$E(\alpha Y+\beta Z\mid X) = \alpha E(Y\mid X) + \beta E(Z\mid X).$$

Soln:
$$\begin{aligned}
E(X^8Y^2) &= E(E(X^8Y^2\mid X)) \\
&= E[X^8\,E(Y^2\mid X)] \\
&= E[X^8(\rho^2X^2 + 1-\rho^2)] \\
&= E[(1-\rho^2)X^8 + \rho^2X^{10}] \\
&= (1-\rho^2)E(X^8) + \rho^2E(X^{10}) \\
&= \rho^2\cdot\frac{10!}{5!\,2^5} + (1-\rho^2)\frac{8!}{4!\,2^4} \\
&= (1.3.5.7.9)\rho^2 + (1.3.5.7)(1-\rho^2) \\
&= (3.5.7)(8\rho^2+1)
\end{aligned}$$

Exc:- Suppose that X and Y have finite second moment and $E(X\mid Y) = Y$, $E(Y\mid X) = X$.

Show that $X = Y$ a.s.

Soln:
$$E[(X-Y)^2] = E[X^2 - XY] + E[Y^2 - XY]$$
$$E[Y^2 - XY] = E(E(Y^2 - XY\mid Y)) = E[Y^2 - YE(X\mid Y)] = 0.$$

Similarly $E(X^2 - XY) = 0$. Thus $E(X-Y)^2 = 0$

i.e. $X = Y$ a.s.

Exc.: Suppose that $(X,Y) \sim BVN$ as before.

Calculate $E(X|X+Y)$

$\cancel{\mathrm{Cov}(Y, X+Y) = \mathrm{Var}(X) + \mathrm{Cov}(X,Y) = 1+\rho}$.

Sol^n: Define $Z = X+Y$.

The Distribution of (X, Z) is BVN with

$E(X) = 0 = E(Z)$, $\mathrm{Var}(X) = 1$, $\mathrm{Var}(Z) = 2(1+\rho)$.

$\mathrm{cov}(X, Z) = 1+\rho$.

The dist^n (Y,Z) is.

$(X, Z) \overset{d}{=} (Y, Z)$.

$E(X|Z) = E(Y|Z) = U$

$\Rightarrow 2U = E(Z|Z) = Z$.

$\Rightarrow U = \frac{Z}{2}$.

Exc.:- Suppose the (X,Y) has joint density.

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \frac{1}{y} e^{\left(-\frac{x^2}{2y^2} - y\right)} I(y>0), \ (x,y) \in \mathbb{R}^2.$$

Calcule $\mathrm{Cov}(X,Y)$.

Sol^n:-

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{y} e^{\left(-\frac{x^2}{2y^2} - y\right)} dx.$$

$$= e^{-y} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, y} e^{-\frac{x^2}{2y^2}} dx.$$

$= e^{-y}$

$\cancel{E(XY) = E(E(XY|Y)) = E(Y E(X|Y)) = 0}$

$\cancel{E(X) = 0}$, $\cancel{E(}$

$$f_{X|y}(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{y} \exp\left(-\frac{x^2}{2y^2}\right), \; x \in \mathbb{R}.$$

$$X|Y \sim N(0, y^2)$$

$$E(XY) = E(Y E(X|Y)) = 0.$$

Also, $E(X) = 0$.

$\therefore$ $Cov(X,Y) = 0$.

$\therefore$ $Corr(X,Y) = 0$.

Exc:- If X and Y are independent and $E|Y| < \infty$, the show that $E(Y|X) = E(Y)$.

Example:- Suppose the conditional Density of Y given $X = x$ is $h(y)$ for all x. Show that the marginal density of Y is h and it is independent of X.

Exc:- Suppose that $X_1, \ldots, X_n$ are i.i.d. with finite mean calculate

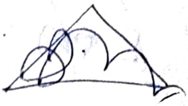
$$E(X_1 | X_1 + \cdots + X_n)$$

Ans: $\frac{1}{n}(X_1 + \cdots + X_n)$

Example:- Suppose that X, Y are i.i.d. from $U(0,1)$, find the conditional distribution of X given $X+Y$.

The joint density of X and Y is

$f(x,y) = 1(0 \le x,y \le 1)$, $x,y \in \mathbb{R}$.



$U := X$

$V := X+Y$.

$\psi(u,v) = (u, v-u)$

$$|J(u,v)| = \left|\det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right| = 1.$$

the function $\phi : (0,1) \times (0,1) \to \{(u,v) \in \mathbb{R}^2, \; 0<u<1, \; u<v<1+u\}$

$g(x,z) = 1(0<x<z<1+x<2)$.

* *

H.W. 7

① ⓐ $(pe^t + q)^n$ $\quad q = 1-p$.

ⓑ $\exp(\lambda(e^t - 1))$

ⓒ $\dfrac{pe^t}{1-qe^t}$, $t < -\ln q$.

ⓓ $\left(\dfrac{pe^t}{1-qe^t}\right)^k$, $t < -\ln q$.

ⓔ $\dfrac{e^t(1-e^{nt})}{n(1-e^t)}$

ⓕ $\dfrac{e^{bt} - e^{at}}{t(b-a)}$.

ⓖ $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

ⓗ $\dfrac{\lambda}{\lambda - t}$, $t < \lambda$.

ⓘ $(1-2t)^{-n/2}$, $t < \frac{1}{2}$.

ⓙ $\left(1 - \frac{t}{\lambda}\right)^{-\alpha}$, $t < \lambda$.

2. (a) 2k-th moment $= \dfrac{\sigma^{2k}(2k)!}{2^k\, k!}$

(b) k th moment $= \dfrac{\alpha(\alpha+1)\cdots(\alpha-k+1)}{\lambda^k}$

(c) $E(Y^{2k}) = (2k)!\binom{n+k-1}{k}$.

(d) 2k th moment

$= \alpha(\alpha+1)\cdots(\alpha+2k-1)$.

(e) (2k) th moment.

$= \dfrac{1}{(k+1)(2k+1)}$.

3. (a). $\alpha = \dfrac{1}{\int_1^\infty e^{-x}x^{-2}\,dx} < \infty$.

(b) finite for $|t| \leq 1$.

4.

(a). $E(X^k) = \dfrac{(2k)!}{2^k\, k!}$

(b) $M(t) = \sum_{k=0}^{\infty} \dfrac{t^k}{k!} \cdot \dfrac{(2k)!}{2^n\, n!}$.

$t = \frac{1}{2}$.

$\sum_{k=0}^{\infty} \dfrac{(2k)!}{(k!)^2\, 4^k}$

by Stirling,

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$$\frac{(2n)!}{n!} \sim \frac{\sqrt{4\pi n}\,(2n/e)^{2n}}{2\pi n\,(n/e)^{2n}} = \frac{4^n}{\sqrt{\pi n}}.$$

$$\Rightarrow \quad \frac{(2n)!}{(n!)4^n} \sim \frac{1}{\sqrt{\pi n}} \quad \text{as} \quad n \to \infty.$$

ratio test gives us. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n} = \infty.$

5.

(a) $(\Rightarrow)$.

$Y \sim \text{Poisson}(\lambda)$.

$$E\left[e^{tY}\right] = \exp\left[\lambda(e^t - 1)\right], \quad \forall t \in \mathbb{R}.$$

$$= \left(E\left[e^{tX_1}\right]\right)^n$$

$$\Rightarrow \quad E\left(e^{tX_1}\right) = \exp\left[\frac{\lambda}{n}(e^t - 1)\right] \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow \quad X_1 \sim \text{Poisson}\left(\frac{\lambda}{n}\right).$$

* *

the marginal density of Z is.

$$g_Z(z) = \int_{-\infty}^{\infty} g(x,z)\,dx.$$

<u>case I</u> $\quad 0 < z \leq 1.$

$$g_Z(z) = \int_0^z 1\,dx = z.$$

Case II $\quad 1 < z < 2$

$$g_Z(z) = \int_{z-1}^{1} 1\,dx = 2 - z.$$

$$\therefore \quad g_Z(z) = \begin{cases} z & ; \; z \in [0,1] \\ 2-z & ; \; z \in (1,2] \\ 0 & ; \; \text{o.w.} \end{cases}$$

The conditional ~~distribution~~ density of X given $Z = z$ is.

$$g_{X|z}(x) = \begin{cases} \dfrac{1\,(0 < x < z)}{z} & ; \; 0 < z \leq 1. \\ \dfrac{1\,(z-1 < x < 1)}{2-z} & ; \; 1 \leq z < 2. \end{cases}$$

fix, $0 \leq z \leq 1$.

$$E(X \mid Z = z) = \int_{-\infty}^{\infty} x\, g_{X|z}(x)\,dx.$$

$$= \frac{1}{z}\int_{0}^{z} x\,dx = \frac{z}{2}.$$

Fix $\quad 1 < z < 2,$

$$E(X \mid Z = z) = \frac{1}{2-z}\int_{z-1}^{1} x\,dx = \frac{z}{2}.$$

FACT:- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ $(b_n \neq 0)$ and

$\sum a_n$ converges, the so does $\sum b_n$.

"this is not true."

Counter example

$a_n = (-1)^n/n$.

$$b_n = \begin{cases} -\frac{1}{n}, & n \text{ odd}, \\ \frac{1}{n}(1+\ln 2n), & n \text{ even}. \end{cases}$$

$\lim_{n\to\infty} \frac{a_n}{b_n} \to 1$.

$\sum a_n$ conv.

$$\sum_{n=1}^{2N} b_n = \sum_{n=1}^{2N} (b_n - a_n) + \sum_{n=1}^{2N} a_n.$$

$$= \underbrace{\sum_{n=1}^{N} \frac{1}{2N \ln 2N}}_{\infty} + \underbrace{\sum_{n=1}^{2N} a_n}_{\text{finite}}.$$

$\sum b_n \to \infty$.

FACT:- If $a_n \downarrow 0$, the $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

HW-9

7. $f(x,y) = e^{-y} \mathbb{1}(0<y<x<y+1)$, $x,y \in \mathbb{R}$.

Fix $y>0$.

Claim: The condition distribution of X given $Y=y$ is Unif $(y, y+1)$.

$$f_Y(y) = \int_{0}^{\infty} f(x,y)\,dx$$

$$= \int_{y}^{y+1} e^{-y}\,dx$$

$$= e^{-y} \int_{y}^{y+1} dx$$

$$= e^{-y}.$$

$$\therefore f_{X|Y} = \frac{f(x,y)}{f_Y(y)} = \mathbb{1}(0<y<x<y+1) \quad \blacksquare$$

$$E(XY) = EE(XY|Y)$$

$$= E(Y\,E(X|Y))$$

$$= E(Y(Y+\tfrac{1}{2}))$$

$$= E(Y^2) + \tfrac{1}{2}E(Y) = 2 + \tfrac{1}{2}\cdot 1 = \tfrac{5}{2}$$

$$E(X) = E(X|Y) = E(Y+\tfrac{1}{2}) = \tfrac{3}{2}.$$

$$\mathrm{Cov}(X,Y) = E(Y^2) + \tfrac{1}{2}E(Y) - E(X)E(Y).$$

$$= \frac{5}{2} - \frac{3}{2}\cdot 1 = 1.$$

8. $\Delta \sim$ Std. Exp.

$$(X \mid \Delta = \lambda) \sim N(\lambda, 1).$$

Conditional density of X given $\Delta = \lambda$ is.

$$f_{X|\lambda}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda)^2} \qquad x \in \mathbb{R}.$$

marginal of Δ is,

$$f_{\Delta}(\lambda) = e^{-\lambda}.$$

$$\therefore f(x,\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2\lambda x + \lambda^2 + 2\lambda)} \cdot 1(\lambda > 0).$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2+\lambda^2)} 1(\lambda>0)}.$$

$\forall\, x, \lambda \in \mathbb{R}$.

$$\therefore f_X(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2+\lambda^2)} 1(\lambda>0) \, d\lambda.$$

$$= e^{-\frac{x^2}{2}} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda^2} d\lambda.$$

$$= \frac{e^{-\frac{x^2}{2}}}{2}$$

$$f_X(x) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\lambda^2 - 2\lambda(x-1) + (x-1)^2 + x^2 - (x-1)^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\lambda-(x-1))^2} \cdot e^{-\frac{1}{2}(2x-1)} \, d\lambda.$$

$$= e^{-\frac{1}{2}(2x-1)} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\lambda-(x-1))^2} d\lambda$$

$$= e^{-x+\frac{1}{2}} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\lambda-(x-1))^2} d\lambda.$$

$$= e^{-x+\frac{1}{2}} [1 - \Phi(1-x)].$$

$$= e^{-x+\frac{1}{2}} \Phi(x-1).$$

the marginal density of X is.

$$f_X(x) = e^{-x+\frac{1}{2}} \Phi(x-1). \quad , x \in \mathbb{R}.$$

$$E(X) = EE(X|\lambda) = E(\Lambda) = 1.$$

13. $E(X\Phi(Y)) = EE[X\Phi(Y)|Y]$

$$= EE(X\Phi(Y)|X). \qquad = E[\Phi(Y)\, E(X|Y)]$$

$$= E(X\, E(\Phi(Y)|X)) \qquad = E[\rho Y \Phi(Y)].$$

$$= \rho\, E[Y\Phi(Y)].$$

$$= \rho \int_{-\infty}^{\infty} y\, \Phi(y)\, \phi(y)\, dy.$$

$$= \rho \int_{-\infty}^{\infty} y\, \phi(y) \int_{-\infty}^{y} \phi(z)\, dz\, dy.$$

$$= \rho \int_{y=-\infty}^{\infty} \int_{z=-\infty}^{y} y\, \phi(y)\, \phi(z)\, dz\, dy.$$

$\rightarrow$

$$\int_{y=-\infty}^{\infty} \int_{z=-\infty}^{y} |y\,\phi(y)\,\phi(z)|\,dz\,dy.$$

$$= \int_{y=-\infty}^{\infty} |y|\,\phi(y) \int_{z=-\infty}^{y} \phi(z)\,dz\,dy.$$

$$\leq \int_{y=-\infty}^{\infty} |y|\,\phi(y)\,dy. < \infty$$

$\therefore E(r)$ exist.

$\therefore$ By Fubini,

$$= \rho \int_{z=-\infty}^{\infty} \int_{y=z}^{\infty} y\,\phi(y)\,\phi(z)\,dy\,dz.$$

$$= \rho \int_{z=-\infty}^{\infty} \phi(z) \int_{y=z}^{\infty} y\,\phi(y)\,dy\,dz.$$

$$= \rho \int_{z=-\infty}^{\infty} \phi(z). \int_{y=z}^{\infty} \frac{y \,.\, e^{-y^2/2}}{\sqrt{2\pi}}\,dy\,dz.$$

$$= \rho \int_{z=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \,.\, \frac{-e^{y^2/2}}{\sqrt{2\pi}}\Big|_{z}^{\infty}\,dz.$$

$$= \rho \int_{z=-\infty}^{\infty} \frac{1}{2\pi} e^{-z^2}\,dz.$$

$$= \frac{\rho}{2\pi\sqrt{2}} \int_{z=-\infty}^{\infty} e^{-\frac{1}{2}(\sqrt{2}z)^2} \sqrt{2}\,dz.$$

$$= \frac{\rho}{2\pi\sqrt{2}} \cdot \sqrt{2\pi}$$

$$= \frac{\rho}{\sqrt{4\pi}} = \frac{\rho}{2\sqrt{\pi}}.$$

14. $E(X^2Y^2)$

$= E(E(X^2Y^2 \mid Y))$

$= E(Y^2 E(X^2 \mid Y))$

$= E(Y^2(1-\rho^2+\rho^2Y^2))$

$= E(Y^2) - \rho^2 E(Y^2) + \rho^2 E(Y^4)$

$= 1 - \rho^2 + 3\rho^2$

$= 2\rho^2 + 1.$

$\mathrm{Cov}(X^2, Y^2) = E(X^2Y^2) - E(X^2)E(Y^2)$

$= (2\rho^2+1) - 1.$

$= 2\rho^2.$

$\mathrm{Var}(X^2) = E(X^4) - E(X) = 3 - 1 = 2.$

$\mathrm{Var}(Y^2) = 2.$

$\therefore \mathrm{Corr}(X^2, Y^2) = \frac{2\rho^2}{\sqrt{2\times 2}} = \rho^2.$

HW-8

1. X has CHF ϕ_X.

$Y = a + X$.

$$\phi_Y(t) = E\left[e^{itY}\right].$$

$$= E\left[e^{it(a+X)}\right].$$

$$= e^{ita} \; E\left[e^{itX}\right] = e^{ita}\phi_X$$

2. X has CHF ϕ. TFAE

(i) ϕ is real.

(ii) $\phi(-t) = \phi(t)$.

(iii) $X \overset{d}{=} -X$.

$$\phi(+t) = E[\cos tX] + i\,E[\sin tX].$$

$$\phi(-t) = E[\cos tX] - i\,E[\sin tX].$$

$$= \overline{\phi(t)}$$

(1) $\Rightarrow$ (2)

$\phi(t)$ real

$\Rightarrow \quad \phi(t) = \overline{\phi(t)} = \phi(-t)$.

(2) $\Rightarrow$ (3). Since $\phi(-t)$ is CHF of $-X$.

uniqueness tells us $X \overset{d}{=} -X$.

(3) $\Rightarrow$ (1). $X \overset{d}{=} -X$.

$$\phi(t) = \phi(-t) = \overline{\phi(t)}$$

$\therefore \phi$ is real.

4.

(a) Binomial (n, p) $\phi(t) = (q + pe^{it})^n$.

(b) Poisson (λ). $\phi(t) = \exp(\lambda(e^{it} - 1))$

(c) Negative Binomial $(k, p) \to \left(\frac{pe^{it}}{1+qe^{it}}\right)^k$.

(d) Uniform (a, b):

$$\phi(t) = \begin{cases} \dfrac{e^{ibt} - e^{iat}}{it(b-a)} & , t \neq 0. \\ 1 & , t = 0. \end{cases}$$

(e) Triangular.

$$\phi(t) = E[\cos(tX)].$$

$$= 2\int_0^1 (1-x)\cos(tx)\,dx.$$

$$= 2\left[\int_0^1 \cos tx\,dx - \int_0^1 x\cos tx\,dx\right]$$

$$= \begin{cases} \dfrac{2(1-\cos t)}{t^2} & , t \neq 0. \\ 1 & , t = 0. \end{cases}$$

(f) Normal (μ, σ^2). $\phi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$.

(g) Exp $(\lambda) \to \dfrac{\lambda}{\lambda - it}$.

(h)

$\chi^2_2 \sim \text{Exp}\left(\frac{1}{2}\right)$.

Let, $X_1, \ldots, X_n$ be i.i.d. from χ^2_2.

$Y = X_1 + \cdots + X_n$ then $Y \sim \chi^2_{2n}$.

CHF of Y is, $\phi_Y(t) = E\left[e^{itX_1}\right]^n$

$$= \left(\frac{1/2}{1/2 - it}\right)^n$$

$$= (1-2it)^{-n}.$$

(i) Double Exponential, $\phi(t) = \frac{1}{1+t^2}$

(f) Cauchy $\phi(t) = e^{-|t|}$.

(5) X has density

$$f(x) = \frac{1}{2^{3/2}\pi^{\frac{1}{2}}} |x|^{-1} \exp\left(-\frac{1}{2}(\ln|x|)^2\right) 1(x \neq 0), \quad x \in \mathbb{R}.$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)\, t^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{E[(iX)^n]\, t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{E[(iX)^{2n}]\, t^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} E(X^{2n})$$

$$E\left[e^{2n \ln|X|}\right]$$

$\&$ $\ln|X| \sim N(0,1)$

$$= e^{-\frac{(2n)^2}{2}}$$

$$= e^{-2n^2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} e^{-2n^2}.$$

(c) Show that for $t \neq 0$,

$$\lim_{n\to\infty} \left| \frac{(-1)^n t^{2n}}{(2n)!} e^{2n^2} \right| = \infty$$

Stirling $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \leq c \quad \forall n \in \mathbb{N}.$$

$$n! \leq c\sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \quad \forall n \in \mathbb{N}.$$

$$\ln |\;| = 2n \ln|t| + 2n^2 - \ln(2n)!.$$

$$\geq 2n \ln|t| + 2n^2 - \ln\left(c\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\right)$$

$$= 2n \ln|t| + 2n^2 - \ln c - \frac{1}{2}\ln(4\pi n) - 2n(\ln 2n - 1)$$

$$= 2n^2\left[\underbrace{\frac{\ln|t|}{n}}_{\to 0} + 1 - \underbrace{\frac{\ln c}{n}}_{\to 0} - \underbrace{\frac{\ln(4\pi n)}{2n^2}}_{\to 0} - \underbrace{\frac{\ln 2n - 1}{n}}_{\to 0}\right]$$

$$\to 2n^2$$

hence, $\lim_{n\to\infty} \left| \frac{(-1)^n t^{2n} e^{2n^2}}{(2n)!} \right| = \infty$.

and hence, the Taylor series diverges for every $t \neq 0$.

6. If X has CHF ϕ, then

$$P(a \le X \le b) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-iat} - e^{-ibt}}{it} \phi(t)\,dt$$

$\forall\ a, b$ such that $P(X \in \{a, b\}) = 0$.

Suppose X and Y have the same CHF.

Then $P(a \le X \le b) = P(a \le Y \le b)$. $(*)$

$\forall\ a, b$ s.t. $P(X \in \{a, b\}) = P(Y \in \{a, b\}) = 0$.

The set $\mathcal{C} = \{x \in \mathbb{R},\ P(X = x) > 0$ or $P(Y = x) > 0\}$ is Countable.

Let, $b \in \mathcal{C}^c$ and $\{a_n\} \subseteq \mathcal{C}^c$ s.t.

$b > a_n \downarrow -\infty$ as $n \to \infty$.

using $(*)$ & $n \to \infty$, we get

$P(X \le b) = P(Y \le b)$. $(**)$

Fix any $x \in \mathbb{R}$ and let $b_n \in \mathcal{C}^c$ s.t. $b_n \downarrow x$.

using $(*)(*)$.

we get.

$$P(X \le b_n) = P(Y \le b_n).$$

Letting $n \to \infty$ & using right continuity of CDF we get,

$$P(X \le x) = P(Y \le x).$$

7.

(a) $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt, \quad x \in \mathbb{R}.$$

$$|f(x)| = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt \right|$$

$$\le \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(t)|\, dt.$$

Claim:

Suppose $g : \mathbb{R} \to [0, \infty)$ is such that $\int_{-\infty}^{\infty} g(t)\, dt < \infty$

then, $$h(x) = \int_{-\infty}^{\infty} e^{-itx} g(t)\, dt, \quad x \in \mathbb{R}.$$

Proof of the claim:

Let, $$\alpha = \int_{-\infty}^{\infty} g(t)\, dt.$$

if $\alpha = 0$, then $h(x) = 0 \;\forall\; x$ and hence nothing to prove. Assume WLOG that, $\alpha > 0$. Then $\alpha^{-1} g$ is a density.

Let, Z be a r RV with density $\alpha^{-1} g$

then $$h(x) = \int_{-\infty}^{\infty} e^{-itx} g(t)\, dt = \alpha E\left[e^{-ixZ}\right]$$

Since, CHF is continuous. so is h.

$$f(x) = \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt.$$

write ϕ as

$$\phi(t) = \left(\psi_1^+(t) - \psi_1^-(t)\right) + i\left(\psi_2^+(t) - \psi_2^-(t)\right).$$

(b) $Z = X + Y$.

$$\phi_z(t) = \phi_x(t)\,\phi_y(t).$$

$$\int_{-\infty}^{\infty} \quad |\phi_z(t)| = |\phi_x(t)\,\phi_y(t)|.$$
$$\leq |\phi_x(t)|.$$

9. Define,

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\,\phi(t)\,dt. \quad \forall x \in \mathbb{R}.$$

problem 7 $\Rightarrow$ g is continuous.

Inversion thm $\Rightarrow$ g is density of X.

& f is also continuous. and density of X.

$\therefore$ $f = g$.

10. $$Y = \left(\sum_{j=1}^{n} |\alpha_j|\right)^{-1} \sum_{k=1}^{n} \alpha_k X_k.$$

$$\phi_Y(t) = E\left[e^{itY}\right].$$
$$= E\left[\exp\left(it\beta \sum_{j=1}^{n} \alpha_j X_j\right)\right].$$
$$= E\left[\exp\left(it \sum_{j=1}^{n} \beta\alpha_j x_j\right)\right].$$
$$= \prod_{j=1}^{n} E\left[e^{it\beta\alpha_j x_j}\right]$$

$$= \prod_{j=1}^{n} e^{-|t\beta\alpha_j|}$$

$$= e^{-\beta|t|\sum_{j=1}^{n}|\alpha_j|}$$

$$= e^{-|t|}.$$

By uniqueness thm, $Y \sim$ Cauchy.

(b) $E\left[e^{it\sum_{j=1}^{n}\alpha_j X_j}\right]$

$$= \prod_{j=1}^{n} E\left[e^{it\alpha_j X_j}\right]$$

$$= \prod_{j=1}^{n} e^{-\frac{t^2\alpha_j^2}{2}}$$

$$= e^{-\frac{t^2}{2}\left(\sum_{j=1}^{n}\alpha_j^2\right)}.$$

12. $M_X(t) < \infty$, $|t| \le \varepsilon$. (for some $\varepsilon > 0$).

and.

$E(X^n) = 0 \quad \forall$ odd $n \in \mathbb{N}$.

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n).$$

$$= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} E(X^{2n}).$$

$\forall |t| < \varepsilon.$

$$M_{-X}(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} E((-X)^{2n}) = M_X(t).$$

$\therefore$ X & $-X$ has same MGF $\forall$ $|t| < \varepsilon$.

i.e. X and $-X$ have the same MGF in $[-\varepsilon, \varepsilon]$, which is also finite

hence $X \overset{d}{=} -X$.

13.

$$f(x) = \begin{cases} \frac{1}{\pi} \frac{1-\cos x}{x^2} & , x \neq 0. \\ \frac{1}{2\pi} & , x = 0. \end{cases}$$

$V \sim$ Triangular distⁿ.

CHF of V is.

$g(x) = (1-|x|) \cdot \mathbb{1}(|x| \le 1)$.

$h_V(x) = \int_{-\infty}^{\infty} e^{-itx} f(t)\, dt$

$g \equiv h_V$.

$E\left[e^{-itX}\right] = (1-|t|)\mathbb{1}(|t| \le 1)$.

$\Rightarrow \quad E\left[e^{itX}\right] = (1-|t|)\mathbb{1}(|t| \le 1)$.

$\phi_Z(t) = E\left[e^{itZ}\right] = (1-|t|)^2 \, \mathbb{1}(|t| \le 1)$.

Inversion formula implies that, density of Z is.

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz} \phi_Z(t)\, dt$$

$$= \frac{1}{\pi}\int_0^1 (1-t)^2 \cos(tz)\, dt$$

$$= \frac{2}{\pi} \frac{z - \sin z}{z^3}$$

14. $(\Rightarrow)$

$Y \sim$ Cauchy.

Let, ϕ be the CHF of X_1.

Then, for all $t \in \mathbb{R}$,

$$\phi_{X_1}\left(\frac{t}{n}\right)^n = E\left[e^{itY}\right] = e^{-|t|}.$$

$$\Rightarrow \phi_{X_1}\left(\frac{t}{n}\right) = e^{-\left|\frac{t}{n}\right|}.$$

Suppose, $n = 2$

$$\phi\left(\frac{t}{2}\right)^2 = e^{-|t|}.$$

$$\phi\left(\frac{t}{2}\right) = \pm e^{-\frac{|t|}{2}}$$

Suppose that,

$$\phi\left(\frac{t}{2}\right) = -e^{-|t|/2} \quad \text{for some } t.$$

$$\phi(0) = 1.$$

then $\phi(\xi) = 0$ for some ξ.

$e^{-|2\xi|} = \phi\left(\frac{\xi}{2}\right)^2 = 0$ which is a contradiction.

Hence, $\phi\left(\frac{t}{2}\right) = e^{-\left|\frac{t}{2}\right|}$

$\Rightarrow X_1 \sim$ Cauchy.

for general n it need complex analysis.

15.

If $X_n \xrightarrow{P} X$ as $n \to \infty$.

$\lim_{n\to\infty} \phi_n(t) = \phi(t)$, $t \in \mathbb{R}$.

$\phi_n(t) = E\left[e^{itX_n}\right]$.

$\phi(t) = E\left[e^{itX}\right]$.

$e^{itX_n} \xrightarrow{P} e^{itX}$.

$|e^{itX_n}| \leq 1$. D.C.T.

Qn:- X_1, X_2 are i.i.d. from Std. uniform.
Y follows triangular $((1-|x|)\mathbb{1}(|x|\leq 1))$.

(a) Show that $X_1 + X_2 \stackrel{d}{=} Y+1$.

(b) Verify using CHF.

(c) If Z_1, Z_2 are iid. $U(-\frac{1}{2}, \frac{1}{2})$, show that $Z_1 + Z_2 \stackrel{d}{=} Y$.

HW-9

6. $(X,Y) \sim N_2(\mu, \Sigma)$.

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$X' = \frac{X-\mu_1}{\sqrt{\sigma_{11}}} \qquad Y' = \frac{Y-\mu_2}{\sigma_{22}}$$

$\therefore$ X' & Y' follows bivariate normal with mean zero, variance 1, and correlation ρ.

$$\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$$

$$Z = \frac{X' - \rho Y'}{\sqrt{1-\rho^2}}$$

Check that Y' and Z are independent.

$$X = \mu_1 + \sqrt{\sigma_{11}}\, X'$$

$$= \mu_1 + \sqrt{\sigma_{11}}\left(\rho Y' + \sqrt{1-\rho^2}\, Z\right).$$

$$= \mu_1 + \rho\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\,(Y-\mu_2) + \sqrt{(1-\rho^2)\sigma_{11}}\, Z.$$

Since, Z is independent of Y, it follows that the conditional distribution of X given Y is normal with mean $\mu_1 + \rho\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}(Y-\mu_2)$ and variance $(1-\rho^2)\,\sigma_{11}$;

10. (X,Y) have joint density

$$f(x,y) = \left[(1+ax)(1+ay) - a\right] e^{-x-y-axy}\, 1(x,y>0), x,y \in \mathbb{R}$$

The marginal density of Y is.

$$f_Y(y) = \int_0^\infty f(x,y)\,dx. \quad , y > 0.$$

$$= (1+ay-a)\,e^{-y} \int_0^\infty e^{-x(1+ay)}\,dx.$$

$$+ a(1+ay)\,e^{-y} \int_0^\infty x\,e^{-x(1+ay)}\,dx.$$

$$= \frac{1+ay-a}{1+ay} \cdot e^{-y} + \frac{a\,e^{-y}}{1+ay}.$$

$$= \frac{1+ay}{1+ay} \cdot e^{-y}$$

$$= e^{-y}.$$

The conditional density of X given $Y=y$ is.

$$f_{X|y}(x) = \left[(1+ax)(1+ay) - a\right] e^{-x(1+ay)}$$

$$\therefore E(X|Y=y) = \int_0^\infty x\left[(1+ax)(1+ay) - a\right] e^{-x(1+ay)}\,dx.$$

$$= (1+ay-a) \int_0^\infty x\,e^{-x(1+ay)}\,dx.$$

$$+ a(1+ay) \int_0^\infty x^2 e^{-x(1+ay)}\,dx.$$

$$= \frac{1+a(y+1)}{(1+ay)^2}$$

$$E(X|Y) = \frac{1+a(1+Y)}{(1+aY)^2}$$

(11). $X \sim \text{Gamma}(a,1)$ $\quad Y \sim \text{Gamma}(a+\frac{1}{2}, 1)$.

<u>Corollary</u>: Suppose U and V are R.V's.

s.t. $\left.\begin{array}{l} E[e^{tU}] < \infty. \\ E[e^{tV}] < \infty \end{array}\right\}$ for all $|t| < \varepsilon$. $(\varepsilon > 0)$.

and $E(U^n) = E(V^n)$, $n \geq 1$.

then $U \stackrel{d}{=} V$

<u>Solⁿ</u>: Let $Z = 2\sqrt{XY}$.

$$E(Z^n) = 2^n E(X^{\frac{n}{2}}) E(Y^{\frac{n}{2}}).$$

$$= 2^n \cdot \frac{\Gamma(a+\frac{n}{2})}{\Gamma a} \cdot \frac{\Gamma(a+\frac{1}{2}+\frac{n}{2})}{\Gamma(a+\frac{1}{2})}.$$

Let $n = 2m$ $\quad = 2^n \dfrac{\Gamma(a+m)}{\Gamma a} \cdot \dfrac{\Gamma(a+m+\frac{1}{2})}{\Gamma(a+\frac{1}{2})}$.

$$= 2^{2m} a(a+1)\ldots(a+m-1)\cdot(a+\tfrac{1}{2})(a+\tfrac{3}{2})\ldots(a+m-\tfrac{1}{2}).$$

$$= 2a(2a+2)\ldots(2a+2m-2)(2a+1)(2a+3)\ldots(2a+2m-1)$$

$$= (2a)(2a+1)\ldots(2a+2m-1).$$

$$= \frac{\Gamma(2a+n)}{\Gamma 2a} = E(U^n).$$

where $U \sim \text{Gamma}(2a)$.

$$E(e^{tU}) < \infty.$$

$$E(e^{tZ}) = E(e^{2t\sqrt{XY}}) \leq E(e^{t(X+Y)})$$

$$= E(e^{tX}) E(e^{tY}) < \infty$$

$$E\left[Z^n\right] = 2^n E\left(X^{\frac{n}{2}}\right) E\left(Y^{\frac{n}{2}}\right).$$

$$= 2^n \frac{\Gamma\left(a+\frac{n}{2}\right)}{\Gamma a} \cdot \frac{\Gamma\left(a+\frac{n}{2}+\frac{1}{2}\right)}{\Gamma a+\frac{1}{2}}$$

$$= 2^n \frac{\Gamma\left(a+\frac{n}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)} \cdot \frac{\Gamma\left(a+\frac{n+1}{2}\right)}{\Gamma a}.$$

14.

$(X, Y) \sim BVN.$

$E\left[\Phi(X)\, \Phi(Y)\right].$

Now, $E\left[\Phi(Y) \mid X=x\right] = \displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}}\, \Phi(y)\, e^{-\frac{1}{2}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2} dy.$

$$= \int_{y=-\infty}^{\infty} \int_{z=-\infty}^{y} \frac{1}{\sqrt{1-\rho^2}}\, \phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) \phi(z)\, dz$$

$Y \sim N(\rho x, 1-\rho^2)$
$Z \sim N(0,1)$ — indep.

$$= P(Z < Y).$$

$$= P(Z - Y < 0).$$

$$= \Phi\left(\frac{\rho x}{\sqrt{2-\rho^2}}\right).$$

$$E\left[\Phi(X)\, \Phi(Y)\right] = E\left[\Phi(X)\, \Phi\left(\frac{\rho X}{\sqrt{2-\rho^2}}\right)\right]$$