ARMA(p, q) processes

Autoregressive Moving Average process of order (p, q)

It is the stationary solution of the equation

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}, \quad \dots\dots (*)$$

if the solution exists, $p, q \geq 0$,

where the following polynomials donot have a common root.

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$$

$$\theta(z) = 1 + \theta_1 z + \dots + \theta_p z^q,$$

and $Z_t \sim WN(0, \sigma^2)$.

So, we can write

$$\phi(B) \cdot X_t = \theta(B) \cdot Z_t, \quad \text{where}$$

B is the backward shift operator.

$$\left[\begin{aligned} B X_t &= X_{t-1} \\ B^j &= X_{t-j} \\ B^{-j} &= X_{t+j} \text{ etc.} \end{aligned} \right]$$

Result:

$\phi(B) X_t = \theta(B) Z_t$

has a stationary solution in X_t if and only if $\phi(z) \neq 0 \ \forall z$ such that $|z| = 1$.

Remark: For ARMA (1, 1), we had a similar condition that $|\phi| \neq 1$.

Proof: Let $\phi(z) \neq 0 \ \forall z$ such that $|z| = 1$

$$\Rightarrow \phi(z) = (1 - a_1 z)(1 - a_2 z) \dots (1 - a_p z),$$

where $a_i \in \mathbb{C}$.

Now, $\forall i$, $\left|\frac{1}{a_i}\right| \neq 1$

i.e., $|a_i| \neq 1$.

So, $\frac{1}{\phi(z)} = \sum_{j=-\infty}^{\infty} x_j z^j$, where

$\sum_{j=-\infty}^{\infty} |x_j| < \infty$, since

$$\frac{1}{1-a_i z_i} = 1 + a_i z + a_i^2 z^2 + \cdots \quad \text{if } |a_i| < 1$$
$$= -a_i^{-1} z^{-1} - a_i^{-2} z^{-2} - a_i^{-3} z_i^{-3} - \cdots \quad \text{if } |a_i| > 1.$$

So, $\phi(B) X_t = \theta(B) Z_t$.

$$\Rightarrow X_t = \frac{\theta(B)}{\phi(B)} \cdot Z_t$$

$$= \cancel{\sum_{j=-\infty}^{\infty} x_j Z^j}$$

$$= \left(\sum_{j=-\infty}^{\infty} x_j B^j\right)\left(\sum_{j=0}^{q} \theta_j B^j\right) Z_t.$$

$\boxed{\theta_0 = 1}$

$$= \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

$$= \psi(B) Z_t.$$

So, we have a unique ~~solution~~ stationary solution in X_t.

'Only if' part

Now, let $\phi(z) = 0$ for some z with $|z| = 1$.

Then, $\phi(B) X_t = \theta(B) Z_t$.

$$X_t = \frac{1}{\phi(B)} \cdot \theta(B) \cdot Z_t .$$

$$= \psi(B) Z_t \quad , \left[\psi(B) = \frac{1}{\phi(B)} \cdot \theta(B)\right]$$

$$= \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} ,$$

with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

$|\,| > 1.$

Putting the roots of $\phi(z)$,
$\phi(z) = 0$, $|z| = 1$,

we have $\frac{1}{\phi(z)} \cdot \theta(z)$ does not exist.

But $|\psi(z)| < \infty$ as $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ with $|z| = 1$.

Question: Argue why the equation should be of the form $\psi(B) Z_t$ if it exists.

Definition (Causality)

ARMA(p,q) process X_t is said to be causal if

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

with $\psi_j = 0$ for $j < 0$.

i.e. $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$.

i.e. X_t depends only on Z_s, $s \le t$, i.e. the past values of $\{Z_t\}$.

Theorem :

ARMA(p, q) process X_t defined by $\phi(B) X_t = \theta(B) Z_t$ is causal if and only if all roots z of $\phi(z)$ have $|z| > 1$, i.e. $\phi(z) \neq 0$ if $|z| \leq 1$.

Proof : 'If part'

Let all roots z have $|z| > 1$.

$$X_t = \frac{1}{\phi(B)} \cdot \theta(B) Z_t .$$

Now,

$$\frac{1}{\phi(z)} = \frac{1}{(1-a_1 z)(1-a_2 z)\cdots(1-a_p z)}$$

$$= \sum_{j=0}^{\infty} \alpha_j z^j \quad , \Big[\text{as}$$

$$\frac{1}{1-a_i z} = 1 + a_i z + a_i^2 z^2 + \cdots \quad \text{as } \left|\frac{1}{a_i}\right| > 1$$

$$\text{ie } |a_i| < 1. \Big]$$

~~So, $X_t = \frac{1}{\phi(B)} \theta(B) Z_t$~~

So,

$$\frac{1}{\phi(z)} \theta(z) = \sum_{j=0}^{\infty} \psi_j z^j$$

$\Rightarrow \quad X_t$ is Causal.

'Only if' part :

Let X_t be causal.

$$\Rightarrow \quad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

i.e. $\frac{1}{\phi(z)} \theta(z) = \sum_{j=0}^{\infty} \psi_j z^j$ . . . . (*1)

If $\phi(z)$ has a root z, with $|z| \le 1$, then as $\sum_{j=0}^{\infty} |\psi_j| < \infty$ at that z, RHS of (*1) is well defined but LHS is not since $\phi(z) = 0$.

ARMA(p, q)

X_t is stationary solution to

$$~~\phi(B)X_t\theta(B)~~$$

$$\phi(B) X_t = \theta(B) Z_t$$

with $Z_t \sim WN(0, \sigma^2)$

Now, $\{X_t\}$ is AR(p) if $q = 0$, i.e. X_t is (autoregressive of order p) is stationary solution to

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t .$$

Again $\{X_t\}$ is MA(q) (moving average of order q) if $p = 0$, i.e. X_t is stationary solution to

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} .$$

So, we have proved —

Theorem 1:

A stationary solution to $\phi(B) X_t = \theta(B) Z_t$ exists if and only if $\phi(z) \ne 0$ for all z such that $|z| = 1$.

Theorem 2

An ARMA (p, q) process $\{X_t\}$ is causal if and only if $\phi(z) \neq 0$ for all z with $|z| \leq 1$.

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An ARMA (p, q) process is stationary solution to

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $Z_t \sim WN(0, \sigma^2)$, and the polynomials $\phi(z)$; $\theta(z)$ have no common factor,

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$$

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

Invertibility

Definition

If we can write

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j},$$

Then $\{X_t\}$ is invertible ARMA process.

Theorem 3

$\{X_t\}$ is invertible if and only if $\theta(z) \neq 0 \quad \forall z \in \mathbb{C}$ and $|z| \leq 1$.

Proof: It is similar to causality condition.

Calculation of ψ_j's in a causal case:

$$\psi(B) Z_t = X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \frac{1}{\phi(B)} \theta(B) Z_t .$$

$$\phi(B)\psi(B) = \theta(B)$$

$$(1 - \phi_1 z - \cdots - \phi_p z^p)(\psi_0 + \psi_1 z + \psi_2 z^2 + \cdots)$$
$$= 1 + \theta_1 z + \cdots + \theta_p z^p$$

Equating coefficients of z^k

$$1 = \psi_0$$
$$\theta_1 = \psi_1 - \phi_1 \psi_0$$
$$\theta_2 = \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0$$
$$\vdots$$
$$\theta_j = \psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2} - \cdots - \phi_j \psi_0 .$$

$$\psi_j = 0, \quad j < 0$$
$$\theta_0 = 1$$
$$\theta_j = 0, \quad j > q$$
$$\phi_j = 0, \quad j > p$$

Calculation to get π_j's in an 'invertible case':

Similarly $\quad \pi(B) X_t = \frac{1}{\theta(B)} \cdot \phi(B) X_t = Z_t$

$$\theta(B) \cdot \pi(B) X_t = \phi(B) X_t$$

That is, $(1 + \theta_1 z + \cdots + \theta_q z^q)(1 + \pi_1 z + \pi_2 z^2 + \cdots) = 1 - \phi_1 z - \cdots - \phi_p z^p$

Q. $\left\{ -\phi_j = \psi_j + \sum_{k=1}^{j} \theta_k \psi_{j-k} \right.$

$-\phi_j = \pi_j + \sum_{k=1}^{j} \theta$

Equating the coefficients of z^j,

$$-\phi_j = \pi_j + \sum_{k=1}^{j} \theta_k \pi_{j-k}$$

$\phi_0 = -1, \quad \pi_j = 0 \quad , j < 0$

$\phi_j = 0 \quad , j > p$

$\theta_j = 0 \quad , j > q$

$\theta_0 = 1$

Question 1

Given $X_t - 0.4 X_{t-1} = Z_t + 0.3 Z_{t-1}$.

Check if (i) stationary X_t exists

(ii) it is unique

(iii) it is causal. Find $\psi(B) Z_t$

(iv) it is invertible. Find $\pi(B) X_t$

(i) Stationary X_t exists if

$$\phi(z) \neq 0 \;\forall\; z \text{ such that } |z| = 1$$

Here $\phi(z) = 1 - 0.4 z$

Indeed $\phi(z) \neq 0 \;\forall\; z$ such that $|z| = 1$

So, stationary X_t exists

(ii) It is unique.

(iii) It is causal if

$\phi(z) \neq 0$ if $|z| \leq 1$.

Here $\phi(z) = 1 - 0.4z$

Indeed $\phi(z) \neq 0$ if $|z| \leq 1$.

So, the process is causal.

$$(1 - 0.4B) X_t = (1 + 0.3B) Z_t$$

$$\Rightarrow \quad X_t = \frac{1}{(1-0.4B)} (1+0.3B) Z_t$$

$$= (1 + 0.4B + 0.4^2 B^2 + \cdots)(1+0.3B) Z_t$$

$$= \sum_{j=0}^{\infty} \psi_j Z_{t-j} ,$$

ψ_j's are real constants,

e.g., $\psi_0 = 1$

~~e.g~~ $\psi_1 = 0.7$ etc.

)Z_t

)X_t

(iv) The process is invertible if

$\theta(z) \neq 0$ if ~~for~~ $|z| \leq 1$.

where $\theta(z) = 1 + 0.3z$

Indeed $\theta(z) \neq 0$ if $|z| \leq 1$.

So, the process is invertible.

$$Z_t = \cancel{0.4B} \frac{1}{(1+0.3B)} (1 - 0.4B) X_t$$

$$= (1 - 0.3B + 0.3^2 B^2 - \cdots)(1 - 0.4B)\, X_t$$

$$= \sum_{j=0}^{\infty} \pi_j X_{t-j} .$$

- Find π_j's explicitly, e.g.

$$\pi_0 = 1 ,$$
$$\pi_1 = -0.7 . \text{ etc.}$$

(2) The equation is

$$X_t + 0.5 X_{t-2} = Z_t + 1.2 Z_{t-1}$$

Check if (i) stationary solution exists
(ii) causality
(iii) invertibility

(i) Stationary solution exists if

$$\phi(z) \neq 0 \quad \forall z \text{ such that } |z| = 1,$$

where $\phi(z) = 1 + 0.5 z^2$

Indeed $\phi(z) \neq 0$ $\forall z$ such that $|z| = 1$.

So, stationary solution exists.

(ii) The process is causal if

$$\phi(z) \neq 0 \quad \text{if } |z| \leq 1$$

Indeed $\phi(z) \neq 0$ if $|z| \leq 1$.

So, the process is causal.

(iii) The process is invertible if

$\theta(z) \neq 0$ if $|z| \leq 1$,

where $\theta(z) = 1 + 1.2z$

Now, $\theta(z) = 0$ for $z = -\frac{1}{1.2} = -\frac{10}{12}$

So, the process is <u>not invertible</u>.

Example AR(2) process.

$$X_t - 0.7X_{t-1} + 0.1X_{t-2} = Z_t$$

$$\phi(z) = 1 - 0.7z + 0.1z^2 = (1-0.5z)(1-0.2z)$$

Roots are 2 and 5 $(|z| > 1)$

$\Rightarrow$ Unique, stationary solution exists and it is causal.

$$(1 - 0.7z + 0.1z^2)(\psi_0 + \psi_1 z + \psi_2 z^2 + \cdots)$$

$$\psi_0 = 1, \quad \psi_1 = 0.7, \quad \psi_2 = 0.7^2 - 1$$

$$\boxed{\psi_j = 0.7\psi_{j-1} - 0.1\psi_{j-2}}$$

We need to solve the above difference equation.

Put $\psi_j = \lambda^j$

$$\lambda^j - 0.7\lambda^{j-1} + 0.1\lambda^{j-2} = 0$$

$$\underline{\lambda^2 - 0.7\lambda + 0.1 = 0}$$

$\Rightarrow \lambda = 0.5$ or 0.2

$$\boxed{\psi_j = \frac{A}{2^j} + \frac{B}{5^j}}$$

Now, $A+B = \psi_0 = 1$

$$\frac{A}{2} + \frac{B}{5} = \psi_1 = 0.7$$

Solve for A and B and get ψ_j.

✓

Theorem 4

$$\phi(B) X_t = \theta(B) Z_t \quad \text{- - - -} \quad (*)$$

If X_t is not causal or invertible, $\exists$ a new polynomials $\phi^*(z)$ and $\theta^*(z)$ and white noise $\{W_t\}$ such that

$$\phi^*(B) X_t = \theta^*(B) W_t$$

gives the same solution as given by $(*)$, provided ~~the~~ stationary solution to $(*)$ exists; and in this equation, X_t is causal and invertible.

Proof: Let $\phi(z) = \phi_1(z) \cdot \phi_2(z)$

$\theta(z) = \theta_1(z) \cdot \theta_2(z)$

$$\left.\begin{array}{l} \phi_1(z) = 0 \\ \theta_1(z) = 0 \end{array}\right\} \text{ only at } |z| > 1$$

$$\left.\begin{array}{l} \phi_2(z) = 0 \\ \theta_2(z) = 0 \end{array}\right\} \text{ only at } |z| < 1$$

Then, $\phi_1(B) \cdot \phi_2(B) X_t = \theta_1(B) \cdot \theta_2(B) Z_t$

$$\Rightarrow \quad \frac{\phi_1(B)}{\theta_1(B)} X_t = \frac{\theta_2(B)}{\phi_2(B)} Z_t = Y_t$$

Then, we can write

$$\frac{\theta_2(B)}{\phi_2(B)} Z_t = \frac{\theta_2^*(B^{-1})}{\phi_2^*(B^{-1})} \cdot (-1)^k B^l Z_t,$$

$\cancel{k, l \geq 0}$ $k, l \in \mathbb{Z}$.

Roots of θ_2^* and ϕ_2^* have modulus > 1.

Rename Y_t as U_{-t}.

So, we ~~get~~ get $\frac{\theta_2^*(B)}{\phi_2^*(B)} W_t$ in place of Y_t with suitable ~~as~~ white noise W_t.

Hence, $\cancel{\phi_1 \phi_2^* X_t}$

$$\phi_1(B)\,\phi_2^*(B)\, X_t = \theta_1(B)\,\theta_2^*(B) W_t$$

is the new equation and roots of $\phi_1, \phi_2^*, \theta_1, \theta_2^*$ have modulus > 1.

[Better proof in "Time Sr: Th. & Methods. Brockwell & Davis]

9th April, 2010

Autocovariance function of ARMA(p,q) process

Equation:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

$$Z_t \sim WN(0, \sigma^2).$$

Assume X_t is causal.

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

$$\gamma(h) = cov(X_t, X_{t-h})$$
$$= \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

Example :

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \quad |\phi| < 1, \quad Z_t \sim WN(0, \sigma^2)$$

$$X_t = Z_t + \sum_{j=1}^{\infty} (\theta+\phi)\phi^{j-1} Z_{t-j}$$

$$\gamma(0) = \sigma^2 \left[1 + (\theta+\phi)^2 \sum_{j=1}^{\infty} \phi^{2j-2} \right]$$
$$= \sigma^2 \left[1 + \frac{(\theta+\phi)^2}{1-\phi^2} \right]$$

$$\gamma(1) = \sigma^2 \left[(\theta+\phi) + \sum_{j=1}^{\infty} (\theta+\phi)^2 \phi^{2j-1} \right]$$
$$\cancel{= \sigma^2 \left[(\theta+\phi) + \sum_{j=1}^{\infty} (\theta+\phi)^2 \phi^{2j-1} \right]}$$
$$= \sigma^2 \left[(\theta+\phi) + \frac{(\theta+\phi)^2 \phi}{1-\phi^2} \right]$$

$$\gamma(h) = \phi^{|h|-1} \gamma(1), \quad |h| > 1$$

Example MA(q)

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}, \quad Z_t \sim WN(0, \sigma^2)$$
$$\gamma(h) = E(X_t \cdot X_{t+h}) = \sigma^2 \sum_{j=1}^{q-|h|} \theta_{j+|h|} \cdot \theta_j .$$

$$\cancel{E(Z_{t-j}^2)}$$

$\theta_0 = 1$

$\theta_j = 0$ for $j < 0$, $j > q$.

① If for a stationary time series,

$$\hat{\gamma}(h) \approx 0, \quad |h| > q \text{ and}$$

$$\hat{\gamma}(0), \ \hat{\gamma}(1), \ \ldots, \ \hat{\gamma}(q)$$

~~then there is~~ are different from '0', then there is high chance that the process is MA(q).

We have also the result that if a stationary time series is q-correlated, i.e.

$$\gamma(h) = 0 \quad \forall \ |h| > q.$$

then the time series is MA(q).

So, to test

H_0 : $\{X_t\}$ is MA(q)

H_1 : $\{X_t\}$ is not q-correlated.

To test this, we should expect all

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \text{ for } |h| > q \text{ ~~be~~ small.}$$

Result:

$\hat{\rho}(h)$ for $|h| > q$ for MA(q) processes are asymptotically $N\left(0, \frac{\omega_{hh}}{n}\right)$

where $\omega_{hh} = 1 + 2\hat{\rho}^2(1) + \cdots + 2\rho^2(q)$.

According to the result, $\hat{\rho}(h)$ lies in the bounds $\pm 1.96 \frac{\sqrt{\omega_{hh}}}{\sqrt{n}}$ with 95% chance.

Otherwise, we reject the null hypothesis.

Calculation of ACVF using ~~ACDF~~ difference equations ;

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

Multiply by X_{t-k} and take expectation.

$$\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) = \sigma^2 \sum_{j=0}^{\infty} \theta_{k+j} \psi_j$$

as $X_{t-k} = \sum_{j=0}^{\infty} \psi_j Z_{t-k-j}$

$0 \leq k < \max(p, q+1)$,
$= m$, say.

$$\ldots\ldots (1\cdot 1)$$

$$\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) = 0 \text{ for } k \geq m. \quad \ldots (1\cdot 2)$$

General solution to equations 1.2 is

$$\gamma(h) = \alpha_1 \xi_1^{-h} + \alpha_2 \xi_2^{-h} + \cdots + \alpha_p \xi_p^{-h} ; \ldots (1\cdot 3)$$

~~assumin~~ where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are unknown, assuming $\xi_1, \xi_2, \ldots, \xi_p$ are different, (are roots of $\phi(z) = 0$)

If some ξ_i's are equal, solution of the equation has similar treatments.

Values of $\gamma(k)$ from equations (1.3) for $k = \cancel{0, 1, 2, \ldots, m-p}$ $m-p, m-p+1, \ldots, m$ are put in equations 1.1 and the set of m equations (1.1) are solved for $\alpha_1, \alpha_2, \ldots, \alpha_p, \gamma(0), \gamma(1), \ldots, \gamma(m-p-1)$; i.e m-many unknowns.

Hence $\gamma(k)$, $k \geq 0$ are obtained.

<u>Example</u> AR(2) causal.

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

$$= (1 - \xi_1^{-1} z)(1 - \xi_2^{-1} z) \quad , \quad |\xi_1|, |\xi_2| > 1$$

$$\phi_1 = \xi_1^{-1} + \xi_2^{-1}$$

$$\phi_2 = -\xi_1^{-1} \cdot \xi_2^{-1} .$$

Here $p = 2$, $q = 0$

$$m = \max(p, q+1) = 2 .$$

~~For~~ $0 \leq k < 2$, i.e. $k = 0, 1$, equations (1.1) become

$$\begin{cases} \gamma(0) - \phi_1 \gamma(1) - \phi_2 \gamma(2) = \sum_{j=0} \theta_{k+j} \psi_j = \sigma^2 \\ \gamma(1) - \phi_1 \gamma(0) - \phi_2 \gamma(1) = 0 \end{cases}$$

For $k \geq 2$, equations 1.2 become

$$\gamma(k) - \phi_1 \gamma(k-1) - \phi_2 \gamma(k-2) = 0$$

General solution ~~to~~ equations (1.2)

$$\gamma(k) = \alpha_1 \xi_1^{-k} + \alpha_2 \xi_2^{-k}$$

equations (1.3) becomes this.

From equations 1.3, putting the ~~val~~ values of $\gamma(0), \gamma(1), \gamma(2)$ in equations 1.1, we get two linear equations in α_1, α_2,

$$(*)\ldots\begin{cases}(\alpha_1+\alpha_2)-\phi_1(\alpha_1\xi_1^{-1}+\alpha_2\xi_2^{-1})-\phi_2(\alpha_1\xi_1^{-2}+\alpha_2\xi_2^{-2})=\sigma^2 \\ (1-\phi_2)(\alpha_1\xi_1^{-1}+\alpha_2\xi_2^{-1})-\phi_1(\alpha_1+\alpha_2)=0\end{cases}$$

From $(*)$, α_1, α_2 are solved and we get $\gamma(k)$ for $k \geq 0$.

Here it is assumed that $AR(2)$ is causal, i.e $|\xi_1^{-1}|, |\xi_2^{-1}| > 1$.

Note that $\gamma(k) = \alpha_1 \xi_1^{-k} + \alpha_2 \xi_2^{-k}$ with $|\xi_1^{-1}|, |\xi_2^{-1}| > 1$.

So, $\gamma(k) \longrightarrow 0$ as $k \longrightarrow \infty$.

<u>Question</u>

Prove that in ~~a causal~~ any ARMA (p,q) process,

$$\gamma(k) \longrightarrow 0 \text{ as } k \to \infty.$$

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<u>Periodogram</u>

$$2\pi f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda}$$

The sample analogue of this will be called 'periodogram'.

$\mathbb{C}^n$

$$\frac{1}{\sqrt{n}}\begin{pmatrix} e^{i\omega_k} \\ e^{i2\omega_k} \\ \vdots \\ e^{in\omega_k} \end{pmatrix} = e_k \,,\quad \omega_k = \frac{2\pi}{n}\cdot k$$

$k = -\left[\frac{n-1}{2}\right], \dots, \left[\frac{n}{2}\right].$

Considering all frequencies $\omega_k \in (-\pi, \pi]$ which is a multiple of $\frac{2\pi}{n}$.

$\{e_1, e_2, \dots, e_n\}$

$$\underbrace{e_j^*}_{\text{conjugate transpose}} \cdot e_k = \begin{cases} \frac{1}{n}(1+1+\dots+1) = 1 & \text{if } j=k \\ \sum_{t=1}^{n} e^{it(\omega_k - \omega_j)} = 0 & \text{o.w.} \end{cases}$$

$$\left(\because e^{i(\omega_k - \omega_j)} = 1^{1/n} \right)$$

$\{e_1, e_2, \dots, e_n\}$ is orthonormal basis for $\mathbb{C}^n$.

observations from the time series $x_1, x_2, \dots, x_n$ given:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$$

x is linear combination of $e_1, e_2, \dots, e_n$.

~~$t = 1, 2, \dots, n$~~

Let $$\underline{x} = \sum_{k=1}^{n} a_k e_k .$$

$t = 1, 2, \ldots, n$, $\quad x_t = \sum_{k=1}^{n} a_k e^{it\omega_k}$

$$a_k = e_k^* \underline{x}$$

$$\sum_{j=1}^{n} a_j e_k^* e_j = a_k .$$

$$a_k = e_k^* \underline{x} = \frac{1}{\sqrt{n}} \sum x_t e^{-it\omega_k},$$

$$e_k = \frac{1}{\sqrt{n}} \begin{pmatrix} e^{i\omega_k} \\ e^{i2\omega_k} \\ \vdots \\ e^{in\omega_k} \end{pmatrix}$$

$$|a_k|^2 = \frac{1}{n} \left| \sum_{t=1}^{n} x_t e^{-it\omega_k} \right|^2 = I_n(\omega_k)$$

Definition

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} x_t e^{-it\lambda} \right|^2$$

~~called~~ This is called periodogram of $\{x_1, x_2, \ldots, x_n\}$ at frequency λ.

$$2\pi \underbrace{f(\lambda)}_{\text{spectral density}} = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-ih\lambda}$$

We ~~shall~~ shall show that

$$I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}(h) e^{-ih\omega_k}$$

Proof: $\sum_{t=1}^{n} e^{it\omega_k} = 0$.

$$I_n(\omega_k) = \frac{1}{n} \left| \sum_{t=1}^{n} (x_t - \bar{x}) e^{-it\omega_k} \right|^2$$

$$= \frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} (x_t - \bar{x})(x_s - \bar{x}) e^{-i\omega_k(t-s)}$$

$$= \sum_{|h|<n} \hat{\gamma}(h) \cdot e^{-ih\omega_k}$$

$$\left[\text{As } \hat{\gamma}(h) = \frac{1}{n} \sum_{j=1}^{n-|h|} (x_j - \bar{x})(x_{j+|h|} - \bar{x}) \right)$$

<u>Result</u> :

Joint distribution function of $I_n(\lambda_1)$, $I_n(\lambda_2)$, … …, $I_n(\lambda_m)$ for $-\pi < \lambda_1 < \lambda_2 < \dots < \lambda_m < \pi$, for large n is given by

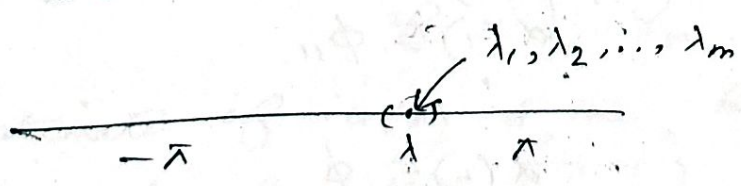
$$F(x_1, x_2, \dots, x_m) = \begin{cases} \prod_{i=1}^{m} \left(1 - \exp\left(-\frac{x_i}{2\pi f(\lambda_i)}\right)\right) & \text{for all } x_i > 0 \\ 0 & \text{o.w.} \end{cases}$$

So they are iid exponential.

$Var(I_n(\lambda_i)) \not\longrightarrow 0$ as $n \to \infty$.

So, $I_n(\lambda_i)$ is not consistent for $2\pi f(\lambda_i)$.

But we are to estimate $f(\lambda_i)$



to estimate $f(\lambda)$

Take the estimate as $\frac{1}{m} \sum_{j=1}^{m} I_n(\lambda_j)$.

This is consistent, provided $m \to \infty$, $\frac{m}{n} \to 0$ as $n \to \infty$.

Take $m_n \to \infty$; $\frac{m_n}{n} \to 0$ as $n \to \infty$.

$$\hat{f}(\lambda) = \frac{1}{2\pi} \sum_{j=-m_n}^{m_n} I_n\left(g(n,\lambda) + \frac{2\pi}{n} \cdot j\right) \Big/ (2m_n+1)$$

$g(n,\lambda)$ is a multiple of $\frac{2\pi}{n}$ that is closest to λ.

This is an estimate of $f(\lambda)$.

Partial Auto correlation function (PACF)

$$P(X_n / X_{n-1}, X_{n-2}, \ldots, X_1, X_0)$$
$$= \sum_{k=1}^{n} \phi_{nk} X_{n-k}$$
$$= \phi_{n1} X_{n-1} + \phi_{n2} X_{n-2} + \cdots + \phi_{n_{n-1}} X_1 + \phi_{n_n} X_0$$

Partial Auto correlation function of order n is defined as

$$\alpha(n) = \phi_{nn}.$$

$$\alpha(0) = 1$$
$$\alpha(1) = \phi_{11}$$
$$\vdots$$
$$\alpha(n) = \phi_{nn}$$

Consider $MA(q)$, ACF $\gamma(n) = 0$ for $n > q$.
This is used as a test for $MA(q)$.
Similarly if X_t is $AR(p)$, then PACF,

$$\alpha(n) = 0 \quad \text{for } n > p.$$

This canbe used as a test for AR(p).

AR(p)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p Z_{t-p} + Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

Predictor of

Now, consider the predictor of X_t

$$P(X_t \mid X_{t-1}, X_{t-2}, \dots, X_1, X_0)$$
$$= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p}$$

$$\phi_{tt} = 0 \quad \text{if} \quad t > p$$

and $\phi_{tt} = \phi_p$ if $t = p$.

So, (*) is proved.

$$(\phi_{n1}, \dots, \phi_{nn}) = \phi_n = \Gamma_n^{-1} \gamma_n$$

where $\gamma_n = (\gamma(1), \gamma(2), \dots, \gamma(n))$

$$\Gamma_n = ((\gamma(i-j)))_{i,j=1}^{n}$$

$\alpha(n) = \phi_{nn}$ is the n^{th} component of $\Gamma_n^{-1} \gamma_n$.

Estimate of $\alpha(n)$ is the $\hat{\alpha}(n) = n^{th}$ component of $\hat{\phi}_n = \hat{\Gamma}_n^{-1} \hat{\gamma}_n$, where $\hat{\Gamma}_n$ and $\hat{\gamma}_n$ are based on sample

$(x_1, x_2, \dots, x_m)$

Result:

$\hat{\alpha}_h$ is approximately $N(0, \frac{1}{m})$ for large ~~mean~~ number of observations m, in an $AR(p)$, for $h > p$.

This is used to test for $AR(p)$.

Exercise:

Show that

ϕ_{nn} = correlation between

$X_0 - P(X_0 | X_1, X_2, \ldots, X_{n-1})$ and $X_n - P(X_n | X_1, \ldots, X_{n-1})$.

Brockwell-Davis

3.11 Show that the value at lag 2 of the PACF of the MA(1) process

$X_t = Z_t + \theta Z_{t-1}$, $t = 0, \pm 1, \pm 2, \ldots$,

where $Z_t \sim WN(0, \sigma^2)$, is

$$\alpha(2) = -\frac{\theta^2}{1+\theta^2+\theta^4}.$$

$$(\phi_{21}, \phi_{22}) = \Gamma_2^{-1} \gamma_2.$$

Now, $\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } h = 0 \\ \theta\sigma^2 & \text{if } h = \pm 1 \\ 0 & \text{o.w.} \end{cases}$

Then $\Gamma_2 = \sigma^2 \begin{pmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{pmatrix}$

$$\gamma_2 = (\theta\sigma^2, 0)$$

$$|\Gamma_2| = \sigma^4\left((1+\theta^2)^2 - \theta^2\right)$$

$$= \sigma^4(1+\theta^2+\theta^4)$$

$$\Gamma_2^{-1} = \frac{\sigma^2}{\sigma^4(1+\theta^2+\theta^4)}\begin{pmatrix} 1+\theta^2 & -\theta \\ -\theta & 1+\theta^2 \end{pmatrix}$$

$$\alpha(2) = \phi_{22} = -\frac{\theta^2}{1+\theta^2+\theta^4}.$$

= Continuation to 3.11

Q. Show that

$$\phi_{hh} = -\frac{(-\theta)^h}{(1+\theta^2+\theta^4+\dots+\theta^{2h})}$$

$w_{ij} =$

$$\sum_{k=1}^{i} \phi^{2i} (\phi^{-k} - \phi^{k})^2 + \sum_{k=i+1}^{\infty} \phi^{2k} (\phi^{-i} - \phi^{i})^2 .$$

$$= \sum_{k=1}^{i} \phi^{2i}(\phi^{-2k} + \phi^{2k} - 2) + \sum_{k=i+1}^{\infty} \phi^{2k} (\phi^{-2i} + \phi^{2i} - 2)$$

$$= \phi^{2i} \sum_{k=1}^{\infty} \phi^{2k} - 2i\phi^{2i} - 2\frac{\phi^{2i+2}}{1-\phi^2}$$

$$+ \phi^{2i} \frac{\phi^{-2}(1-\phi^{-2i})}{1-\phi^{-2}} + \frac{\phi^2}{1-\phi^2}$$

$$= \frac{\phi^{2i+2}}{1-\phi^2} - 2i\phi^{2i} - 2\frac{\phi^{2i+2}}{1-\phi^2}$$

$$+ \frac{(\phi^{2i}-1)}{\phi^2-1} + \frac{\phi^2}{1-\phi^2}$$

$$= \frac{\phi^2 + (1-\phi^{2i}) - \phi^{2i+2}}{1-\phi^2} - 2i\phi^{2j}$$

$$= (1-\phi^2)^{-1}(1+\phi^2)(1-\phi^{2j}) - 2i\phi^{2j}$$