~~hence~~
i.e. $\lim_{n\to\infty} E(X_n) \geq E(X)$.

Thm:- If $X, Y \geq 0$, then,

$$E(X+Y) = E(X) + E(Y).$$

Proof:- First concider the case that $X$ & $Y$ are simple.

If $X$ takes values $x_1, x_2, \dots, x_m$ & $Y$ takes values $y_1, y_2, \dots, y_n$, then.

$$E(X+Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_i + y_j) P(X = x_i, Y = y_j).$$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} P(X = x_i, Y = y_j) + \sum_{j=1}^{n} y_j \sum_{i=1}^{m} P(X = x_i, Y = y_j).$$

$$= \sum_{i=1}^{m} x_i P(X = x_i) + \sum_{j=1}^{n} y_j P(Y = y_j).$$

$$= E(X) + E(Y).$$

Ex $0 \leq X \leq 1$.

$X_n = 2^{-n} \lfloor 2^n X \rfloor$, $n \geq 1$.

Check if $X_n$ are increasing.

Suppose $X$ & $Y$ are bounded and non-negative R.V.

Define for $n \geq 1$.

$$X_n = 2^{-n}\lfloor 2^n X \rfloor, \quad Y_n = 2^{-n}\lfloor 2^n Y \rfloor.$$

Clearly, $X_n$ and $Y_n$ are simple and non-negative, furthermore, $0 \le X_1 \le X_2 \le \dots$ and $0 \le Y_1 \le Y_2 \le Y_3 \le \dots$

$$2^n X - 1 \le \lfloor 2^n X \rfloor \le 2^n X$$

$$\Rightarrow X - 2^{-n} \le 2^{-n} \lfloor 2^n X \rfloor \le X$$

i.e. $|X - X_n| \le 2^{-n}$ and hence $X_n \uparrow X$.

MCT ~~to~~ tells us

$$\lim_{n\to\infty} E(X_n) = E(X).$$

Similarly $\lim_{n\to\infty} E(Y_n) = E(Y)$

Also, $0 \le X_n + Y_n \uparrow X+Y$.

Using MCT once again,

$$E(X+Y) = \lim_{n\to\infty} E(X_n + Y_n)$$

$$= \lim_{n\to\infty} [E(X_n) + E(Y_n)] \quad \left[\text{as } X_n \,\&\, Y_n \text{ are simple R.V.}\right]$$

$$= \lim_{n\to\infty} E(X_n) + \lim_{n\to\infty} E(Y_n).$$

$$= E(X) + E(Y).$$

Finally, let $X$ & $Y$ be any non-negative RV.

Observe that, $\cancel{0 \le X_n \min(X \& n) \uparrow X}$.

~~the proof fo~~ $\cancel{\min X}$

$$X_n := \min(X \,\&\, n)$$

$$0 \le X_n \uparrow X.$$

& $X_n$ is bdd.

the proof follows by applying MCT like before.

Exc:- If $X \geq 0$ and $\alpha \in [0, \infty)$, then

$E(\alpha X) = \alpha E(X)$.

Solⁿ:-

$$E(\alpha X) = \int_0^\infty P(\alpha X > x)\,dx$$

$$= \int_0^\infty P\left(X > \frac{x}{\alpha}\right) dx.$$

$$= \int_0^\infty P(X>y)\,\alpha\,dy.$$

$$= \alpha \int_0^\infty P(X>y)\,dy.$$

$$= \alpha\, E(X).$$

Conventions regarding $\pm\infty$

$$x \cdot \infty = \begin{cases} +\infty & ; x > 0 \\ -\infty & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$

$$x \cdot (-\infty) = (-x) \cdot \infty.$$

$x + \infty = \infty \quad \forall\, x \in \mathbb{R} \cup \{\infty\}$

$x - \infty = -\infty \quad \forall\, x \in \mathbb{R} \cup \{-\infty\}$

$\infty - \infty$ is ~~NO~~ NOT defined.

Defⁿ:- Let, $X$ be any ~~ra~~ RV not ~~necss~~ necessarily non-negative. Define,

$X^+ = X \vee 0$, $\quad [a \vee b = \max(a, b)]$

$X^- = (-X) \vee 0$.

we define,

$$E(X) = E(X^+) - E(X^-).$$

when atleast one of them is finite.

Theorem:

1. The expectation of a RV $X$ is defined and finite iff $E(|X|) < \infty$.

2. If ~~expectation~~ $E(X)$ is defined, the $|E(X)| \leq E(|X|)$

3. If $E(X)$ is defined, then $E(X) = \int_0^\infty P(X > x)dx$

$$- \int_{-\infty}^0 P(X \leq x)dx.$$

4. If $X$ is a discrete R.V. taking values $x_1, x_2, \ldots$ then $E(X)$ is defined and finite iff

$$\sum_n |x_n| P(X = x_n) < \infty.$$

in that case, $E(X) = \sum_n x_n P(X = x_n)$.

5. If $X$ has density $f$, then $E(X)$ is defined finitely iff $\int_{-\infty}^{\infty} |x| f(x)dx < \infty$

and in that case:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Proof:

1. $E(X)$ is defined finitely iff both $E(X^+)$ and $E(X^-)$ are finite.

$\Leftrightarrow$ $E(X^+) + E(X^-) < \infty$

$\Leftrightarrow$ $E(X^+ + X^-) < \infty$

$\Leftrightarrow$ $E(|X|) < \infty$

2.

$$E(X) = E(X^+) - E(X^-) \leq E(X^+) + E(X^-) = E(|X|).$$

$$-E(X) = -E(X^+) + E(X^-) \leq E(X^+) + E(X^-) = E(|X|)$$

$$|E(X)| \leq E(|X|).$$

3. $E(X^+) = \int_0^\infty P(X^+ > x)\,dx = \int_0^\infty P(X > x)\,dx$.

Ily,

$E(X^-) = \int_{-\infty}^0 P(X < x)\,dx$.

$\therefore E(X) = E(X^+) - E(X^-)$

$= \int_0^\infty P(X > x)\,dx - \int_{-\infty}^0 P(X < x)\,dx$.

4. We first claim that for a non-negative discrete RV $X$ taking distinct values $x_1, x_2, \dots$

$$E(X) = \sum_n x_n P(X = x_n).$$

$X_n = X . 1(X \in \{x_1, x_2, \dots, x_n\}),\ n \geq 1$.

Clearly $0 \leq X_n \uparrow X$ and hence by MCT,

$$E(X) = \lim_{n\to\infty} E(X_n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i P(X = x_i) \quad \left(\text{because } x_1, x_2, \dots \text{ are distinct}\right).$$

$$= \sum_{i=1}^{\infty} x_i P(X = x_i).$$

For any discrete RV $X$ taking distinct values $x_1, x_2, \dots$

$$E(X^+) = \sum_{i : x_i \geq 0} x_i P(X = x_i)$$

$$E(X^-) = \sum_{i : x_i < 0} (-x_i) P(X = x_i).$$

Adding the two equation, we get

$$E(|X|) = \sum_i |x_i| P(X = x_i).$$

and hence,

$E(X)$ is defined finitely iff $E(|X|) < \infty$.

and in that case,

$$E(X) = E(X^+) \mp E(X^-)$$

$$= \sum_i x_i P(X = x_i).$$

5.

<u>FACT</u>:- (Tonelli's Thm).

For a function $f : \mathbb{R}^2 \to [0, \infty)$,

$$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x,y)\, dx\, dy = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x,y)\, dy\, dx.$$

Now,

$$E(X^+) = \int_0 P(X > x)\, dx$$

$$= \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(y)\, dy\, dx.$$

$$\boxed{\cancel{= \int_{y=0}^{\infty} \int_{x=0}^{\infty} f(y)\, dx\, dy}}$$

$$= \int_{x=0}^{\infty} \int_{y=0}^{\infty} f(y)\, 1(x < y)\, dy\, dx.$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{\infty} f(y)\, 1(x < y)\, dx\, dy.$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{y} f(y)\, dx\, dy.$$

$$= \int_{y=0}^{\infty} f(y) \int_{x=0}^{y} 1 \, dx \, dy.$$

$$= \int_{y=0}^{\infty} f(y) \cdot y \cdot dy.$$

$$= \int_{0}^{\infty} y f(y) \, dy$$

Similarly,

$$E(X^-) = -\int_{-\infty}^{0} x f(x) \, dx.$$

$$\therefore \quad E(|X|) = E(X^+) + E(X^-).$$

$$= \int_{0}^{\infty} x f(x) \, dx - \int_{-\infty}^{0} x f(x) \, dx.$$

$$= \int_{-\infty}^{\infty} |x| f(x) \, dx.$$

$\therefore$ thus $E(X)$ is defined iff

$$\int_{-\infty}^{\infty} |x| f(x) \, dx < \infty.$$

and in that case.

$$E(X) = E(X^+) - E(X^-)$$

$$= \int_{0}^{\infty} x f(x) \, dx + \int_{-\infty}^{0} x f(x) \, dx.$$

$$= \int_{-\infty}^{\infty} x f(x) \, dx.$$

Theorem:- If $X$ and $Y$ have finite expectation then,

1. $X+Y$ has finite expectation, and $E(X+Y) = E(X) + E(Y)$

2. For any $\alpha \in \mathbb{R}$ $\alpha X$ has finite expectation and $E(\alpha X) = \alpha E(X)$.

Proof:-

1. $E(|X+Y|) \leq E(|X| + |Y|)$

$= E(|X|) + E(|Y|) < \infty.$

$\because E(X)$ & $E(Y)$ are finite.

$\therefore$ $E(X+Y)$ is finite.

$(X+Y)^+ - (X+Y)^- = X+Y$

$= (X^+ - X^-) + (Y^+ - Y^-)$

$\Rightarrow (X+Y)^+ + X^- + Y^- = X^+ + Y^+ + (X+Y)^-$

$= Z$ (say).

Thus.

$E[(X+Y)^+] + E(X^-) + E(Y^-) = E(Z).$

$= E[(X+Y)^-] + E(X^+) + E(Y^+).$

$\therefore E[(X+Y)^+] - E[(X+Y)^-] = E(X^+) - E(X^-) + E(Y^+) - E(Y^-)$

$\Rightarrow E(X+Y) = E(X) + E(Y).$

2. $(\alpha X)^+ = \begin{cases} \alpha X^+ , & \alpha \geq 0. \\ -\alpha X^- , & \alpha < 0. \end{cases}$

$(\alpha X)^- = \begin{cases} \alpha X^- , & \alpha \geq 0 \\ -\alpha X^+ , & \alpha < 0. \end{cases}$

$E(\alpha X) =$

Show that, if $X \leq Y$, then $E(X) \leq E(Y)$.

Solⁿ: $\because X \leq Y$

then $X^+ \leq Y^+$

$X^- \geq Y^-$

$X^+ + Y^- \leq Y^+ + X^-$

$E(X^+ + Y^-) \leq E(Y^+ + X^-)$

$\Rightarrow E(X^+) + E(Y^-) \leq E(Y^+) + E(X^-)$

$\Rightarrow E(X) \leq E(Y).$

Exc:- If $0 < r < s$ and $E(|X|^s)$ is finite, then. show that $E(|X|^r) < \infty$.

2. If $1 \leq m \leq n$ are integers and $X^n$ has a finite expectation the $X^m$ has a finite expectation. prove it.

Soln:-

1.

$$|X|^r \leq 1 + |X|^s$$

$$E(|X|^r) \leq 1 + E(|X|^s) < \infty.$$

2.

$$|X^m| \leq |X^n|.$$

$$E(|X|^m) \leq E(|X|^n) < \infty$$

(Fatou's Lemma)
Thm:- For $X_n \geq 0$,

$$E\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} E(X_n).$$

Proof:- Define,

$$Y_n = X_n \wedge X_{n+1} \wedge X_{n+2} \cdots, \quad n \geq 1.$$

If $Z = \liminf_{n\to\infty} X_n$, then

$0 \leq Y_n \uparrow Z$ (definition of liminf).

applying MCT

$$E(Z) = \lim_{n\to\infty} E(Y_n).$$

$$\leq \liminf_{n\to\infty} E(X_n) \quad (Y_n \leq X_n)$$

which completes the proof.

EXM:-

Let, X follow the Cauchy distribution

1. $X_n = \frac{1}{n}|X|$.

2. $Y_n = X 1(X \geq n)$

Calculate $E(\liminf_{n\to\infty} X_n)$, $\liminf E(X_n)$ and Ily for $Y_n$.

Soln:-

1. $\liminf_{n\to\infty} X_n = 0$.

$E(\liminf_{n\to\infty} X_n) = 0$

$E(X_n) = \frac{1}{n} E(|X|) = \infty$ [$\because$ distribution is cauchy]

$\therefore \liminf_{n\to\infty} E(X_n) = \infty$.

$\therefore 0 < \infty$

2. $\liminf_{n\to\infty} Y_n = 0$.

$E(\liminf_{n\to\infty} Y_n) = 0$

$\liminf_{n\to\infty} E(Y_n) = \infty$

Thm:- (Dominated convergence theorem (DCT)).

If $X_n$ ~~is~~ converges to $X$ & $|X_n| \le Y$.
where $Y$ is a non-negative random variable such that $E(Y)$ is finite, then $E(X_n) \to E(X)$.

Proof:- $$|X| = \left|\lim_{n\to\infty} X_n\right| = \lim_{n\to\infty} |X_n| \le Y.$$

$\therefore$ $E(|X|) \le E(Y) < \infty$, and hence,

$E(X)$ is finite.

$$X_n + Y \ge 0.$$

$$\begin{aligned} E(X) + E(Y) &= E(X+Y) \\ &= E\left(\liminf_{n\to\infty} (X_n+Y)\right) \\ &\le \liminf_{n\to\infty} E(X_n+Y) \quad [\text{Fatou's Lemma}] \\ &= E(Y) + \liminf_{n\to\infty} E(X_n). \end{aligned}$$

$$\Rightarrow \quad E(X) \le \liminf_{n\to\infty} E(X_n) \quad [\because E(Y) < \infty]$$

$(*)$

$$Y - X_n \ge 0.$$

$$\begin{aligned} E(Y) - E(X) &\le E(Y) + \liminf_{n\to\infty} E(-X_n). \\ &= E(Y) - \limsup_{n\to\infty} E(X_n). \end{aligned}$$

$$\Rightarrow \quad E(X) \ge \limsup_{n\to\infty} E(X_n). \quad (**)$$

$$E(X) \le \liminf_{n\to\infty} E(X_n) \le \limsup_{n\to\infty} E(X_n) \le E(X).$$

The proof follows from $(*)$ & $(**)$.

**Corollary**:- (**Bounded convergence theorem**):

If $X_n \to X$ and $|X_n| \leq c$ where $c$ is a finite constant; then,

$$E(X_n) \to E(X).$$

**Exc**:- Suppose that $X_n \to 0$

show that $\lim_{n\to\infty} P(X_n \leq x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$

**Soln**:- Fix $x > 0$

$$Y_n = 1(X_n \leq x).$$

Now, $Y_n$ converges to 1 $\because \exists\, n \in \mathbb{N}$ s.t. $X_n(\omega) \leq x$ $\because X_n(\omega) \to 0$

$Y_n \to 1$ and $|Y_n| \leq 1$.

BCT implies that,

$$E(Y_n) \to 1$$
$$\parallel$$
$$P(X_n \leq x)$$

**Exc**:- Suppose $X_n \downarrow X \geq 0$. Then $\lim_{n\to\infty} E(X_n)$ exists and equals either $E(X)$ or $\infty$.

**Soln**:- $E(X_1) \geq E(X_2) \geq \cdots$

$\lim_{n\to\infty} E(X_n)$ exists.

Case I :-

$E(X_n) = \infty \quad \forall n.$

Case II :-

$E(X_n) < \infty$ for some $n$.

in case I, trivially $\lim_{n\to\infty} E(X_n) = \infty$.

in case II, $E(X_{n_0}) < \infty$ for some $n_0$.

For every $n \geq n_0$ $\quad |X_n| = X_n \leq X_{n_0}$

Applying DCT to $\{X_n : n \geq n_0\}$ we ~~take~~ get that

$$\lim_{n\to\infty} E(X_n) = E(\lim_{n\to\infty} X_n) = E(X).$$

Exc:- Suppose that $X$ has CDF.

$$F(x) = \begin{cases} \alpha_1 - \exp(-e^x) & , x < 0. \\ \alpha_2 & , 0 \leq x < 1. \\ \alpha_3 - e^{-x} & , x \geq 1. \end{cases}$$

(a) If $X$ is a continuous RV, Calculate $\alpha_1, \alpha_2, \alpha_3$ and a density of $X$ if it exists.

$\because F(x)$ is CDF

$$\lim_{x\to-\infty} F(x) = 0.$$

$$\Rightarrow \lim_{x\to-\infty} \alpha_1 - e^{-e^x} = 0.$$

$$\Rightarrow \alpha_1 - 1 = 0.$$

$$\Rightarrow \alpha_1 = 1.$$

$\lim_{x\to\infty} F(x) = 1 \Rightarrow \lim_{x\to\infty} \alpha_3 - e^{-x} = 1 \Rightarrow \alpha_3 = 1.$

For continuity of $X$.

$$\lim_{x \to 1^-} F(x) = F(1).$$

$$\Rightarrow \quad \alpha_2 = 1 - e^{-1}.$$

ⓑ do the same problem without assuming that $X$ is a continuous RV. Since $F$ is non-decreasing,

$$\alpha_2 \geq \lim_{x \uparrow 0} F(x) = 1 - e^{-1}$$

$$\alpha_2 \leq F(1) = 1 - e^{-1}.$$

Ex:- Suppose a CDF is defined as.

$$F(x) = \begin{cases} \alpha_0 + F_1(x), & x < 0 \\ \alpha_1 + F_2(x), & 0 \leq x < 1. \\ \alpha_2 + F_3(x), & x \geq 1. \end{cases}$$

where $F_1, F_2$ & $F_3$ are diff'ble in their respective domains.

Let $f_i = F_i'$, $i = 1(1)3$.

and $f(x) = f_1(x)\, 1(x<0) + f_2(x)\, 1(0 \leq x < 1) + f_3(x)\, 1(x \geq 1)$. $x \in \mathbb{R}$.

ⓐ If $\int_{-\infty}^{\infty} f(x)\, dx = 1$, then show that there is a unique choice for $\alpha_0, \alpha_1$ & $\alpha_2$ which makes $F$ a CDF.

$\lim_{x\to-\infty} F_1(x) = -\alpha_0.$

$\lim_{x\to\infty} F_3(x) = 1-\alpha_2.$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f_1(x)dx + \int_{0}^{1} f_2(x)dx + \int_{1}^{\infty} f_3(x)dx = 1$$

$$\Rightarrow F_1(0^-) - F_1(-\infty) + F_2(1^-) - F_2(0) + F_3(+\infty) - F_3(1) = 1.$$

$$\Rightarrow -F_1(-\infty) + F_1(0) - F_2(0) = 1 - F_3(+\infty) + F_3(1) - F_2(1^-).$$
$$= k \text{ (say)}.$$

$$\alpha_1 \geq \alpha_0 + F_1(0^-) - F_2(0).$$
$$= -F_1(-\infty) + F_1(0^-) - F_2(0) = k.$$

$$\alpha_1 \leq 1 - F_3(+\infty) + F_3(1) - F_2(1^-) = k$$

$\therefore$ $k \leq \alpha_1 \leq k$

$\Rightarrow \alpha_1 = k$

$\therefore$ For F to be CDF there is only one choice for $\alpha_0, \alpha_1$ & $\alpha_2$.

(b) Show that, in that case f is density of X.

(c) If $\int_{-\infty}^{\infty} f(x)dx < 1$ then show that choice of $\alpha_1$ is not unique, and F is continuous.

HW-1.

Exc:-

6.

(b).

$$(1-X) \leq 2X.$$

$$\Rightarrow X \leq \frac{1}{3}.$$

$$\therefore P\left(X \leq \tfrac{1}{3}\right) = 1 - P\left(X > \tfrac{1}{3}\right).$$

$$= 1 - P\left(X \geq \tfrac{1}{3}\right)$$

$$= 1 - \frac{2}{3}.$$

$$= \frac{1}{3}$$

7. $X \sim$ Cauchy dist$^n$.

$$\therefore f(x) = \frac{1}{\pi(1+x^2)} \quad \forall x \in \mathbb{R}.$$

$$\cancel{G(x)}\ P(X^2 \leq x).$$

$$= P(-\sqrt{x} \leq X \leq \sqrt{x}).$$

$$= 2\int_0^{\sqrt{x}} f(t)\,dt.$$

$$= 2\int_0^{\sqrt{x}} \frac{\cancel{dx}\ dt}{\pi(1+t^2)}.$$

$$= 2\tan^{-1}\sqrt{x}.$$

$g(x^2) =$

density of $X^2$ is.

$$= \frac{2}{1+x} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{\pi}$$

$$= \frac{1}{\pi\sqrt{x}(1+x)}.$$

HW-2 [illegible].

2. CDF of $Z$ is.

$$P(Z \le z)$$
$$= P(\mu + \sigma Y \le z)$$
$$= P\left(Y \le \frac{z-\mu}{\sigma}\right).$$
$$= \frac{1}{\pi} \tan^{-1}\left(\frac{z-\mu}{\sigma}\right) + \frac{1}{2}.$$

Density of $Z$

$$f(z) = \frac{1}{\pi\sigma\left[1+\left(\frac{z-\mu}{\sigma}\right)^2\right]} \quad -\infty < z < \infty.$$

3.

| | Density | CDF | Mean | Var |
|---|---|---|---|---|
| $\lvert U \rvert$ | $1(0 \le x \le 1)$ | $\begin{cases} 0, & x<0 \\ x, & 0 \le x \le 1 \\ 1, & x>1. \end{cases}$ | $\frac{1}{2}$ | $\frac{1}{12}$ |
| $U^2$ | $\frac{1}{2\sqrt{x}} 1(0<x\le 1)$ | $\begin{cases} 0, & x<0 \\ \sqrt{x}, & 0 \le x \le 1 \\ 1, & x>1. \end{cases}$ | $\frac{1}{3}$ | $\frac{4}{45}$. |
| $U^3$ | $\frac{1}{6} x^{-2/3} 1[v \in [-1,1] - \{0\}]$ | $\begin{cases} 0 & x<-1 \\ \frac{\operatorname{sgn}(x)\lvert x \rvert^{1/3}+1}{2} & -1 \le x \le 1. \\ 1 & x>1 \end{cases}$ | $0$ | $\frac{1}{7}$. |

5. $E(|X|) = E\left(\frac{1}{|Z|}\right)$

$$= \int_{-\infty}^{\infty} \frac{1}{|z|} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

$$= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{z} e^{-\frac{z^2}{2}} dz.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{z^2}{2}} z^{-1} dz.$$

$$\geq \sqrt{\frac{2}{\pi}} \int_0^1 \frac{1}{z} e^{-\frac{z^2}{2}} dz.$$

$$\geq \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}} \int_0^1 \frac{dz}{z}.$$

$$= \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}} \ln z \Big|_0^1$$

$$= \infty.$$

$\therefore$ $E(X)$ does not exist.

6.

ⓐ Let $\lim_{x \downarrow 0} f(x) = c\ (>0)$.

$\exists\ \delta > 0$, s.t. $\forall\ x \in (0, \delta)$. $|f(x) - c| < \frac{c}{2}$.

$\Rightarrow f(x) > \frac{c}{2}$.

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8.

ⓐ $E(X^2) = \int_a^b x^2 \left(\frac{\cancel{xxx}}{b-a}\right) dx \quad = \frac{1}{b-a}\int_a^b (x^3 - a x^2) dx$

$= x^4$

$$E(X^2) = \int_a^b \frac{x^2 dx}{b-a} = \frac{1}{b-a} \left. \frac{x^3}{3} \right|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3}$$

$$= \frac{b^2 + ab + a^2}{3}$$

ⓑ $\cancel{E}$: $Var(X) = \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$

$$= \frac{(b-a)^2}{12}$$

9. $|X| \leq c$

$E(|X|) \leq c < \infty$

10. Define $Y = |X - \mu|$

Thm:- Let $X$ has density $f$. For a function $g : \mathbb{R} \to \mathbb{R}$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

if either $g(x) \geq 0 \ \forall x$ or

$$\int_{-\infty}^{\infty} |g(x)| f(x)\,dx < \infty.$$

Proof:- The proof will be given only for a bounded continuous function $g$. This ensures that

$$\int_{-\infty}^{\infty} |g(x)| f(x)\,dx < \infty.$$

Step 1. The claim holds if there exist $a < b$ such that $g$ is a constant on $(-\infty, a]$ and on $[b, \infty)$.

Pf: For $n \geq 1$, define

$$Y_n = g(a)\, 1(X \leq a) + g(b)\, 1(X > b) \cancel{+ \sum_{i=1}^{n} g\left(a + \frac{(b-a)(i-1)}{n}\right)}$$
$$+ \sum_{i=1}^{n} \left[ g\left(a + \frac{(b-a)i}{n}\right) 1\left(a + \frac{(b-a)(i-1)}{n} \leq X < a + \frac{(b-a)i}{n}\right)\right].$$

$$E(Y_n) = \cancel{g(a) +} g(a)\, P(X \leq a) + g(b)\, P(X > b).$$
$$+ \sum_{i=1}^{n} g\left(a + \frac{(b-a)i}{n}\right) \int_{a+\frac{(b-a)(i-1)}{n}}^{a+\frac{(b-a)i}{n}} f(x)\,dx.$$

$$|Y_n - g(X)| \le \sup_{x,y \in [a,b] : |x-y| \le \frac{b-a}{n}} |g(x) - g(y)| =: \varepsilon(n).$$

Since, $g$ is continuous it is uniformly continuous on $[a,b]$ and hence. $\lim_{n \to \infty} \varepsilon(n) = 0$.

$\therefore$ $Y_n \to g(X)$. as $n \to \infty$. Since $G$ is assumed to be bounded, BCT applies and hence

$$\lim_{n\to\infty} E(Y_n) = E[g(X)].$$

$$\int_{-\infty}^{\infty} g(x) f(x)\,dx = g(a) \int_{-\infty}^{a} f(x)\,dx + g(b) \int_{b}^{\infty} f(x)\,dx + \sum_{i=1}^{n} \int_{a + \frac{(b-a)(i-1)}{n}}^{a + \frac{(b-a)i}{n}} g(x) f(x)\,dx$$

Therefore,

$$\left| E(Y_n) - \int_{-\infty}^{\infty} g(x) f(x)\,dx \right| = \left| \sum_{i=1}^{n} \int_{a + \frac{(b-a)(i-1)}{n}}^{a + \frac{(b-a)i}{n}} \left[ g\left(a + (b-a)\tfrac{i}{n}\right) - g(x) \right] f(x)\,dx \right|$$

$$\le \sum_{i=1}^{n} \int_{a + \frac{(b-a)(i-1)}{n}}^{a + \frac{(b-a)i}{n}} \left| g\left(a + \frac{(b-a)i}{n}\right) - g(x) \right| f(x)\,dx.$$

$$\le \varepsilon(n) \int_{a}^{b} f(x)\,dx \le \varepsilon_n \to 0,\ n \to \infty.$$

hence,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

Step 2:- Claim holds for any bdd cont. $g$.

Pf:- For $n \geq 1$, define

$$g_n(x) = \begin{cases} g(-n) & , x \leq -n \\ g(x) & , -n < x < n \\ g(n) & , x \geq n \end{cases}$$

By step 1,

$$E[g_n(X)] = \int_{-\infty}^{\infty} g_n(x) f(x) dx.$$

$$= g(-n) \int_{-\infty}^{-n} f(x)\,dx + \int_{-n}^{n} g(x) f(x)\,dx + g(n) \int_{n}^{\infty} f(x)\,dx. \quad \ldots (*)$$

Since, $g$ is bdd, the RHS of $(*)$ converges to $\int_{-\infty}^{\infty} g(x) f(x)\,dx$ ; as $n \to \infty$.

Note that for every $x \in \mathbb{R}$,

$$g_n(x) = g(x) \quad \forall\ n > |x|.$$

Hence, $g_n(X) \to g(X)$.

Also, $|g| \leq c$, then $|g_n| \leq c$. Once again BCT yields that LHS of $(*) \to E[g(X)]$ as $n \to \infty$

$$\therefore E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

## Definition:-

For any 2 RV $X$ & $Y$ they are independent if $P(X \le x, Y \le y) = P(X \le x) P(Y \le y)$ for every $x, y \in \mathbb{R}$.

**Thm:-** If $X$ and $Y$ are independent, then

$$P(a < X \le b,\ c < Y \le d) = P(a < X \le b) P(c < Y \le d)$$

$$\forall\ a < b \ \& \ c < d.$$

**Pf:-** $P(a < X \le b,\ c < Y \le d)$.

$$= P(X \le b, Y \le d) - P(X \le a, Y \le d) - P(X \le b, Y \le c) + P(X \le a, Y \le c).$$

$$= P(X \le b) P(Y \le d) - P(X \le a) P(Y \le d) - P(X \le b) P(Y \le c) + P(X \le a) P(Y \le c).$$

$$= (P(X \le b) - P(X \le a)) P(Y \le d) - (P(X \le b - P(X \le a)) P(Y \le c).$$

$$= (P(X \le b) - P(X \le a))(P(Y \le d) - P(Y \le c)) = P(a < X \le b) - P(c < Y \le d)$$

**Thm:-** For independent non-negative RV $X$ and $Y$, $E(XY) = E(X) E(Y)$.

**Proof:-** The case that either of $E(X)$ and $E(Y)$ is zero is trivial. Hence, WLOG, assume that $E(X) > 0$, and $E(Y) > 0$.

$$X_n := \sum_{k=0}^{\infty} \frac{k}{2^n} 1\left(\frac{k}{2^n} \le X \le \frac{k+1}{2^n}\right) = 2^{-n} \lfloor 2^n X \rfloor$$

$$Y_n := 2^{-n} \lfloor 2^n Y \rfloor.$$

$$0 \le X_n \uparrow X, \quad 0 \le Y_n \uparrow Y \ldots\ (*)$$

$P(X_n = 2^{-n}i,\ Y_n = 2^{-n}j)$

$= P(2^{-n}i \le X < 2^{-n}(i+1),\ 2^{-n}j \le Y < 2^{-n}(j+1))$.

$= P(2^{-n}i \le X < 2^{-n}(i+1))\, P(2^{-n}j \le Y < 2^{-n}(j+1))$.

$= P(X_n = 2^{-n}i)\, P(Y_n = 2^{-n}j)$.

Thus $X_n$ & $Y_n$ are independent.

$E(X_n Y_n) = E(X_n)\,E(Y_n)$, $n \ge 1$.

An appeal to (*) and MCT gives us.

$0 \le E(X_n) \uparrow E(X)$, $0 \le E(Y_n) \uparrow E(Y)$.

and $0 \le X_n Y_n \uparrow XY$.

hence $0 \le E(X_n Y_n) \uparrow E(XY)$.

$\therefore$ $E(XY) = E(X)\,E(Y)$.

Thm:- If $X$ and $Y$ are independent each having finite expectation then $XY$ has a finite expectation given by

$E(XY) = E(X)\,E(Y)$.

Pf:- Show that $|X|$ and $|Y|$ are independent. when $X$ & $Y$ are independent.

Therefore, by the previous result,

$E(|XY|) = E(|X|)\,E(|Y|) < \infty$.

$\therefore$ $E(XY)$ is finite.

[Hint: $X = (X^+ - X^-)$]

$(XY)^+ = X^+Y^+ + X^-Y^-$

$(XY)^- = X^+Y^- + X^-Y^+$

$E((XY)^+) = E(X^+)E(Y^+) + E(X^-)E(Y^-)$.

$E((XY)^-) = E(X^+)E(Y^-) + E(X^-)E(Y^+)$.

$\therefore E((XY)^+) - E((XY)^-) = E(X^+)(E(Y^+) - E(Y^-))$
$\qquad - E(X^-)(E(Y^+) - E(Y^-))$.
$\qquad = (E(X^+) - E(X^-))(E(Y^+) - E(Y^-))$

$\Rightarrow E(XY) = E(X)E(Y)$.

Thm:- (Cauchy-Schwarz).
For any two RV $X$ and $Y$,

$$E(|XY|) \leq \sqrt{E(X^2)E(Y^2)}$$

furthermore, if the LHS is finite, then

$$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$$

Proof:- For the first claim, there is nothing to prove if $E(X^2), E(Y^2) = \infty$. So, WLOG assume that $E(X^2)E(Y^2) < \infty$. Then, either both $E(X^2) < \infty$ and $E(Y^2) < \infty$, or at least one of them is zero. In the latter case, both sides of the desired inequality are zero. So, we furthermore assume that $E(X^2) < \infty$ and $E(Y^2) < \infty$. observe that $(a+b)^2 \leq 2(a^2+b^2)$.
For any $\lambda \in \mathbb{R}$,

$$E[(|X| + \lambda|Y|)^2] \leq 2E(X^2 + \lambda^2 Y^2) < \infty.$$

$$E(X^2) + 2\lambda E(|XY|) + \lambda^2 E(Y^2) \geq 0 \quad \forall \lambda \in \mathbb{R}.$$

Analysis on discriminant yields that

$$E(|XY|) \leq \sqrt{E(X^2)E(Y^2)}.$$

If $E(|XY|)$ is finite, then

$$|E(XY)| \leq E(|XY|) \leq \sqrt{E(X^2)E(Y^2)}.$$

Defn:- If $X$ and $Y$ have finite means $\mu_X$ & $\mu_Y$ respectivly then their Covariance is.

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)].$$

whenever the expectation on the RHS is defined.

Exc:- If $X$ and $Y$ have a finite variance, then $E(|XY|) < \infty$ and

$$Cov(X,Y) = E(XY) - E(X)E(Y).$$

Soln:

$$|XY| = |(X-\mu_X) + \mu_X||(Y-\mu_Y) + \mu_Y|.$$

$$\leq (|X-\mu_X| + |\mu_X|)(|Y-\mu_Y| + |\mu_Y|).$$

$$= |(X-\mu_X)(Y-\mu_Y)| + |\mu_Y||X-\mu_X| + |\mu_X||Y-\mu_Y| + |\mu_X\mu_Y|.$$

$\therefore E(|XY|)$ is finite.

Exc: If $X$ and $Y$ have finite variances, then,

1. $Cov(X,Y)$ is finite,

and

2. $X+Y$ has a finite variance and $Var(X+Y) = Var(X) + Var(Y) + 2\,cov(X,Y)$.

Furthermore, if X and Y are indep, then
$Var(X+Y) = Var(X) + Var(Y)$.

Definition:-
Let ~~1~~, X an

1. by Cauchy Schwarz inequality,

$$E(|(X-\mu_x)(Y-\mu_Y)|) \le \sqrt{E((X-\mu_x)^2)\,E((Y-\mu_x)^2)} < \infty$$

[$\because$ Finite Var]

$$\therefore |E((X-\mu_x)(Y-\mu_Y))| \le E(|(X-\mu_x)(Y-\mu_Y)|) < \infty.$$

2. $Var(X+Y)$.

$$= E((X+Y-\mu_x-\mu_Y)^2)$$

$$= E(((X-\mu_x)+(Y-\mu_Y))^2)$$

$$= E((X-\mu_x)^2) + E((Y-\mu_Y)^2) + E(2(X-\mu_x)(Y-\mu_x))$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y).$$

if X & Y independent.

then, $E(XY) = E(X)E(Y)$

$\Rightarrow Cov(XY) = 0.$

$\therefore Var(X+Y) = Var(X) + Var(Y).$

**Definition:-**

Let X and Y be random variable with finite and positive variances. Then correlation between X and Y is defined as,

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}.$$

Exc:- Show that, correlation always lies in $[-1, 1]$. furthermore if $\text{corr}(X,Y) = \pm 1$, then

$$\frac{X-E(X)}{\sqrt{\text{Var}(X)}} = \text{Cor}(X,Y)\,\frac{Y-E(Y)}{\sqrt{\text{Var}(Y)}}.$$

$$E\left(\frac{X-E(X)}{\sqrt{\text{Var}(X)}} - \text{cor}(X,Y)\,\frac{Y-E(Y)}{\sqrt{\text{Var}(Y)}}\right)$$

$$= E\left(\frac{X-E(X)}{\sqrt{\text{Var}(X)}}\right) - E\left(\text{Cor}(X,Y)\,\frac{Y-E(Y)}{\sqrt{\text{Var}(Y)}}\right)$$

$$= \frac{E(X)-E(X)}{\sqrt{\text{Var}(X)}} - \text{cor}(X,Y)\,\frac{E(Y)-E(Y)}{\sqrt{\text{Var}(Y)}} = 0.$$

$$\text{Var}\left(\frac{X-E(X)}{\sqrt{\text{Var}(X)}} - \text{Cor}(X,Y)\,\frac{Y-E(Y)}{\sqrt{\text{Var}(Y)}}\right)$$

$$= \frac{\text{Var}(X-E(X))}{\text{Var}(X)} + (\text{Cor}(X,Y))^2\,\frac{\text{Var}(Y-E(Y))}{\text{Var}(Y)} - 2\,\text{cor}(X,Y)\,\frac{\text{cov}(X-E(X),\,Y-E(Y))}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

$$= 1 - \text{cor}^2(X,Y)$$

$$= 0 \quad [\because \text{cor}^2(X,Y) = \pm 1.]$$

## Bivariate Random Variable:-

<u>Def$^n$:-</u> For Random variable $X$ and $Y$, their joint CDF is a function $F : \mathbb{R}^2 \to [0,1]$ defined by,

$$F(x,y) = P(X \le x, Y \le y) \quad , \forall (x,y) \in \mathbb{R}^2 .$$

<u>Caution:-</u> Joint CDF of $X$ and $Y$ is not same as joint CDF of $Y$ and $X$.

<u>Thm:-</u> If $F$ is the joint CDF of $X$ & $Y$ then

1. for all $a < b$ and $c < d$,

$$F(b,d) - F(a,b) - F(b,c) + F(a,c) \ge 0 .$$

2. For all fixed $x \in \mathbb{R}$.

$$\lim_{y \to -\infty} F(x,y) = 0 .$$

and for fixed $y \in \mathbb{R}$

$$\lim_{x \to -\infty} F(x,y) = 0 .$$

3. $$\lim_{x \to \infty, y \to \infty} F(x,y) = 1 .$$

and 4. for every $x, y \in \mathbb{R}$

$$\lim_{u \downarrow x, v \downarrow y} F(u,v) = F(x,y) .$$

<u>Pf:-</u>

1. The claim follows from the observation

2. We know that,

$$\lim_{y \to -\infty} P(Y \le y) = 0.$$

Since for any $x \in \mathbb{R}$,

$0 \le F(x,y) \le P(Y \le y)$, it follows that

$$\lim_{y \to -\infty} F(x,y) = 0.$$

~~Claim~~

3. All we need to show is whenever $x_n \uparrow \infty$ and $y_n \uparrow \infty$,

$$F(x_n, y_n) \to 1.$$

Fix $x_n$ and $y_n$ as above,

$$[X \le x_n, Y \le y_n] \uparrow \Omega.$$

$$\therefore P(X \le x_n, Y \le y_n) \uparrow P(\Omega)$$

$$\therefore F(x_n, y_n) \uparrow 1.$$

4. Suffices to show, if $u_n \downarrow x$ & $v_n \downarrow y$. then $F(u_n, v_n) \to F(x,y)$.

Fix $u_n$ & $v_n$ as above. and note that.

$$[X \le u_n, Y \le v_n] \downarrow [X \le x, Y \le y].$$

$$\therefore P(X \le u_n, Y \le v_n) \downarrow P(X \le x, Y \le y).$$

$$F(u_n, v_n) \downarrow F(x,y).$$

FACT:- If a function $F: \mathbb{R}^2 \to [0,1]$ satisfies 1-4 of the previous theorem then $\exists$ R.V. $X$ & $Y$ whose joint CDF is $F$.

Defn:- If $F$ is the joint CDF of $X$ & $Y$ and if there exist a function $f: \mathbb{R}^2 \to [0,\infty)$ such that

$$\int_{x=-\infty}^{a} \int_{y=-\infty}^{b} f(x,y)\,dy\,dx = F(a,b) \text{ for every } a,b \in \mathbb{R}$$

then $f$ is the joint density (PDF) of $X$ and $Y$.

Theorem:-

Let $F$ and $f$ be the joint CDF and PDF of $(X,Y)$, respectively.

1. The marginal CDF of $X$ is

$$F_X(x) = \lim_{y\to\infty} F(x,y), \quad \forall x \in \mathbb{R}.$$

and the marginal CDF of $Y$ is

$$F_Y(y) = \lim_{x\to\infty} F(x,y), \quad \forall y \in \mathbb{R}.$$

2. The (marginal) density of $Y$ is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx, \quad \forall y \in \mathbb{R}.$$

and (marginal) density of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} F(x,y)\,dy, \quad \forall x \in \mathbb{R}.$$

3. For $a<b$ & $c<d$,

$$P(a \le X \le b, c < Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Proof:-

1. Fix $x \in \mathbb{R}$ and $y_n \uparrow \infty$.

Since, $[X \leq x, Y \leq y_n] \uparrow [X \leq x]$;

$\therefore P(X \leq x, Y \leq y_n) \uparrow P(X \leq x)$.

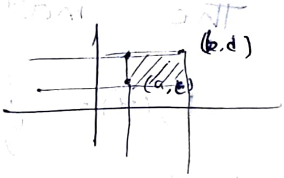
$\therefore \lim_{y \to \infty} F(x,y) = F_X(x)$.

2. To show, for $\forall\, t \in \mathbb{R}$.

$$P(Y \leq t) = \int_{-\infty}^{t} f_Y(y)\,dy.$$

$$\text{L.H.S.} = \int_{y=-\infty}^{t} \int_{x=-\infty}^{\infty} f(x,y)\,dx\,dy = P(Y \leq t, X \leq \infty)$$
$$= P(Y \leq t).$$

3. $P(a < X \leq b,\ c < Y \leq d)$.




$$= F(b,d) - F(a,d) - F(b,c) + F(a,c).$$
$$= [F(b,d) - F(a,d)] - [F(b,c) - F(a,c)].$$
$$= \int_{a}^{b} \int_{-\infty}^{d} f(x,y)\,dy\,dx - \int_{a}^{b} \int_{-\infty}^{c} f(x,y)\,dy\,dx.$$
$$= \int_{a}^{b} \left[ \int_{-\infty}^{d} f(x,y)\,dy - \int_{-\infty}^{c} f(x,y)\,dy \right] dx$$
$$= \int_{a}^{b} \int_{c}^{d} f(x,y)\,dy\,dx.$$

FACT:- Let, f be the joint density of X and Y

1. For $A \subseteq \mathbb{R}^2$,

$$P((X,Y) \in A) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) 1((x,y) \in A)\, dy\, dx.$$

2. For $g : \mathbb{R}^2 \to \mathbb{R}$.

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dy\, dx$$

if $g \geq 0$ or $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x,y)| f(x,y)\, dy\, dx < \infty$.

Thm:-
If X and Y are independent, and have densities f and g, respectively, then (X,Y) has joint density

$$h(x,y) = f(x) g(x), \quad x,y \in \mathbb{R}.$$

Pf:- Need to show that for $a,b \in \mathbb{R}$,

$$P(X \leq a, Y \leq b) = \int_{-\infty}^{a}\int_{-\infty}^{b} h(x,y)\, dy\, dx.$$

LHS $= P(X \leq a) P(Y \leq b)$ [independence].

$$= \left(\int_{-\infty}^{a} f(x)\, dx\right)\left(\int_{-\infty}^{b} g(y)\, dy\right)$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{b} f(x) g(y)\, dy\, dx.$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{b} h(x,y)\, dy\, dx.$$

Exc:- Suppose that ~~X and Y follow~~ $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ independent of each other, Calculate $P(X<Y)$.

Solⁿ:- The joint density of X and Y is.

$$f(x,y) = \begin{cases} \lambda e^{-\lambda x} \mu e^{-\mu y} & ; x,y>0, \\ 0 & , \text{else}. \end{cases}$$

$$\therefore P(X<Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu\lambda\, e^{-(\lambda x+\mu y)}\, 1.(x<y)\, dy\, dx$$

$$= \int_{0}^{\infty}\int_{x}^{\infty} \mu\lambda\, e^{-(\lambda x+\mu y)}\, dy\, dx$$

$$= \int_{0}^{\infty} \mu\lambda\, e^{-\lambda x} \int_{x}^{\infty} e^{-\mu y}\, dy\, dx$$

$$= \int_{0}^{\infty} \left(\cancel{\mu}\lambda\, e^{-\lambda x} \cdot \frac{e^{-\mu x}}{\cancel{\mu}}\right) dx.$$

$$= \int_{0}^{\infty} \lambda e^{-(\lambda+\mu)x}\, dx$$

$$= \frac{\lambda}{\lambda+\mu}.$$

**Defⁿ :-** R.V. $X_1, \ldots, X_n$ are independent if

$$P(X_1 \le x_1, \ldots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i), \ \forall\ x_1, \ldots, x_n \in \mathbb{R}$$

**Exc :-** Let, $X_1, \ldots, X_n$ be iid. from standard uniform. Define,

$$U = X_1 \wedge \ldots \wedge X_n.$$

$$V = X_1 \vee \ldots \vee X_n.$$

Find the joint density of $(U, V)$.

$\cancel{P(U \le u, V \le v)}$.

$\Rightarrow \{(u,v) : 0 < u < v < 1\}$

Fix $0 < u < v < 1$.

$$P(U \le u, V \le v) = P(V \le v) - P(u < U, V \le v).$$

$$= P(X_1 \le v, \ldots, X_n \le v) - P(u < X_1 \le v, \ldots, u < X_n \le v)$$

$$= \prod_{i=1}^{n} P(X_i \le v) - \prod_{i=1}^{n} P(u < X_i \le v).$$

$$= v^n - (v-u)^n.$$

$$\frac{\delta^2 P(U \le u, V \le v)}{\delta u\, \delta v} = n(n-1)(v-u)^{n-2}.$$

<u>Claim</u> :- the joint density of $U, V$ is

$$f(u,v) = n(n-1)(v-u)^{n-2}\, 1(0 < u < v < 1), \quad u, v \in \mathbb{R}$$

$$\int_{-\infty}^{u}\int_{-\infty}^{v} n(n-1)(y-x)^{n-2}\,dy\,dx$$

$$= \int_{0}^{u} n(n-1)\left.\frac{(y-x)^{n-1}}{n-1}\right|_{0}^{u} dx$$

$$= \int_{0}^{u} n(v-x)^{n-1} - (-x)^{n-1}\,dx$$

$$= \frac{n(v-x)^{n}}{n}\cdot(-1) - \frac{(-x)^{n}}{n}\cdot(-1)\Big|_{0}^{u}$$

$$= u^{n}$$

$$P(U \le u, V \le v) = \int_{-\infty}^{u}\int_{-\infty}^{v} f(x,y)\,dy\,dx$$

$$= \int_{x=0}^{u}\int_{y=x}^{v} n(n-1)(y-x)^{n-2}\,dy\,dx$$

$$= \int_{x=0}^{u} \left.\frac{n(n-1)(y-x)^{n-1}}{(n-1)}\right|_{x}^{v} dx$$

$$= \int_{x=0}^{u} n(v-x)^{n-1}\,dx$$

$$= -(v-x)^{n}\Big|_{0}^{u} = v^{n} - (v-u)^{n}.$$

Exc:- Let $X \sim$ Standard Normal.

$Y = X$.

Find the joint CDF of $X$ and $Y$.

~~P(X ≤ y, X ≤~~

$F(x,y)$
$= P(X \le x, Y \le y)$ ~~= P(X ≤ x, X ≤ y)~~.

$= P(X \le x \wedge y)$

$= \Phi(x \wedge y)$.

~~Check that f~~

Let, $f(x,y)$

??

Exc:- Let $X \sim$ Bernouli $(\frac{1}{2})$ Let, $Y = 1-X$.

Denote, the joint CDF of $X$ and $Y$ by $F$.

1. Calculate $F(1,1)$

$F(1,1) = P(X \le 1, Y \le 1)$

$= P(X = 1, Y = 0) + P(X = 1, Y = 0)$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$.

2. Calculate, $\lim_{x \uparrow 1\, y \uparrow 1} F(x,y) = 0$

3. $P(X=1, Y=1) = 0.$

Ex

**Exc:-** Let, $X$ and $Y$ be i.i.d. from $N(0,1)$. Find the PDF of $X+Y$.

**Sol^n:** For $k \in \mathbb{R}$

$$P(X+Y \leq k)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{k-x} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \, dy \, dx.$$

$$= \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left( \int_{y=-\infty}^{k-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \right) dx.$$

$$= \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, \Phi(k-x) \, dx$$

$Z := X+Y.$

for $z \in \mathbb{R}$.

$$P(X+Y \leq z) := \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} \phi(x) \, \phi(y) \, dy \, dx.$$

Step I: Fix $x$ and substitute for $y$ so that the inner integral is till $z$.

Step II: Use Tonelli.

$$= \int_{x=-\infty}^{\infty} \phi(x) \int_{y=-\infty}^{z-x} \phi(y)\, dy\, dx$$

put $u = x+y$ fix, $x$.

$du = dy$.

$$= \int_{x=-\infty}^{\infty} \phi(x) \int_{u=-\infty}^{z} \phi(u-x)\, du$$

(Tonelli) $$\int_{-\infty}^{z}\int_{-\infty}^{\infty} \phi(x)\,\phi(u-x)\, dx\, du.$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-x)^2}{2}}\, dx\, du.$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\left(\frac{2x^2+u^2-2ux}{2}\right)}\, dx\, du.$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}\left(\left(\sqrt{2}x-\frac{u}{\sqrt{2}}\right)^2+\frac{u^2}{2}\right)}\, dx\, du.$$

$$= \int_{-\infty}^{z} \frac{1}{2\pi} e^{-\frac{u^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{u}{\sqrt{2}}\right)^2}\, dx\, du$$

$$= \int_{-\infty}^{z} \frac{1}{2\pi} e^{-\frac{u^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\frac{u}{2}}{\frac{1}{\sqrt{2}}}\right)^2}\, dx\, du$$

$$\left[\text{put} \quad \sqrt{2}x - \frac{u}{\sqrt{2}} = v \,;\quad \Rightarrow \sqrt{2}\,dx = dv.\right]$$

$$= \int_{-\infty}^{z} \frac{1}{2\pi} e^{-\frac{u^2}{4}} \int_{-\infty}^{\infty} e^{-v^2/2} \frac{dv}{\sqrt{2}} \cdot du$$

$$= \int_{-\infty}^{z} \frac{1}{2\pi} e^{-\frac{u^2}{4}} \cdot \sqrt{\pi}\, du.$$

$$= \int_{-\infty}^{z} \frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}} du$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\cdot\frac{(u-0)^2}{2}} du.$$

$$= \Phi\left(\frac{z}{\sqrt{2}}\right).$$

$\therefore Z \sim N(0,2)$

Exc:- If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ independently, then show that $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

Exc:- Suppose that X and Y have joint density f. Calculate density of X+Y.

Let, $Z = X+Y$.

For $z \in \mathbb{R}$,

$$P(Z \le z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f(x,y)\, dy\, dx.$$

put $u = x+y$ fixing $x$.

$du = dy$.

$$\therefore P(Z \le z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z} f(x,u-x)\, du\, dx.$$

$$= \int_{u=-\infty}^{z} \underbrace{\int_{x=-\infty}^{\infty} f(x,u-x)\, dx}_{g(u)}\, du.$$

$$= \int_{u=-\infty}^{z} g(u)\, du.$$

$\therefore$ density of Z is

$$g(u) = \int_{x=-\infty}^{\infty} f(x,u-x)\, dx \quad \forall u \in \mathbb{R}$$

Check that,

$$g(u) = \int_{-\infty}^{\infty} g(u-y,y)\, dy.$$

Exc (Convolution Formula):- If $X$ and $Y$ are independent random variable having respective densities $f_X$ and $f_Y$, then the density of $X+Y$ is

$$g(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

Exc:- Let, $X_1, \ldots, X_n$ be i.i.d. with CDF and PDF with $F$ and $f$; respectively. Find the joint density of

$$U = \min(X_1, \ldots, X_n)$$
$$V = \max(X_1, \ldots, X_n)$$

The joint CDF of $U$ and $V$ is.

$$P(U \le u, V \le v) = \begin{cases} F(v)^n - [F(v) - F(u)]^n, & u \le v \\ F(v)^n, & u > v \end{cases}$$

$$\frac{\delta^2}{\delta u\, \delta v} P(U \le u, V \le v)$$

$$= -n [F(v) - F(u)]^{n-1} \cdot -f(u)$$

$$= n f(u) [F(v) - F(u)]^{n-1}$$

$$= n f(u)(n-1)[F(v) - F(u)]^{n-2} f(v).$$

Claim: Then the joint density of $(U,V)$. is.

$$g(u,v) = n(n-1) f(u) f(v) [F(v) - F(u)]^{n-2} \, 1(u < v),$$
$u, v \in \mathbb{R}$

~~To pro~~

Show that $\displaystyle\int_{s=-\infty}^{u}\int_{t=-\infty}^{v} g(s,t)\,dt\,ds = P(U\le u, V\le v).$

$\forall\ u,v\in\mathbb{R}.$

Case I $\quad u\le v.$

$$\int_{s=-\infty}^{u}\int_{t=s}^{v} n(n-1)\,f(s)\,f(t)\left[F(t)-F(s)\right]^{n-2} dt\,ds.$$

$$= \int_{s=-\infty}^{u} n(n-1)\,f(s)\int_{t=s}^{v} f(t)\left[F(t)-F(s)\right]^{n-2} dt\,ds.$$

$$= \int_{s=-\infty}^{u} n(n-1)\,f(s)\ \frac{\left[F(t)-F(s)\right]^{n-1}}{n-1}\Bigg|_{t=s}^{v} ds.$$

$$= \int_{s=-\infty}^{u} n\,f(s)\ \left(F(v)-F(s)\right)^{n-1} ds.$$

$$= \int_{s=-\infty}^{u} n\left[F(v)-F(s)\right]^{n-1} f(s)\,ds.$$

$$= -\left[F(v)-F(s)\right]^{n}\Big|_{s=-\infty}^{u}.$$

$$= -\left[F(v)-F(u)\right]^{n} + \left[F(v)-F(-\infty)\right]^{n}.$$

$$= F(v)^{n} - \left[F(v)-F(u)\right]^{n}.$$

$$= P(U\le u,\ V\le v).$$

Case II $\quad u > v.$

$$\int_{-\infty}^{u}\int_{-\infty}^{v} g(s,t)\,dt\,ds.$$

$$= \int_{-\infty}^{v}\int_{-\infty}^{v} g(s,t)\,dt\,ds.$$

$$= \int_{-\infty}^{u'}\int_{-\infty}^{v} g(s,t)\,dt\,ds \qquad [u' = v].$$

$$= P(U \le u', V \le v).$$

$$= P(V \le v) = F(v)^n.$$

Exc: Let $X_1, X_2, \dots, X_n$ be i.i.d. from $\text{Exp}(\lambda)$. Calculate the density of $X_1 + \dots + X_n$.

Soln:- ~~For~~ ~~n=2~~,

Let, $f_n$ be the density of $X_1 + X_2 + \dots + X_n$; $n \ge 1$. for $n = 2$, and $x > 0$,

$$f_2(x) = \int_{-\infty}^{\infty} f_1(u)\, f_1(x-u)\,du.$$

$$= \int_{0}^{x} \lambda e^{-\lambda u} \cdot \lambda e^{-\lambda(x-u)}\,du$$

$$= \int_{0}^{x} \lambda^2 e^{-\lambda x}\,du$$

$$= \lambda^2 e^{-\lambda x} \int_{0}^{x} du = \lambda^2 x e^{-\lambda x}.$$

Thus,

$$f_2(x) = \lambda^2 x e^{-\lambda x} 1(x>0), \quad x \in \mathbb{R}.$$

Claim: $f_n(x) = \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} 1(x>0), \quad x \in \mathbb{R}.$

We have already verified for $n=2$.

Induction hypothesis: The claim holds for $k$.

using convolution formula one again,

$$f_{k+1}(x) = \int_{-\infty}^{\infty} f_k(u) f_1(x-u)\, du$$

$$= \int_0^x \frac{\lambda^{n+1} u^{n-1} e^{-\lambda x}}{(n-1)!} du.$$

$$= \frac{\lambda^{n+1} e^{-\lambda x}}{(n-1)!} \int_0^x u^{n-1} du$$

$$= \frac{\lambda^{n+1} x^n e^{-\lambda x}}{n!}$$

$$= \frac{1}{n!} \cdot \lambda^{n+1} x^n e^{-\lambda x}.$$

$$\therefore f_{k+1}(x) = \frac{1}{n!} \lambda^{n+1} x^n e^{-\lambda x} 1(x>0) \quad x \in \mathbb{R}.$$

Defn:- The distribution whose density is.

$$f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x} 1(x>0), \quad x \in \mathbb{R}$$

for some $\lambda, \alpha > 0$ is called $\text{Gamma}(\alpha, \lambda)$.

Exc: Check that the above is a density and find its mean and variance.

$$E(X) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma\alpha} \cdot x^\alpha \cdot e^{-\lambda x}\, dx.$$

$$= \frac{1}{\lambda\Gamma\alpha} \int_0^\infty (\lambda x)^\alpha e^{-\lambda x} \cdot \lambda dx.$$

$$= \frac{1}{\lambda\Gamma\alpha} \cdot \Gamma\alpha+1$$

$$= \frac{\alpha\Gamma\alpha}{\lambda\Gamma\alpha} = \frac{\alpha}{\lambda}.$$

$$E(X^2) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha+1} e^{-\lambda x}\, dx.$$

$$= \frac{1}{\lambda^2\Gamma\alpha} \int_0^\infty (\lambda x)^{\alpha+1} e^{-\lambda x} \cdot \lambda dx.$$

$$= \frac{1}{\lambda^2\Gamma\alpha} \Gamma\alpha+2.$$

$$= \frac{(\alpha+1)\alpha\cancel{\Gamma\alpha}}{\lambda^2\cancel{\Gamma\alpha}}$$

$$\therefore Var(X) = \frac{\alpha^2+\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\alpha}{\lambda^2}$$

Exc. Suppose that the life time in hours of light bulbs of a certain brand follow Exp($\lambda$). In a room with a single bulb holder, the bulb is replaced with a new one the moment it fuses. Find the distribution of the number of bulbs that will be fused in $t$ hours.

Soln:- Let $N$ be the number of bulbs that will fuse by time $t$.

$N$ takes values $0, 1, 2, \ldots$

Let, $X_1, X_2, X_3, \ldots$ denote life times of the respective bulbs.

$$P(N=0) = P(X_1 > t) = e^{-\lambda t}$$

$$P(N \leq 1) = P(X_1 + X_2 > t) = \int_t^\infty \lambda^2 x e^{-\lambda x} dx$$

$$= \int_{\frac{t}{\lambda}}^{\infty} t e^{-t} dt$$

$$= e^{-\lambda t}(\lambda t + 1).$$

$$\therefore P(N=1) = P(N \leq 1) - P(N=0)$$

$$= e^{-\lambda t} \lambda t.$$

Claim:- $N \sim$ Poisson $(\lambda t)$.

Induction hypothesis:- $P(N=k) = e^{-\lambda t} \dfrac{(\lambda t)^k}{k!}$

$k = 0, \ldots, n.$

$$P(N \le n+1) = P(X_1 + \cdots + X_{n+2} > t).$$

$$= \int_t^\infty \frac{\lambda^{n+2}}{(n+1)!} x^{n+1} e^{-\lambda x}\, dx.$$

$$= \frac{\cancel{\lambda^{n+2}}}{(n+1)!} \int_t^\infty (\lambda x)^{n+1} e^{-\lambda x} \lambda\, dx$$

$$= \frac{1}{(n+1)!} \int_{\lambda t}^\infty y^{n+1} e^{-y}\, dy$$

$$= \frac{1}{(n+1)!} \left[ -y^{n+1} e^{-y} \Big|_{\lambda t}^{\infty} + \int_{\lambda t}^\infty (n+1) y^n e^{-y}\, dy \right]$$

$$= \frac{1}{(n+1)!} \left[ (\lambda t)^{n+1} e^{-\lambda t} + (n+1) \int_{\lambda t}^\infty y^n e^{-y}\, dy \right]$$

$$= \frac{1}{(n+1)!} (\lambda t)^{n+1} e^{-\lambda t} + \frac{1}{n!} \int_t^\infty \lambda^{n+1} x^n e^{-\lambda x}\, dx$$

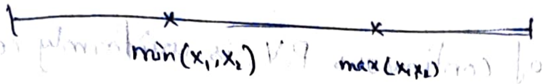
$$= \frac{1}{(n+1)!} (\lambda t)^{n+1} e^{-\lambda t} + P(N \le n).$$

$$\Rightarrow P(N = n+1) = e^{-\lambda t} \frac{(\lambda t)^{n+1}}{(n+1)!}$$

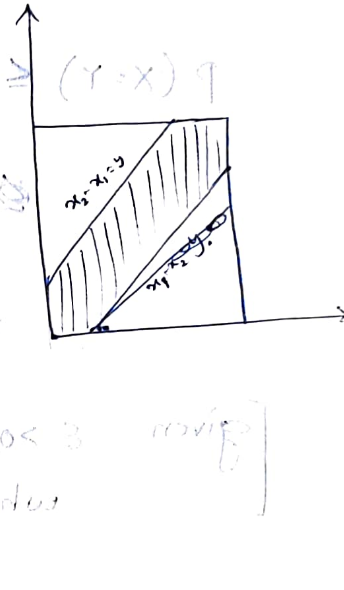
Exc: A stick of lenth one meter is broken at two points chosen uniformly at random. Find the distribution of the middle part.

Let, $X_1$ & $X_2$ denote the two break points.

$Y$ = Length of the middle part.




$$\therefore Y = X_1 \vee X_2 - X_1 \wedge X_2$$
$$= |X_1 - X_2|.$$




$$P(Y \le y) = P(|X_1 - X_2| \le y)$$
$$= 1 - (1-y)^2$$

$P(|X_1 - X_2| \le y)$

$$= 1 - P(|X_1 - X_2| > y).$$
$$= [1 - (P(X_2 > X_1 + y) + P(X_1 > X_2 + y))]$$

$P(X_2 > X_1 + y)$.

$$= \int_{x_1=0}^{1-y} \int_{x_2=x_1+y}^{1} 1 \, dx_2 \, dx_1 = \int_{x_1=0}^{1-y} (1 - x_1 - y) \, dx_1$$

$$= (1-y)x_1 - \frac{x_1^2}{2} \Big|_0^{1-y}.$$

$$= (1-y)^2 - \frac{(1-y)^2}{2}$$

$$= (1-y)^2/2$$

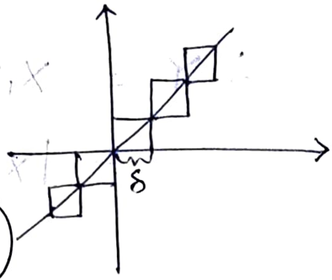
Exc:- Suppose that X and Y are independent RV. each having a density. Calculate $P(X=Y)$.

Ans: 0.

Exc:- Suppose that X and Y are independent RV. If one of them is continuous, show that $P(X=Y)=0$.

FACT: (Show that,) the CDF of continuous RV is uniformly continuous.

Let, X has CDF, F



$$P(X=Y) \le P\left(\bigcup_{k\in\mathbb{Z}} [k\delta < X, Y < (k+1)\delta]\right)$$

$$= \sum_{k\in\mathbb{Z}} P(k\delta < X, Y < (k+1)\delta).$$

$$= \sum_{k\in\mathbb{Z}} [F((k+1)\delta) - F(k\delta)]\, P(k\delta < Y \le (k+1)\delta).$$

[given $\varepsilon > 0$, $\exists$ there exist $\delta > 0$ s.t. $|F(x)-F(y)| \le \varepsilon$ whenever $|x-y| \le \delta$.]

$$\le \varepsilon \sum_{k\in\mathbb{Z}} P(k\delta < Y \le (k+1)\delta) < \varepsilon.$$

Exc:- Suppose that $X_1, \dots, X_n$ are independent and $X_i \sim \text{Exp}(\lambda_i)$ Find the distribution of.

$Y = X_1 \wedge X_2 \wedge \dots \wedge X_n$.

$P(Y \le y)$

$= 1 - P(Y \ge y) = 1 - P(\min_{i \in 1(1)n} \{X_i\} \ge y)$.

$= 1 - \prod_{i=1}^{n} P(X_i \ge y)$.

$= 1 - \prod_{i=1}^{n} (1 - P(X_i \le y))$

$= 1 - e^{-\sum_{i=1}^{n} \lambda_i}$.

Exc:- It is known that people take Exp ($\lambda$) minutes to complete a call in a phone booth. A person arriving at the booth finds both the phones busy. He is told that the phone on the left is busy for the last two minutes, while one on the right is busy for 5 minutes. Given this information, and ~~assu~~ assuming that the times taken by different people are indepl, find out which phone is more likely to be free sooner.

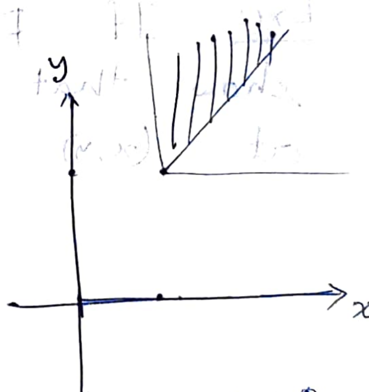
Solⁿ:- Let, X and Y be the times taken to complete the calls at the left and the right phones, respectively.

Conditional probability that the phone on the left will be free sooner ~~equals~~

$$= P\left[(X-2) < (Y-5) \mid (X>2, Y>5)\right]$$

$$= P\left[(X-Y < -3) \mid (X>2, Y>5)\right]$$

$$= \frac{P(X>2, Y>5, X+3<Y)}{P(X>2, Y>5)}$$




$$= \frac{\int_2^\infty \int_{x+3}^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda y}\, dy\, dx}{P(X>2)\,P(Y>5)}$$

$$= \frac{\int_2^\infty \lambda^2 e^{-\lambda x} \int_{x+3}^\infty e^{-\lambda y}\, dy\, dx}{e^{-2\lambda} \cdot e^{-5\lambda}}$$

$$-\frac{\lambda e^{-\lambda y}}{\lambda}\Big|_{x+3}^{\infty} \quad 0 + e^{-\lambda(x+3)}$$

$$= \frac{\int_2^\infty \lambda e^{-\lambda x} e^{-\lambda(x+3)}\, dx}{e^{-7\lambda}} = \frac{\frac{1}{2} e^{-3\lambda} \int_2^\infty 2\lambda e^{-2\lambda x}\, dx}{e^{-7\lambda}}$$

$$= \frac{1}{2}$$

Exc:- For i.i.d. RV $X_1, X_2, \ldots$ from Pareto

Calculate $\lim_{n\to\infty} P\left(n^{-1/\alpha}(X_1 \vee \cdots \vee X_n) \le x\right)$.

For $x \le 0$ the limit is clearly 0,

For $x > 0$,

$$P(X_1 \vee \cdots \vee X_n \le n^{1/\alpha} x)$$

$$= P(X_1 \le n^{1/\alpha} x)^n$$

$$= \left[1 - (n^{1/\alpha} x)^{-\alpha}\right]^n \quad \text{for large } n.$$

$$= \left[1 - \frac{x^{-\alpha}}{n}\right]^n$$

$$= e^{-x^{-\alpha}}$$

Def$^n$:- A R.V. $Z$ follows Freche't $(\alpha)$ for $\alpha$

if its CDF is

$$F(x) = \begin{cases} \exp(-x^{-\alpha}) & , x > 0. \\ 0 & , x \le 0 \end{cases}$$

Exc:- If $F$ is the joint CDF of $X$ and $Y$ then show that for any $(x,y) \in \mathbb{R}^2$, $F$ is continuous at $(x,y)$ iff $P[(X-x) \vee (Y-y) = 0] = 0$.

FACT:- (Jacobian theorem)

Suppose that $(X, Y)$ has density $f$ and takes values in an open set $U \subseteq \mathbb{R}^2$ that is $P[(X,Y) \in U] = 1$. Assume that,

$\phi : U \to V$ is a bijection for some open set $V \subseteq \mathbb{R}^2$. denote

$$(W, Z) = \phi(X, Y)$$

and $\psi = \phi^{-1} = (\psi_1, \psi_2)$.

Furthermore we assume that $\psi_1, \psi_2$ is continuously diff'ble function from $V$ to $\mathbb{R}$, such that

$$J(w,z) = \det\left(\begin{bmatrix} \frac{\delta \psi_1}{\delta w}(w,z) & \frac{\delta \psi_2}{\delta w}(w,z) \\ \frac{\delta w_1}{\delta z}(w,z) & \frac{\delta w_2}{\delta z}(w,z \end{bmatrix}\right) \neq 0.$$

$\forall (w, z) \in V$.

Then density of $(W, Z)$ is

$$g(w,z) = f \circ \psi(w,z) \, |J(w,z)| 1.((w,z) \in V), \quad (w, z) \in \mathbb{R}^2.$$

EXM:- Let, $X, Y$ be iid. from standard uniform. Let, $A$ be a $2\times2$ invertible matrix. Denote

$$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}.$$

and find the density of $(W, Z)$.

Sol$^n$:- $U = (0,1) \times (0,1)$.

Let, $V$ be the interior of the parallelogram with vertices $(0,0)$, $(a,c)$, $(b,d)$, $(a+b, c+d)$.

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$(W, Z) = \phi(X, Y)$ where $\phi(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$.

hence, $\phi^{-1}(w, z) = A^{-1}\begin{bmatrix} w \\ z \end{bmatrix} = \psi(w, z)$.

Let, $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$\therefore \psi(w, z) = (pw + qz, rw + sz)$

the jacobian matrix is, $\begin{bmatrix} p & q \\ r & s \end{bmatrix} = A^{-1}$.

Jacobian theorem implies that the density of $(W, Z)$ is

$$g(w, z) = |\det(A^{-1})| \, 1.[(w, z) \in V]$$

$$= \frac{1}{\det(A)} \, 1.[(w, z) \in V].$$

Exm:- Let, $X$ & $Y$ be i.i.d. from standard normal. $W = \frac{X}{Y}$ & $Z = Y$.

Find the joint density of $W$ and $Z$.

Solⁿ:-

Define, the sets,

$$V = U = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$$

Clearly, these are open sets and $P[(X, Y) \in U] = 1$, and

$\phi : U \to V$ defined by.

$\phi(x,y) = (\frac{x}{y}, y)$ is a bijection. Its inverse is $\psi : V \to U$ defined by $\psi(w,z) = (wz, z)$.
The jacobian matrix of $\chi$ is $\begin{bmatrix} z & w \\ 0 & 1 \end{bmatrix}$, whose determinant is $|z|$ and which is non-zero in $V$.

$\therefore$ the joint density of $(w,z)$ is.

$$g(w,z) = f \circ \psi(w,z)\, |J(w,z)|\, \mathbb{1}[(w,z) \in V], \text{ where,}$$

$$f(u,v) = \frac{1}{2\pi} e^{-(u^2+v^2)/2}, \quad J(w,z) = z.$$

$$\therefore g(w,z) = \frac{1}{2\pi} \exp\left[-\frac{1}{2} z^2(1+w^2)\right] |z|\, \mathbb{1}(z \neq 0).,$$
$$(w,z) \in \mathbb{R}^2.$$

The marginal density of $w$ is.

$$g_w(w) = \int_{-\infty}^{\infty} g(w,z)\, dz$$

$$= 2\int_0^{\infty} \frac{1}{2\pi} z \cdot e^{-\frac{z^2}{2}(1+w^2)}\, dz.$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\frac{z^2(1+w^2)}{2}} z\, dz.$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-u(1+w^2)}\, du.$$

$$= \frac{1}{\pi(1+w^2)}$$

· if $X$ and $Y$ are i.i.d. $N(0,1)$ then

$\frac{X}{Y} \sim$ Cauchy

$X$ & $Y$ are indep

$$E\left(\left|\frac{X}{Y}\right|\right) = E(|X|)\, E\left(\frac{1}{|Y|}\right).$$

$$= \infty.$$

$$g_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} z\, e^{-\frac{1}{2}z^2(1+w^2)}\, dw.$$

$$= \frac{z}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2(1+w^2)}\, dw.$$

~~put $1+w^2 = t$~~

~~$2w\,dw = dt$~~

~~$\Rightarrow dw = \frac{dt}{2\sqrt{t-1}}$~~

$$= \frac{z}{2\pi} e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z^2w^2}{2}}\, dw.$$

put. $\frac{z^2w^2}{2} = t$

$w z^2\, dw = dt.$

$\Rightarrow dw = \frac{dt}{z^2\sqrt{t}}.$

$$= \frac{1}{2\pi z} e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} e^{-t} \frac{dt}{\sqrt{t}}.$$

$$= \frac{1}{2\pi z} e^{-\frac{z^2}{2}}\; 2\sqrt{\pi}.$$

· if X and Y are i.i.d. N(0,1) then

$\frac{X}{Y} \sim$ Cauchy

X & Y are indep

$E\left(\left|\frac{X}{Y}\right|\right) = E(|X|)\, E\left(\frac{1}{|Y|}\right).$

$= \infty.$

$$g_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} z\, e^{-\frac{1}{2}z^2(1+w^2)}\, dw.$$

$$= \frac{z}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2(1+w^2)}\, dw.$$

~~put $1+w^2 = t$.~~

~~$2w\,dw = dt$.~~

~~$\Rightarrow dw = \frac{dt}{2\sqrt{t-1}}$~~

$$= \frac{z}{2\pi} e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{z^2w^2}{2}}\, dw.$$

put. $\frac{z^2w^2}{2} = t$.

$w z^2 dw = dt$.

$\Rightarrow dw = \frac{dt}{z^2\sqrt{t}}$.

$$= \frac{1}{2\pi z} e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} e^{-t} \frac{dt}{\sqrt{t}}.$$

$$= \frac{1}{2\pi z} e^{-\frac{z^2}{2}}\, 2\sqrt{\pi}.$$

Exm:- Let $(X,Y)$ have joint density
$$f(x,y) = \frac{1}{2\pi} e^{-\sqrt{x^2+y^2}}, \quad x,y \in \mathbb{R}.$$

Let $(R, \Theta)$ denote the polar transformation of $(X,Y)$.
That is, $R = \sqrt{X^2+Y^2}$ (*) &

$$\Theta = \begin{cases} \tan^{-1}\frac{Y}{X}, & X>0, Y\neq 0. \\ \pi + \tan^{-1}\frac{Y}{X}, & X<0, Y>0. \\ -\pi + \tan^{-1}\frac{Y}{X}, & X<0, Y<0. \end{cases} \quad (**)$$

Let, $U = \{(x,y) : (x,y) \in \mathbb{R}^2 \ \& \ xy \neq 0\}$.

$$V = (0,\infty) \times \left((-\pi,\pi) \setminus \{-\tfrac{\pi}{2}, 0, \tfrac{\pi}{2}\}\right)$$

$P[(X,Y) \in U] = 1.$

$\phi : U \to V$ defined by $\phi(x,y) = (r,\theta)$ where $r, \theta$ defined in (*) & (**) is a bijection.

The jacobian matrix is.

$$\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

whoose det. is $r$

$\therefore$ the joint density of $R$ and $\Theta$ is.

$$g(r,\theta) = f(r\cos\theta, r\sin\theta)\,|r|\, \mathbb{1}[(r,\theta)\in V]$$
$$= \frac{1}{2\pi} e^{-r} r\, \mathbb{1}(r>0)\, \mathbb{1}\left(\theta \in (-\pi,\pi)\setminus\{0, \pm\tfrac{\pi}{2}\}\right).$$

Check that $R \sim \text{Gamma}(2,1)$ and
$\Theta \sim \text{Unif}(-\pi,\pi)$, independently.

Exc: Suppose that the joint density f of $(X,Y)$ has the following form:

$$f(x,y) = g(x)h(y) \quad , (x,y) \in \mathbb{R}^2 .$$

1. Show that X and Y are independent.
2. Is it necessary that g and h are the densities of X and Y resp?
3. Under what condition, is the answer to 2 is 'Yes'?

Exm:- Let, X and Y ~ Exp($\lambda$) independently. Find the joint density of X and X+Y.

$U = X$
$V = X+Y$.

$$(U,V) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \phi(X,Y).$$

$X = U$
$Y = V-U$. $\quad \psi(u,v) = (u, v-u)$

$$f(x,y) = \lambda^2 e^{-\lambda(x+y)}$$

$$J(u,v) = \left| \left( \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right) \right| = 1.$$

$\therefore g(u,v) = \lambda^2 e^{-\lambda v} 1(0<u<v). \quad (u,v) \in \mathbb{R}^2.$

$V \sim$ Gamma $(2,\lambda)$ check.

Exm:- Let, X ~ Gamma($\alpha,\lambda$) Find density of $\frac{X}{X+Y}$.
Y ~ Gamma($\beta,\lambda$).

Sol$^n$: Let, $U = \frac{X}{X+Y}$, $V = X+Y$.

$X = UV \quad Y = V-UV.$

$f(x,y) =$

$$J(u,v) = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v(1-u) + uv = v.$$

Defn:- ~~for some a, b > 0, then we say~~, If density of X is,

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} 1(0<x<1), x \in \mathbb{R}.$$

Exm:- Let, $X \sim$ Gamma $(\alpha, 1)$, $Y \sim$ Gamma $(\beta, 1)$.
Find the density of $\frac{X}{X+Y}$.

Sol$^n$:-

Let, $U = \frac{X}{X+Y}$, $V = X+Y$. $U \in [0,1]$, $V \in [0, \infty)$

$X = UV$ $\quad$ $Y = V - UV$.

$\psi(u,v) = (uv, v - uv)$

$\therefore$ the Jacobian Matrix is given by, $\begin{pmatrix} v & u \\ -v & 1-u \end{pmatrix}$

$$\therefore j(u,v) = \left\| \begin{matrix} v & u \\ -v & 1-u \end{matrix} \right\| = |v(1-u) + uv| = |v|.$$

Let, $f$ be the joint density of $X, Y$ then,

$$f(x,y) = \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x} \cdot \frac{1}{\Gamma\beta} y^{\beta-1} e^{-y} \, 1.(x>0, y>0)$$

$$= \frac{1}{\Gamma\alpha \Gamma\beta} e^{-(x+y)} x^{\alpha-1} y^{\beta-1}.$$

$\therefore$ density of $U, V$ is,

$$g(u,v) = \frac{1}{\Gamma\alpha \Gamma\beta} e^{-v} \cdot (uv)^{\alpha-1} (v-uv)^{\beta-1} \cdot v \, 1(u \in [0,1], v \in [0,\infty))$$

$$\therefore g_U(u) = \int_{-\infty}^{\infty} \frac{1}{\Gamma\alpha \Gamma\beta} e^{-v} (uv)^{\alpha-1} (v-uv)^{\beta-1} v . 1(u \in [0,1], v \in [0,\infty)) dv$$

$$= \int_0^\infty \frac{1}{\Gamma\alpha \Gamma\beta} u^{\alpha-1} (1-u)^{\beta-1} e^{-v} v^{\alpha-1+\beta-1+1} dv.$$

$$= \frac{1}{\Gamma\alpha \Gamma\beta} u^{\alpha-1} (1-u)^{\beta-1} \int_0^\infty e^{-v} v^{\alpha+\beta-1} dv.$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha \Gamma\beta} u^{\alpha-1} (1-u)^{\beta-1} \, 1(u \in [0,1]).$$

## Definition:-

If density of a Random Variable $X$ is,

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \cdot 1(x \in [0,1]), \quad x \in \mathbb{R}.$$

then, $X \sim \beta eta\ (a,b)$.

**EXM:-** Let, $X$ and $Y$ be i.i.d. from standard Normal

Let $-1 < \rho < 1$. Define

$$W = X$$

$$Z = \rho X + \sqrt{1-\rho^2}\, Y.$$

1. Find the density of $W$ and $Z$.

**Soln:-** $(W, Z) = \phi(X, Y)$

Let, $\psi = \phi^{-1}$

$$\psi(w,z) = \left( w, \frac{z - \rho w}{\sqrt{1-\rho^2}} \right)$$

$$J(w,z) = \det \begin{pmatrix} 1 & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{pmatrix} = \frac{1}{\sqrt{1-\rho^2}}$$

$\therefore$ The joint density of $X, Y$ is.

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, \quad x, y \in \mathbb{R}.$$

$\therefore$ The density of $w, z$ is,

$$g(w,z) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(w^2 + \frac{(z-\rho w)^2}{1-\rho^2}\right)} \cdot \frac{1}{\sqrt{1-\rho^2}}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{w^2 - \rho^2 w^2 + z^2 - 2\rho w z + \rho^2 w^2}{1-\rho^2}\right)}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(w^2+z^2-2\rho wz)\right], \quad (w,z)\in\mathbb{R}^2.$$

2. Calculat correlation of W, Z.

$\cancel{\text{cor}\ (X, \rho X + \sqrt{1-\rho^2} Y)}$

$\cancel{= \rho\, Var(X)}$

$$Cov(W, Z) = Cov(X, \rho X + \sqrt{1-\rho^2}\, Y).$$

$$= \rho\, Var(X) = \rho$$

$Var(W) = 1$

$Var(Z) = \rho^2 Var(X) + (1-\rho^2) Var(Y) = \rho^2 + 1 - \rho^2 = 1.$

$Corr(W, Z) = \rho.$

3. Fix $\mu_1, \mu_2 \in \mathbb{R}$ & $\sigma_1, \sigma_2 > 0$ define

$V_1 = \mu_1 + \sigma_1 W.$

$V_2 = \mu_2 + \sigma_2 Z,$

Find the joint density of $(V_1, V_2)$.

$\cancel{f(v_1,v_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} e}$

$$\phi(w, z) = (\mu_1+\sigma_1 w, \mu_2+\sigma_2 z).$$

$$\phi^{-1} = \psi(v_1, v_2) = \left(\frac{v_1-\mu_1}{\sigma_1}, \frac{v_2-\mu_2}{\sigma_2}\right).$$

$$J(v_1, v_2) = \det\begin{bmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{bmatrix} = \frac{1}{\sigma_1\sigma_2}.$$

$$g(v_1, v_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\sqrt{1-\rho^2}}\left(\left(\frac{v_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{v_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{v_1-\mu_1}{\sigma_1}\right)\left(\frac{v_2}{\sigma_2}\right.\right.\right.$$

~~Corrtupu~~

4. $\text{Corr}(V_1, V_2) = \text{Corr}(W, Z) = \rho$

Definition:-

Random Variables $(V_1, V_2)$ follow Bivariate normal with means $(\mu_1, \mu_2)$, variances $(\sigma_1^2, \sigma_2^2)$ and correlation coefficient $(\rho)$ if their density is,

$$h(v_1, v_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \, e^{-\frac{1}{2}\left(\frac{\left(\frac{v_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{v_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{v_1-\mu_1}{\sigma_1}\right)\left(\frac{v_2-\mu_2}{\sigma_2}\right)}{1-\rho^2}\right)}$$

Exc: ~~For~~ Let $(V_1, V_2)$ be as in the above ~~dem~~ definition.

1. for $\alpha, \beta \in \mathbb{R}$ find the distribution of

$$\alpha V_1 + \beta V_2 .$$

Soln:- Let, $Y_1 = \dfrac{V_1 - \mu_1}{\sigma_1}$, $Y_2 = \dfrac{V_2 - \mu_2}{\sigma_2}$

Let, $X_1 = Y_1$, $X_2 = \dfrac{Y_2 - \rho Y_1}{\sqrt{1-\rho^2}}$

Check that $X_1$ & $X_2$ are i.i.d. from $N(0,1)$.

$$X_1 = \frac{V_1 - \mu_1}{\sigma_1}, \quad X_2 = \frac{\frac{V_2 - \mu_2}{\sigma_2} - \rho\,\frac{V_1 - \mu_1}{\sigma_1}}{\sqrt{1-\rho^2}}$$

$\alpha = \beta = 0$ then $\alpha V_1 + \beta V_2 = 0$.

assume that at least one of $\alpha$ and $\beta$ is non-zero.

$$\alpha V_1 + \beta V_2 = \alpha(\mu_1 + \sigma_1 Y_1) + \beta(\mu_2 + \sigma_2 Y_2).$$

$$= \alpha(\mu_1 + \sigma_1 X_1) + \beta(\mu_2 + \sigma_2(\rho X_1 + \sqrt{1-\rho^2} X_2)).$$

$$= \alpha\mu_1 + \alpha\sigma_1 X_1 + \beta\mu_2 + \beta\sigma_2\rho X_1 + \beta\sigma_2\sqrt{1-\rho^2} X_2.$$

$$= \alpha\mu_1 + \beta\mu_2 + (\alpha\sigma_1 + \beta\sigma_2\rho)X_1 + \beta\sigma_2\sqrt{1-\rho^2} X_2.$$

and $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$.

$\therefore$

$$E(\alpha V_1 + \beta V_2) = \alpha\mu_1 + \beta\mu_2 + 0 + 0.$$

$$Var((\alpha\sigma_1 + \beta\sigma_2\rho)X_1) = (\alpha\sigma_1 + \beta\sigma_2\rho)^2$$

$$Var(\beta\sigma_2\sqrt{1-\rho^2} X_2) = \beta^2\sigma_2^2(1-\rho^2)$$

$$\therefore Var(\alpha V_1 + \beta V_2) = \alpha^2\sigma_1^2 + \cancel{\beta^2\sigma_2^2\rho^2} + 2\alpha\beta\sigma_1\sigma_2\rho + \beta^2\sigma_2^2 - \cancel{\beta^2\sigma_2^2\rho^2}$$

$$= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + 2\alpha\beta\sigma_1\sigma_2\rho.$$

$$\therefore \alpha V_1 + \beta V_2 \sim N(\alpha\mu_1 + \beta\mu_2,\ \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + 2\alpha\beta\sigma_1\sigma_2\rho).$$

2. What are the marginal ~~density~~ distribution of $V_1$ & $V_2$?

Ans $V_1 \sim N(\mu_1, \sigma_1^2)$ [put $\alpha = 1, \beta = 0$]

$V_2 \sim N(\mu_2, \sigma_2^2)$ [put $\beta = 1, \alpha = 0$].

3. Show that, $V_1$ & $V_2$ are independent iff the are uncorrelated. independence $\Rightarrow$ uncorrelation
uncorrelation $\Rightarrow$ ~~math~~ $\rho = 0 \Rightarrow$ write density.

4. If above true under the only assumption that marginals of $V_1$ & $V_2$ are normal.

Exm:- Let $V_1 \sim N(0,1)$.

$X$ takes values $-1$ & $1$, with prob. $\frac{1}{2}$. independently of $V_1$

$V_2 = XV_1$.

$$P(V_2 \leq v)$$
$$= P(V_2 \leq v, X=+1) + P(V_2 \leq v, X=-1)$$
$$= P(V_1 \leq v, X=1) + P(V_1 \geq -v, X=-1)$$
$$= \Phi(v) \cdot \frac{1}{2} + (1 - \Phi(-v))\frac{1}{2}$$
$$= \Phi(v).$$

$\therefore V_2 \sim N(0,1)$.

$$P(V_2 \leq v / V_1 \leq v) = \frac{\frac{1}{2}\Phi(v)}{\Phi(v)} = \frac{1}{2}.$$

$$P(V_2 \leq v) = \Phi(v).$$

$$P(V_2 \leq v / V_1 \leq v) \neq P(V_2 \leq v).$$

$\therefore V_1, V_2$ are not independent.

$$\text{cov}(V_1, V_2) = E(V_1 V_2) - E(V_1)E(V_2) = E(V_1 V_2) = E(XV_1^2) = E(X)E(V_1^2)$$
$$= 0.$$

$\therefore V_1, V_2$ are uncorrelated.

Theorem:- Suppose that $X$ is a random variable with $P(X \leq 0) < 1$. If $X$ has the memoryless property, i.e.

$$P(X > s+t) = P(X>s)\,P(X>t) \text{ for every } s,t>0,$$

then $X \sim Exp(\lambda)$ for some $\lambda > 0$.

Pf: The function $x \mapsto P(X>x)$ is right continuous [$\because$ CDF is right conti. and $P(X>x) = 1 - \cancel{P}CDF(x)$]. The hypothesis implies that $P(X>0) > 0$. By right continuity, $\exists\, \varepsilon > 0$ such that, $P(X>\varepsilon) > 0$.

by memoryless property,

~~the identity~~

$$P(X > n\varepsilon) = P(X>\varepsilon)^n > 0 \quad \text{for } \forall n \in \mathbb{N}.$$

Thus $P(X>t) > 0$ for every $t>0$.

Define, $f(t) := \log(P(X>t))$, $t>0$.

$\therefore$ by memoryless property.

$$f(t+s) = f(t) + f(s), \quad s,t>0.$$

Letting $\lambda = -f(1)$, it follows that for every $r \in \mathbb{Q} \cap (0,\infty)$, $f(r) = -\lambda r$.

For any $x \geq 0$,

$\exists$ rationals $r_n > 0$ such that $r_n \downarrow x$

Right continuity implies,

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} -\lambda r_n = -\lambda x.$$

$$\therefore P(X>x) = e^{-\lambda x}, \quad x \geq 0.$$

This implies that $P(X \leq x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0. \\ 0, & x < 0. \end{cases}$

as desired.

Thm:- Suppose that you are in the middle of a desert searching for water. Let $(X,Y)$ denote the coordinates of the droplet of water closest to the origin which is your current location. We make the following assumptions ① If A and B are disjoint subset of $\mathbb{R}^2$, then the events that A has water ~~and~~ is independent of the event that B has water. ② If ~~P~~ A & B has the same area then probability A has water is same as the probability that B has water.

Furthermore, assume that the LHS is strictly positive if and only if area$(A) > 0$.
③ If $(R, \Theta)$ is the polar transformation of $(X,Y)$ then $\Theta \sim \text{Unif}(-\pi,\pi)$ independently of $R$.

Then $X$ and $Y$ are i.i.d. from $N(0,\sigma^2)$ for some $\sigma > 0$.

Pf:- Clearly, R takes values in $[0,\infty)$ and

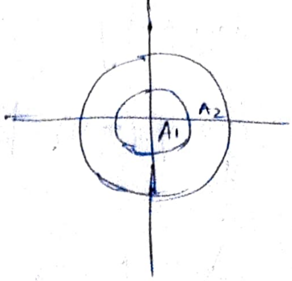
$P(R=0) = 0$.

$P(R > r+s)$
$= P$ [(no water in the circle of radius $r$) $\cap$ (no water in the annulus of length $s$ around the circle of radius $r$)]

$=$ P [no water in the circle of radius $r$] P [no water in the annulus of length $s$ around the circle of radius $r$].

P(a region A does not have water)
$= f(\text{area}(A))$.

$P(R^2 > r+s)$




$= P(R > \sqrt{r+s}) = f(\pi(r+s))$

$= P(\text{no water in } A_1)\, P(\text{no water in } A_2)$

$= f(\pi r)\, f(\pi s)$.

$= P(R^2 > r)\, P(R^2 > s)$

$\therefore$ $R^2$ has memoryless property & $P(R^2 \le 0) = 0 < 1$.

Therefore, $R^2 \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

Therefor Density of $R$ is,

$$f(r) = 2\lambda r e^{-\lambda r^2} 1(r>0).$$

Given that $\Theta \sim \text{Unif}(-\pi,\pi)$ independent of $R$.

Hence joint density of $R$ and $\Theta$ is,

$$g(r,\theta) = \frac{\lambda}{\pi} r e^{-\lambda r^2} 1(r>0, \theta \in [-\pi,\pi]),\quad (r,\theta) \in \mathbb{R}^2.$$

$X = r\cos\theta$

$Y = r\sin\theta$

$$\phi(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

The inverse of above transformation is.

$$\psi : [(x,y) : xy \neq 0] \to (0,\infty) \times [(-\pi,\pi) \setminus \{-\pi/2, 0, \pi/2\}].$$

$$\psi(x,y) = (r,\theta)$$

$$r = \sqrt{x^2+y^2}$$

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x}, & QI \,\&\, QIV \\ \pi + \tan^{-1}\frac{y}{x}, & QII \\ -\pi + \tan^{-1}\frac{y}{x}, & QIII \end{cases}$$

$$j(x,y) = \det\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = \frac{1}{\sqrt{x^2+y^2}}.$$

$1 + \frac{y^2}{x^2}$ $\quad x^2$

$\frac{-y}{x^2+y^2}$

$\frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$

$\therefore$ joint density of X and Y is,

$$h(x,y) = g \circ \varphi(x,y)\, |J(x,y)|\, 1(xy \neq 0).$$

$$= \frac{\lambda}{\pi} \sqrt{x^2+y^2} \cdot e^{-\lambda(x^2+y^2)} \frac{1}{\sqrt{x^2+y^2}} 1(xy \neq 0)$$

$$= \frac{\lambda}{\pi} e^{-\lambda(x^2+y^2)}.$$

Let, define $\sigma := \sqrt{\frac{1}{2\lambda}} \Rightarrow \sigma^2 = \frac{1}{2\lambda} \Rightarrow \lambda = \frac{1}{2\sigma^2}$.

$$\therefore h(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)} 1(xy \neq 0).$$

Therefore X and Y are i.i.d. from $N(0,\sigma^2)$.

HM-3

3. $E(X^2) = E\left(\lim_{n\to\infty} X_n^2\right)$

$$\leq \liminf_{n\to\infty} E(X_n^2).$$

$$= \lim_{n\to\infty} E(X_n^2) < \infty.$$

DCT implies

$$\lim_{n\to\infty} E(X_n) = E(X).$$

$$\Rightarrow \lim_{n\to\infty} E(X_n)^2 = E(X)^2$$

$$\therefore Var(X) = E(X^2) - E(X)^2$$

$$\leq \lim_{n\to\infty} E(X_n^2) - \lim_{n\to\infty} E(X_n)^2$$

$$= Var(X_n).$$

$E(Y^2) < \infty$.

$X_n^2 \le Y \qquad X_n^2 \to X.$

$\therefore \lim_{n\to\infty} E(X_n^2) = E(X^2)$ (DCT implies.).

4. A $|X_n - X| = X_n + X - 2(X_n \wedge X)$.

$E(|X_n - X|) = E(X_n) + E(X) - 2E[\min(X_n, X)]$

Since, $\lim_{n\to\infty} (X_n \wedge X) = X$ and $0 \le X_n \wedge X \le X$.

DCT applies., which implies that,

$\lim_{n\to\infty} E(X_n \wedge X) = E(X)$.

Hence,

$\lim_{n\to\infty} E(|X_n - X|) = E(X) + E(X) - 2E(X) = 0$.

(Sheffe's Theorem)

6. $E(X) = \int_0^\infty P(X > x)\,dx$.

Let, if possible, $E(X) = 0$ ~~and~~ but $P(X=0) < 1$.

$P(X > 0) > 0$ & by right continuity ~~the~~ $\exists\, \varepsilon > 0$.

such that, $P(X > \varepsilon) > 0$.

$E(X) = \int_0^\infty P(X > x)\,dx \ge \int_0^\varepsilon P(X > x)\,dx \ge \int_0^\varepsilon P(X > \varepsilon)\,dx$.

$= \varepsilon P(X > \varepsilon) > 0$.

$(\Rightarrow \Leftarrow)$

7.

b: $\mathrm{Var}(X+Y)$

$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(XY)$

$\le \mathrm{Var}(X) + \mathrm{Var}(Y) + 2|\mathrm{cov}(XY)|$.

$\longrightarrow$

$$\leq Var(X) + Var(Y) + 2\sqrt{Var(X)\,Var(Y)}$$

$$= \left(\sqrt{Var(X)} + \sqrt{Var(Y)}\right)^2$$

Exc:- Let, $\Sigma$ be a $2\times 2$ positive definite matrix. Show that there exists a $2\times 2$ symmetric matrix A such that $\Sigma = A^2$

$$\Sigma = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad a > 0 \text{ and } c > 0.$$

and $|b| < \sqrt{ac}$ [$\because$ determinant positive]

$$A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

$$\Sigma = A^2 = \begin{bmatrix} x^2+y^2 & xy+yz \\ xy+yz & y^2+z^2 \end{bmatrix}$$

$x^2+y^2 = a.$
$y^2+z^2 = c.$
$xy+yz = b.$

Three equation
three variable.
unique solution.

Thm:- Let, $\Sigma_{2\times 2}$ be a p.d. matrix and $\mu \in \mathbb{R}^2$. Then there exists Random Variable $X_1$ and $X_2$ which are jointly normal (bivariate normal) and.

$E(X_i) = \mu_i$, $i = 1,2.$

and $cov(X_i, X_j) = \sigma_{ij}$ $\quad i,j = 1,2.$

where $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$

Proof:-

By the preceeding exercise there exists a $2\times 2$ symmetric matrix $A$ such that $\Sigma = A^2$.

Since, $\Sigma$ is non-singular it ensures that $A$ is also non-singular.

Let, $Z_1$ and $Z_2$ be i.i.d. from standard normal. Define,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{\mu} + A \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

The joint density of $Z_1$ and $Z_2$ is,

$$f(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2 + z_2^2)} \quad , (z_1, z_2) \in \mathbb{R}^2$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\vec{z}'\vec{z}} \quad , \quad \vec{z} \in \mathbb{R}^2 .$$

Clearly,

$(Z_1, Z_2) = \psi(X_1, X_2)$ where.

$\psi : \mathbb{R}^2 \to \mathbb{R}^2$

$$A^{-1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} .$$

$$\psi(\vec{x}) = A^{-1}(\vec{x} - \vec{\mu}) .$$

$$\psi(\vec{x}) = \begin{bmatrix} a(x_1 - \mu_1) + b(x_2 - \mu_2) \\ b(x_1 - \mu_1) + c(x_2 - \mu_2) \end{bmatrix}$$

$$\therefore J(\vec{x}) = \det \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \det(A^{-1})$$

$\therefore$ joint density of $X_1$ and $X_2$ is,

$$g(\vec{x}) = |\det(A^{-1})| \, f \circ \psi(\vec{x}) \quad , \quad \vec{x} \in \mathbb{R}^2$$

$$= \frac{1}{|\det(A)|} \frac{1}{2\pi} e^{-\frac{1}{2}(A^{-1}(\vec{x} - \vec{\mu}))'(A^{-1}(\vec{x} - \vec{\mu}))} ,$$

$$= \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})'(A^{-1})'A^{-1}(\vec{x}-\vec{\mu})\right]$$

$$= \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})' A^{-2}(\vec{x}-\vec{\mu})\right] \quad [\because (A^{-1})' = A^{-1}$$

$$= \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})\right].$$

Clearly $E(X_i) = \mu_i \quad i = 1, 2.$

$A = \begin{bmatrix} p & q \\ q & s \end{bmatrix}$ then $X_1 = \mu_1 + p\cancel{z} Z_1 + q Z_2$

$X_2 = \mu_2 + q Z_1 + s Z_2$

$Var(X_1) = p^2 + q^2$

$Cov(X_1, X_2) = pq + qs.$

$$\therefore \quad A^2 = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix}.$$

Denote, $\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$.

$\therefore \quad corr(X_1, X_2) = \rho.$

Hence, $|\rho| \leq 1.$

Since, $X_1$ and $X_2$ has a joint density, it follows that $|\rho| \leq 1$. Let, ~~B~~ $h$ be the joint density of BV Normal with means $\mu_1$ & $\mu_2$ variances $\sigma_{11}$ and ~~$\sigma_{22}$~~ $\sigma_{22}$, and correlation coefficient $\rho$.

~~$h(x_1, x_2) = \frac{1}{2\pi\sqrt{\rho_1\rho_2(1-}}$~~

$$h(x_1,x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho^2)}} \exp\left[-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} + \frac{(x_2-\mu_2)^2}{\sigma_{22}} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sqrt{\sigma_{11}\sigma_{22}}}\right]\right].$$

$$g(x_1,x_2) = (\text{Const}) \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})\right]$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{\det(\Sigma)} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}.$$

$$= \frac{1}{\sigma_{11}\sigma_{22}(1-\rho^2)} \begin{bmatrix} \sigma_{22} & -\rho\sqrt{\sigma_{11}\sigma_{22}} \\ -\rho\sqrt{\sigma_{11}\sigma_{22}} & \sigma_{11} \end{bmatrix}$$

$$\vec{x}'\Sigma^{-1}\vec{x} = \frac{1}{\sigma_{11}\sigma_{22}(1-\rho^2)}\left(\sigma_{22}x_1^2 + \sigma_{11}x_2^2 - 2\rho\sqrt{\sigma_{11}\sigma_{22}}\,x_1x_2\right).$$

$$(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})$$

$$= \frac{1}{\sigma_{11}\sigma_{22}(1-\rho^2)}\left(\sigma_{22}(x_1-\mu_1)^2 + \sigma_{11}(x_2-\mu_2)^2 - 2\rho\sqrt{\sigma_{11}\sigma_{22}}(x_1-\mu_1)(x_2-\mu_2)\right)$$

$$= \frac{1}{1-\rho^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} + \frac{(x_2-\mu_2)^2}{\sigma_{22}} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sqrt{\sigma_{11}\sigma_{22}}}\right]$$

$\therefore$ exp part of $g$ and $h$ matches and both are density, i.e. integrated to 1.

$\therefore$ $h$ and $g$ are same.

i.e. $g(\vec{x}) = h(\vec{x}) \quad \forall\, \vec{x} \in \mathbb{R}^2$.

**Thm:- (Box-Muller Transformation)**

Let, $U$ and $V$ be i.i.d. from standard uniform.

Define, $X = \sqrt{-2\log U}\,\cos(2\pi V)$.

$Y = \sqrt{-2\log U}\,\sin(2\pi V)$.

Then, $(X,Y) \sim N_2(\vec{0}, I_2)$, that is,

$X$ and $Y$ are i.i.d. from standard normal.

Pf:- Let, $R = \sqrt{-2\log U}$.

and $\Theta = 2\pi V$.

$\Theta \sim \text{Unif}(0, 2\pi)$.

$R^2 \sim \text{Exp}(\frac{1}{2})$.

independently.

[Now, $R > 0$ and $R^2 \sim \text{Exp}(\lambda)$ and $\Theta \sim \text{Unif}(-\pi, \pi)$ independent of $R$, then, $R\cos\Theta$ and $R\sin\Theta$ are i.i.d. from $N(0, \frac{1}{2\lambda})$.]

$\therefore$ $X = R\cos(\Theta - \pi)$

$Y = R\sin(\Theta - \pi)$

[Observe that negative sign doesn't changes anything. as $X \sim N(0,1) \Rightarrow -X \sim N(0,1)$.]

$\therefore$ $X, Y$ are i.i.d. from $N(0,1)$

$\therefore$ $(X,Y) \sim N_2(\vec{0}, I_2)$.

Exc:- Suppose that $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta$ independently then what is the distribution of $X+Y$.

$$f(z) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-x} \cdot \frac{\lambda^\beta}{\Gamma\beta} (z-x)^{\beta-1} e^{-(z-x)}\,dx$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma\alpha\,\Gamma\beta} e^{-z} z^{\alpha+\beta-1} \int_0^\infty \left(\frac{x}{z}\right)^{\alpha-1} \left(1-\frac{x}{z}\right)^{\beta-1} \frac{dx}{z} = \frac{\lambda^{\alpha+\beta}}{\Gamma\alpha+\beta} e^{-z} z^{\alpha} \ldots 1(z>0$$

~~Soln~~ Ans:- $X+Y \sim$ Gama $(\alpha+\beta, \lambda)$.

Exc:- Now suppose $X_1, \dots, X_n$ are i.i.d. from standard normal. Find the distribution of.

$$Y = X_1^2 + \dots + X_n^2 .$$

Soln:- For $z>0$

$$P(X_1^2 \le z)$$

$$= P(-\sqrt{z} \le X_1 \le \sqrt{z}).$$

$$= 2\Phi(\sqrt{z}) - 1.$$

$\therefore$ density of $X_1^2$ is,

$$f(z) = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z}{2}} \cdot \frac{1}{2\sqrt{z}}$$

$$= \frac{1}{\sqrt{2\pi}} z^{\frac{1}{2}-1} e^{-z/2} \cdot 1(z>0).$$

$\therefore X_1^2 \sim$ Gamma $(\frac{1}{2}, \frac{1}{2})$.

$\therefore Y \sim$ Gamma $(\frac{n}{2}, \frac{1}{2})$.

$\therefore$ The density of $Y$ is,

$$g(y) = \frac{(\frac{1}{2})^{n/2}}{\Gamma\frac{n}{2}} \cdot e^{-\frac{x}{2}} x^{\frac{n}{2}-1} \quad 1(x>0).$$

$$= \frac{1}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} e^{-\frac{x}{2}} x^{\frac{n}{2}-1}.$$

Definition:- The distribution ~~whoo~~ whose density is $f$, defined as above, is called "chi-squared with n degrees of freedom" or $\chi_n^2$. This is same as Gamma $(\frac{n}{2}, \frac{1}{2})$.

Exc:- Let, $X$ and $Y$ follow $\chi^2_m$ and $\chi^2_n$ independently resp. Find the density of $Z = \frac{X/m}{Y/n}$

Ans: The density of $Z$ is,

$$f(z) = \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}}}{\beta\left(\frac{m}{2},\frac{n}{2}\right)} z^{\frac{m}{2}-1}\left(1+\frac{m}{n}z\right)^{-\frac{m+n}{2}} 1.(z>0).$$

$z \in \mathbb{R}$.

Defn:- The distribution with the above density is "Fisher's F - ~~dist~~ distribution" with degrees of freedom $m$ and $n$" or simply $F_{m,n}$.

Exc:- Let, $X \sim N(0,1)$ and $Y \sim \chi^2_n$ independently. Find the density of, $Z = \frac{X}{\sqrt{Y/n}}$

Soln:- Clearly, $Z^2 \sim F_{1,n}$

$[z>0]$.

$P(Z \le z) = P(Z \le 0) + P(0 < Z \le z)$.

[$\because$ $Z$ has the same distribution as $-Z$].

$$\therefore P(Z \le z) = \frac{1}{2} + \frac{1}{2} P(-z \le Z \le z),$$

$$= \frac{1}{2} + \frac{1}{2} P(Z^2 \le z^2)$$

$$= \frac{1}{2} + \frac{1}{2}\int_0^{z^2} \frac{1}{n^{\frac{1}{2}}\beta\left(\frac{1}{2},\frac{n}{2}\right)} t^{-\frac{1}{2}}\left(1+\frac{t}{n}\right)^{-\frac{n+1}{2}} dt.$$

$\therefore$ Density of $Z$ is,

$$f(z) = \frac{1}{\cancel{2}n^{\frac{1}{2}}\beta\left(\frac{1}{2},\frac{n}{2}\right)} \cancel{(z^2)^{-\frac{1}{2}}}\left(1+\frac{z^2}{n}\right)^{-\frac{n+1}{2}}.\cancel{2z}$$

$$= \frac{1}{n^{\frac{1}{2}}\beta\left(\frac{1}{2},\frac{n}{2}\right)}\left(1+\frac{z^2}{n}\right)^{-\frac{n+1}{2}}, \quad z \in \mathbb{R}.$$

Definition:- The distribution with the above density is "student's t-distribution with n degrees of freedom" or $t_n$. It was discovered by Gosset who used 'Student' as pen name.

HW-3

11. $P[\Phi(Z) \le z] = P(Z \le \Phi^{-1}(z)) = \Phi(\Phi^{-1}(z)) = z.$

$\therefore \Phi(Z) \sim Unif(0,1).$

$E(\Phi(Z)) = \frac{1}{2}.$

15. (a) $E(Z) = \int_0^\infty P(X > x)\, dx.$

$= \int_0^1 P(X > x)\, dx.$

$Z = YX^2 + (1-Y)X.$

$E(Z) = E(Y)E(X^2) + E(X) - E(Y)E(X).$

$= \frac{1}{2}E(X^2) + \frac{1}{2}E(X).$

(b) $P(Y=1 / Z \le \frac{1}{2}).$

$$= \frac{P(Z \le \frac{1}{2} \mid Y=1)\, P(Y=1).}{P(Z \le \frac{1}{2} \mid Y=1) + P(Z \le \frac{1}{2} \mid Y=0)\, P(Y=0)}$$

$$= \frac{P(X^2 \le \frac{1}{2}).}{P(X^2 \le \frac{1}{2}) + P(X \le \frac{1}{2})}$$

$$= \frac{1}{1 + \frac{P(X^2 \le \frac{1}{2})}{P(X \le \frac{1}{2})}} = \frac{1}{1 + \frac{\pi}{4}\sin^{-1}\frac{1}{2t}}.$$

HW-4

1. (a) Let, $B \sim$ Bernoulli $(\lambda)$ independently of $X$ and $Y$.

then, $Z = BX + (1-B)Y$.

(b) $Z = \max(X, Y) = (X \vee Y)$

(c) $Z = \min(X, Y) = (X \wedge Y)$.

2. Claim:- $F: \mathbb{R} \to \mathbb{R}$ is a monotone function then its set of discontinuities is countable.

For every $x$, discontinuity $F(x-) \neq F(x+)$.

$D = \{x \in \mathbb{R} : F(x-) \neq F(x+)\}$.

then consider $\phi: D \to \mathbb{Q}$ such that,

$F(x-) \wedge F(x+) < \phi(x) < F(x-) \vee F(x+)$.

Altn:-

$\{x : P(X = x) > 0\}$

$= \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : P(X = x) > \frac{1}{n}\}$

$\#\{\ \} = n$.

5. $F(x - \frac{1}{n}, y - \frac{1}{n}) = P(X \leq x - \frac{1}{n}, Y \leq y - \frac{1}{n})$.

$[X \leq x - \frac{1}{n}, Y \leq y - \frac{1}{n}] \uparrow [X < x, Y < y]$.

$\therefore \lim_{n \to \infty} F(x - \frac{1}{n}, y - \frac{1}{n}) = P(X < x, Y < y)$.

due to continuity,

$\lim_{n \to \infty} [F(x, y) - F(x - \frac{1}{n}, y - \frac{1}{n})] = 0$.

$\Rightarrow P(X \leq x, Y \leq y) - P(X < x, Y < y) = 0$

$\Rightarrow P[(X - x) \vee (Y - y) = 0] = 0$.

Conversly, $P[(X - x) \vee (Y - y) = 0] = 0$.

$\longrightarrow$