HW-4

1. Let us define.

$$\bigcup_{k=1}^{n} A_k = B_n$$

Then $B_n \uparrow B_\infty = \bigcup_{k=1}^{\infty} A_k \Rightarrow P(B_n) \uparrow P(B_\infty)$

And $P(B_n) \leq \sum_{k=1}^{n} P(A_k) \ \forall n.$

Letting $n \to \infty$ we get $P(B_\infty) \leq \sum_{k=1}^{\infty} P(A_k)$.

$$\Rightarrow P\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} P(A_k).$$

2. $X \sim P(\lambda)$.

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda} \lambda^x}{x!} & ; \ x = 0, 1, 2, \ldots \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\therefore E(X) = \sum_{x=0}^{\infty} x . \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= 0 + \sum_{x=1}^{\infty} \frac{e^{-\lambda} . \lambda . \lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda . \sum_{(x-1)=0}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!}.$$

$$= e^{-\lambda} \lambda . e^{\lambda}$$

$$= \lambda.$$

a. Now, $Var(X) = E(X^2) - E^2(X)$.

$$E(X^2) = \sum_{x=0}^{\infty} x^2 . \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} (x^2 - x) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} x . \frac{e^{-\lambda} \lambda^x}{x!}$$

$\longrightarrow$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + E(X).$$

$$= 0 + 0 + \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda} \cdot \lambda^2 \cdot \lambda^{x-2}}{x \cdot (x-1) \cdot (x-2)!} + E(X).$$

$$= \sum_{(x-2)=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^2 \cdot \lambda^{(x-2)}}{(x-2)!} + E(X)$$

$$= e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda} + \lambda.$$

$$= \lambda^2 + \lambda.$$

$$\therefore Var(X) = E(X^2) - E^2(X).$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda.$$

6.

(b) Let, $Z = \min(X, Y)$.

$$P(Z \geq z) = P(X \geq z, Y \geq z)$$

$$= P(X \geq z) P(Y \geq z)$$

$$= q_1^{z-1} \cdot q_2^{z-1}$$

$$P(Z \geq z+1) = (q_1 q_2)^z.$$

$$\therefore P(Z = z) = (q_1 q_2)^{z-1} \cdot (1 - q_1 q_2).$$

$$Z \sim Geo(1 - q_1 q_2).$$

(c) $W = \operatorname{Max}\{X, Y\}$.

$$P(W \le w) = P(X \le w, Y \le w)$$

$$= P(X \le w)\, P(Y \le w)$$

$$= (1 - q_1^w)(1 - q_2^w).$$

$$P(W = w) = \begin{cases} (1-q_1^{w})(1-q_2^{w}) - (1-q_1^{w-1})(1-q_2^{w-1}), & w \in \\ 0 & \text{otherwise.} \end{cases}$$

$$E(W) = \sum_{n=0}^{\infty} P(W > n)$$

$$= \sum_{n=0}^{\infty} 1 - P(W \le n).$$

$$= 0 + \sum_{n=1}^{\infty} 1 - (1-q_1^n)(1-q_2^n).$$

$$= \sum_{n=1}^{\infty} \left[q_1^n + q_2^n - (q_1 q_2)^n \right]$$

7. $\lim_{n\to\infty} \operatorname{Var}(X_n)$ bdd

$$= \lim_{n\to\infty} \left[\underbrace{\frac{1}{n} \left[\left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n} \right]}_{\to 0} - \underbrace{\left(1 - \frac{1}{n}\right)}_{1} \left[\underbrace{\left(1 - \frac{2}{n}\right)^n}_{e^{-2}} - \underbrace{\left(1 - \frac{1}{n}\right)^{2n}}_{(e^{-1})^2} \right] \right]$$

$$= 0$$

9. Let, U and T denote the digits in the units place and tens place respectively of the number.

$$P(X=x,\ Y=y) = \begin{cases} \dfrac{2}{100}, & x \in \{0,\dots,18\},\ y \in \{0,\dots,81\} \text{ and } x^2-4y \text{ is a perfect square and strictly +ve.} \\ \dfrac{1}{100}, & x \in \{0,\dots,18\},\ y \in \{0,\dots,18\} \text{ and } x^2 = 4y. \\ 0, & \text{otherwise.} \end{cases}$$

~~10. Let, $Z_1 = 1$~~

~~Let, Z_2 = be the number of throws~~

10.

Let, Y_k be the number of throws needed for exactly k boxes to be occupied.

Let, $Z_1 = Y_1$

$Z_2 = Y_2 - Y_1$

$\vdots$

$Z_k = Y_k - Y_{k-1}$.

$Z_1 \equiv 1$.

$Z_2 \sim \text{Geo}\left(\frac{N-1}{N}\right)$

$Z_3 \sim \text{Geo}\left(\frac{N-2}{N}\right)$.

$\vdots$

$Z_k \sim \text{Geo}\left(\frac{N-k+1}{N}\right)$.

$E(X) = E(Y_k)$

$= E(Z_1 + Z_2 + \cdots + Z_k)$

$= 1 + \frac{N}{N-1} + \frac{N}{N-2} + \cdots + \frac{N}{N-k+1}$.

Check that for $1 \le i < j \le k$; Z_i and Z_j are indep.

then $\mathrm{Var}(X) = \mathrm{Var}(Y_k) = \sum_{i=1}^{k} \mathrm{Var}(Z_i)$

13.

(e) $P(i \le X, Y \le j) = \left(\frac{j-i+1}{N}\right)^n$ Fix $1 \le i \le j \le N$.

If $i = j$ $P(X = i, Y = j) = N^{-n}$.

Fix $1 \le i < j \le N$.

$P(X=i, Y=j) = P(X = i, Y \le j) - P(X = i, Y \le j-1)$.

$= P(i \le X, Y \le j) - P(i+1 \le X, Y \le j)$.

$= P(i \le X, Y \le j) - P(i+1 \le X, Y \le j) + P(i+1 \le X, Y \le j-1)$

$- P(i \le X, Y \le j-1)$

$= \left(\frac{j-i+1}{N}\right)^n - \left(\frac{j-i}{N}\right)^n - \left(\frac{j-i}{N}\right)^n + \left(\frac{j-i-1}{N}\right)^n$

$= \left(\frac{j-i+1}{N}\right)^n - 2\left(\left(\frac{j-i}{N}\right)_*\right)^n$

$+ \left(\left(\frac{j-i-1}{N}\right)_*\right)^n$

where $x_* = \begin{cases} x & , x \ge 0 \\ 0 & , x < 0 \end{cases}$ || $\forall\ 1 \le i \le j \le n$.

14. $Var(Y) = \sum_{i=1}^{n} \sigma_i^2$ [as $cov(X_i, X_j) = 0$]

$Y = \sum_{i=1}^{n} X_i$.

$Var(Y) = \sum_{i=1}^{n} \sigma_i^2$

$E(Y) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \mu_i$.

Midsem,

4. For $1 \le i \le W$, define.

$$X_i = \begin{cases} 1, & \text{if no black ball is drawn before the white ball labeled } i, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \le j \le B$.

$$Z_j = \begin{cases} 1, & \text{if no white ball is drawn before the black ball labeled } j, \\ 0, & \text{otherwise.} \end{cases}$$

$$Y = \sum_{i=1}^{W} X_i + \sum_{j=1}^{B} Z_j + 1.$$

$$E(Y) = \sum_{i=1}^{W} E(X_i) + \sum_{j=1}^{B} E(Z_j) + 1.$$

$$= \sum_{i=1}^{W} \frac{1}{B+1} + \sum_{j=1}^{B} \frac{1}{W+1} + 1$$

$$= \frac{W}{B+1} + \frac{B}{W+1} + 1$$

Conditional Distribution and Expectation

Def^n: Let, X and Y be random variable defined on same sample space Ω. Suppose that $A \subseteq \Omega$ with $P(A) > 0$.

The conditional distribution of X and Y given A is.

$$p_A(x,y) = P(X = x, Y = y / A).$$

$$= \frac{P([X = x] \cap [Y = y] \cap A)}{P(A)}.$$

Example:-(5.1) An urn contains 5 balls each of 3 colors. white, black & red. 10 balls are drawn with replacement. Let X & Y be the number of black balls and white balls drawn respectively.

(a) Suppose that the 1st draw is red. Find the conditional distribution of X & Y given this information.

Ans: Let, A be the event that the first draw is a red.

$X = 0, 1, \ldots, 9$

$Y = 0, 1, \ldots, 9$.

$$P(X = i; Y = j / A) = \begin{cases} \dfrac{9!}{i!\, j!\, (9-i-j)!} \cdot \left(\dfrac{5}{15}\right)^{9} & ; \text{if } i,j \in \{0, \ldots, 9\} \\ 0 & ; \text{otherwise.} \end{cases}$$

(b) Find the conditional distribution of X given that ~~atleast a~~ one ~~white ball is~~ $X > 0$.

Fix $i \in \{1, \ldots, 10\}$.

$$P(X = i \mid X > 0) = \frac{P([X = i] \cap [X > 0])}{P(X > 0)}$$

$\cancel{0 \times P(X>0)}$

~~$X \sim \text{Bin}(n, p)$~~

~~$E(X) = np$~~

$$= \frac{P(X = i)}{P(X > 0)}$$

$$= \frac{\binom{10}{i}\left(\frac{1}{3}\right)^{i}\left(\frac{2}{3}\right)^{10-i}}{1 - \left(\frac{2}{3}\right)^{10}}$$

(c) given that $X = 3$, claculate conditional distribution of Y.

$Y = 0, 1, \ldots, 7$.

$$P(Y = i \mid X = 3) = \frac{P(Y = i) \cap P(X = 3)}{P(X = 3)}$$

$$= \cancel{\frac{\frac{10!}{3!\, i!\, (7-i)!} \cdot \left(\frac{1}{3}\right)^{10}}{\binom{10}{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{7}}}$$

$$= \begin{cases} \binom{7}{i}\left(\frac{1}{2}\right)^{7} & ; \ i = 0, 1, \ldots, 7 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

Defn.:- Let X be a random variable having an expectation and A be an event of positive probability. The conditional expectation of X given A is

$$E(X/A) = \sum_n x_n P(X = x_n / A) \quad (5.2).$$

where, $x_1, x_2, \ldots$ are the possible values of X.

Exc:- Show that if X has an expectation then the sum as the RHS of (5.2) is absolutely summable.

Soln.

$$\sum_n |x_n P(X = x_n | A)| = \sum_n |x_n| P(X = x_n | A)$$

$$= \sum_n |x_n| \frac{P([X = x_n] \cap A)}{P(A)}$$

$$= \frac{1}{P(A)} \sum_n |x_n| P([X = x_n] \cap A).$$

$$\leq \frac{1}{P(A)} \underbrace{\sum_n |x_n| P(X = x_n)}.$$

finite; as X has expectation.

In Ex. (5.1) Calculate the following.

$A \subseteq B$.

$P(A|B) = \frac{np}{1-q^n}$

$= \frac{P(A)}{}$

(i) $E(X/A) = 3$

$X/A \sim \text{Bin}(9, \frac{1}{3})$.

$P(A \cap B)$

$= P(A)$.

(ii) $E(X/ X>0) = \dfrac{10/3}{1-\left(\frac{2}{3}\right)^{10}} = \dfrac{\sum_{i=0}^{10} i P(X=i)}{P(X>0)}$

(iii) $E(Y | X=i)$

Fix $i \in (0, 1, \cdots, 10)$

$E(Y/X=i) = \frac{10-i}{2}$

$Y/X=i \sim Bin(10-i, 2)$.

<u>Theorem</u>: Let X & Y be jointly distributed random variables with E(X) being defined. The $E(X) = \sum_j E(X/Y=y_j)\, P(Y=y_j)$

where $y_1, y_2, \ldots$ are the possible values

(Formula of Double Expectation)

Pf: Let, $x_1, x_2, \ldots$ be the possible values of

$\sum_j E(X/Y=y_j)\, P(Y=y_j)$

$= \sum_j \sum_i x_i\, P(X=x_i/Y=y_j)\, P(Y=y_j)$.

(*) $= \sum_i \sum_j x_i P(X=x_i/Y=y_j)\, P(Y=y_j)$

$= \sum_i x_i \sum_j P(X=x_i/Y=y_j)\, P(Y=y_j)$.

$= \sum_i x_i \sum_j P(X=x_i, Y=y_j)$.

$= \sum_i x_i\, P(X=x_i)$

$= E(X)$.

$$\boxed{\begin{array}{l}\text{Fact: } \sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij} \text{ if.} \\ \text{either } \sum_i \sum_j |a_{ij}| < \infty \text{ or } a_{ij} \geq 0 \ \forall i,j.\end{array}}$$

Exc:- There are two coins each showing head with probability p. Both the coins are tossed simultaneously repeatedly until the first coin gives a tails. If X is the number of heads for the second coin, calculate $E(X)$.

Ans: Let, Y be the number of tosses.

$$E(X \mid Y=i) = ip, \quad i = 1, 2, \ldots \quad [\because (X \mid Y=i) \sim Bin(i,p)]$$

By formula of Double expectation is.

$$E(X) = \sum_{i=1}^{\infty} E(X \mid Y=i)\, P(Y=i)$$

$$= p \sum_{i=1}^{\infty} i\, P(Y=i) = p\, E(Y) = \frac{p}{1-p}.$$

Exc:- The number of road accidents in Kolkata follows Poisson with mean λ. Suppose that each accident is reported to the Police with prob. p indep of the other accident. Let X & Y denote the number of reported and unreported accidents, respectively. Find joint distribution of X and Y.

Ans. Let, Z be the random variable denoting number of road accident in Kolkata.

$$Z \sim P(\lambda).$$

$$P(X=i, Y=k-i \mid Z=k) = \binom{k}{i} p^i q^{k-i} \quad [0 \le i \le k].$$

$$[q = 1-p]$$

Fix $i, j \in \{0, 1, 2, \dots\}$.

$$P(X=i, Y=j) = P(X=i, Y=j, Z=i+j).$$

$$= P(X=i, Y=j / Z=i+j) P(Z=i+j)$$

$$= \binom{i+j}{i} p^i q^j \cdot e^{-\lambda} \frac{\lambda^{(i+j)}}{(i+j)!}$$

$$= \frac{e^{-\lambda p} (\lambda p)^i}{i!} \frac{e^{-\lambda(1-p)} (\lambda(1-p))^j}{j!}$$

Check that $X \sim Poi(\lambda p)$, $Y \sim Poi(\lambda(1-p))$.
and X, Y are independent.

Exc:- Let X be a random variable having an expectation, and Let $A_1, A_2, \dots$ be disjoint & exhaustive events with positive probability. Show that

$$E(X) = \sum_n E(X | A_n) P(A_n).$$

$$E(X) = \sum_{i \geq 1} x_i P(X = x_i)$$

$$= \sum_{i \geq 1} x_i \sum_{n \geq 1} P([X = x_i] \cap A_n)$$

$$= \sum_{i \geq 1} x_i \sum_{n \geq 1} P(X = x_i / A_n) P(A_n).$$

$$= \sum_{i\geq 1} \sum_{n\geq 1} x_i \, P(X = x_i / A_n)\, P(A_n)$$

$$= \sum_{n\geq 1} \sum_{i\geq 1} x_i \, P(X = x_i / A_n)\, P(A_n)$$

$$= \sum_{n\geq 1} P(A_n) \sum_{i\geq 1} x_i \, P(X = x_i / A_n)$$

$$= \sum_{n\geq 1} P(A_n)\, E(X/A_n).$$

Alt. Let, $Y = n$ if A_n occurs.

$$E(X \mid Y = n) = E(X / A_n)$$

$$E(X) = \sum_n E(X \mid Y = n) P(Y = n)$$

$$= \sum_n E(X \mid A_n) P(A_n).$$

(2) A Fair die is rolled till the first 6 is observed. Given that this needed 10 rolls, Find the conditional distribution of the number of 1's observed.

$\boxed{(X/A) \sim \text{Bin}\left(9, \frac{1}{5}\right)}$ → observation.

Sol: Let X be the no. of 1's observed.

Let Y be the no. of throughs needed to observe the 1st 6.

$$\therefore P(X = i \mid Y = 10) = \frac{P([X = i] \cap [Y = 10])}{P(Y = 10)}$$

$$= \frac{\binom{9}{i} \left(\frac{1}{6}\right)^i \left(\frac{4}{6}\right)^{9-i} \frac{1}{6}}{P(Y = 10)}$$

→

since, $\sum_{i=0}^{9} P(X=i \mid Y=10) = 1$.

$\Rightarrow \quad c = \left(\frac{6}{5}\right)^9$

$P(X=i \mid Y=10) = \binom{9}{i}\left(\frac{1}{5}\right)^i\left(\frac{4}{5}\right)^{9-i}$.

③ A fair coin is tossed untill 2 consequtive heads. Calculate expected no. of tosses.

~~HH~~

T~~H~~H.

$2.\frac{1}{2^2} + 3.\frac{1}{2^3} + \cdots$

$2.\frac{1}{2^3} + \cdots$

$2.\frac{1}{2^2} +$

$\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$\frac{1}{2^2} + \frac{\frac{1}{4}}{1-\frac{1}{2}}$

$\frac{1}{2} S_n = \frac{1}{4} + \frac{1}{2}$

$S_n = \frac{3}{\cancel{4}2}$

$A_1 = \{$First toss is H, second T$\}$

$A_2 = \{$first toss is H, Second H$\}$.

$A_3 = \{$First toss is T$\}$.

Let, X be the number of tosses.

~~$A_1 = \{$First toss H, second T$\}$.~~

~~$A_2 = \{$ " "~~

$$E(X) = E(X \mid A_1)\frac{1}{4} + E(X \mid A_2)\cdot\frac{1}{4} + E(X \mid A_3)\frac{1}{2}.$$

$$= (2 + E(X))\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + (1 + E(X))\frac{1}{2}.$$

Exc:- Suppose X & Y are independent random variable, having the same distribution. and E(X) & E(Y) defined. If z is a real number such that.

$P(X+Y = z) > 0$.

the ~~calcu~~ calculate $E(X \mid X+Y = z)$.

~~$E(X \mid X+Y = z)$~~

$= \Sigma$

Claim: For any x,

$$P(X = x, X+Y = z) = P(Y = x, X+Y = z).$$

$$\text{L.H.S.} = P(X = x, Y = z - x)$$

$$= P(X = x)\,P(Y = z - x).$$

$$= P(Y = x)\,P(X = z - x).$$

$$= P(Y = x, X = z - x) = \text{R.H.S.}$$

$$\Rightarrow \quad E(X \mid X+Y=z) = E(Y \mid X+Y=z) = \alpha \text{ (say)}.$$

$$\Rightarrow \quad 2\alpha = E(X \mid X+Y=z) + E(Y / X+Y=z)$$

$$= E(X+Y \mid X+Y=z)$$

$$= z.$$

$$\Rightarrow \quad \alpha = \frac{z}{2}.$$

(5) Suppose $X \sim \text{Geo}(p)$ for a fixed $k \in \{0,1,2,\ldots\}$ find the conditional distribution of $X-k$ given that $X > k$.

$P(X-k \mid X>k)$. [Memoryless property]

i.e. $[X-k \mid X>k] \overset{d}{=} X$.

$$= \cancel{\sum_{i \geq 1} P(X-k=i \mid X>k)}$$

$$= \frac{P[(X-k=i) \cap (X>k)]}{P(X>k)}$$

$$= \frac{q^{k+i-1}\, p}{q^k}$$

$$= q^{i-1}\, p.$$

(6) X & Y are 2 independent random variable, s.t. $X \sim \text{Bin}(m,p)$ $Y \sim \text{Bin}(n,p)$. respectivly, Given that $X+Y=k \in \mathbb{N} \cup \{0\}$, find the conditional dist^n of X.

~~x / X+Y~~

Solⁿ:-

$$P(X=i / X+Y=k) = \frac{P([X=i]\cap[Y=k-i])}{P(X+Y=k)}$$

$$= \frac{\binom{m}{i}\binom{n}{j} p^k q^{n+m-k}}{\binom{m+n}{k} p^k q^{n+m-k}}$$

$$= \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}}$$

$$= \frac{\binom{m}{i}\binom{n}{k-i}}{\binom{m+n}{k}}$$ [where $\binom{m}{n}=0$, $s\le -1$ or $s\ge r+$]

(7) A fair coin is tossed till you get HT. Calculat expected no. of tosses.

~~$E(X) = E(X/HT)\,P(HT) + E(X|HH)\,P(HH)$~~ $+E($

Solⁿ:- Let X denote the number of tosses to get HT.
Let Y denote the number of tosses to get H.

For a fixed $k \in \mathbb{N}$, the conditional distribution of $X-Y$ given $Y=k$ is Geo (1/2).

$E(X|Y=k) = E(Y|Y=k) + E(X-Y/Y=k)$.
$= k + 2$

$$E(X) = \sum_{k=1}^{\infty} (k+2)\, P(Y=k). \quad \left[\text{Formula of double expectation.}\right.$$

$$= E(Y) + 2.$$

$$= 2+2 = 4.$$

Ex:- If conditional distribution of Z given $Y=k$ does not depend on k for every possible values y of Y, show that Y and Z are independent.

Several Random Variables:-

Def^n:- For random variable $X_1, \dots, X_n$ defined on the same sample space, their joint PMF is a function $p : \mathbb{R}^n \to [0,1]$ defined by

$$p(x) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \ , \ x \in \mathbb{R}^n .$$

Theorem:- If p is the joint PMF of $(X_1, \dots, X_n)$ then for a fixed $i \in \{1, \dots, n\}$, and $x \in \mathbb{R}$

$$P(X_i = x) = \sum_{x_1 : P(X_1 = x_1) > 0} \cdots \sum_{x_{i-1} : P(X_{i-1} = x_{i-1}) > 0} \ \sum_{x_{i+1} : P(X_{i+1} = x_{i+1}) > 0} \cdots \sum_{x_n : P(X_n = x_n) > 0} P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n).$$

Theorem:- Suppose that p is the joint PMF of $(X_1, \dots, X_n)$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is such that $f(X_1, \dots, X_n)$ has an expectation.

Then,

$$E\left[f(X_1, \dots, X_n)\right] = \sum_{x \in \mathbb{R}^n ; \, p(x) > 0} f(x) p(x) .$$

Example:-

1. An urn contains 10 white balls, 15 red balls and 20 black balls. Five balls are drawn without replacement. Denote by X, Y and Z the number of white, red & black balls drawn respectivly. Find join PMF of X, Y & Z.

$$p(i,j,k) = \begin{cases} \dfrac{\binom{10}{i}\binom{15}{j}\binom{20}{k}}{\binom{45}{5}} & ; \ i,j,k \geq 0, \ i+j+k = 5 . \\ 0 & ; \ \text{otherwise}. \end{cases}$$

Def^n:- For $n \geq 2$, $X_1, \dots, X_n$ are independent if

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_1) P(X_2 = x_2) \dots P(X_n = x_n)$$
$$\forall (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

i.e. the joint PMF is the product of the marginal PMF's.

Example:- A fair coin is tossed till we get n heads. Let X_1 denote the number of tosses for the first heads, X_2 denote the nos. of tosses after the first head to get the second heads. Similarly $X_3, \dots, X_n$ are defined.

Fix $x_1, \dots, x_n \in \mathbb{N}$.

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \left(\frac{1}{2}\right)^{\sum_{i=1}^{n} x_i}$$

Check that, $P(X_i = x_i) = \left(\frac{1}{2}\right)^{x_i}$.

$\therefore$ $X_1, \dots, X_n$ are independent.

Furthermore, each $X_i \sim \text{Geo}\left(\frac{1}{2}\right)$.

Def^n:- Random variables $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) if they are independent and $X_i \overset{d}{=} X_j$ for every $i, j = 1, \dots, n$

Thm:- Suppose that $X_1, \dots, X_n$ are independent.

① For any permutation π of $\{1, \dots, n\}$, $X_{\pi(1)}, \dots, X_{\pi(n)}$ are independent.

② For any m with $2 \leq m < n$, $X_1, \dots, X_m$ are independent.

3. If $A_1, \ldots, A_m \subseteq \mathbb{R}$, then the events $[X_1 \in A_1], \ldots, [X_m \in A_m]$ are independent.

4. If f and g are functions from $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, resp., to $\mathbb{R}$ then $f(X_1, \ldots, X_m)$ and $g(X_{m+1}, \ldots, X_n)$ are independent.

Proofs:-

2. WLOG assume $m < n$.

Fix $(x_1, \ldots, x_m) \in \mathbb{R}^m$.

$$P(X_1 = x_1, \ldots, X_m = x_m)$$

$$= \sum_{x_{m+1}: P(X_{m+1} = x_{m+1}) > 0} \cdots \sum_{x_n: P(X_n = x_n) > 0} P(X_1 = x_1, \ldots, X_n = x_n).$$

$$\sum_{x_{m+1}: P(X_{m+1} = x_{m+1}) > 0} \cdots \sum_{x_n: P(X_n = x_n) > 0} P(X_1 = x_1) \ldots P(X_n = x_n).$$

$$= P(X_1 = x_1) P(X_2 = x_2) \ldots P(X_m = x_m) \sum_{x_{m+1}} \cdots \sum_{x_n} P(X_{m+1} = x_{m+1}) \ldots P(X_n = x_n)$$

$$= P(X_1 = x_1) P(X_2 = x_2) \ldots P(X_m = x_m) \left(\sum_{x_{m+1}} P(X_{m+1} = x_{m+1})\right) \ldots \left(\sum_{x_n} P(X_n = x_n)\right).$$

$$= P(X_1 = x_1) P(X_2 = x_2) \ldots P(X_m = x_m).$$

3. Fix $1 \leq i_1 < \ldots < i_k \leq m$.

The $P\left(\bigcap_{j=1}^{k} [X_{i_j} \in A_{i_j}]\right)$.

$$= \sum_{x_{i_1} \in A_{i_1}} \cdots \sum_{x_{i_k} \in A_{i_k}} P(X_{i_1} = x_{i_1}, \ldots, X_{i_k} = x_{i_k}).$$

By 1 & 2, $X_{i_1}, \dots, X_{i_k}$ are independent.

Hence, $P\left(\bigcap_{j=1}^{k} [X_{i_j} \in A_{i_j}]\right)$.

$$= \sum_{x_{i_1} \in A_{i_1}} \cdots \sum_{x_{i_k} \in A_{i_k}} \prod_{j=1}^{k} P(X_{i_j} = x_{i_j}).$$

$$= \prod_{j=1}^{k} P(X_{i_j} \in A_{i_j}).$$

Hence, the events $[X_1 \in A_1], \dots, [X_m \in A_m]$ are independent.

4. $P[f(X_1, \dots, X_m) = y,\ g(X_{m+1}, \dots, X_n) = z]$

$$= \sum_{x \in \mathbb{R}^m : f(x) = y} \ \sum_{\omega \in \mathbb{R}^{n-m} : g(\omega) = z} P(X_1 = x_1, \dots X_m = x_m, X_{m+1} = \omega_1, \dots, X_n = \omega_{n-m})$$

~~Applyi~~

$$= \sum_{\substack{x \in \mathbb{R}^m: \\ f(x) = y}} \ \sum_{\substack{\omega \in \mathbb{R}^{n-m}: \\ g(\omega) = z}} P(X_1 = x_1) P(X_2 = x_2) \cdots P(X_m = x_m) P(X_{m+1} = \omega_1) \cdots P(X_n = \omega_{n-m}).$$

Applying 1 & 3 we get $X_1, \dots, X_m$ are indep & so are $X_{m+1}, \dots, X_{n-m}$

$$= \sum_{\substack{x \in \mathbb{R}^m \\ : f(x) = y}} \ \sum_{\substack{\omega \in \mathbb{R}^{n-m} \\ : g(\omega) = z}} P(X_1 = x_1, \dots, X_m = x_m)\, P(X_{m+1} = \omega_1, \dots, X_n = \omega_{n-m})$$

$$= \left(\sum_{\substack{x \in \mathbb{R}^m \\ : f(x) = y}} P(X_1 = x_1, \dots, X_m = x_m)\right) \left(\sum_{\substack{\omega \in \mathbb{R}^{n-m} \\ : g(\omega) = z}} P(X_{m+1} = \omega_1, \dots, X_n = \omega_{n-m})\right)$$

$$= P[f(x) = y] \cdot P[g(w) = z]$$

$= P[f(X_1, \dots, X_m) = y]\, P[g(X_{m+1}, \dots X_n) = z]$.

$\therefore f$ & g are independent.

Theorem:- Let $X_1, X_2, \dots, X_n$ be independent random variable each having an expectation then,

$E(X_1, \dots, X_n) = E(X_1) \dots E(X_n)$.

furthermore if each of them have variance, then $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$.

Proof: by 4,

X_1 is independent of $X_2 . X_3 . \dots X_n$ Invoting the result for $n=2$

we get,

$E(X_1 \dots X_n) = E(X_1)\, E(X_2 X_3 \dots X_n)$.

Proceeding inductively, the first claim follows.

The 2nd claim is immediate. $\because Cov(X_i, X_j) = 0\ \forall i, j$.

Definition:-

For $n \geq 2$, $X_1, \dots, X_n$ are pairwise independent if for every $1 \leq i < j \leq n$, X_i and X_j are indep.

EXC:- Let $X_1, \dots, X_n$ be i.i.d. with mean μ & variance σ^2. Calculate $Var(X_1 \dots X_n)$.

$Var(X_1 \dots X_n) = E(X_1^2 \dots X_n^2) - E^2(X_1 \dots X_n)$.

$= E(X_1^2) \dots E(X_n^2) - E^2(X_1) \dots E^2(X_n)$.

$= (\sigma^2 + \mu^2)^n - (\mu^2)^n$.

Exc. Let $X_1, \dots, X_n$ be i.i.d. such that e^{X_1} has an expectation. Show that:

$$E\left[e^{X_1+X_2+\dots+X_n}\right]$$

$$= E\left[e^{X_1}\dots e^{X_n}\right]$$

$$= \prod_{i=1}^{n} E\left(e^{X_i}\right). \quad [\text{independence}).$$

$$= \left(E\left(e^{X_i}\right)\right)^n \quad [\text{identically distributed}].$$

Defⁿ:- Infinitely many random variables $X_1, X_2, \dots$ are independent if so are $X_1, \dots X_n$ for every finite $n \geq 2$.

Exc:- Let $N, X_1, X_2, \dots$ be i.i.d. from a Poisson (λ) distribution. Calculate $E(X_1 + \dots + X_N)$.

Solⁿ:-

Let. $Z = X_1 + \dots + X_N$.

Fix ~~n≥0~~. $n \geq 1$

$$E(Z \mid N=n) = E\left(\sum_{i=1}^{n} X_i \mid N=n\right).$$

$$= \sum_{i=1}^{n} E(X_i \mid N=n).$$

$$= \sum_{i=1}^{n} E(X_i) \quad [\because \text{independent}]$$

$$= n\lambda.$$

Applying double expectation rule,

$$E(Z) = \sum_{n=0}^{\infty} E(Z \mid N=n) P(N=n).$$

$$= \sum_{n=0}^{\infty} n\lambda P(N=n) = \lambda \sum_{n=0}^{\infty} n P(N=n) = \lambda E(N)$$
$$= \lambda^2.$$

Random Walks:-

Def^n:- Let $X_1, X_2, \ldots$ be i.i.d. random variables taking value $+1$ & -1 each with probability $\frac{1}{2}$.

Define $S_n = X_1 + X_2 + \cdots + X_n$, $n \geq 1$.

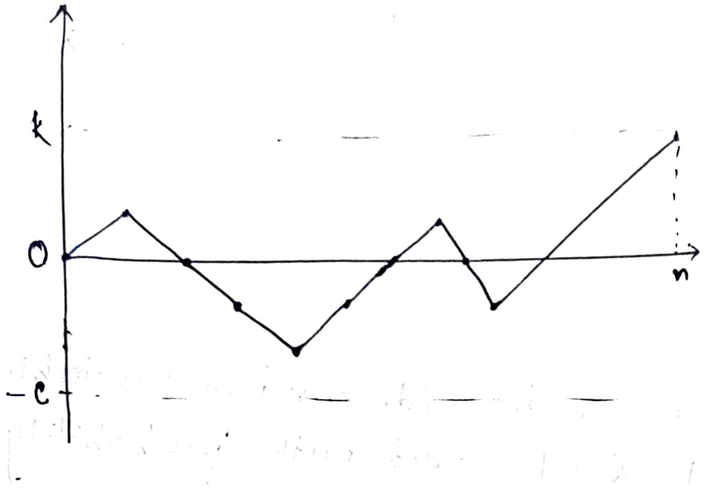
$\{S_n\}$ is called a "symmetric random walk".

Observation:

1. S_n can take those values in $\{-n, \ldots, n\}$ which have the same parity as n.

2. $$P(S_n = k) = \begin{cases} \binom{n}{\frac{n+k}{2}} \frac{1}{2^n} & ; k \in \{-n, \ldots, n\}. \text{ and has same parity as } n. \\ 0 & ; \text{ otherwise.} \end{cases}$$

Exc:- Consider a gambler entering a Casino with c rupees ($c \in \mathbb{N}$). Suppose that after n games, he has won k rupees. What is the conditional probability that he didn't run out of money in the middle?



doesn't touches (-c).

Notice that the desired probability equals

$$P\left(\min_{0\le i\le n} S_i > -c \;\middle|\; S_n = k\right).$$

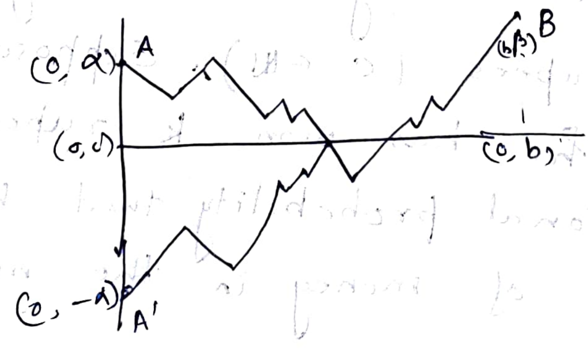
$$= \frac{P\left[\left(\min_{0\le i\le n} S_i > -c\right) \cap (S_n = k)\right]}{P(S_n = k)}.$$

***

<u>Theorem (Reflection principle)</u>

Let, $A = (0, \alpha)$ & $B = (b, \beta)$ be points such that $\alpha, \beta, b \in \mathbb{N}$. Then the number of paths (means for each step +1 or -1) from A to B ~~with~~ which touch the horizontal axis atleast once equals the number of paths from $A' = (0, -\alpha)$ to B.

Proof:-



Consider a path from A to B which touches the horizontal axis at least once. Let T be the left most point where the path touches the horizontal axis. The segment of the path to the left of T is reflected along the horizontal axis. A' and the remaining ~~path~~ part of the path left unchanged. What results is a path from A' to B. It's elementary to check that the above function is a bijection from set of paths from A to B which touch the horizontal axis to the set of paths ~~which~~ from A' to B. Hence the proof follows.

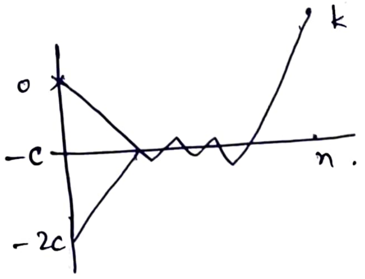
***

$$= 1 - P\left(\min_{0\le i\le n} S_i \le -c \mid S_n = k\right).$$

$$= 1 - \frac{P\left(\min_{0\le i\le n} S_i \le -c,\ S_n = k\right)}{P(S_n = k)}.$$

$$= 1 - \frac{\binom{n}{\frac{n+k}{2}+c}\frac{1}{2^n}}{\binom{n}{\frac{n+k}{2}}\frac{1}{2^n}}$$

$$= 1 - \frac{\binom{n}{\frac{n+k}{2}+c}}{\binom{n}{\frac{n+k}{2}}}$$



By reflection principle

paths from (0,0) to (n,k) that hit the line (-c)

= # paths from (0,-2c) to (n,k)

$$= \binom{n}{\frac{n+k}{2}+c}\frac{1}{2^n}.$$

Exc :- Consider another gambler in the same casino. She plans to leave the casino the moment she has won K rupees. Find the probability that she achives this in exactly n games.

Soln :- Let, (S_n) be the RW. The required probability is,

$$P\left(S_n = K, \max_{0 \le i \le n-1} S_i < K\right)$$

$= \frac{1}{2^n}$ (# paths from $(0,0)$ to (n,K) that never touch the line K before touching (n,K))

$= \frac{1}{2^n}$ (# paths from $(0,0)$ to $(n-1, K-1)$ that never touch the line K.)

$= 1 - \left[\frac{1}{2^n}\right.$ (# paths from $(0, 2k)$ to $(n-1, K-1)$ $\left.)\right]$

$$= 1 - \frac{1}{2^n}\binom{n-1}{\frac{n+k}{2}}$$

$$\boxed{\begin{aligned} p+q &= n-1. \\ p-q &= -k-1. \\ &\text{solve for } p \,\&\, q \end{aligned}}$$

Exc :- Find the probability that a random walk starting at 0 hits 1 for the first time at time n.

Theorem (Ballot Theorem):-

Consider an election run by two candidates A and B. Given that they polled p and q respectively, where $p > q$, the probability that candidate A was strictly ahead through out the counting is $\frac{p-q}{p+q}$.

Proof:- Let $n = p+q$.

***

required probability is

$$= P\left(\min_{1 \le i \le n} S_i \ge 1 \,/\, S_n = p-q\right).$$

$$= \frac{P\left(\min_{1 \le i \le n} S_i \ge 1, S_n = p-q\right)}{P(S_n = p-q)}$$

$$P\left(\min_{1 \le i \le n} S_i \ge 1, \ S_n = p-q\right).$$

$= 2^{-n}$ (# Paths from $(0,0)$ to $(n, p-q)$ which do not touch the horizontal axis after time 0).

$= 2^{-n}$ (# paths from $(1,1)$ to $(n, p-q)$ which do not touch the horizontal ~~axis after~~ axis).

$= 1 - 2^{-n}$ (# paths from $(1,1)$ to $(n, p-q)$ that touch the horizontal axis.

$= 1 - 2^{-n}$ (# paths from $(1,-1)$ to $(n, p-q)$ that touch the horizontal axis)

$$= 1 - 2^{-n} \cdot \binom{n-1}{\frac{n+p-q}{2}}$$

$$= 1 - 2^{-n} \binom{n-1}{p}$$

$\frac{n+p-q}{2}$

$\frac{p+q+p-q}{2}$

p.

$\therefore$ Required probability is,

$$= \frac{1 - 2^{-n} \binom{n-1}{p}}{2^{-n} \binom{n}{p}}. \quad \leftarrow \text{Wrong}$$

$$P\left(\min_{1 \le i \le n} S_i \ge 1 \,/\, S_n = p-q\right).$$

$$= \frac{P\left(\min_{1 \le i \le n} S_i \ge 1,\ S_n = p-q\right)}{P(S_n = p-q)}.$$

$$= \frac{2^{-n}(\#\text{ path from } (0,0) \text{ to } (n, p-q) \text{ that do not touch horizontal axis.})}{P(S_n = p-q)}$$

$$\neq \frac{2^{-n}\left[\binom{n}{p} - \#\text{ path from } (0,0) \text{ to } (n,p-q) \text{ that touch the horizontal } \cancel{\text{axis after } 0}\right.}{2^{n}\binom{n}{p}}.$$

$$= \frac{\binom{n}{p} - \binom{n-1}{p}}{\binom{n}{p}}$$

$$= 1 - \frac{\binom{n-1}{p}}{\binom{n}{p}}$$

$$= 1 - \frac{(n-1)!}{p!\,(n-p-1)!} \cdot \frac{p!\,(n-p)!}{n!}$$

$$= 1 - \frac{n-p}{n}$$

$$= \frac{n-n+p}{n}$$

$$= \frac{p}{n}$$

$$= \frac{p}{p+q}.$$

*** $P\left(\min_{1 \le i \le n} S_i \geq 1 \,/\, S_n = p-q\right)$ $\left[\begin{matrix} n = p+q \\ (S_n) \to RW \end{matrix}\right]$.

$\Rightarrow P\left(\min_{1 \le i \le n} S_i \geq 1,\ S_n = p-q\right).$

$= 2^{-n}$ (# Paths from $(0,0)$ to $(n, p-q)$ that do not touch the horizontal axis)

~~$\Rightarrow 2^{-n}\left[\binom{n}{p}\right.$ − # Paths from $(0,0)$ to $(n,p-q)$ that touch the horizontal axis$\left.\right]$.~~

$= 2^{-n}$ (# paths from $(1,1)$ to $(n, p-q)$ that do not touch horizontal axis)

$= 2^{-n}$ [(# all paths from $(1,1)$ to $(n, p-q)$)

− (# paths from $(1,1)$ to $(n, p-q)$ that touch horizontal axis)]

$\longrightarrow$

$\stackrel{\text{refl.}}{=} 2^{-n}\left[\binom{n-1}{p-1} - \text{(\# paths from } (1,-1) \text{ to } (n, p-q))\right]$

$= 2^{-n}\left[\binom{n-1}{p-1} - \binom{n-1}{p}\right]$

$= 2^{-n}\left[\frac{(n-1)!}{(p-1)!\,q!} - \frac{(n-1)!}{p!\,(q-1)!}\right]$

$= 2^{-n}\left[\frac{(n-1)!}{p!\,q!}[p-q]\right]$

$\therefore P\left[\min_{1\le i\le n} S_i \geq 1 \,/\, S_n = p-q\right]$

$$= \frac{P\left[\min_{1\le i\le n} S_i \geq 1,\ S_n = p-q\right]}{P\left[S_n = p-q\right]}$$

$$= \frac{2^{-n}\left(\frac{(n-1)!}{p!\,q!}(p-q)\right)}{2^{-n}\binom{n}{p}}.$$

$= \frac{p-q}{n}$

$= \frac{p-q}{p+q}.$

Exc:- A gambler enters a casino (simple & fair) with c rupees ($c \in \mathbb{N}$). Given that after n games he has won k rupees, find the probability that he was NOT out of money at middle.

Solⁿ:- Let, S_n be a RW. The desired probability equals,

$$P\left(\min_{1\leq i\leq n} S_i \geq 1-c \,/\, S_n = k\right).$$

$$= 1 - P\left(\min_{1\leq i\leq n} S_i \leq -c \,/\, S_n = k\right)$$

$$= 1 - \frac{P(\#\text{ paths from } (0,0) \text{ to } (n,k) \text{ that touch } -c)}{\binom{n}{\frac{n+k}{2}}}$$

$$\overset{\text{refl. princ.}}{=} 1 - \frac{(\#\text{ paths from } (0,-2c) \text{ to } (n,k))}{\binom{n}{\frac{n+k}{2}}}$$

$$= 1 - \frac{\binom{n}{\frac{n+k}{2}+c}}{\binom{n}{\frac{n+k}{2}}}.$$

Exc:- Another gambler, in the same casino plans to leave the moment she has won K rupees, where $K \in \mathbb{N}$. Find the probability that she achives this in n games.

Solⁿ:- Let, S_i denote the ammount she has won, after i games.

Req. Probability $= P\left(\max_{0\leq i\leq n-1} S_i < K,\ S_n = K\right).$

$= 2^{-n}$ (# paths from $(0,0)$ to (n,K) that do not touch K till $(n-1)$)

$= 2^{-n}$ (# paths from $(0,0)$ to $(n-1, K-1)$ which do not touch K)

$= 2^{-n}$ [(# paths from $(0,0)$ to $(n-1, K-1)$)

— (# paths from $(0,0)$ to $(n-1, K-1)$ which touch)]

$\overset{\text{refl. prin.}}{=} 2^{-n} \left[\binom{n-1}{\frac{n-k}{2}} - (\text{\# paths from } (0,2K) \text{ to } (n-1, K-1)) \right]$

$$= 2^{-n} \left[\binom{n-1}{\frac{n-k}{2}} - \binom{n-1}{\frac{n+k}{2}} \right]$$

Exc:- Find the probability that a RW hits 0 for the first time after 0 at time $2n$.

Solⁿ:- $P(S_i \neq 0 \ \forall i = 1(1)2n-1 ;\ S_{2n} = 0)$.

$$= 2P\left(\min_{1 \le i \le 2n-1} S_i \ge 1,\ S_{2n} = 0\right).$$

$= 2 \cdot 2^{-2n}$ (# paths from $(1,1)$ to $(2n,0)$ which does not touch 0 before $2n$)

$= 2^{-2n+1}$ (# paths from $(1,1)$ to $(2n-1, 1)$ which do not touch 0 ~~before 2n~~).

$$= 2^{-2n+1}\left[\binom{2n-2}{n-1} - (\text{\# paths from } (1,-1) \text{ to } (2n-1,1))\right]$$

$$= 2^{-2n+1}\left[\binom{2n-2}{n-1} - \binom{2n-2}{n}\right].$$

$$= 2^{-2n+1}\left[\frac{(2n-2)!}{(n-1)!\,(n-1)!} - \frac{(2n-2)!}{n!\,(n-2)!}\right].$$

$$= 2^{-2n+1}\frac{(2n-2)!}{n!\,(n-1)!}\left[n - (n-1)\right] = 2^{-2n+1}\frac{(2n-2)!}{n!\,(n-1)!}.$$

$x_1 + x_2 + x_3 + \cdots + x_n = r$ (non-ve) $\binom{n+r-1}{r}$

$\frac{n+r-1}{(n+1)!\,r!}$

Exc:- Suppose n left brackets & n right brackets are randomly permuted. What is probability that a legitimate arrangement is obtained?

$p - q = k$

$p + q = n$

Sol:- The required probability is,

$$P\left(\min_{1\le i\le 2n} S_i \ge 0 \mid S_{2n} = 0\right)$$

$$= \frac{P\left(\min_{1\le i\le 2n} S_i \ge 0, S_{2n} = 0\right)}{P(S_{2n} = 0)}$$

$(x + \cdots + x^6)^{20}$ $(1 + \cdots + x^i)$

$$= \frac{(\#\text{ paths from } (0,0) \text{ to } (2n,0)) - (\#\text{ paths from } (0,0) \text{ to } (2n,0) \text{ which touch } (-1))}{\binom{2n}{n}}$$

$$= \frac{\binom{2n}{n} - (\#\text{paths from } (0,-2) \text{ to } (2n,0))}{\binom{2n}{n}}$$

$$= \frac{\binom{2n}{n} - \binom{2n}{n+1}}{\binom{2n}{n}} = \frac{\frac{(2n)!}{n!\,n!} - \frac{2n!}{(n+1)!(n-1)!}}{\frac{(2n)!}{n!\,n!}}$$

$$= \frac{\frac{(2n)!}{n!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}} = \frac{1}{n+1}.$$

$4!, 3!$

$?4$

$a(1 - P(X=0))$

$P(X=1) + P(X=2) + \dots + P(X=n) =$

$a \sum 1 - q^n$

$\frac{7!}{4!}$

$1 - P(X=0)$

Thm:- Let (S_n) be a RW. Denote $M_n = \max_{0 \le i \le n} S_i$, for a fixed n. Then for $r \geq 0$

$$P(M_n = r) = P(S_n = r) + P(S_n = r+1)$$

Pf:- $P(M_n < r)$

$$= \sum_{j=-n}^{r-1} P(M_n < r, S_n = j).$$

$$= \sum_{j=-n}^{r-1} [P(S_n = j) - P(M_n \geq r, S_n = j)]$$

$$\overset{\text{refl}^n \text{ prin}^l}{=} \sum_{j=-n}^{r-1} [P(S_n = j) - P(S_n = j - 2r)].$$

$$= \sum_{j=-n}^{r-1} P(S_n = j) - \sum_{k=-n-2r}^{-r-1} P(S_n = k) = P(S_n < r) - P(S_n < -r). \ast\ast$$

$\ast\ast$ $P(M_n = r)$

$= P(M_n < r+1) - P(M_n < r)$

$= [P(S_n < r+1) - P(S_n < -(r+1))] + [P(S_n < r) - P(S_n < -r)]$

$= [P(S_n < r+1) - P(S_n < r)] + [P(S_n < -r) - P(S_n < -r-1)]$

$= P(S_n = r) + P(S_n = -r-1)$

$\overset{\$}{=} P(S_n = r) + P(S_n = r+1)$

Exc:- Suppose n left brackets & n right brackets are randomly permuted. What is probability that a legitimate arrangement is obtained?

Solⁿ:-

legitimate arrangement ≡ for any position number of left brackets before it is greater than or equal to number of right brackets before it.

∴ the required probability

$$= P\left(\min_{1\le i\le 2n} S_i \ge 0 \mid S_{2n} = 0\right)$$

$$= \frac{P\left(\min_{1\le i\le 2n} S_i \ge 0,\ S_{2n} = 0\right)}{P(S_{2n}=0)}$$

$$= \frac{\frac{1}{2^{2n}}\left[(\#\text{ paths from } (0,0) \text{ to } (2n,0)) - (\#\text{paths from } (0,0) \text{ to } (2n,0) \text{ which touch } (-1))\right]}{\frac{1}{2^{2n}}\cdot\binom{2n}{n}}$$

$$= \frac{\binom{2n}{n} - \binom{2n}{n+1}}{\binom{2n}{n}}$$

$$= \frac{\frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n+1)!\,(n-1)!}}{\frac{(2n)!}{n!\,n!}} = 1 - \frac{n}{n+1} = \frac{1}{n+1}.$$

Thm:- Let (S_n) be a random walk. Denote

$$M_n = \max_{0 \le i \le n} S_i \text{ , for a fixed } n.$$

Then for $r \geq 0$.

$$P(M_n = r) = P(S_n = r) + P(S_n = r+1).$$

Pf:- $P(M_n < r)$

$$= \sum_{j=-n}^{r-1} P(M_n < r, S_n = j)$$

$$= \sum_{j=-n}^{r-1} \left[P(S_n = j) - P(M_n \geq r, S_n = j) \right]$$

$$= \sum_{j=-n}^{r-1} P(S_n = j) - \sum_{j=-n}^{r-1} P(M_n \geq r, S_n = j)$$

$$= \sum_{j=-n}^{r-1} P(S_n = j) - \sum_{j-2r=-n-2r}^{-r-1} P(S_n = j - 2r)$$

$$= \sum_{j=-n}^{r-1} P(S_n = j) - \sum_{k=-n-2r}^{-(r+1)} P(S_n = k).$$

$$= \sum_{j=-n}^{r-1} P(S_n = j) - \sum_{k=-n}^{-(r+1)} P(S_n = k).$$

$$= P(S_n < r) - P(S_n < -r).$$

$\therefore$ $P(M_n = r)$

$$= P(M_n < r+1) - P(M_n < r)$$

$$= P(S_n < r+1) - P(S_n < -(r+1)) - P(S_n < r) + P(S_n < -r)$$

$$= P(S_n = r+1) + P(S_n = -r) = P(S_n = r) + P(S_n = r+1).$$

Thm: Let, $(S_n : n \geq 0)$ be a random walk and $M_n = \max(S_0, \dots, S_n)$; $n \geq 0$.

Then for $n, r \geq 0$.

$$P(M_n \geq r) = P(S_n = r) + 2P(S_n > r).$$

Pf:- Using the formula proved previously,

$$P(M_n \geq r) = \sum_{j=r}^{n} P(M_n = j)$$

$$= \sum_{j=r}^{n} \left[P(S_n = j) + P(S_n = j+1)\right]$$

$$= \sum_{j=r}^{n} P(S_n = j) + \sum_{j=r+1}^{n} P(S_n = j).$$

$$= P(S_n = r) + P(S_n > r) + P(S_n > r)$$

$$= P(S_n = r) + 2P(S_n > r).$$

Alt. pf.

$$P(M_n \geq r) = P(M_n \geq r, S_n \geq r) + P(M_n \geq r, S_n < r)$$ [Thm of total probability]

$$= P(S_n \geq r) + P(S_0 = 2r, S_n < r).$$

[$\because [S_n \geq r] \cap [M_n \geq r] = [S_n \geq r]$] [By reflection principal]

$$= P(S_n \geq r) + P(S_0 = 0, S_n < -r)$$

$$= P(S_n \geq r) + P(S_n < -r).$$

$$= P(S_n \geq r) + P(S_n > r)$$

$$= P(S_n = r) + 2P(S_n > r).$$

Exc:- Let, S_n be a random walk and fix an odd $n \geq 1$. Define $M = \max(S_0, \ldots, S_n)$

① Find PMF of M_n

② Calculate $E(M_n - |S_n|)$.

1. Clearly, M_n takes values $0(1)n$.

for an odd $r \in \{0, 1, \ldots, n\}$.

$$P(M_n = r) = P(S_n = r) + P(S_n = r+1)$$

$$= \binom{n}{\frac{n+r}{2}} 2^{-n} + 0.$$

$$= \binom{n}{\frac{n+r}{2}} 2^{-n}$$

for an even $r \in \{0, 1, \ldots, n\}$

$$P(M_n = r) = P(S_n = r) + P(S_n = r+1)$$

$$= 0 + \binom{n}{\frac{n+r+1}{2}} 2^{-n}$$

$$= \binom{n}{\frac{n+r+1}{2}} 2^{-n}.$$

$\therefore$ PMF of M_n is.

$$P(M_n = r) = \begin{cases} \binom{n}{\frac{n+r}{2}} 2^{-n} & ; \ r = \text{odd and } r \in 0(1)n \\ \binom{n}{\frac{n+r+1}{2}} 2^{-n} & ; \ r = \text{even and } r \in (0(1)n) \\ 0 & ; \ \text{otherwise.} \end{cases}$$

2. $E(M_n)$

$= \sum_{r=0}^{n} P(M_n > r)$.

$= \sum_{r=1}^{n} P(M_n \geq r)$.

$= \sum_{r=1}^{n} [P(S_n = r) + 2P(S_n > r)]$

$= \sum_{r=1}^{n} P(S_n = r) + \sum_{r=1}^{n} 2P(S_n > r)$

$= P(S_n > 0) + \sum_{r=1}^{n} [P(S_n > r) + P(S_n < -r)]$

$= P(S_n > 0) + \sum_{r=1}^{n} P(|S_n| > r)$. $\left[\because P(|S_n| > r) = P(S_n > r) + P(S_n < -r)\right]$

$= P(S_n > 0) + \sum_{r=0}^{n} P(|S_n| > r) - P(|S_n| > 0)$

$= \frac{1}{2} + E(|S_n|) - 1$ $\left[\because n \text{ is odd } P(S_n = 0) = 0 \Rightarrow P(S_n > 0) = \frac{1}{2}\right]$

$\Rightarrow E(M_n) - E(|S_n|) = -\frac{1}{2}$.

$\Rightarrow E(M_n - |S_n|) = -\frac{1}{2}$.

☒ The last visit to origin till time $2n$ is defined by $\max \{0 \leq k \leq 2n : S_k = 0\}$

Let, $L_n = \frac{1}{n} \max. \{0 \leq k \leq n : S_{2k} = 0\}$

then $0 \leq L_n \leq 1$.

Thm: (Arc sine law for the last visit to origin)

For fixed $0 < x < y < 1$,

$$\lim_{n \to \infty} P(x < L_n \leq y) = \frac{2}{\pi}\left(\sin^{-1}\sqrt{y} - \sin^{-1}\sqrt{x}\right)$$

P.T.O.

Observation:

L_n takes values $\frac{i}{n}$; $i \in 0(1)n$

Lemma:
$P(L_n = \frac{k}{n}) = P(S_{2k}=0)\, P(S_{2n-2k}=0) \quad \forall\, k \in 0(1)n.$

Pf:- $P(L_n = \frac{k}{n}) = P(\max\{0 \le j \le n \,;\, S_{2j}=0\} = k)$

$= P(S_{2k}=0, S_{2k+1} \ne 0, \dots, S_{2n} \ne 0).$

Define $\tilde{S}_j = S_{2k+j} - S_{2k}$, $0 \le j \le 2n-2k.$

$\therefore P(L_n = \frac{k}{n}) = P(S_{2k}=0, \tilde{S}_1 \ne 0, \dots, \tilde{S}_{2n-2k} \ne 0).$

$= P(S_{2k}=0)\, P(\tilde{S}_1 \ne 0, \dots, \tilde{S}_{2n-2k} \ne 0).$ (by principle 1)

$= P(S_{2k}=0)\, P(S_1 \ne 0, \dots, S_{2n-2k} \ne 0).$

$P(S_1 \ne 0, \dots, S_{2n-2k} \ne 0).$

$= 2P(S_1 = -1, S_2 \ne 0, \dots, S_{2n-2k} \ne 0)$

$= 2P(S_1 = -1, \tilde{S}_1 \ne 0, \dots, \tilde{S}_{2n-2k-1} \ne 0)$ ~~(by principle 1)~~

$= 2P(S_1 = -1) \cdot P(\tilde{S}_1 \ne 0, \dots, \tilde{S}_{2n-2k-1} \ne 0)$ (by principle 1)

$= 2P(S_1 = -1)\, P(S_1 \le 0, \dots, S_{2n-2k-1} \le 0)$ (by principle 2)

$= P(\max_{0 \le i \le 2n-2k-1} S_i = 0)$

$= P(M_{2n-2k-1} = 0)$

$= P(S_{2n-2k-1} = 0) + P(S_{2n-2k-1} = 1)$ (From previous result.)

$= P(S_{2n-2k-1} = 1).$

$= P(S_{2n-2k} = 0).$

Some results needed for the proof:-

- **Definition:-** Two collection of random variables $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are independent if

$P[X_1 \in A_1, X_2 \in A_2, \dots, X_m \in A_m, Y_1 \in B_1, \dots, Y_n \in B_n]$

$= P[X_1 \in A_1, X_2 \in A_2, \dots, X_m \in A_m]\, P[Y_1 \in B_1, Y_2 \in B_2, \dots, Y_n \in B_n]$

for all choices of $A_1, A_2, \dots, A_m, B_1, \dots, B_n \subseteq \mathbb{R}.$

Fact:- The collection $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are independent iff for functions $f: \mathbb{R}^m \to \mathbb{R}$ and $g: \mathbb{R}^m \to \mathbb{R}$, $f(X_1, \dots, X_m)$ and $g(Y_1, \dots, Y_m)$ are independent random variables.

Proof is out of Syllabi

Back to Random Walk:-

Let, S_n is a random walk & $\tilde{S}_j = S_{k+j} - S_k$

Claim: $(S_1, \dots, S_k)$ and $(\tilde{S}_1, \dots, \tilde{S}_{n-k})$ are independent.

Pf:- Follows from observation that each S_i is the function of $(X_1, \dots, X_k)$ and for $1 \le i \le k$, and each $\tilde{S}_j$ is a function of $(X_{k+1}, \dots, X_n)$ for $1 \le j \le n-k$.

Priciple 1:- The increments after time k are independent of the walk in time k.

Principle 2:- For any $r \ge 1$,
$(\tilde{S}_1, \dots, \tilde{S}_r) \stackrel{d}{=} (S_1, \dots, S_r)$

In other words, $(\tilde{S}_j\ j \ge 1)$ is also a simple random walk.

Lemma 2:- As $n \to \infty$

$$P(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}.$$

For sequence a_n, b_n with $b_n \neq 0$ for all n, we write $a_n \sim b_n$ if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$

~~Now, $P(x \le L_n \le y) = \sum_{j=[nx]+1}^{[ny]} P(L_n = \frac{j}{n})$~~

- **Fact** (Stirling's Formula)

As $n \to \infty$ $\quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Now,
$$P(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$$
$$= \frac{2n!}{(n!)^2} \cdot 2^{-2n}$$
$$\sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n} \cdot 2^{-2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\right)^2}$$
$$= \frac{2\sqrt{\pi n}}{2\pi n}$$
$$= \frac{1}{\sqrt{\pi n}}$$

Now,
$$P(x < L_n \le y) = \sum_{j=[nx]+1}^{[ny]} P(L_n = \tfrac{j}{n})$$
$$= \sum_{j=[nx]+1}^{[ny]} P(S_{2j} = 0)\, P(S_{2n-2j} = 0).$$
$$= \sum_{j=[nx]+1}^{[ny]} \frac{1}{\sqrt{\pi j}} \frac{1}{\sqrt{\pi (n-j)}}$$
$$= \frac{1}{\pi} \sum_{j=[nx]+1}^{[ny]} \frac{1}{\sqrt{j(n-j)}}$$
$$= \frac{1}{\pi} \frac{1}{n} \sum_{j=[nx]+1}^{[ny]} \frac{1}{\sqrt{\frac{j}{n}(1-\frac{j}{n})}}$$

Fact:- If f is a continuous function from an interval $[a,b]$ (such that $-\infty < a < b < \infty$) to $\mathbb{R}$, then
$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=[na]+1}^{[nb]} f\left(\frac{j}{n}\right) = \int_a^b f(t)\,dt$$

Fact:- (Fundamental thm of integral calculus)
If $F : [a,b] \to \mathbb{R}$ is continuously differentiable (i.e. diff'ble & derivative is continuous), then
$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

$$\therefore P(x < L_n \le y) = \frac{1}{\pi} \int_x^y \frac{dt}{\sqrt{t(1-t)}}$$
$$= \frac{1}{\pi}\left(\sin^{-1}\sqrt{y} - \sin^{-1}\sqrt{x}\right)$$

Now, $\lim_{n\to\infty} \sqrt{\pi n}\, P(S_{2n} = 0) = 1$ (by previous Lemma).

$\longrightarrow$

Fix $0 < \varepsilon < 1$. There exists N such that for all $n \geq N$, it holds that.

$$(1-\varepsilon)(\pi n)^{-\frac{1}{2}} \leq P(S_{2n}=0) \leq (1+\varepsilon)(\pi n)^{-\frac{1}{2}}.$$

Fix $0 < x < y < 1$.

$$P(x < L_n \leq y) = \sum_{j=[nx]+1}^{[ny]} P\left(L_n = \frac{j}{n}\right).$$

$$= \sum_{j=[nx]+1}^{[ny]} P(S_{2j}=0)P(S_{2n-2j}=0).$$

If $n \geq \left\lceil \frac{N}{x} \right\rceil \wedge \left\lceil \frac{N}{1-y} \right\rceil$, then $\forall j \in [nx, ny]$

$j \geq N$; $n-j \geq N$.

$$(1-\varepsilon)^2 \frac{1}{\pi\sqrt{j(n-j)}} \leq P(x < L_n \leq y) \leq (1+\varepsilon)^2 \frac{1}{\pi\sqrt{j(n-j)}}$$

$$= \sum_{j=[nx]+1}^{[ny]}$$

4. Let, X takes values $x_1, x_2, \dots$

$E(X \mid A)$

$$= \sum_j x_i P(X = x_j \mid A) = \sum_{j: x_j \in A} x_j \frac{P([X=x_j] \cap A)}{P(A)}.$$

$$= \sum_{j: x_j \in A} \frac{x_j P(X=x_j)}{P(A)}.$$

$$P(1_A = 1) = P(A)$$

$$E(X 1_A) = \sum_j x_j P(X=x_j, 1_A = 1) = \sum_{j: x_j \in A} x_j P(X=x_j)$$

Probability Generating Function:-

X takes values $0, 1, 2, 3, \dots$

$$\boxed{f(s) = \sum_{n=0}^{\infty} P(X=n) s^n \quad ; \ |s| \leq 1} \quad \text{PGF}.$$

Digression:-

Let, $\{a_n : n \geq 0\}$ be a sequence of real numbers. The power series corresponding to $\{a_n : n \geq 0\}$ is

$$f(s) = \sum_{n=0}^{\infty} a_n s^n.$$

$$R = \frac{1}{\limsup_{n \to \infty} |a_n|^{\frac{1}{n}}}, \text{ with the convention that } \frac{1}{0} = \infty, \ \frac{1}{\infty} = 0.$$

FACT:- $\forall$ s with $|s| < R$, the series is absolutely summable.

We can talk about the function,

$$f : (-R, R) \longrightarrow \mathbb{R}.$$

FACT :- The function f is infinitely many times diff'ble on $(-R, R)$. The k-th derivative of f is given by

$$\underbrace{D^k f(s)}_{\text{kth derivative of } s} = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n s^{n-k},$$

PGF :-

Unless stated otherwise, the random variables talked about in this topic will take values in $0, 1, 2, \dots$, i.e. non-negative integers.

Def^n :- For a Random variable X its probability generating function (PGF) is

$$f(s) = \sum_{n=0}^{\infty} P(X=n) s^n, \quad |s| \le 1. \qquad \text{—(1)}$$

Remark 1 :- The series in (1) is absolutly convergent $\forall\ |s| \le 1$ because $\sum_{n=0}^{\infty} |P(X=n)s^n| \le \sum_{n=0}^{\infty} P(X=n) = 1$.

Remark 2 :- Clearly $\limsup_{n\to\infty} P(X=n)^{\frac{1}{n}} \le 1$, and hence the radius of convergence of f is at least 1.

Remark 3 :- $f(s) = E(s^X)$, $|s| \le 1$.

Example :- 1. $X \sim \text{Bin}(n,p)$. PGF of f is.

$$f(s) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} \cdot s^x$$

$$= (ps+q)^n.$$

Alt :- $X = X_1 + \cdots + X_n$ $\quad X_i$'s are iid. $\forall\ i = 1(1)n$.

$$E(s^X) = E(s^{X_1 + X_2 + \cdots + X_n}) = \left(E(s^{X_1})\right)^n = (ps+q)^n.$$

$$\left(\frac{\lambda s}{n} + 1 - \frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty}\left(1 - \frac{\lambda(1-s)}{n}\right)^n.$$

$$\left(\left(1 - \frac{\lambda(1-s)}{n}\right)^{\frac{n}{\lambda(1-s)}}\right)^{\lambda(1-s)}$$

$$e^{\lambda(1-s)}$$

Ex. 2 :- $X \sim P(\lambda)$. PGF of f is

$$f(s) = \sum_{x=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^x}{x!} \cdot s^x$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda s)^x}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda s}$$

$$= e^{-\lambda(1-s)}.$$

Ex. 3 :- $X \sim \text{Geo}(p)$. PGF of f is.

$$f(s) = \sum_{x=1}^{\infty} (1-p)^{x-1} p \cdot s^x$$

$$= \frac{p}{q} \sum_{x=0}^{\infty} qs^x = \frac{p}{q} \cdot \frac{qs}{1-qs} = \frac{ps}{1-qs}.$$

Thm:- Let, f be the PGF of X. Then f is continuous on $[-1,1]$ and infinitely many times diff'ble on $(-1,1)$; Furthermore for every $n \geq 0$,

$$P(X=n) = \frac{1}{n!} D^n f(0);$$

where $D^n f$ is nth derivative of f and $D^0 f$ is just f.

Pf:- Since the radius of convergence of f is at least 1, it follows from the stated facts that f is infinitely diff'ble on $(-1,1)$ and

$$D^n f(s) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, P(X=n)\, s^{k-n};\ |s|<1.$$

$$\Rightarrow D^n f(0) = n!\, P(X=n).$$

$$\Rightarrow P(X=n) = \frac{1}{n!} D^n f(0).$$

All that remains to be shown is that f is continuous at ± 1.

We prove continuity at 1. Fix $\varepsilon > 0$.

Let, N be such that

$$\sum_{N=N+1}^{\infty} P(X=n) < \frac{\varepsilon}{3}.$$

The polynomial $\sum_{n=0}^{N} P(X=n) s^n$ is a continuous function. Hence $\exists\ \delta \in (0,1)$ such that,

$$\left| \sum_{n=0}^{N} P(X=n) s^n - \sum_{n=0}^{N} P(X=n) \right| < \frac{\varepsilon}{3}.$$

whenever $1-\delta \leq s \leq 1$.

Therefore,

$$|f(s) - f(1)| \leq \left| \sum_{n=N+1}^{\infty} P(X=n) s^n \right| + \left| \sum_{n=0}^{N} P(X=n) s^n - \sum_{n=0}^{N} P(X=n) \right| + \left| \sum_{n=N+1}^{\infty} P(X=n) \right|$$

(by choice of N)

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad \left[\because \left|\sum_{n=N+1}^{\infty} P(X=n) s^n\right| \leq \left|\sum_{n=N+1}^{\infty} P(X=n)\right| \right]$$

$$= \varepsilon.$$

this proves continuity of f at 1.

Ily for (-1).

Exc:- PGF of $\mathrm{Bin}(n,p)$ is $(ps+q)^n$. Suppose that $\{p_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} n p_n = \lambda \in (0,\infty)$.

Let, $X_n \sim \mathrm{Bin}(n, p_n)$.

The PGF of X_n is:

$$f_n(s) = (p_n s + 1 - p_n)^n, \quad |s| \leq 1.$$

$$= (1 + (s-1) p_n)^n$$

$$= \left(1 + \frac{\lambda(s-1)}{n}\right)^n.$$

$$= e^{\lambda(s-1)}.$$

Thm:- Let, $X_1, X_2, \dots, X_\infty$ be random variables taking non-negative integer values, with respective PGF's $f_1, \dots, f_\infty$.

Then, $\lim_{n\to\infty} P(X_n = k) = P(X_\infty = k)$, $\forall\, k \in \mathbb{N} \cup \{0\}$,

if and only if

$\lim_{n\to\infty} f_n(s) = f_\infty(s) \quad \forall\, s \in [-1,1]$

Proof:- $(\Rightarrow)$

$$|f_n(s) - f_\infty(s)| = \left| \sum_{k=0}^{\infty} \left(P(X_n=k) - P(X_\infty=k)\right) s^k \right|$$

$$\leq \sum_{k=0}^{\infty} |P(X_n=k) - P(X_\infty=k)|\,|s^k|.$$

fix $\varepsilon > 0$. and s such that $|s| < 1$.

Let, K be such that $\sum_{n=k+1}^{\infty} |s|^n \leq \frac{\varepsilon}{3}$.

Suppoce that $\{a_n\}$ and $\{b_n\}$ are bounded sequences of real numbers. Show that $\exists$ $1 \leq n_1 < n_2 < \dots$ such that $\{a_{n_k}\}$ and $\{b_{n_k}\}$ converge.

Pf:- ~~We start with proving the only if part,~~

~~i.e. we assume that~~

~~$\lim_{n\to\infty} f_n(s) = f_\infty(s)$ for every~~ $|s|<1$.

~~fix $\varepsilon > 0$ and s such that $|s| < 1$.~~

Let, ~~$N \in \mathbb{N}$ be~~ such that,

~~$|s|^{N+1} \leq \frac{1}{2} \varepsilon (1-|s|)$~~

Fact:-

Sol^n:- For $m \in \mathbb{N}$, $\{a_{mn} ; n \geq 1\}$ is a bdd. sequence of real numbers. Then there exists natural numbers $1 \leq n_1 \leq n_2 \leq \dots$ such that for every fixed m, $\{a_{mn_k} : k \geq 1\}$ converges.

Proof: By Bolzano - Weirtrass thm there exists $1 \leq r_{11} \leq r_{12} \leq \dots$ such that $\{a_{1r_{1r}}\}$ converges as $r \to \infty$.

Clearly $\{a_{2k_{1r}} : k \geq 1\}$ is bdd. Applying Bolzano - weirtrass again we get a further subsequence $\{a_{2r_{2k}} : k \geq 1\}$ which converges.

Proceeding similarly, we get natural numbers $\{r_{ij} : i,j \geq 1\}$ with the following property.

(i) For every i, $\{a_{ir_{ij}} ; j \geq 1\}$ converges.

(iii) $\{r_{i+1,j} ; j \geq 1\}$

Branching Process :-

Def^n:- Let, $p_0, p_1, \ldots$ be non-negative numbers adding to 1. Suppose that $(Y_{nj} : n, j \geq 1)$ is a family of i.i.d. random variables with

$$P(Y_{nj} = k) = p_k \quad , k = 0, 1, 2, \ldots$$

Let $X_0 = 1$ and define X_n recursively as follows

$$X_n = \sum_{j=1}^{X_{n-1}} Y_{nj}.$$

Then family of random variables $(X_n : n \geq 0)$ is called a branching process with progeny distribution $(p_k : k \geq 0)$.

Remark: For every $n \geq 1$, X_{n-1} is independent of $(Y_{n1}, Y_{n2}, \ldots)$.

Unless stated otherwise, $(X_n : n \geq 0)$ is always a branching process with progeny distribution $(p_k : k \geq 0)$.

Thm:- If ϕ is the PGF of the progeny distribution i.e.

$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k \quad , |s| \leq 1,$$

then for every $n \geq 1$, the PGF of X_n is.

$$\phi^{(n)} = \phi . \phi \ldots \phi \text{ (n times)}.$$

Pf:- Since $X_1 = Y_{11}$ it is immediate that the PGF of X_1 is ϕ and hence the claim is true for $n = 1$. let it be true for n. As induction hypothesis

For a fixed $t \in [-1, 1]$.

$$E(t^{X_{n+1}}) = E\left(t^{\sum_{j=1}^{X_n} Y_{n+1,j}}\right).$$

$$= \sum_{k=0}^{\infty} E\left(t^{\sum_{j=1}^{X_n} Y_{n+1 j}} \,/\, X_n = k\right) P(X_n = k).$$

$$= \sum_{k=0}^{\infty} E\left(t^{\sum_{j=1}^{k} Y_{n+1,j}}\right) P(X_n = k) \cdot \left[\because Y_{n+1,j}\text{'s are indep of } X_n\right]$$

$$= \sum_{k=0}^{\infty} \phi(t)^k P(X=k).$$

$$= E\left(\phi(t)^{X_n}\right) = E(s^{X_n}).$$

$\left[\text{clearly } s = \phi(t) \in [-1,1]\right]$

$$= \phi^{(n)}(s) \text{ (by induction hypothesis)}$$

$$= \phi^{(n)}(\phi(t))$$

$$= \phi^{n+1}(t).$$

Def^n:- The event of extinction is defined as

$$E = \bigcup_{n=1}^{\infty} [X_n = 0].$$

The extinction probability of $(X_n : n \geq 0)$ is defined as $p_e = P(E)$.

Now, quite obviously $[X_n = 0] \subseteq [X_{n+1} = 0]$

$$\Rightarrow \lim_{n\to\infty} P(X_n = 0) = P(E).$$

Thm:- $\lim_{n\to\infty} \phi^n(0) = p_e = \min\{0 \leq s \leq 1 : \phi(s) = s\}$.

proof:- $\forall n \geq 1$ define $A_n = [X_n = 0]$

clearly $A_1 \subseteq A_2 \subseteq \dots \subseteq A_N \subseteq A_{N+1} \dots$

& $\bigcup A_n = E$.

i.e. $A_n \uparrow E$ and hence $P(A_n) \uparrow P(E)$.

In other words,

$$p_e = P(E)$$
$$= \lim_{n\to\infty} P(A_n).$$
$$= \lim_{n\to\infty} P(X_n = 0).$$
$$= \lim_{n\to\infty} \phi^{(n)}(0).$$

It remains to show that $p_e = \min\{0 \leq s \leq 1 \mid \phi(s) = s\}$

~~RHS is nor~~

Denote the RHS by s_0, Since ϕ is continuous and $\phi^{(n)}(0) \to p_e$, it follows that

$$\phi(\phi^{(n)}(0)) \to \phi(p_e).$$

However, $\phi(\phi^{(n)}(0))$ ~~is nothing~~

$$= \phi^{(n+1)}(0) \to p_e.$$

$\Rightarrow \phi(p_e) = p_e$. and hence $p_e \geq s_0$.

Note that $0 \leq s_0$. Since ϕ is non-decreasing on $[0,1]$, we get

$$\phi(0) \leq \phi(s_0) = s_0.$$
$$\Rightarrow \phi(\phi(0)) \leq \phi(s_0) = s_0.$$

inductively $\phi^{(n)}(0) \leq s_0$, $n \geq 1$.

$\Rightarrow p_e \leq s_0$. (Letting $n \to \infty$).

$\therefore p_e = s_0$. proved.

Example:- Consider the progeny distribution $(p_k : k \geq 0)$ where $p_1 = 1$. In this case $\{0 \leq s \leq 1;\ \phi(s) = s\}$
$= [0,1]$.

Thm:- If $p_1 \neq 1$, then $\#\{0 \leq s \leq 1 : \phi(s) = s\} \leq 2$.

Pf:- Case I. $p_0 + p_1 = 1$. Since, $p_1 < 1$.
the equation $p_0 + p_1 s = s$ has exactly 1 root which is $s = 1$.

Case II. $p_0 + p_1 < 1$.

Assume for the sake of contradiction that

$$\#\{0 \leq s \leq 1\ \phi(s) = s\} \geq 3.$$

that $\exists\ 0 \leq s_1 < s_2 < s_3 \leq 1$ such that

$$\phi(s_i) = s_i\ ;\ i = 1,2,3.$$

Define

$$f(s) = \phi(s) - s.$$

$$Df(s) = D\phi(s) - 1.$$

$$D^2 f(s) = D^2 \phi(s).$$

Applying Rolle's theorem thrice we get that there exists $t \in (s_1, s_3)$ such that

$$D^2 f(t) = 0.$$

However

$$D^2 f(t) = D^2 \phi(t).$$

$$= \sum_{n=0}^{\infty} n(n-1) p_n t^{n-2} > 0$$

$\because p_n > 0$ for some $n \geq 1$
$\because p_0 + p_1 < 1$.

$(\Rightarrow \Leftarrow)$

HW-6.

9. $A = \{$All the rolls till the first 6 given even numbers$\}$.

$P(A) = \{$The first 6 is before the first odd number$\}$.

$= \sum_{k=1}^{\infty} P($6 comes in the kth roll and the ones before that yield 2 or 4$)$.

$$= \sum_{k=1}^{\infty} \frac{1}{6} \left(\frac{2}{6}\right)^{k-1}.$$

$$= \frac{1/6}{1 - \frac{2}{6}} = \frac{1}{4}.$$

Now, $P(A) = P(A \mid \text{First roll gives } 6) \cdot \frac{1}{6} + P(A \mid \text{First roll gives } 1,3 \text{ or } 5) \frac{3}{6}$

$+ P(A \mid \text{first roll is } 2 \text{ or } 4) \frac{2}{6}$.

$$= 1 \cdot \frac{1}{6} + 0 \cdot \frac{3}{6} + P(A) \cdot \frac{2}{6}.$$

$$= \frac{1}{6} + \frac{2P(A)}{6}$$

$\Rightarrow 4P(A) = 1$

$\Rightarrow P(A) = \frac{1}{4}$.

10. $$P(Y_i = y_i \mid X = k) = \frac{P(Y = y_i, X = k)}{P(X = k)}.$$

$$= \frac{}{\left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}}.$$

$(~~\phi = y~~ \mid X = k) \sim Bin(k,$

Distribution of the number of times 1 is obtained till the first 6.

Ans: it will follow $(X+1) \sim Geo(\frac{1}{2})$.

$~~P(X=0)~~ = P(X=0 \mid Y=k)$

$= \left(\frac{4}{6}\right)^{k-1} \cdot \frac{1}{6}$.

$\therefore P(X=0) = \sum_{k=1}^{\infty} \left(\frac{4}{6}\right)^{k-1} \cdot \frac{1}{6}$.

$= \dfrac{\frac{1}{6}}{1-\frac{4}{6}} = \frac{1}{2}$.

$~~P(X=1 \mid Y=k)~~$.

13.(b) T is the number of tosses needed.

$E(T) = q(1+E(T)) + p^2 2 + pq(2+E(T))$.

$(\longleftarrow)_2$

Let, $(X_n : n \geq 0)$ be a branching process with progeny distribution $(p_k : k \geq 0)$. Its extinction probability is denoted by p_e.

Exc:- Suppose that X_1 has expectation say m. Then calculate $E(X_n)$ for each n.

Solⁿ:- $E(X_1) = m$

$E(X_2) = E\left(\sum_{j=1}^{X_1} Y_{2j}\right)$.

Fix $k \geq 0$.

$E(X_2 \mid X_1 = k) = E\left(\sum_{j=1}^{k} Y_{2j} \mid X_1 = k\right)$.

$= \sum_{j=1}^{k} E(Y_{2j})$ [because X_1 is indep. of $(Y_{2j}$'s$)$]

$= k \cdot m$. $[\because E(Y_{2j}) = E(X_1)]$.

$\Rightarrow E(X_2) = \sum_{k=0}^{\infty} E(X_2 \mid X_1 = k)\, P(X_1 = k)$.

$= \sum_{k=0}^{\infty} km\, P(X_1 = k)$.

$= m \sum_{k=0}^{\infty} k\, p_k$

$= m^2$

Thm. I

1. If $p_0 = 0$, then $p_e = 0$.
2. If $p_0 > 0$, and $\sum_{n=1}^{\infty} n p_n \leq 1$, then $p_e = 1$.
3. If $p_0 > 0$, and $\sum_{n=1}^{\infty} n p_n > 1$, then $0 < p_e < 1$.

$\sum n p_n$ \ p_0	0	> 0
< 1	Impossible.	$p_e = 1$.
$= 1$	(only possible case, $p_1 = 1$) $p_e = 0$.	$p_e = 1$.
> 1	$p_e = 0$	$0 < p_e < 1$.

Proof 1:- If ϕ denotes the PGF of the progeny distribution then $\phi(0) = 0$.

$\because p_e = \min\{0 \leq s \leq 1 : \phi(s) = s\}$ it follows that $p_e = 0$.

Proof 2:- Let, $p_0 > 0$, $\sum_{n=1}^{\infty} n p_n \leq 1$.

Case I:- $p_0 + p_1 = 1$. In this case.

$\phi(s) = p_0 + p_1 s$ and hence $\phi(s) = s$.

has exactly 1 solution $s = 1$, thus $p_e = 1$.

Case II: $p_0 + p_1 < 1$.

Let, if possible let $p_e < 1$.

$\phi(p_e) = p_e$ and $\phi(1) = 1$.

Thus $\exists$ there exists $\xi \in (p_e, 1)$ such that

$$D(\phi)(\xi) = 1.$$

Since $p_0 + p_1 < 1$, $\exists$ there exists $n_0 \geq 2$ such that $p_{n_0} > 0$.

hence, $D\phi(s) = \sum_{n=1}^{\infty} n p_n s^{n-1}$ is strictly increasing in $(0,1)$. Then,

$$\sum_{n=1}^{\infty} n p_n \geq \sum_{n=1}^{\infty} n p_n \left(\frac{1+\xi}{2}\right)^{n-1} > \sum_{n=1}^{\infty} n p_n \xi^{n-1} = 1. \quad [\because D(\phi)(\xi) = 1]$$

$$\left[\frac{1+\xi}{2} < 1\right]$$

$$\sum_{n=1}^{\infty} n p_n \geq \sum_{n=1}^{\infty} n p_n \left(\frac{1+\xi}{2}\right)^{n-1} \quad \left[\because \frac{1+\xi}{2} < 1\right]$$

$$> \sum_{n=1}^{\infty} n p_n (\xi)^{n-1}$$

$$= D\phi(\xi)$$

$$= 1. \quad (\Rightarrow \Leftarrow)$$

Which proves that $p_e = 1$.

Proof: 3 Let, $p_0 > 0$ and $\sum_{n=1}^{\infty} n p_n > 1$.

Need to show that $0 < p_e < 1$.

Clearly, $p_e > 0$ because $p_0 = \phi(0) > 0$.

as p_e is the minimum root of $\phi(s) = s$.

Our first claim is that $\exists\ \xi \in (0,1)$ such that $D\phi(\xi) \geq 1$

Since, $\sum_{n=1}^{\infty} np_n > 1$,

$\exists\ N \in \mathbb{N}$ such that

$$\sum_{n=1}^{N} np_n > 1.$$

Clearly $\lim_{s\to 1} \sum_{n=1}^{N} np_n s^{n-1} = \sum_{n=1}^{N} np_n > 1.$

therefore $\exists\ \xi \in (0,1)$ such that

$$1 \leq \sum_{n=1}^{N} np_n \xi^{n-1} \leq D\phi(\xi).$$

Since we are trying to prove that $p_e < 1$, assume for the sake of contradiction that $p_e = 1$, that is $\phi(s) > s$ for every $s \in [0,1)$.

In particular $\phi(\xi) > \xi$.

The mean value theorem implies that

$$\frac{\phi(1) - \phi(\xi)}{1-\xi} = D\phi(\theta) \quad \text{for } \theta \in (\xi, 1).$$

Cleary, $D\phi(\theta) \geq D\phi(\xi) \geq 1.$

$$\begin{aligned} \phi(1) &= \phi(1) - \phi(\xi) + \phi(\xi). \\ &= (1-\xi)\, D\phi(\theta) + \phi(\xi). \\ &\geq 1 - \xi + \phi(\xi) \quad [\because D\phi(\theta) \geq 1] \\ &> 1 \quad [\because \phi(\xi) > \xi] \end{aligned}$$

$\Rightarrow \phi(1) > 1$ $\Rightarrow\Leftarrow$

$\Rightarrow p_e < 1.$

EXM:- Calculate the extinction probability for a branching process with progeny distribution Bernoulli (p) for all choices of $p \in [0,1]$.

$\Rightarrow$

$$\phi(s) = ps + (1-p).$$

$$\therefore p_e = \begin{cases} 0, & p = 0 \\ 1, & p > 0 \end{cases}$$

EXM:- Consider an organism, starting with one "good" cell, and evolving as follows. A good cell gives birth to another cell which is good or bad with probability p and $(1-p)$ respectively. The moment a bad cell is born from a good cell, both the cells kill each other. The above procedure is followed by all the good cells independently. Calculate the probability that at some point no good cell is left.

Let, X_n denote the number of good cells in the nth generation.

Clearly X_n $(n \geq 0)$ is a branching process with progeny distribution $(p^k(1-p) : k \geq 0)$

$$\phi(s) = E(s^k).$$

$$= \sum_{k=0}^{\infty} p^k s^k (1-p)$$

$$= \frac{1-p}{1-ps}.$$

$$\frac{(1-p)}{1-ps} = s.$$

$$\Rightarrow 1-p = s - ps^2$$

$$\Rightarrow ps^2 - s - (p-1) = 0.$$

$$\Rightarrow s = \frac{1-p}{p}, 1.$$

$$\therefore p_e = \begin{cases} 1, & 0 \le p \le \frac{1}{2} \\ \frac{1-p}{p}, & \frac{1}{2} \le p \le 1 \end{cases}$$

Exm:- Concider the progeny distribution $(p_n : n \ge 0)$.

with $p_n = \frac{1}{(n+1)(n+2)}$ $n \ge 0$.

Calculate $p_e = ?$.

Calculate $D\phi(s)$ and $\lim_{s \uparrow 1} D\phi(s)$.

$$\sum_{n=1}^{\infty} n p_n = \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)} = \infty$$

$$\Rightarrow \lim_{s\uparrow 1} D\phi(s) \text{ does not exist.}$$

Theorem:- Let, X_n be a branching process with progeny distribution $(p_k : k \ge 0)$ satisfying

$$m = \sum_{n=1}^{\infty} n p_n < 1.$$

Let, the survival time be defined by

$$S = \sup \{n \ge 0 : X_n \ge 1\}$$ then. $E(S)$ exists an

$$E(S) \le \frac{m}{1-m}.$$

Pf:-

$$\sum_{n=1}^{\infty} P(S \ge n).$$

$$= \sum_{n=1}^{\infty} P(X_n \ge 1).$$

$$\le \sum_{n=1}^{\infty} E(X_n)$$

$$= \sum_{n=1}^{\infty} m^n$$

$$= \frac{m}{1-m}$$

Exc:- Check this for Bernoulli(p).

$\ln(1-s)$.

$-s - \frac{s^2}{2} - \frac{s^3}{3}$

$$\sum_{k\ge 0} \frac{s^k}{(k+1)(k+2)} = \sum_{k\ge 0} \frac{s^k}{k+1} - \frac{s^k}{k+2}$$

$$= -\frac{1}{s}\sum_{k\ge 0} \frac{s^{k+1}}{k+1} + \frac{1}{s^2}\sum \frac{s^{k+2}}{k+2}$$

$$= \cancel{\frac{1}{s^2}\left(\ln(1-s) + s\right)}$$

$$- \frac{1}{s}\left(\ln(1-s)\right).$$

$\ln(1-s) + s - s\ln(1-s) = s^3$ $= -\left[\frac{1}{s}\sum_{k\ge 0}\frac{-s^{k+1}}{k+1} - \frac{1}{s^2}\sum_{k\ge 0}\frac{-s^{k+2}}{k+2}\right]$

$(1-s)\ln(1-s) = -s(1-s)(1+s)$

$\frac{1}{s^2}\cancel{\ln(1-s)} = \frac{1}{s}\ln(1-s)$ $= -\left[-\frac{1}{s}\ln(1-s) + \frac{1}{s^2}\left(\ln(1-s) + s\right)\right]$

$\frac{1}{s^2}\ln(1-s) + \frac{1}{s} - \frac{1}{s}\ln(1-s) = s.$

HW-7

1

$$E(T_n) = E\left(\sum_{j=1}^{n} 1(S_j = 0)\right).$$

$$= \sum_{j=1}^{n} P(S_j = 0)$$

$$= \sum_{j=1}^{[\frac{n}{2}]} P(S_{2j} = 0).$$

$$\Rightarrow \sum_{j=\infty}^{[\frac{n}{2}]} \frac{1}{\sqrt{\pi j}}$$

$$P(S_{2j} = 0) \sim \frac{1}{\sqrt{\pi j}} \quad (*)$$

Claim:

$E(T_n) \sim \sqrt{\frac{2}{\pi}}\, n^{\frac{1}{2}}$ as $n \to \infty$.

Fix $0 < \varepsilon < 1$; from $(*)$ we get $N \geq 2$. Such that,

$$\left| \sqrt{\pi j}\, P(S_{2j} = 0) - 1 \right| \leq \varepsilon, \quad j \geq N!$$

For $n \geq 2N$,

$$E(T_n) = \sum_{j=1}^{[n/2]} P(S_{2j} = 0).$$

$$= \sum_{j=1}^{N-1} P(S_{2j} = 0) + \sum_{j=N}^{[n/2]} P(S_{2j} = 0).$$

$$\leq \sum_{j=1}^{N-1} P(S_{2j} = 0) + \frac{1+\varepsilon}{\sqrt{\pi}} \sum_{j=N}^{[n/2]} \frac{1}{\sqrt{j}}$$

$$\leq \sum_{j=1}^{N-1} P(S_{2j} = 0) + \frac{1+\varepsilon}{\sqrt{\pi}} \sum_{j=N}^{[\frac{n}{2}]} \int \frac{1}{\sqrt{t}}$$

$$= \sum_{j=1}^{N-1} P(S_{2j} = 0) + \frac{1+\varepsilon}{\sqrt{\pi}} \int_{N-1}^{[n/2]} \frac{dt}{\sqrt{t}}.$$

$$E(T_n) = \sum_{j=1}^{N-1} P(S_{2j} = 0) + \frac{1+\varepsilon}{\sqrt{\pi}} \left[2\sqrt{[n/2]} - 2\sqrt{N-1} \right]$$

$$\therefore \frac{E(T_n)}{\sqrt{n}} \leq \frac{K}{\sqrt{n}} + \frac{1+\varepsilon}{\sqrt{\pi}}\, 2\sqrt{\frac{[n/2]}{n}}.$$

where $k = \sum_{j=1}^{N-1} P(S_{2j} = 0) - 2\frac{(1+\varepsilon)}{\sqrt{\pi}}\sqrt{N-1}$.

$$\limsup_{n \to \infty} \frac{E(T_n)}{\sqrt{n}} \leq \sqrt{\frac{2}{\pi}}\,(1+\varepsilon).$$

Since, ε is arbitrary, we get

$$\limsup_{n \to \infty} \frac{E(T_n)}{\sqrt{n}} \leq \sqrt{\frac{2}{\pi}}.$$

A similar argument shows that the liminf $\geq \sqrt{\frac{2}{\pi}}$.

Hence, $E(T_n) \sim \sqrt{\frac{2}{\pi}}\sqrt{n}$, $n \to \infty$.

2.

a. $P(T > 2n) = P(S_1 \neq 0, \ldots, S_{2n} \neq 0).$

$$= P(S_{2n} = 0).$$

$$= \binom{2n}{n} 4^{-n}.$$

2.b. $E = \{$ ~~Eventually~~ the walk never hits 0 after first step $\}$.

$[T > 2n] \downarrow E$

$\Rightarrow$ ~~$P[T > 2n]$~~ $P(E) = \lim_{n\to\infty} P(T > 2n).$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\pi n}} = 0.$$

2\. c. by ratio test

$$\sum_{n=1}^{\infty} P(T > 2n) = \infty$$

and hence T does not have an expectation.

3\. b. $$E(M) = \sum_{r \geq 0} P(M > r).$$

$$= \sum_{r \geq 1} P(M \geq r)$$

$$= \sum_{r \geq 1} \left[P(S_n = r) + 2P(S_n > r) \right]$$

$$= \sum_{r \geq 1} P(S_n = r) + \sum_{r \geq 1} 2P(S_n \geq r)$$

$$= P(S_n > 0) + \sum_{r \geq 1} \left[P(S_n > r) + P(S_n < -r) \right]$$

$$= \frac{1}{2} (P(S_n \neq 0)) + \sum_{r \geq 1} P(|S_n| > r).$$

$$= \frac{1}{2} P(S_n \neq 0) + \sum_{r \geq 0} P(|S_n| > r)$$
$$- P(|S_n| > 0).$$

$$= \frac{1}{2} P(S_n \neq 0) + E(|S_n|)$$
$$- P(S_n \neq 0).$$

$$\Rightarrow E(M - |S_n|) = -\frac{1}{2} P(S_n \neq 0).$$

$$= -\frac{1}{2^{n+1}} \binom{n}{\frac{n}{2}}$$

10\.
b. Toss a coin with prob. of head p indep of X & Y.

Define, $$W = \begin{cases} X & \text{if heads occurs} \\ Y & \text{if Tails occurs.} \end{cases}$$

11\. $$\sup_{|s| \leq 1} |f_n(s) - f_\infty(s)|$$

$$= \sup_{|s| \leq 1} \left| \sum_{k=0}^{\infty} P(X_n = k) s^k - \sum_{k=0}^{\infty} P(X_\infty = k) s^k \right|.$$

$$\leq \sup_{|s| \leq 1} \sum_{k=0}^{\infty} |P(X_n = k) - P(X_\infty = k)| \, |s^k|$$

$$\leq \sum_{k=0}^{\infty} |P(X_n = k) - P(X_\infty = k)| \quad \left[|a-b| = a+b-2\min(a,b) \right]$$

$$= \sum_{k=0}^{\infty} \left[P(X_n = k) + P(X_\infty = k) - 2\min\left(P(X_n = k), P(X_\infty = k)\right) \right]$$

$$= 1 + 1 - 2 \sum_{k=0}^{\infty} \min\left(P(X_n = k), P(X_\infty = k)\right)$$

$$\leq 2 - 2 \sum_{k=0}^{N} \min\left[P(X_n = k), P(X_\infty = k)\right]$$

$$\therefore \lim_{n \to \infty} \sup_{|s| \leq 1} |f_n(s) - f_\infty(s)| \leq 2 - 2 \sum_{k=0}^{N} P(X_\infty = k). \quad \forall N \in \mathbb{N}$$

In other words,

$$\limsup_{n \to \infty} \sup_{|s| \leq 1} \cancel{|f_n(s) - f_\infty(s)|} \leq$$

letting $N \to \infty$ [$\because$ true for $\forall N \in \mathbb{N}$

$$\lim_{n\to\infty} \sup_{|s|\le 1} | f_n(s) - f_\infty(s)| \le 2\left[1 - \sum_{k=0} P(X_\infty = k)\right]$$

$$= 0.$$

12.

($\Rightarrow$)

$$\lim_{n\to\infty} P(X_n = k) = P(Z=k), \quad k = 0,1,2,\dots$$

$\Rightarrow$ Let, f is the PGF of Z, then,

$$\lim \sup_{|s|\le 1} |\phi_n(s) - f(s)| = 0 \quad \text{(by previous result)}.$$

and $\phi_n(s) \to \phi(s)$, $n\to\infty$, $|s|<1$. (given).

~~(by previous result)~~.

and $\lim_{s\to 1} f(s) = 1$. [$\because$ it is PGF.]

$$f(s) = \phi(s) \quad |s|<1.$$

Hence, $\lim_{s\uparrow 1} \phi(s) = \lim_{s\uparrow 1} f(s) = f(1) = 1.$

($\Leftarrow$) $\lim_{n\to\infty} \phi_{(n)}(s) = \phi(s)$.

and $\lim_{s\uparrow 1} \phi(s) = 1$.

shall show $\exists$ random variable Z taking values $0,1,2,\dots$ such that

$$\lim_{n\to\infty} P(X_n = k) = P(Z=k), \; k = 0,1,2,\dots$$

Repeating the proof of the said result we can get $1 \le n_1 < n_2 < \dots$ such that

$$p_k = \lim_{n\to\infty} P(X_{n_r} = k) \text{ exists.}$$

following the subsequence steps of the same proof, we get

$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k, \quad |s|<1.$$

$\because \lim_{s\uparrow 1} \phi(s) = 1$. (given).

we get. $\sum_{k=0}^{\infty} p_k = 1$.

13. $X_1, X_2, \dots$ are i.i.d. $\mathbb{N}\cup\{0\}$ valued for $n\ge 1$.

$Y_1, Y_2, \dots, Y_n$ are iid. Bernoulli $(1/n)$,

indep of $(X_1, X_2, \dots)$.

$$Z_n := \sum_{i=1}^{n} X_i Y_i.$$

Let, the PGF of X_i's be $\phi(s)$.

$$\therefore E(s^{Z_n}) = E\left(s^{\sum_{i=1}^n X_i Y_i}\right).$$

$$=$$

$$=$$

$$= \left[1 + \frac{\phi(s)-1}{n}\right]^n.$$

$$\therefore f_n(s) = \left[1 + \frac{\phi(s)-1}{n}\right]^n.$$

$$\lim_{n\to\infty} f_n(s) = \left[1 + \frac{\phi(s)-1}{n}\right]^n$$

$$= \exp(\phi(s) - 1).$$

clearly $\lim_{s\uparrow 1} f(s) = 1$. [$\because \phi(1) = 1$].

$\longrightarrow$

$\therefore \exists$ a random variable Z with $\lim_{n\to\infty} P(Z_n = k)$
$= P(Z=k), \; k = 0, 1, \dots]$ and with PGF f.

ⓑ $P(Z=0) = f(0)$.
$= e^{\phi(0)} - 1$.
$= e^{p_0 - \text{bop}}$

$P(Z=1) = p_1 e^{p_0 - 1}$.

14. Let ϕ & ψ be the PGF of X_1 and Y_1, resp.

ⓐ $\phi(s)^k = \psi(s)^k$, $0 \le s \le 1$.
Since, $\phi(s), \psi(s) \in [0,1]$, is holds that.
$\phi(s) = \psi(s)$, $0 \le s \le 1$.

ⓑ Let, f be the PGF of N.
$f(\phi(s)) = f(\psi(s))$, $0 \le s \le 1$.
Since, $P(N=0) < 1$, f is strictly increasing on $[0,1]$. and hence. 1-1. Therefore $\phi(s) = \psi(s)$, $0 \le s \le 1$.

H.W. 8

1. $p_e = \begin{cases} \dfrac{2-p-\sqrt{(2-p)^2 - 4q^2}}{2p}, & \frac{1}{3} < p \le 1. \\ 1, & 0 \le p \le \frac{1}{3}. \end{cases}$

If mean of the progeny distribution ≤ 1. and $p_0 > 0$, then $p_e = 1$.

5. $P(E \mid X_1 = 5)$.
$= \lim_{n\to\infty} P(X_n = 0 \mid X_1 = 5)$. $= \lim_{n\to\infty} (\phi^{(n)}(0))^5$
$= p_e^5$.

$P(X_1 = 0 \mid X_1 = 5) = 0$.
$P(X_2 = 0 \mid X_1 = 5) = (\phi(s))^5$.
$P(X_3 = 0 \mid X_1 = 5) = (\phi^{(2)}(s))^5$.

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Qn:- A random sample of size n is drawn from $\{1, 2, \dots, N\}$. What is the probability that there ~~exist~~. are exactly k distinct units?.

$= \binom{N}{k} P(\text{Each of unit } 1, \dots, k \text{ is drawn and nothing else is drawn})$.

$A_i = \{\text{Unit } i \text{ is chosen and nothing other than } \{1, \dots, k\} \text{ is chosen}\}$, $1 \le i \le k$.

$B = \{\text{Nothing other than } 1, \dots, k \text{ is chosen}\}$
~~P(Each of unit,~~ $\underline{1, \dots, k \text{ is chosen \& nothing else}})$
$\ne P(A_1 \cap \dots \cap A_k \cap B)$.

$B \setminus (A_1 \cap \dots \cap A_n) = \bigcup_{i=1}^{k} E_i$.

$E_i = \{$ Unit $i, k+1, \dots, N$ is not in the sample.

$\therefore P(A_1 \cap \dots \cap A_k) = P(B) - P(B \setminus (A_1 \cap \dots \cap A_k))$

$$= P(B) - P\left(\bigcup_{i=1}^{k} E_i\right).$$

$$= P(B) - \sum_{i=1}^{k} P(E_i) + \sum_{1 \le i < j \le k} P(E_i \cap E_j)$$

$$= \left(\frac{k}{N}\right)^n - k \cdot \left(\frac{k-1}{N}\right)^n + \binom{k}{2}\left(\frac{k-2}{N}\right)^n \dots$$

$$= \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \left(\frac{k-j}{N}\right)^n.$$

Continuous Random Variable:-

Choose a number at random from $[0,1]$.

$\Omega = [0,1]$.

Ideally, would like a function.

$$P : \mathcal{P}(\Omega) \to [0,1].$$

Satisfying the following:

1. $P(\Omega) = 1$.
2. For disjoint $A_1, A_2, \dots \subseteq \Omega$,
$$P\left(\bigcup_{n \ge 1} A_n\right) = \sum_{n \ge 1} P(A_n).$$
3. $P([a,b]) = b-a$, $0 \le a \le b \le 1$.

For technical reasons, such a P does not exist!

So, ~~in~~ instead of $\mathcal{P}(\Omega)$, we work with a subset $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ which has the following properties.

0. $\Omega \in A$.
1. If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$. (Closed under countable unions).
2. If $A_1, A_2, \dots \in \mathcal{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. (Closed under countable unions).

Thankfully, a collection $\mathcal{A}$ of sets that are of interest, a function $P : \mathcal{A} \to [0,1]$ exists. satisfying the desired axioms.

this $\mathcal{A}$ is called "Sigma field".

Definition:-

A function $F:\mathbb{R}\to[0,1]$ is a cummulative distribution function (CDF) if F satisfies the following :

1. For $x<y \Rightarrow F(x)\le F(y)$, i.e. F is non-decreasing.
2. The function F is right continuous.
3. As $x\to-\infty$ $F(x)\to 0$.
4. As $x\to+\infty$ $F(x)\to 1$.

FACT

Fundamental Theorem of Probability

For any F satisfying 1-4, in the above definition, there exists a random variable X defined on some sample space such that

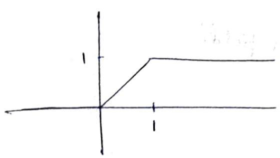
$$P(X\le x)=F(x)$$

for every real number x.

Example:-

Choose a number at random from $[0,1]$. Let us denote it by X. If X were a random variable then its CDF would be.

$$F(x)=\begin{cases}0 & x<0.\\ x & 0\le x\le 1\\ 1 & 1.\end{cases}$$



~~The~~ ~~hence~~ F is non decreasing, continuous (hence right continuous) and $\lim_{x\to-\infty}F(x)=0$. $\lim_{x\to\infty}F(x)=1$.

hence, fundamental theorem of probability guarantees the existence of a random ~~Defn~~ variable whose CDF is F.

Defn 1. A random variable X is discrete if there exists a countable set $\{x_1,x_2,\dots\}\subseteq\mathbb{R}$ such that $\sum_n P(X=x_n)=1$.

2. A random variable X is continuous if. $P(X=x)=0$. $\forall x\in\mathbb{R}$.

Thm:- A random variable is continuous ~~iff~~ iff it's CDF is a continuous function.

Proof:-

Recall that if F is the CDF of X then for any $x\in\mathbb{R}$.

$$P(X=x)=F(x)-\underbrace{\lim_{y\to x^-}F(y)}_{F(x-)}.$$

$\therefore$ for a fixed $x\in\mathbb{R}$.

$$P(X=x)=0 \Leftrightarrow F(x)=F(x-).$$

$\Leftrightarrow$ F is left continuous at x

$\Leftrightarrow$ F is continuous at x. [$\because$ F is CDF hence, right continuous.]

Defn:- A function f is $f:\mathbb{R}\to[0,\infty)$ is a density (probability density function, or PDF) of a random variable X if

$$P(X\le x)=\int_{-\infty}^{x}f(t)\,dt \quad \forall x\in\mathbb{R}.$$

EXM:- Let X be a number chosen at random from $[0,1]$. The function f defined on $\mathbb{R}$ by.

$$f(x) = \begin{cases} 1, & x \in [0,1]. \\ 0, & \text{otherwise}. \end{cases}$$

is a density of X.

EXM:- Let, X be as before. Define $Y = X^2$.

The CDF of Y is,

$$G(y) = \begin{cases} 0 & y<0, \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & y>1. \end{cases}$$

Define,

$$g(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & , \text{otherwise}. \end{cases}$$

Is that true that,

$$\int_{-\infty}^{y} g(t)dt = G(y) \quad \forall y \in \mathbb{R}.$$

Yes, hence g is density of Y.

Words of Caution:-

1. For continuous random variable X, $P(X=x)=0$ for every x, unlike discrete random variable.

2. Unlike ~~PDF~~ PMF a density can exceed 1 and in fact be unbounded. However density is never -ve. (see the preceding example).

3. Not all continuous random variable has a density.

4. A density need not be unique. For example, if X be a random from $[0,1]$ then.

$$\tilde{f}(x) = \begin{cases} 2018 & , x = \frac{2}{3}. \\ 1 & , x \in [0,1]-\{\frac{2}{3}\} \\ 0 & , x \in [0,1]^c. \end{cases}$$

is also a density of X. However a density is "essentially" unique.

5. A random variable may neither discrete nor continuous. Let X be as before and set,

$$Y = \min\{X, \tfrac{1}{2}\}.$$

$$P(Y \ge y) := \begin{cases} 0 & y<0. \\ y & 0 \le y < \frac{1}{2} \\ 1 & y \ge \frac{1}{2}. \end{cases}$$

So, Y is neither discrete nor continuous.

Random Variable:-

A random variable is a function, $X: \Omega \to \mathbb{R}$ such that.

$$\{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{A}, \text{ for } \forall x \in \mathbb{R}.$$

$\phi(c) = a$

$\phi(d) = b$

FACT (Change of variables):

If $\phi: (c,d) \to (a,b)$ is a one-to-one, onto and differentiable function, then,

$$\int_a^b f(x)dx = \int_c^d f \circ \phi(y) |\phi'(y)| dy.$$

for any $f: (a,b) \to \mathbb{R}$. for which the integral makes sense.

Caution:- The above formula is false without the modulus. or if f is not one-to-one.

Defn:- If X is a random variable with density f, then the expectation or mean of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx.$$

which is defined whenever

$$\int_{-\infty}^{\infty} |x| f(x)\,dx < \infty.$$

Example:- Define

$f: \mathbb{R} \to [0,\infty)$ by.

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a,b]. \\ 0, & \text{otherwise.} \end{cases}$$

It is immediate that f is a density, and hence

$F: \mathbb{R} \to [0,1]$ defined by.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

is a CDF.

If X is a random variable whose CDF is F, then we say that $X \sim$ Uniform(a,b).

When, $a=0$, $b=1$, this is called "Standard Uniform"

If $X \sim \text{Unif}(a,b)$. Calculate $E(X)$.

$$E(X) = \int_a^b \frac{x\,dx}{b-a}.$$

$$= \left[\frac{x^2}{2}\right]_a^b \cdot \frac{1}{b-a}.$$

$$= \frac{a+b}{2}.$$

Example:- Suppose that X is a number chosen at random from $[2,3]$. Calculate the density, CDF and mean of X.

Soln. The density of X is.

$$f(x) = \begin{cases} c, & x \in [2,3] \\ 0, & \text{otherwise.} \end{cases}$$

$\because f$ is a density, $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow c = 1$.

hence,

$F: \mathbb{R} \to [0,1]$ defined by.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

is a CDF.

$$F(x) = \begin{cases} 0, & x < 2 \\ x-2, & 2 \le x \le 3. \\ 1, & x > 3. \end{cases}$$

$E(X) = 2.5$

Example:- For a constant $\lambda > 0$, define $F : \mathbb{R} \to [0,1]$ by.

$$F(x) = \begin{cases} 0, & x < 0. \\ 1 - e^{-\lambda x} & x \geq 0. \end{cases}$$

Its density is.

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}(x \geq 0).,\quad x \in \mathbb{R}.$$

If X has this density, then

$X \sim \text{Exponential}(\lambda)$.

Suppose $X \sim \text{Exp}(\lambda)$. Calculate $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$= \int_0^{\infty} \lambda x e^{-\lambda x}\, dx. \qquad \left|\ \begin{array}{l} y = \lambda x \\ dy = \lambda dx. \end{array}\right.$$

$$= \int_0^{\infty} \frac{y}{\lambda} e^{-y}\, dy.$$

$$= \frac{1}{\lambda} \int_0^{\infty} y e^{-y}\, dy.$$

$$= \frac{1}{\lambda}\left[\left[-ye^{-y}\right]_0^{\infty} + \int_0^{\infty} e^{-y}\, dy\right]$$

$$= \frac{1}{\lambda}\left[\int_0^{\infty} e^{-y}\, dy\right]$$

$$= \frac{1}{\lambda}\left[-e^{-y}\right]_0^{\infty} = \frac{1}{\lambda}.$$

Caution:

Exponential with mean λ is not same as Exponential with parameter λ.

Suppose $X \sim \text{Exp}(\lambda)$. For $s, t > 0$.

$$P(X > s+t \mid X > s).$$

$$= \frac{P(X > s, t+t)}{P(X > s)} = \frac{1 - F(s+t)}{1 - F(s)}.$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = P(X > t).$$

This is called the "memoryless property".

Example:-

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = 2\int_0^{\infty} e^{-\frac{x^2}{2}}\, dx$$

$$= 2\left[\int_0^1 e^{-\frac{x^2}{2}}\, dx + \int_1^{\infty} e^{-\frac{x}{2}}\, dx\right]$$

$$\leq 2\left[\int_0^1 1\, dx + \int_1^{\infty} e^{-\frac{x}{2}}\, dx\right].$$

$$= 2\left[1 + 2e^{-\frac{1}{2}}\right] < \infty.$$

FACT:- (Polar transformation) If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that, the following integral make sense, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

$\longrightarrow$

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right).$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx\, dy.$$

Transforming Polar coordinate $= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}((r\sin\theta)^2+(r\cos\theta)^2)} r\,dr\,d\theta.$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr\,d\theta.$$

$$= \left(\int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr\right)\left(\int_0^{2\pi} 1\,d\theta\right)$$

$$= 2\pi\left(\int_0^{\infty} e^{-s} ds\right) \qquad s = \frac{r^2}{2} \Rightarrow ds = r\,dr.$$

$$= 2\pi.$$

$\therefore I = \sqrt{2\pi}$ i.e.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \; x \in \mathbb{R}$$

is a density.

The distribution whose ~~def~~ density is ϕ is called standard Normal or Standard Gaussian. The CDF of Standard Normal is usually denoted by Φ.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt; \quad x \in \mathbb{R}.$$

$$\int_{-\infty}^{\infty} \frac{|x| e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} 2\int_0^{\infty} x \cdot e^{-\frac{x^2}{2}} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}} dx.$$

$$= \sqrt{\frac{2}{\pi}}.$$

If $X \sim$ Standard Normal.

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = 0. \quad [\because \text{odd function \& integral make sense}]$$

Example: Define $F: \mathbb{R} \to [0,1]$ by

$$F(x) = \frac{1}{\pi} \tan^{-1} x + \frac{1}{2}$$

~~Recall that~~

Its a CDF. Calculate the PDF:

$$f(x) = \frac{1}{\pi(1+x^2)}$$

$$\int_{-\infty}^{\infty} \frac{|x|\,dx}{\pi(1+x^2)} = \int_0^{\infty} \frac{2x\,dx}{\pi(1+x^2)}$$

$$= \frac{1}{\pi}\left[\ln(1+x^2)\right]_0^{\infty} = \infty.$$

$\therefore E(X)$ does not exist.

This is the "Cauchy distn."

Exc class

HW. 1

1.a. x has density f.

$$P(a \le X \le b)$$
$$= P(a < X \le b) \quad [\text{because } P(X=a)=0]$$
$$= P(X \le b) - P(X \le a)$$
$$= F(b) - F(a) \quad [\text{where } F \text{ is the CDF of } X]$$
$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$
$$= \int_{a}^{b} f(x)\,dx.$$

1.(b) Fix $\delta > 0$.

since, f is continuous at x_0, $\exists\, n > 0$ s.t.

$$|f(x) - f(x_0)| \le \delta \quad \forall\, x \in [x_0 - n, x_0 + n]$$

$$f(x_0) - \delta \le f(x) \le f(x_0) + \delta, \; x \in [x_0 - n, x_0 + n].$$

If $\varepsilon \in (0, n]$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} [f(x_0) - \delta]\,dx \le \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\,dx \le \int_{x_0-\varepsilon}^{x_0+\varepsilon} [f(x_0) + \delta]\,dx.$$

$$\Rightarrow 2\varepsilon [f(x_0) - \delta] \le \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\,dx \le 2\varepsilon [f(x_0) + \delta]$$

$$\Rightarrow [f(x_0) - \delta] \le \frac{1}{2\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\,dx \le [f(x_0) + \delta]$$

$$\Rightarrow \left| \frac{1}{2\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\,dx - f(x_0) \right| \le \delta \quad \forall\, \varepsilon \in [0, n].$$

$$\therefore \; \cancel{P(|X - x_0| \le \varepsilon)} \; |P(|X - x_0| \le \varepsilon) - f(x_0)| \le \delta.$$

2\. a. $X \sim \text{Exp}(\lambda)$

Clearly, Y takes values $1, 2, 3, \ldots$

for $k \in \mathbb{N}$

$$P(Y=k) = P(k-1 < X \leq k)$$

$$= e^{-\lambda(k-1)} - e^{-\lambda k}$$

$$= e^{-\lambda(k-1)}(1-e^{-\lambda}).$$

$\therefore Y \sim \text{Geo}(1-e^{-\lambda}).$

2\. c. $Z \sim \text{Geo}(p)$

Define $\lambda = \ln \frac{1}{1-p} > 0 \; [\because p < 1]$

Let, $X \sim \text{Exp}(\lambda)$

By par (a)

$$\lceil X \rceil \overset{d}{=} Z$$

$$P(Z > m+n \mid Z > m)$$

$$= P(\lceil X \rceil > m+n \mid \lceil X \rceil > m)$$

$$= P(X > m+n \mid X > m).$$

$$= P(X > n).$$

$$= P(\lceil X \rceil > n)$$

$$= P(Z > n).$$

3\. X takes values in $(-1, 1)^c$

Let $x < -1$.

$$P(X \leq x) = P\left(\frac{1}{U} \leq x\right).$$

$$= P\left(\frac{1}{x} \leq U < 0\right)$$

$$= -\frac{1}{2x}$$

$\therefore$ The CDF of X is.

$$F(x) = \begin{cases} -\frac{1}{2x} & , x < -1 \\ \frac{1}{2} & , -1 \le x \le 1. \\ \frac{1}{2x} & , x > 1. \end{cases}$$

$\therefore$ The density of X is.

$$f(x) = \frac{1}{2x^2} \cdot \mathbb{1}(|X| \geq 1), \quad x \in \mathbb{R}.$$

4.

b. $$F(x) = \begin{cases} 0 & , x < 0. \\ \frac{2}{\pi} \sin^{-1}\sqrt{x} & , 0 \le x \le 1. \\ 1 & , x > 1. \end{cases}$$

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \cdot \mathbb{1}(0 < X < 1) \quad x \in \mathbb{R}.$$

$$\int_{-\infty}^{\infty} |x| f(x)\, dx = \int_0^1 x f(x)\, dx \le \int_0^1 f(x)\, dx = 1.$$

$$\therefore I = \int_0^1 x \cdot \frac{1}{\pi\sqrt{x(1-x)}}\, dx$$

$y = 1-x.$
$dy = -dx.$

$$= \int_0^1 (1-y)\, f(1-y)\, dy$$

$$= \int_0^1 (1-y)\, f(y)\, dy \quad \left[\because \text{distribution is sym. about } \tfrac{1}{2}.\right]$$

$$= \int_0^1 f(y)\, dy - \int_0^1 y f(y)\, dy = 1 - I.$$

$$\Rightarrow I = \frac{1}{2}.$$

If density is sym. around a point then the expectation is same as the value.

15.01.18

Theorem: (Change of Density)

Let, X be a random variable having density f and taking values in (a,b) $[-\infty < a, b < \infty]$, i.e. $P(a < X < b) = 1$

Let, $\phi\ (a,b) \to (c,d)$ be a differentiable function and a bijection such that ϕ' is also differentiable. If ψ denotes ϕ^{-1}, then the density of $Y = \phi(X)$ is

$$g(y) = f \circ \psi(y) \, |\psi'(y)| \cdot 1\{c < Y < d\} \quad [y \in \mathbb{R}]$$

Proof:-

Since, ϕ is continuous and one-one, it is either strictly increasing or strictly decreasing. Let, us assume the Later to be the case. The proof of the former being similar.

Clearly $P(c < Y < d) = 1$ & it sulfices to show that $\forall\ y \in (c,d)$.

$$P(Y \leq y) = \int_c^y g(t)\,dt.$$

$$P(Y \leq y) = P(X \geq \psi(y)) \quad [\text{because } \phi \text{ is strictly decreasing.}$$

$$= \int_{\psi(y)}^{b} f(t)\,dt.$$

Let, us substitute $t = \psi(u)$. then.

$dt = |\psi'(u)|du$ and.

$\psi : (c,y) \to (\psi(y), b)$ is a bijection.

$\therefore \quad P(Y \le y) = \int_c^y f(\psi(u)) \, |\psi'(u)| \, du$

$= \int_c^y g(u) \, du$ as desired.

Exc:- Let $U \sim$ standard Uniform and $V = -\ln U$. Find density of V.

$V = -\ln U$

$\Rightarrow U = e^{-V}$

$\psi(V) = e^{-V}$.

$\therefore \quad |\psi'(v)| = |-e^{-v}|$

$\therefore \quad g(v) = f(\psi(v)) \, |\psi'(v)|$

$= 1 \cdot e^{-v} = e^{-v}$

Density of V is $g(v) = e^{-v} \cdot 1(v \ge 0)$

In other words $V \sim \operatorname{Exp}(1)$

Exc:- Let $Z \sim$ Std Normal. Find density of $X = \frac{1}{Z}$.

$Z = X^{-1}$

$dz = |-X^{-2}| \, dX$

$= X^{-2} \, dX$.

$\frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2x^2}} \frac{dx}{x^2}$

$\therefore \quad f(x) = \frac{1}{x^2 \sqrt{2\pi}} e^{-\frac{1}{2x^2}}$

[this solution is not theoretically correct. Since $\phi(x)$ is not diff'ble at $x = 0$.]

Actual Sol^n:-

For $x > 0$,

$$P(X \le x) = P(Z < 0) + P(Z \ge \tfrac{1}{x})$$

$$= \tfrac{1}{2} + 1 - \Phi(\tfrac{1}{x}).$$

$$= \tfrac{3}{2} - \Phi(\tfrac{1}{x}).$$

For $x < 0$

$$P(X \le x) = P(\tfrac{1}{x} \le Z < 0)$$

$$= \tfrac{1}{2} - \phi(\tfrac{1}{x})$$

$$f(z) = -\phi(\tfrac{1}{x})(-\tfrac{1}{x^2}).$$

$$= \frac{1}{x^2}\phi(\tfrac{1}{x}) = \frac{1}{\sqrt{2\pi}\,x^2} e^{-\frac{1}{2x^2}}.$$

Fact:- Suppose that X has density f and takes values in an open set U. Let $\phi : U \to V$ be differentiable function which is a bijection onto an open set V such that $\psi = \phi^{-1}$ is differentiable. Then $\phi(X)$ has density.

$$g(y) = f \circ \psi(y) \, |\psi'(y)| \, 1(y \in V).$$

Previous problem

$V = U = \mathbb{R} - \{0\}.$

we just remove the point $x = 0$.

Thm:- 3 Let X has density f. If $g: \mathbb{R} \to \mathbb{R}$ is such that $g(X)$ has an expectation, then

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

Proof:- We shall prove this only in the case such that g is one-one and diff'ble such that g^{-1} is also diff'ble on its domain.

Let $(a,b) = g(\mathbb{R})$, and $h: (a,b) \to \mathbb{R}$ be its inverse.

The density of $g(x) = f \circ h(y) \, |h'(y)| \, 1.(y \in (a,b))$

$$\therefore \quad E[g(X)] = \int_{-\infty}^{\infty} y \, f_Y(y)\, dy.$$

$$= \int_{-\infty}^{\infty} y \, f(h(y)) \, |h'(y)| \, dy.$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

$g(x) = y$
$x = h(y)$
$dx = |h'(y)| dy$

Thm:- 4 If X has an expectation, then $E(aX) = aE(X)$.

Proof: Follows from the previous result.

Fact:- If X & Y each have an expectation then $E(X+Y) = E(X) + E(Y)$

Thm:- 5 If $E(X^2)$ exists, then so does $E(X)$.

Proof:- Let, f be the density of X.

Need to show that, $\int_{-\infty}^{\infty} |x| f(x)\, dx < \infty$.

To that end, fix $T > 0$. and define

$$\mu_T = \int_{-T}^{T} |x| \, f(x)\, dx.$$

$$0 \le \int_{-T}^{T} (|x| - \mu_T)^2 f(x)\,dx.$$

$$= \int_{-T}^{T} (x^2 + \mu_T^2) f(x)\,dx - 2\mu_T \int_{-T}^{T} |x| f(x)\,dx.$$

$$= \int_{-T}^{T} (x^2 + \mu_T^2) f(x)\,dx - 2\mu_T^2.$$

$$\le \int_{-\infty}^{\infty} (x^2 + \mu_T^2) f(x)\,dx - 2\mu_T^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + \mu_T^2 \int_{-\infty}^{\infty} f(x)\,dx - 2\mu_T^2.$$

$$= E(X^2) - \mu_T^2.$$

Thus $\mu_T \le \sqrt{E(X^2)}$.

Letting $T \to \infty$, we get.

$$\int_{-\infty}^{\infty} |x| f(x)\,dx \le \sqrt{E(X^2)}.$$

This shows that $E(X)$ exist, whenever $E(X^2)$ exists.

also $E(|X|) \le \sqrt{E(X^2)}$.

Another method (But not useful for finding bound)

$T > 1$.

$$\int_0^T |x| f(x)\,dx = \int_0^1 x f(x)\,dx + \int_0^T x f(x)\,dx.$$

$$\leq \int_0^1 f(x)\,dx + \int_1^T x^2 f(x)\,dx.$$

$$\leq 1 + E(x^2).$$

Def^n:- For a R.V. X for which $E(x^2)$ exists. its variance is defined as,

$$Var(x) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx.$$

where $\mu = E(X)$.

standard deviation $= \sqrt{Var(x)}$

Thm:- 6 If $E(x^2)$ exists, then the following hold.

1. $Var(x) = E(x^2) - [E(x)]^2$
2. for any $a \in \mathbb{R}$.

   $Var(x+a) = Var(x)$
3. for any $a \in \mathbb{R}$.

   $Var(ax) = a^2 Var(x)$.
4. $|E(x)| \leq \sqrt{E(x^2)}$
5. $Var(x) \leq E[(X-c)^2] \ \forall c \in \mathbb{R}$.

Pf:- 1. $(X-\mu)^2 = X^2 - 2\mu X + \mu^2$.

Since, $E(X^2)$ exists so does $E(X)$.

And hence, the quantity above has an expectation, by a stated fact

$$E[(X-\mu)^2] = E(X^2) - E(2\mu X) + \mu^2$$

$$= E(X^2) - \mu^2, \text{ as desired.}$$

5. Letting $\mu = E(X)$.

$$(x-c)^2 = [(x-\mu)+(\mu-c)]^2$$

$$= (x-\mu) + (\mu-c)^2 + 2(x-\mu)(\mu-c)$$

$$E[(x-c)^2] = Var(x) + (\mu-c)^2 + 2(\mu-c)\underbrace{E(X-\mu)}_{0}$$

$$\geq Var(X).$$

Example:- Let $Z \sim N(0,1)$. Fix $\mu \in \mathbb{R}$ and $\sigma > 0$, and define $X = \mu + \sigma Z$

Solⁿ:-

$$Z = \frac{X-\mu}{\sigma}$$

$$dz = \frac{1}{\sigma}dx$$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

CDF $F(x) = P(X \leq x)$

$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right).$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right).$$

$$E(X) = \mu$$

$$Var(X) = \sigma^2$$

This distribution of X is known as $N(\mu, \sigma^2)$

$\equiv$ Normal with mean μ & variance σ^2

$\equiv$ Normal " " μ and s.d. σ.

Exm:- $X \sim \text{Pareto}(\alpha)$, $\alpha > 0$. if its density is

$$f(x) = \alpha x^{-\alpha-1} \cdot 1(x \geq 1), \quad x \in \mathbb{R}$$

Calculate $\text{Var}(X)$.

$$E(X^2) = \int_1^{\infty} x^2 \alpha x^{-\alpha-1}\, dx.$$

$$= \int_1^{\infty} \alpha\, x^{-\alpha+1}\, dx$$

$$= \alpha \left[\frac{x^{-\alpha+2}}{-\alpha+2} \right]_1^{\infty}$$

$$= \frac{\alpha}{2-\alpha} \quad \text{for } \alpha > 2.$$

$$E(X) = \int_1^{\infty} x \alpha x^{-\alpha-1}\, dx.$$

$$= \int_1^{\infty} \alpha\, x^{-\alpha}\, dx$$

$$= \alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^{\infty}$$

$$= \frac{\alpha}{1-\alpha}.$$

$$\therefore \text{Var}(X) = \frac{\alpha}{2-\alpha} - \frac{\alpha^2}{(1-\alpha)^2}$$

$$= \frac{\alpha}{(\alpha-1)^2(\alpha-2)}, \quad \alpha > 2.$$

General Definition of expectation:-

Defn: A random variable is simple if it takes finitely many values.

For a simple r.v. X taking values $x_1, x_2, \ldots,$

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i). \quad (2.1.)$$

Defn:- For a non-negative random variable X define

$$E(X) = \sup\{E(Y) : 0 \le Y \le X ;\ Y \text{ simple RV}\}.$$

including the case when RHS $= \infty$.

Exc:- Check that for a simple non-negative X, the above defn. matches with that in (2.1).

Thm:- For $0 \le X \le Y$

$$E(X) \le E(Y).$$

Proof:- Since

$$\{E(Z) : Z \text{ simple RV}, 0 \le Z \le X\}$$
$$\subseteq \{E(Z) : Z \text{ simple RV}, 0 \le Z \le Y\}.$$
$$\Rightarrow E(X) \le E(Y).$$

Thm:- For any R.V. $X \ge 0$,

$$E(X) = \int_0^\infty P(X > x)\,dx.$$

Pf:- Let X be non-negative & simple RV, taking values $x_1, \ldots, x_n$ which are in ascending order WLOG.

$$\int_0^\infty P(X>x)\,dx = \int_0^{x_n} P(X>x)\,dx$$

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} P(X>x)\,dx. \quad [x_0=0]$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \sum_{j=i}^{n} P(X=x_j).$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{j} (x_i - x_{i-1})\, P(X=x_j).$$

$$= \sum_{j=1}^{n} P(X=x_j) \sum_{i=1}^{j} (x_i - x_{i-1}).$$

$$= \sum_{j=1}^{n} P(X=x_j)(x_j - 0)$$

$$= \sum_{j=1}^{n} x_j\, P(X=x_j)$$

$$= E(X).$$

Suppose X is any non-negative random variable not ~~nece~~ necessarily simple.

Let, Y be a simple random variable such that $0 \le Y \le X$.

$$E(Y) = \int_0^\infty P(Y>x)\,dx$$

$$\le \int_0^\infty P(X>x)\,dx.$$

In other words.

$$E(X) = \sup\{E(Y) : 0 \le Y \le X,\ Y \text{ simple}\} \le \int_0^\infty P(X>x)\,dx$$

Fix $\alpha < \int_0^\infty P(X>x)\,dx$

Since, $\int_0^\infty P(X>x)\,dx = \lim_{T\to\infty} \int_0^T P(X>x)\,dx$

$\exists\ T \in (0, \infty)$ s.t. $\displaystyle\int_0^T P(X > x)\,dx > \alpha$.

$\exists\ n \in \mathbb{N}$ s.t.

$$\sum_{i=1}^{n} \frac{T}{n} P\left(X > \frac{iT}{n}\right) > \alpha$$

Define

$$Y = \begin{cases} \dfrac{iT}{n}, & \dfrac{iT}{n} \le X < \dfrac{(i+1)T}{n} \\ & \text{for some } i \in \{0, \dots, n-1\} \\ T ; & \text{if } X \ge T. \end{cases}$$

$\frac{nY}{T}$ takes values in $\{0, \dots, n\}$.

$$E\left(\frac{nY}{T}\right) = \sum_{i=1}^{\infty} P\left(\frac{nY}{T} \ge i\right).$$

$$= \sum_{i=1}^{n} P\left(\frac{nY}{T} \ge i\right).$$

$$= \sum_{i=1}^{n} P\left(Y \ge \frac{iT}{n}\right).$$

$$= \sum_{i=1}^{n} P\left(X \ge \frac{iT}{n}\right).$$

$$E(Y) = \sum_{i=1}^{n} \frac{T}{n} P\left(X \ge \frac{iT}{n}\right).$$

$$\ge \sum_{i=1}^{n} \frac{T}{n} P\left(X > \frac{iT}{n}\right) > \alpha.$$

Thus, we can find a simple R.V. Y whose expectation is greater than α.

$\therefore\ E(X) > \alpha$ whenever $\alpha < \int_0^\infty P(X > x)\,dx$

i.e. $E(X) \geq \int_0^\infty P(X > x)\,dx$

$$\therefore\ E(X) = \int_0^\infty P(X > x)\,dx$$

Thm: (Monotone Convergence Theorem (MCT)).

If $0 \leq X_n \uparrow X$, then $E(X_n) \uparrow E(X)$.

Proof:- A previous result shows us, that

$$E(X_1) \leq E(X_2) \leq \cdots$$

and hence, $\lim_{n\to\infty} E(X_n)$ exists.

furthermore, $\lim_{n\to\infty} E(X_n) \leq E(X)$.

Let, $\alpha < E(X) := \int_0^\infty P(X > x)\,dx$.

Like in the previous proof $\exists\ T \in (0, \infty)$ and $k \in \mathbb{N}$ s.t.

$$\sum_{i=1}^{k} \frac{T}{k} P\left(X > \frac{iT}{k}\right) > \alpha.$$

For any $x \in \mathbb{R}$, $[X_n > x] \uparrow [X > x]$, and hence,

$$\lim_{n\to\infty} P(X_n > x) = P(X > x) \quad (*)$$

hence, $\lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} \int_0^\infty P(X_n > x)\,dx.$

$$\lim_{n\to\infty} \int_0^\infty P(X_n > x)\,dx \geq \liminf_{n\to\infty} \sum_{i=1}^{k} \frac{T}{k} P\left(X_n > \frac{iT}{k}\right)$$

$$= \sum_{i=1}^{T} \frac{T}{k} P\left(X > \frac{iT}{k}\right) \quad [\text{by } (*)].$$

$$> \alpha$$