Assume for the sake of contradiction that F is not continuous at (x,y).

Then $\exists \{(x_n,y_n)\} \to (x,y)$ such that

$$\liminf_{n\to\infty} |F(x_n,y_n) - F(x,y)| > 0. \quad (*)$$

Either $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \downarrow x$

or $x_{n_k} < x \ \forall k$ and $x_{n_k} \uparrow x$.

Case I :- $x_{n_k} \downarrow x$.

$\exists$ $1 \le m_1 < m_2 < \cdots$ such that

$\{m_1, m_2, \ldots\} \subseteq \{n_1, n_2, \ldots\}$ and either

$y_{m_k} \downarrow y$ [Subcase 1.1]

or $y_{m_k} \uparrow y$ and $y_{m_k} < y$. [Subcase 1.2]

In subcase 1.1 we have $x_{m_k} \downarrow x$ and $y_{m_k} \downarrow y$.

hence $F(x_{m_k}, y_{m_k}) \downarrow F(x,y)$ which contradicts $(*)$.

Subcase (1.2)

$F(x_{m_k}, y_{m_k})$

$= F(x_{m_k}, y_{m_k}) - F(x_{m_k}, y) + F(x_{m_k}, y)$.

$= -[F(x_{m_k}, y) - F(x_{m_k}, y_{m_k})] + F(x_{m_k}, y)$

$= -P(X \le x_{m_k}, y_{m_k} < Y \le y) + F(x_{m_k}, y)$.

The events $[X \le x_{m_k}] \downarrow [X \le x]$

and $[y_{m_k} < Y \le y] \downarrow \phi$

$\therefore [X \le x_{m_k}, y_{m_k} < Y \le y] \downarrow \phi$

$\therefore F(x_{m_k}, y_{m_k}) \longrightarrow F(x,y)$.

Case II.

$\exists \quad 1 \le m_1 \le m_2 < \dots$ such that

$\{m_1, m_2, \dots\}$

6.

Let, $F : A \to B$ be any function.

$F^{-1}(B_0) = \{x \in A : F(x) \in B_0\}$ for any $B_0 \subseteq B$.

$\therefore P[H_1(X_1) \le x_1, \dots, H_n(X_n) \le x_n]$

$= P[X_1 \in H_1^{-1}((-\infty, x_1]), \dots, X_n \in H_n^{-1}((-\infty, x_n])]$.

$= \prod_{i=1}^{n} P(X_i \in H_i^{-1}((-\infty, x_i]))$

$= \prod_{i=1}^{n} P[H_i(X_i) \le x_i]$.

Hence, $H_1(X_1), \dots, H_n(X_n)$ are independent.

7. a. Let, $Y := X_1 \wedge \dots \wedge X_n$.

$\therefore E(Y) = \int_0^\infty P(Y \ge y)\, dy = \int_0^1 (1-y)^n\, dy$.

$= \frac{1}{n+1}$

b. $E\left[\max_{1\le i,j\le n} |X_i - X_j|\right]$

$$= E\left[\left(\max_{1\le j\le n} X_j\right) - \left(\min_{1\le j\le n} X_j\right)\right]$$

c. $E\left[\dfrac{X_1}{X_1+X_2+\cdots+X_n}\right] = \dfrac{1}{n}$.

$$\therefore\ 1 = \sum_{i=1}^{n} E\left(\frac{X_i}{X_1+X_2+\cdots+X_n}\right).$$

d. $P(X_1 \le X_2 \le \cdots \le X_n)$.

$$P(X_1 = X_2) = 0.$$

$$P(X_1 \le X_2) = P(X_1 < X_2) = \frac{1}{2}\left[P(X_1 < X_2) + P(X_1 > X_2)\right]$$

g. $E\left[\dfrac{1}{X_1}\min_{1\le j\le n} X_j\right] = \displaystyle\int_0 P(Y>y)\,dy$.

Fix $0<y<1$.

$$P[Y>y] = P\left[\min_{1\le j\le n} X_j > yX_1\right]$$

$$= P\left[\min_{2\le j\le n} X_j > yX_1\right] =$$

$$\min_{2\le j\le n} X_j \sim \operatorname{Exp}(n-1)$$

$$yX_1 \sim \operatorname{Exp}\left(\frac{1}{y}\right).$$

$Z_1 \sim \text{Exp}(\lambda_1)$

$Z_2 \sim \text{Exp}(\lambda_2)$.

$$P(Z_1 < Z_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

$$\therefore P\left(\min_{2\le j\le n} X_j > y X_1\right) = \frac{\frac{1}{y}}{\frac{1}{y}+n-1} = \frac{1}{1+(n-1)y}$$

$$E(Y) = \int_0^1 \frac{dy}{1+(n-1)y} = \frac{\log n}{n-1}.$$

10. Gamma $(n\alpha, \lambda)$.

14.

Dist.	Mean	Variance.
Gamma(α,λ)	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Beta(a,b)	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$

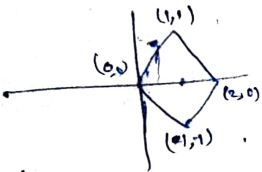
16. $P(Y^2 > 4Z)$

$$= \int_{y=0}^{1}\int_{z=0}^{\frac{y^2}{4}} f(y,z)\,dz\,dy.$$

$$= \int_{y=0}^{1}\int_{z=0}^{\frac{y^2}{4}} 1\,dz\,dy = \int_0^1 \frac{y^2}{4}\,dy = \frac{1}{12}.$$

17. The joint density of $(U+V, U-V)$ is.

$$f(x,y) = \frac{1}{2}1.[(x,y) \in ABCD]$$



$$= \int_0^2 \left(\int_{-x}^{x} \frac{1}{2}\,dy + \int_{1-x}^{x+1} \frac{1}{2}\,dy\right) dx$$

$P(X \le x) =$

15.

(b)

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = P \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\Rightarrow \psi(y) = P^T \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}. \quad [\because P^{-1} = P^T]$$

Therefore, the joint density of X_1 & X_2 is.

$$f(\vec{x}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\vec{x}'\vec{x}}.$$

$$|j(\vec{y})| = 1. \quad [\because \text{orthogonal}].$$

$\therefore$ The joint density of Y_1 and Y_2 is.

$$f(\vec{y}) = f \circ \psi(\vec{y}) \, |j(y)|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(P'\vec{y})'(P'\vec{y})} \cdot 1$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\vec{y}'\vec{y}}.$$

$\therefore$ Y_i are again i.i.d. standard Normal.

20.

d. $$E(e^X) = \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} \exp\left[x - \frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} \exp\left[\frac{2\sigma_x^2 x - x^2 + 2\mu_x x - \mu_x^2}{2\sigma_x^2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{-\infty}^{\infty} \exp -\left[\frac{x^2 - 2 \cdot x(\mu_x + \sigma_x^2) + (\mu_x + \sigma_x^2)^2}{2\sigma_x^2} - \frac{\sigma_x^4 + 2\sigma_x^2 \mu_x}{2\sigma_x^2}\right].$$

$$= e^{-\left(\frac{1}{2}\sigma_x^2 + \mu_x\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{1}{2}\left(\frac{x - (\mu_x + \sigma_x^2)}{\sigma_x}\right)^2\right] dx.$$

$$= e^{\frac{1}{2}\sigma_x^2 + \mu_x}.$$

20.
e. $\mathrm{Cov}(e^X, e^Y)$

$$= E(e^{X+Y}) - E(e^X)E(e^Y).$$

$$= e^{\mu_X + \mu_Y + \frac{1}{2}(\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y)} - e^{\mu_X + \mu_Y + \frac{1}{2}(\sigma_X^2 + \sigma_Y^2)}.$$

20.
(f) $P[(X-\mu_X)(Y-y_X) < 0]$

$$= P\left[\underbrace{\left(\frac{X-\mu_X}{\sigma_X}\right)}_{Z_1}\underbrace{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)}_{Z_2} < 0\right]$$

$$= P(Z_1 Z_2 < 0).$$

$Z_1 = U$
$Z_2 = \rho U + \sqrt{1-\rho^2}\,V$

$\Rightarrow$ $U = Z_1$
$V = \dfrac{Z_2 - \rho Z_1}{\sqrt{1-\rho^2}}$. Check U, V i.i.d. $N(0,1)$.

$P(U(\rho U + \sqrt{1-\rho^2}\,V)) < 0.$

$$= P(\rho U^2 + \sqrt{1-\rho^2}\,UV < 0)$$

$$= P\left(-\frac{V}{U} < \frac{\rho}{\sqrt{1-\rho^2}}\right).$$

$\left[-\frac{V}{U} \sim \text{Cauchy.}\right]$ $= 1 - F\left(\frac{\rho}{\sqrt{1-\rho^2}}\right).$

$$= 1 - \left[\frac{1}{\pi}\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}.$$

$$
\left[
\begin{array}{l}
\rho \in [-1, 1] \\
\rho = \cos\theta \qquad \theta \in [0, \pi]. \\
\sin\theta > 0. \\
\sin\theta = \sqrt{1-\rho^2}. \\
\cot\theta = \dfrac{\rho}{\sqrt{1-\rho^2}}. \\
\tan\left(\frac{\pi}{2}-\theta\right) = \dfrac{\rho}{\sqrt{1-\rho^2}}.
\end{array}
\right]
$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\frac{\pi}{2} - \theta\right].$$

$$= \frac{1}{\pi}\theta.$$

$$= \frac{1}{\pi}\cos^{-1}\rho.$$

Distributions upto Midsem:-

1. Uniform Distribution:- (Parameter "a, b")

CDF:-

$$F(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , x \in [a,b] \\ 1 & , x > b \end{cases}$$

Density:-

$$f(x) = \frac{1}{b-a} \cdot \mathbb{1}(x \in [a,b])$$

["Stadard Uniform" when $a=0$, $b=1$.]

Expectation:-

$$E(X) = \frac{a+b}{2}$$

Variance:-

$$Var(X) = \frac{(b-a)^2}{12}$$

2. Exponential Distribution:- (Parameter "λ")

CDF:-

$$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\lambda x} & , x \geq 0 \end{cases}$$

Density

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}(x \geq 0)$$

Expectation:-

$$E(X) = \frac{1}{\lambda}$$ [Caution:- Exp with parmeter λ and mean λ are not the same]

Variance:-

$$E(X^2) = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} dx$$

$$\frac{y^2}{e^y} = \frac{2y}{e^y} = \frac{2}{e^y}$$

$$= \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y} dy$$

$$= \frac{1}{\lambda^2} \left(\left[-y^2 e^{-y} \right]_0^\infty + \int_0^\infty 2y e^{-y} dy \right)$$

$$= \frac{1}{\lambda^2} \left[0 + 0 + 2 \int_0^\infty y e^{-y} dy \right]$$

$$= \frac{1}{\lambda^2} 2 \left[-y e^{-y} \Big|_0^\infty + \int_0^\infty 1 \cdot e^{-y} dy \right]$$

$$= \frac{1}{\lambda^2} \times 2 \times \left(0 + -e^{-y} \Big|_0^\infty \right) = \frac{1}{\lambda^2} \times 2 \times (-0+1) = \frac{2}{\lambda^2}$$

P.T.O. →

$\therefore \; Var(X) = E(X^2) - E^2(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$

$\therefore \; \boxed{Var(X) = \frac{1}{\lambda^2}.}$ for $X \sim Exp(\lambda)$.

"Standard Exp." when $\lambda = 1$.

3. **Normal Distribution:-** (Parameter "μ", "σ^2").

CDF:- $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Density:- $$\phi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \forall\, x \in \mathbb{R}.$$

Expectation:-

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left(\frac{x-\mu}{\sigma}\right) dx + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$\cancel{\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y} dy}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot y \, dy + \mu \cdot \left[\because \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] = 1 \quad \because \text{it is CDF.}\right]$$

$$= 0 + \mu \quad \left[\because e^{-\frac{y^2}{2}} y. \text{ is odd function.}\right]$$

$$= \mu.$$

Variance:-

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}} dy.$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \cdot 2 \cdot \int_{0}^{\infty} y^2 e^{-\frac{y^2}{2}} dy.$$

$$\left[\text{put } t = \frac{y^2}{2} \Rightarrow dt = y\,dy \Rightarrow \frac{dt}{\sqrt{2t}} = dy \right]$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \cdot 2 \cdot \int_0^{\infty} 2t \cdot e^{-t^2} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} \sqrt{t}\, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\frac{3}{2}. \quad \text{[using Gamma function]}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2}$$

$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \frac{1}{\cancel{2}} \cdot \cancel{\sqrt{\pi}} \quad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

$$= \sigma^2.$$

"Standard Normal" if $\mu = 0,\ \sigma^2 = 1$.

4. Cauchy Distribution:-

CDF:-

$$F(x) = \frac{1}{\pi} \tan^{-1} x + \frac{1}{2} \quad \forall x \in \mathbb{R}.$$

Density:- $$f(x) = \frac{1}{\pi(1+x^2)} \quad \forall x \in \mathbb{R}.$$

Expectation:-

$E(X)$ does not exist.

No Question of Variance.

5. Arc-sine Distribution:-

CDF:-

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{2}{\pi}\sin^{-1}\sqrt{x} & , 0 \le x \le 1, \\ 1 & , x > 1 \end{cases}$$

Density:-

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \mathbb{1}(x \in (0,1)).$$

Expectation:-

$$E(|X|) = \int_{-\infty}^{\infty} |x| f(x) dx.$$

$$= \int_0^1 x \cdot \frac{1}{\pi\sqrt{x(1-x)}} \cdot dx.$$

$$= \int_0^1 (1-y) \cdot \frac{1}{\pi\sqrt{(1-y)y}} dy \quad [\text{put, } x = 1-y]$$

$$\therefore 2E(|X|) = \int_0^1 x \frac{1}{\pi\sqrt{x(1-x)}} dx + \int_0^1 (1-x) \frac{1}{\pi\sqrt{x(1-x)}} dx$$

$$= \int_0^1 \frac{1}{\pi\sqrt{x(1-x)}} dx = 1 \quad [\because \text{PDF}]$$

$$\Rightarrow E(|X|) = \frac{1}{2}.$$

hence, $E(X)$ exists

and $E(X) = \frac{1}{2}$.

Variance:-

$$Var(X) = \frac{1}{8}.$$

6. Pareto Distribution:- (Parameter "α")

$\alpha > 0$

CDF:-

$$F(x) = \begin{cases} 0 & , x < 1 \\ 1 - x^{-\alpha} & , x \geq 1 \end{cases}$$

Density:-

$$f(x) = \alpha x^{-\alpha-1} 1(x \geq 1).$$

Expectation:-

$$E(X) = \int_{-\infty}^{\infty} x \alpha x^{-\alpha-1} 1(x \geq 1) dx.$$

$$= \int_{1}^{\infty} x \alpha x^{-\alpha-1} dx$$

$$= \int_{1}^{\infty} \alpha x^{-\alpha} dx.$$

$$= \alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty$$

$$= \alpha \quad 0 - \frac{1}{-\alpha+1}.$$

$$= \frac{\alpha}{\alpha - 1}.$$

$$\therefore \boxed{E(X) = \frac{\alpha}{\alpha-1}}$$

Variance:-

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \alpha x^{-\alpha-1} 1(x \geq 1) dx.$$

$$= \int_{1}^{\infty} \alpha x^{-\alpha+1} dx.$$

$$= \alpha \left. \frac{x^{-\alpha+2}}{-\alpha+2} \right|_1^\infty$$

$$= -\frac{\alpha}{2-\alpha} \cdot \text{ exists for } \alpha > 2.$$

$$E(X^2) - E^2(X) = \frac{\alpha}{2-\alpha} - \frac{\alpha^2}{(1-\alpha)^2}$$

$$= -\left[\frac{\alpha(1-\alpha)^2 + \alpha^2(2-\alpha)}{(2-\alpha)(1-\alpha)^2}\right]$$

$$= \frac{\alpha - 2\alpha^2 + \alpha^3 + 2\alpha^2 - \alpha^3}{(\alpha-2)(\alpha-1)^2}$$

$$= \frac{\alpha}{(\alpha-2)(\alpha-1)^2}$$

$$\therefore \boxed{Var(X) = \frac{\alpha}{(\alpha-2)(\alpha-1)^2}}$$

7. Gamma Distribution:- (Parameter "α", "λ")

CDF:- $F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x}\, dx \quad \forall x \in \mathbb{R}.$

$\left[\text{where } \int_a^b f(x)\,dx = 0 \text{ if } b \leq a\right]$

Density:- $f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x}\, 1(x \geq 0).$

Expectation:- $E(X) = \frac{\alpha}{\lambda}.$

Variance:- $Var(X) = \frac{\alpha}{\lambda^2}.$

~~8. Frechet Distribution:-~~

~~CDF:-~~ $\cancel{F(x) = e^{-x^{-\alpha}}}$;

~~Density:-~~ $\cancel{f(x) = e^{-x^{-\alpha}}\, 1(x > 0)}$

~~Expectation:-~~

8. Frechet Distribution:- (Parameter "α")

CDF:- $F(x) = \begin{cases} e^{-x^{-\alpha}} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$

Density:-

$$f(x) = -e^{-x^{-\alpha}} \cdot -\alpha x^{-\alpha-1} \cdot 1(x>0).$$

$$\Rightarrow \boxed{f(x) = \alpha x^{-\alpha-1} e^{-x^{-\alpha}} 1(x>0)}.$$

9. Beeta Distribution:- (Parameter "α", "β")

CDF:-

$$F(x) = \int_{-\infty}^{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, 1(x \in (0,1)) \, dx.$$

Density:-

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, 1(x \in (0,1)).$$

Expectation:-

$$E(X) = \int_{0}^{1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1} \, dx.$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$\boxed{E(X) = \frac{\alpha}{\alpha+\beta}}.$$

Variance:-

$$E(X^2) = \int_{0}^{1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1} \, dx.$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\therefore \mathrm{Var}(X) = \frac{\alpha}{(\alpha+\beta)}\left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right) = \frac{\alpha(\cancel{\alpha^2}+\cancel{\alpha}+\cancel{\alpha\beta}+\beta-\cancel{\alpha^2}-\cancel{\alpha\beta}-\cancel{\alpha}}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\therefore \boxed{\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

10. Chi-Squared Distribution:- (with n degrees of freedom)

Actually Gamma $(\frac{n}{2}, \frac{1}{2})$.

CDF:-

$$F(x) = \int_0^x \frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} dx.$$ [where $\int_a^b f(x)dx = 0$ if $b<a$

Density:-

$$f(x) = \frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} 1(x>0)$$

Expectation:-

$$E(X) = n.$$

Variance:-

$$Var(X) = 2n.$$

11. Fisher F Distribution:- (with degrees of freedom m & n)

Density:-

$$f(x) = \frac{(\frac{m}{n})^{\frac{m}{2}}}{\beta(\frac{m}{2}, \frac{n}{2})} z^{\frac{m}{2}-1} (1+\frac{m}{n}z)^{-\frac{m+n}{2}} 1(z>0)$$

Expectation:-

$$E\left(\frac{nX}{mY}\right) = \frac{n}{m} E(X) \cdot E\left(\frac{1}{Y}\right).$$

$$= \frac{n}{m} \cdot m \cdot \frac{1}{n-1} = \frac{n}{n-1}.$$

12. Student t distribution:- (with degress of freedom n).

Density:-

$$f(x) = \frac{(\frac{1}{n})^{\frac{1}{2}}}{\beta(\frac{1}{2}, \frac{1}{n})} \left(1+\frac{z^2}{n}\right)^{-\frac{n+1}{2}}, \quad \forall z \in \mathbb{R}.$$

Midsem Question:-

3.

$$P(Y \le y) = P(\text{at least two of } X_1, X_2, X_3 \le y).$$

$$= P(\text{exactly two of } X_1, X_2, X_3 \le y) + P(X_1 \vee X_2 \vee X_3 \le y).$$

$$= \binom{3}{2} y^2(1-y) + y^3.$$

$$= 3y^2 - 3y^3 + y^3.$$

$$= 3y^2 - 2y^3.$$

$$\cancel{= y^2(3-2y)}.$$

. The density of Y is.

$$f(y) = 6y(1-y)\ 1\,(0<y<1).$$

Law of Large Number:-

"A coin has probability of Heads p"

"If tossed a large number of times, proportion of heads $\approx p$".

Suppose that coin is tossed infinitely many times. Let X_n denote the observed proportion of heads after n tosses. X_n is a Random variable.

Qn:- What does it mean to say $X_n \to p$?

Definition:-

A random variable X_n converges "almost surely" to a random variable X if

$$P(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1.$$

DCT:-

Suppose that,

$$X_n \to X \quad \text{a.s.}$$

and $|X_n| \le Y$ where Y is such that $E(Y) < \infty$.
Then show that

$$\lim_{n\to\infty} E(X_n) = E(X).$$

Sol^n:-

Let, $A := \{\omega \in \Omega : X_n(\omega) \to X(\omega)\}$.

Then by hypothesis implies that $P(A) = 1$.

Define,
$$\tilde{X}_n(\omega) = \begin{cases} X_n(\omega) & ; \omega \in A \\ 0 & ; \omega \in A^c \end{cases}$$

similarly
$$\tilde{X}(\omega) = \begin{cases} X(\omega) & ; \omega \in A \\ 0 & ; \omega \in A \end{cases}$$

Check : ① $\tilde{X}_n(\omega) \to \tilde{X}(\omega) \quad \forall \omega \in \Omega$

② $|\tilde{X}_n(\omega)| \le Y$ for every n.

③ $E(\tilde{X}_n) = E(X_n)$ & $E(\tilde{X}) = E(X)$.

①, ② & DCT implies:
$$\lim_{n\to\infty} E(\tilde{X}_n) = E(\tilde{X}).$$

then by ③
$$\lim_{n\to\infty} E(X_n) = E(X). \blacksquare$$

Hence, almost sure convergence is as good as convergence of every ω for all practical purposes.

Theorem:- For random variables X_n and X,
$X_n \to X$ a.s. if and only if
$$\lim_{N\to\infty} P\left(\sup_{n\ge N} |X_n - X| > \varepsilon\right) = 0 \quad \forall\, \varepsilon > 0.$$

~~Sol~~ Proof:-
($\Rightarrow$) $X_n \to X$ a.s. (given).

Fix $\varepsilon > 0$.

Define, $A_N := \{\omega \in \Omega \mid \sup_{n\ge N} |X_n(\omega) - X(\omega)| > \varepsilon\}$.
for every $N \ge 1$.

$$A_1 = [\omega \in \Omega \mid \sup_{n\ge 1} |X_n(\omega) - X(\omega)| > \varepsilon].$$

$$A_2 = [\omega \in \Omega \mid \sup_{n\ge 2} |X_n(\omega) - X(\omega)| > \varepsilon].$$

Clearly, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$

Let, $A_\infty = \bigcap_{N=1}^{\infty} A_N$.

$$A_\infty = [\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon \text{ for infinitely many } n\text{'s}]$$

$$\subseteq [\omega \in \Omega : \limsup_{n\to\infty} |X_n(\omega) - X(\omega)| \ge \varepsilon]$$

$$\subseteq [\omega \in \Omega : X_n(\omega) \not\to X(\omega)\}.$$

Hence, $P(A_\infty) \le 0$

$\Rightarrow P(A_\infty) = 0$. ~~and~~

($\Leftarrow$) $\lim_{n\to\infty} P(\sup_{n\ge N} |X_n - X| > \varepsilon) = 0 \quad \forall\, \varepsilon > 0.$

shall show, $X_n \to X$.

Fix $\varepsilon > 0$ for every $N \ge 1$,

$$\left[\limsup_{n\to\infty} |X_n - X| \le \varepsilon\right] \supseteq \left[\sup_{n\ge N} |X_n - X| \le \varepsilon\right].$$

Hence, $P(\limsup_{n\to\infty} |X_n - X| \le \varepsilon)$
$\ge P(\sup_{n \ge N} |X_n - X| \le \varepsilon) \quad \forall N \ge 1.$

Letting $N \to \infty$.

$$P(\limsup_{n\to\infty} |X_n - X| \le \varepsilon) \ge \lim_{N\to\infty} P(\sup_{n\ge N} |X_n - X| \le \varepsilon).$$
$$= 1 - \lim_{N\to\infty} P(\sup_{n\ge N} |X_n - X| > \varepsilon).$$
$$= 1. \quad \text{[By hypothesis]}.$$

Show that if $B_1, B_2, \ldots$ have probability 1,
then $P(\bigcap_{n=1}^{\infty} B_n) = 1$.

Soln:- $P((\bigcap_{n=1}^{\infty} B_n)^c) = P(\bigcup_{n=1}^{\infty} B_n^c) \le \sum_{n=1}^{\infty} P(B_n^c) = 0$

$$P(X_n \to X) = P(\limsup_{n\to\infty} |X_n - X| = 0)$$
$$= P(\bigcap_{k=1}^{\infty} [\limsup_{n\to\infty} |X_n - X| \le \tfrac{1}{n}]) = 1.$$

$\therefore X_n \to X$ a.s. proved

Exc:- Show that $X_n \to X$ almost surely iff
$P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon \text{ for infinitely many } n\}) = 0.$
for every $\varepsilon > 0$.

Definition:-

A random variable X_n converges to X in probability ("$X_n \xrightarrow{P} X$") if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Exc:- Show that almost sure convergence implies convergence in probability.

Exc:- Show that if $X_n \xrightarrow{P} X$ and also $X_n \xrightarrow{P} X'$, then $P(X = X') = 1$.

Soln:- Fix $\varepsilon > 0$.

$$P(|X - X'| > \varepsilon)$$

Claim:-

$$[|X - X'| < \varepsilon] \subseteq [|X - X_n| > \tfrac{\varepsilon}{2}] \cup [|X' - X_n| > \tfrac{\varepsilon}{2}].$$

Pf:- Suppose that $\omega \notin$ RHS.

i.e. $$|X(\omega) - X_n(\omega)| \le \frac{\varepsilon}{2}.$$

and

$$|X_n(\omega) - X'(\omega)| \le \frac{\varepsilon}{2}.$$

$$\Rightarrow |X(\omega) - X'(\omega)| \le \varepsilon$$

hence $\omega \notin$ LHS.

$$\therefore P(|X - X'| \geq \varepsilon) \le P(|X - X_n| > \tfrac{\varepsilon}{2}) + P(|X_n - X| > \tfrac{\varepsilon}{2})$$

$$P(|X - X'| > \varepsilon) \le \lim_{n\to\infty} P(|X - X_n| > \tfrac{\varepsilon}{2}) + \lim_{n\to\infty} P(|X_n - X| > \tfrac{\varepsilon}{2})$$

$$= 0.$$

$$\therefore P(|X - X'| > \varepsilon) = 0.$$

$$\Rightarrow P(X = X') = 1.$$

$\therefore X = X'$ is a.s.

Cor:- For a sequence of random variables, sh... that an almost sure limit, if exists, is unique upto a set of probability zero.

Exm:- Let, U be stadard uniform random varia...

$X_1 = 1(0 < U \leq \frac{1}{2})$.

$X_2 = 1(\frac{1}{2} < U \leq 1)$.

$X_3 = 1(0 < U \leq \frac{1}{4})$.

$X_4 = 1(\frac{1}{4} < U \leq \frac{1}{2})$.

$X_5 = 1(\frac{1}{2} < U \leq \frac{3}{4})$.

$X_6 = 1(\frac{3}{4} < U \leq 1)$. and so on.

$\vdots$

Claim:- $X_n \xrightarrow{P} 0$ as $n \to \infty$.

Fix $\varepsilon \in [0,1)$:

$P(|X_1| > \varepsilon) = P(X_1 = 1) = \frac{1}{2}$.

$P(|X_2| > \varepsilon) = \frac{1}{2}$.

$P(|X_3| > \varepsilon) = \frac{1}{4}$.

Clearly, $P(|X_n| > \varepsilon) \longrightarrow 0$ as $n \to \infty$ and hence the claim follows.

Claim:- For any ω such that $0 < U(\omega) \leq 1$, $\exists$ $1 \leq n_1 \leq n_2 < n_3 < \dots$ such that $X_{n_k}(\omega) = 1$ for every k.

Hence, $P(X_n \not\to 0) \geq P(0 < U \leq 1) = 1$.

i.e. $P(X_n \to 0) = 0$ and therefore X_n does not converge to 0 almost surely.

The above notion shows that convergence in probability is a strictly weaker notion than a.s convergence.

Theorem:- (Weak law of Large numbers) (WLLN).

Let, $X_1, X_2, \ldots$ be i.i.d. random variables with mean μ and finite variance. Then,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu. \quad \text{as } n \to \infty$$

Exc:- Show that WLLN implies that if a coin with probability of heads p is tossed infinitely many times and Z_n denotes the proportion of heads in the first n tosses, then,

$$Z_n \xrightarrow{P} p \quad \text{as } n \to \infty.$$

Lemma (Chebyshev's inequality):

For a random variable Z with finite variance,

$$P(|Z - E(Z)| > a) \le \frac{1}{a^2} \operatorname{Var}(Z).$$

Pf. of WLLN:-

We have $X_1, X_2, \ldots$ are iid with mean μ and variance σ^2. Fix n and define

$$Z = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Applying Chebyshev for a fixed $\varepsilon > 0$,

$$P(|Z - \mu| > \varepsilon) \le \frac{1}{\varepsilon^2} \operatorname{Var}(Z).$$

$$= \frac{\sigma^2}{n\varepsilon^2}.$$

Thus, for fixed $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \varepsilon\right) = 0.$$

Theorem:- (Markov Inequality)

For a non-negative Random variable X and

$$P(X \geq a) \leq \frac{1}{a} E(X).$$

Pf:-

$$E(X) \geq E\left[X 1(X \geq a)\right] \geq E\left[a 1(X \geq a)\right]$$
$$= a P(X \geq a)$$

Exc:- Prove Markov Inequality by using the form

$$E(X) = \int_0^\infty P(X > x)\, dx.$$

Corollary:- (Chebyshev's Inequality)

A random variable Z with finite variance,

$$P\left[|Z - E(Z)| \geq a\right] \leq \frac{1}{a^2} \operatorname{Var}(Z), \quad a > 0.$$

Pf:-

$$X = (Z - E(Z))^2.$$

$$P\left[|Z - E(Z)| \geq a\right]$$
$$= P(X \geq a^2) \leq \frac{1}{a^2} E(X) = \frac{1}{a^2} \operatorname{Var}(Z).$$

Lemma:- (Borel-Cantelli Lemma)

If $A_1, A_2, \ldots$ are events such that

$$\sum_{n=1}^{\infty} P(A_n) < \infty,$$

then $P(\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}) = 0.$

Pf:- We start with the observation that

$$\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$$
$$= \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right).$$

suppose that, $\omega \in LHS$.

Fix $n \geq 1$.

Then $\omega \in A_k$ for some $k \geq n$. Thus.

$$\omega \in \bigcup_{j=n}^{\infty} A_j .$$

hence this is true for every n, $\omega \in RHS$.

Conversely,

suppose $\omega \in RHS$.

$\Rightarrow$ ~~$\omega \in$ to~~ for $\forall n \in \mathbb{N}$, $\exists\ k_n \geq n$ such that

$\omega \in A_{k_n}$

Clearly the set $\{k_1, k_2, \dots\}$ is infinite, even though it's not necessary that they are all distinct. Hence. $\omega \in LHS$, which proves the set theoretic equality.

Let, $B_n = \bigcup_{k=n}^{\infty} A_k$, $n \geq 1$.

Clearly $P(B_n) \leq \sum_{k=1}^{\infty} P(A_k)$.

Since, $\sum_{n=1}^{\infty} P(A_n) < \infty$, it follows that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} P(A_k) = 0$$

$$\lim_{n\to\infty} P(B_n) = 0$$

Now, $\{\omega \in \Omega : \omega \in A_k$ for infinitely many $k\} = \bigcap_{n=1}^{\infty} B_n$.

$B_1 \supseteq B_2 \supseteq \dots \supseteq B_n$

$\therefore\ B_n \downarrow \bigcap_{n=1}^{\infty} B_n$.

$\Rightarrow P(B_n) \downarrow P(\bigcap_{n=1}^{\infty} B_n)$.

$\therefore\ P(\bigcap_{n=1}^{\infty} B_n) = 0$.

Theorem:- If $X_n \xrightarrow{P} X$ as $n \to \infty$, then X_n has a subsequence $\{X_{n_k}\}$ such that:
$X_{n_k} \to X$ a.s., as $k \to \infty$.

Pf:- $\exists\ N \geq 1$ such that

$$P(|X_N - X| > 1) < 1.$$

Let, $n_1 = N$.
$\exists\ N_2$ such that
$P(|X_n - X| > \frac{1}{2}) \leq \frac{1}{2^2}$ $\forall\ n \geq N_2$.

Define $n_2 := (n_1 + 1) \vee N_2$.

Thus, $n_2 > n_1$ and.

$$P(|X_{n_2} - X| > \tfrac{1}{2}) \leq \frac{1}{2^2}.$$

Proceeding inductively we get,
$1 \leq n_1 < n_2 < n_3 < \cdots$
such that, $P(|X_{n_k} - X| > \frac{1}{k}) \leq \frac{1}{k^2}$ $\forall k \in \mathbb{N}$.

$$\therefore \sum_{k=1}^{\infty} P(|X_{n_k} - X| > \tfrac{1}{k}) < \infty$$

By, Borel-Cantelli lemma

$$P(\underbrace{\{w \in \Omega : |X_{n_k}(w) - X(w)| > \tfrac{1}{k} \text{ for infinitely many } k\}}_{A}) = 0.$$

$P(A) = 0$. $\therefore P(A^c) = 1$.

Clearly for $w \in A^c$
$|X_{n_k}(w) - X(w)| > \frac{1}{k}$ for at most finitely many k's.

i.e. $\exists$ K, depend on ω,

such that

$$|X_{n_k}(\omega) - X(\omega)| \leq \frac{1}{k} \quad \forall\ k \geq K.$$

and hence the LHS above goes to 0 as $k \to \infty$

Since, $P(A^c) = 1$, it follows that

$$X_{n_k} \to X \text{ a.s.}, \text{ as } k \to \infty.$$

Exc:- Suppose, that $X_n \xrightarrow{P} X$ and $|X_n| \leq Y$ for some Y such that $E(Y) < \infty$.
Then, show that X has a finite expectation and $\lim_{n \to \infty} E(X_n) = E(X)$.

Sol^n:- By the previous thm, $\{X_n\}$ has a subsequence $\{X_{n_k}\}$ such that

$$X_{n_k} \to X \text{ a.s. as } k \to \infty.$$

$\&$ $|X_{n_k}| \leq Y$.

$\therefore$ $E(|X|)$ is finite.

$\therefore$ X has finite expectation.

Recall that a sequence $\{\alpha_n\}$ of reals converges to α iff any subsequence of α_n has a further subsequence converging to α.

In order to show that $E(X_n) \to E(X)$. Let $1 \leq n_1 < n_2 < n_3 < \dots$ Now, All we need to do is that $E(X_{n_k})$ has a further subsequence which converges to $E(X)$.

Since, $X_n \xrightarrow{P} X$ it follows that

$$X_{n_k} \xrightarrow{P} X \quad \text{as } k \to \infty.$$

Using the previous thm, it follows that.

that $\{X_{n_k}\}$ has a further subsequence $\{X_{n_{k_l}}\}$ such that

$$X_{n_{k_l}} \xrightarrow{a.s.} X \quad \text{as} \quad l \to \infty.$$

Since, $|X_{n_{k_l}}| \le Y \ \forall\, l$ and $E(Y) < \infty$,

it follows that

$$\lim_{l \to \infty} E(X_{n_{k_l}}) = E(X).$$

FACT:- For a sequence $\{x_n\}$ of reals, $x_n \to x \ (\in \mathbb{R})$ if and only if every subsequence of x_n has a further subsequence converging to x.

Pf:- Since the "only if" part is trivial, the "if" part is the only one that will be proved.

~~Let us assume that.~~

Let, if possible $x_n \not\to x$.

i.e. $\exists\ \varepsilon > 0$ such that $\forall\, n \in \mathbb{N}$, $|x_n - x| \ge \varepsilon$

Therefore, $|x_n - x| \ge \varepsilon$ for infinitely many n's.

Let's say $n_1 < n_2 < n_3 < \cdots$ are such that

$$|x_{n_k} - x| \ge \varepsilon \ \text{for every } k \ge 1.$$

~~However, other by~~

i.e. no further subsequence of $\{x_{n_k}\}$ can converge to x. $(\Rightarrow\Leftarrow)$.

Exc:- For a sequence of real $\{\alpha_n\}$ and $\alpha \in [-\infty, \infty]$ it holds that $\lim_{n\to\infty} \alpha_n = \alpha$

iff every subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ has a further subsequence $\{\alpha_{n_{k_l}}\}$ such that

$$\lim_{l\to\infty} \alpha_{n_{k_l}} = \alpha .$$

Exc:- Show that, $X_n \xrightarrow{P} X$ iff every subsequence of X_n has a further subsequence which converges to X in probability.

Exc:- ~~Show~~ Is it true that $X_n \to X$ a.s. iff every subsequence of X_n has a further subsequence which converges to X a.s.

Exc:- For r.v. X_n and X show that TFAE

1. $X_n \xrightarrow{P} X$ as $n \to \infty$.
2. Every subsequence of X_n has further subseq. which conv. to X a.s.
3. Every subsequence of X_n has further subseq. which conv. to X in probability.

Soln:-

$$\begin{array}{ccc} & 1 & \\ \text{follows from analytic fact} \nearrow & & \searrow \text{Follows from the prev. Theorem.} \\ 3 & \xleftarrow{\text{trivial}} & 2 \end{array}$$

Theorem:- (Strong Law of Large numbers; for finite variance.)

Let, $X_1, X_2, \ldots$ be i.i.d. random variables with finite variance. As $n \to \infty$,

$$\frac{1}{n} \sum X_i \longrightarrow E(X_1) \quad \text{a.s.}$$

Lemma:- If $x_1, x_2, x_3, \ldots \geq 0$ are such that

$$\frac{s_{n_k}}{n_k} \longrightarrow \theta \quad \text{s.t. } \theta \in \mathbb{R} \quad \text{as } k \to \infty, \text{ where}$$

$$s_n = x_1 + \cdots + x_n, \quad n \geq 1.$$

$$n_k = k^2, \quad k \geq 1.$$

then, $\frac{s_n}{n} \to \infty$ as $n \to \infty$.

Proof:-

~~Clearly, $\theta \geq 0$.~~

~~Fix $\varepsilon > 0$ we need to find N such that~~

$$\cancel{\left|\frac{S_N}{N} - \theta\right| \leq \varepsilon \quad \forall\, n \geq N}$$

Fix $n \in \mathbb{N}$.

Let, k be such that $k^2 \leq n \leq (k+1)^2$ then

$$s_{n_k} \leq s_n \leq s_{n_{k+1}}.$$

and hence,

$$\frac{s_{n_k}}{n_{k+1}} \leq \frac{s_n}{n} \leq \frac{s_{n_{k+1}}}{n_k}$$

$$\therefore \quad \frac{n_k}{n_{k+1}} \cdot \frac{s_{n_k}}{n_k} \leq \frac{s_n}{n} \leq \frac{s_{n_{k+1}}}{n_k} \cdot \frac{n_{k+1}}{n_k}.$$

As $n \to \infty$, then k for which $n_k \le n \le n_{k+1}$ also goes to ∞. (Let, $k = \lfloor \sqrt{n} \rfloor$).

$\therefore$ letting $n \to \infty$, the extreme left and right side go to 0.

This completes the proof.

Proof of SLLN:-

Step 1:-

Let, us assume that, $X_1 \geq 0$ a.s.

Then, X_2, X_3, etc. are all non-negative a.s.

Let, $S_n = X_1 + \cdots + X_n$, $n \geq 1$.

Denote $\mu = E(X_1)$

$\sigma^2 = Var(X_1)$.

Chebyshev's inequality implies that, for a fixed $\varepsilon > 0$,

$$P\left(\left|\frac{1}{n} S_n - \mu\right| > \varepsilon\right) = \frac{1}{\varepsilon^2} Var\left(\frac{1}{n} S_n\right) = \frac{\sigma^2}{n \varepsilon^2}, \quad n \geq 1.$$

Let, $n_k = k^2$, $k = 1, 2, \ldots$

then, $\sum_{k=1}^{\infty} P\left(\left|\frac{1}{n_k} S_{n_k} - \mu\right| > \varepsilon\right)$

$$\leq \sum_{k=1}^{\infty} \frac{\sigma^2}{n_k \varepsilon^2}$$

$$= \frac{\sigma^2}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

Borel-Cantelli implies that,

$$P\left(\left|\frac{1}{n_k} S_{n_k} - \mu\right| > \varepsilon \text{ for infinitely many } k\right) = 0.$$

Therefore, $\dfrac{S_{n_k}}{n_k} \to \mu$ almost surely. as $k \to \infty$

If ω is such that,

$$\frac{S_{n_k}(\omega)}{n_k} \to \mu, \text{ then,}$$

the previous lemma will tell us that,

$$\frac{S_n(\omega)}{n} \to \mu.$$

Therefore,

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) \geq P\left(\lim_{k\to\infty} \frac{S_{n_k}}{n_k} = \mu\right) = 1.$$

Hence, $\frac{S_n}{n} \xrightarrow{a.s.} \mu.$

<u>Step 2:-</u>

For a general X_1 (not necessarily non-neg).

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu.$$

<u>Pf:-</u>

Let,

$$\left.\begin{aligned} X_n^+ &= \max\{X_n, 0\} \\ X_n^- &= \max(-X_n, 0) \end{aligned}\right\} n \geq 1.$$

now, $0 \leq X_n^+ \leq |X_1|$

$$\therefore E((X_n^+)^2) \leq E(X_1^2) < \infty.$$

Hence, X_1^+ has finite variance. By step 1, it follows that,

$$\left.\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} (X_i^+) &\xrightarrow{a.s.} E(X_1^+) \quad n \to \infty \\ \text{and } \frac{1}{n}\sum_{i=1}^{n} X_i^- &\xrightarrow{a.s.} E(X_1^-) \quad n \to \infty \end{aligned}\right\}$$

$$\therefore \frac{1}{n}\sum_{i=1}^{n} X_i^+ - X_i^- \to E(X_1^+) - E(X_1^-).$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i \to E(X_1) \text{ a.s.}$$

Homework set. 6

3. (a) X_n ~~Bin~~

(b) ~~Every~~

Pf: (a ⇒ b)

$X_n \xrightarrow{P} X$

Let, X_{n_k} be any subsequence of (X_n) then

$X_{n_k} \xrightarrow{P} X$

then, every sequence converging in probability has a subsequence converging almost surely.

This shows (b).

Pf: (b ⇒ c), trivial.

Pf: (c ⇒ a).

Fix $\varepsilon > 0$ and consider

$$P(|X_n - X| > \varepsilon) = \alpha_n \text{ (say)}.$$

∴ the hypothesis (c) is equivalent to every subsequence of α_n has a further subsequence going to 0. i.e. (a) holds.

hence, $\alpha_n \to 0$,

5. $$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \to \mathrm{Var}(X_1) \text{ a.s.}$$

as $n \to \infty$.

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2$$

SLLN: $\bar{X}_n \xrightarrow{a.s.} E(X_1)$. [because $X_1^2, X_2^2, \dots$ iid with finit Variance

& $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{a.s.} E(X_1^2)$

$$\therefore\ S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 \xrightarrow{a.s.} E(X_1^2) - (E(X_1))^2.$$

$$= \text{Var}(X_1).$$

8.

(a) $P(Z_n \le z)$

$= P(\text{at most one of } U_1, \ldots, U_n \text{ is greater than } z)$

$= P(\text{exactly one of } U_1, \ldots, U_n > z) + P(\text{none of them} > z).$

$= n(1-z)z^{n-1} + z^n.$

$= z^{n-1}(n - nz + z)$

$= (n - (n-1)z) z^{n-1}.$

$\therefore$ Density of Z_n is.

$$f(z) = \left(n(n-1)z^{n-2}(1-z) - nz^{n-1} + nz^{n-1}\right) 1(0<z<1).$$

$$= n(n-1)z^{n-2}(1-z)\, 1(0<z<1).$$

(b) $P(|Z_n - 1| > \varepsilon) = P(Z_n \le 1-\varepsilon)$. | fix $\varepsilon \in (0,1)$.

$$= n\varepsilon(1-\varepsilon)^{n-1} + (1-\varepsilon)^n.$$

Applying ratio test,

$$\sum_{k=1}^{\infty} \left[n\varepsilon(1-\varepsilon)^{n-1} + (1-\varepsilon)^n\right] < \infty.$$

hence, $Z_n \xrightarrow{a.s.} 1$ as $n \to \infty$.

15. $X_n \sim N(0, \sigma_n^2)$.

(a) ($\Leftarrow$) $X_n \xrightarrow{P} 0$ iff $\sigma_n \to 0$.

Suppose, that $\sigma_n \to 0$. For a fixed $\varepsilon > 0$,

$$P(|X_n| > \varepsilon) \leq \frac{1}{\varepsilon^2} \operatorname{Var}(X_n)$$

$$= \frac{\sigma_n^2}{\varepsilon^2} \to 0.$$

Hence, ~~P(|X_n| > ε)~~ $X_n \xrightarrow{P} 0$

($\Rightarrow$) assume $X_n \xrightarrow{P} 0$

$$P(|X_n| > 1) = P\left(\left|\frac{X_n}{\sigma_n}\right| > \frac{1}{\sigma_n}\right).$$

$$= 2 \int_{\frac{1}{\sigma_n}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Let, if possible, $\sigma_n \not\to 0$.

i.e. $\exists$ a subsequence (σ_{n_k}) ~~which is bdd away from~~ such that.

$$\liminf_{k \to \infty} \sigma_{n_k} \geq \varepsilon > 0.$$

$$\limsup_{k \to \infty} \frac{1}{\sigma_{n_k}} \leq \frac{1}{\varepsilon} < \infty.$$

hence, ~~P(|X_n| > 1)~~

$$\liminf_{n \to \infty} P(|X_n| > 1) \geq 2 \int_{\frac{1}{\varepsilon}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx > 0.$$

($\Rightarrow \Leftarrow$)

(b) Exc.

(20). $\lim_{n\to\infty} E(|X_n - Y|) = 0$.

and $X_n \xrightarrow{a.s.} Z$, $n \to \infty$.

to show $Y = Z$ a.s.

$$P(|X_n - Y| > \varepsilon) \le \frac{1}{\varepsilon} E(|X_n - Y|) \longrightarrow 0.$$

$$X_n \xrightarrow{P} Y$$
$$X_n \xrightarrow{P} Z$$

Probability Transform:-

Moment Generating function (MGF):-

Defⁿ:- The MGF of a random variable X (or its distribution) is defined as

$$M(t) = E(e^{tX}) \quad , \quad \forall t \in \mathbb{R}.$$

Example:-

(1) Suppose that $X \sim Bin(n,p)$.

i.e. $\exists$ i.i.d. Bernoulli p random variables such that, $Y_1, \dots, Y_n$ s.t.

$$X \overset{d}{=} Y_1 + \dots + Y_n.$$

$\therefore$ for $t \in \mathbb{R}$,

$$E(e^{tX}) = E(e^{t\sum_{i=1}^{n} Y_i}) = \prod_{i=1}^{n} E(e^{tY_i})$$

$$= \prod_{i=1}^{n} (pe^t + q)$$

$$= (pe^t + q)^n.$$

$\therefore$ Thus, the MGF of Binomial (n,p) is.

$$M(t) = (pe^t + q)^n \quad , t \in \mathbb{R}.$$

$\cancel{(1-p+pe^t)^n}$

$\cancel{(1-p(e^t-1))^n}$

(2) MGF of Poisson (λ).

$$E(e^{tX}) = \sum e^{tx} \cdot e^{-\lambda} \frac{\lambda^x}{x!}.$$

$$= \sum e^{-\lambda} \frac{(e^t \lambda)^x}{x!}.$$

$$= e^{-\lambda} \cdot e^{e^t \lambda}$$

$$= e^{-\lambda(1-e^t)}.$$

$$= e^{\lambda(e^t - 1)}.$$

3. MGF of Exp(λ).

Let, $X \sim Exp(\lambda)$ for $t \in \mathbb{R}$,

$$E\left[e^{tX}\right] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx.$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx.$$

$$= \begin{cases} \frac{\lambda}{\lambda - t}, & t < \lambda, \\ \infty, & t \geq \lambda \end{cases}$$

4. Suppose $X \sim \chi^2_n$:

$$E(e^{tX}) = (1-2t)^{-\frac{n}{2}}.$$

$$M(t) = \begin{cases} (1-2t)^{-\frac{n}{2}}, & t < \frac{1}{2} \\ \infty, & t \geq \frac{1}{2} \end{cases}$$

$Y_1, \dots, Y_n \overset{i.i.d.}{\sim} N(0,1)$.

$$X \overset{d}{=} \sum_{i=1}^n Y_i^2$$

$$E(e^{tX}) = \prod_{i=1}^n E(e^{tY_i^2})$$

$$= \prod_{i=1}^n \int_{-\infty}^\infty e^{ty^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

$$= \prod_{i=1}^n \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2(1-2t)}{2}} dy.$$

for $t < \frac{1}{2}$. $E(e^{tX}) = \prod_{i=1}^n \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{y^2}{(1-2t)^{-1}}} dy$

$$= \prod_{i=1}^n (1-2t)^{-\frac{1}{2}} = (1-2t)^{-\frac{n}{2}}$$

(5) Let, X have density,

$$f(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{tx+x} dx + \frac{1}{2} \int_{0}^{\infty} e^{(t-1)x} dx$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{(t+1)x} dx + \frac{1}{2} \int_{0}^{\infty} e^{(t-1)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{(t+1)x}}{t+1} \Big|_{-\infty}^{0} + \frac{e^{(t-1)x}}{(t-1)} \Big|_{0}^{\infty} \right]$$

$$= \begin{cases} \frac{1}{1-t^2}, & |t| < 1 \\ \infty, & |t| \geq 1 \end{cases}$$

Theorem:- If $X_1, \dots, X_n$ are independent random variables with respective MGF's $M_1, M_2, \dots, M_n$, then MGF of $Y = X_1 + \dots + X_n$ is

$$M_Y(t) = M_1(t) M_2(t) \dots M_n(t), \quad t \in \mathbb{R}.$$

Theorem:- Suppose that M is the MGF of X

Let $\alpha = \inf \{t \in \mathbb{R} : M(t) < \infty\}$.

$\beta = \sup \{t \in \mathbb{R} : M(t) < \infty\}$.

Then, $\alpha \leq 0 \leq \beta$ and if $\alpha < \beta$, the M is finite on (α, β).

Proof:- That $\alpha \leq 0 \leq \beta$ follows from the ~~[illegible]~~ fact $M(0) = 1$.

Suppose $\alpha < \beta$ and $t \in (\alpha, \beta)$.

Let, $\exists$ ~~[illegible]~~ $a < t$ be such that $M(a) < \infty$.

and $b > t$ be such that $M(b) < \infty$.

Note that,

$$e^{tX} \le e^{bX} 1(X \ge 0) + e^{aX} 1(X < 0).$$

$$\le e^{bX} + e^{aX}.$$

Hence,

$$M(t) = E(e^{tX}) \le E(e^{bX}) + E(e^{aX})$$

$$= M(a) + M(b) < \infty.$$

Thus, M is finite on (α, β).

Word of Caution:-

We don't know anything about $M(\alpha)$ and $M(\beta$

HW-5

3.

(b) $X \sim F_{m,n}$.

$E(X) = ?$

if $Y_1, \ldots, Y_m$, $Z_1, \ldots, Z_n$ are i.i.d. $N(0,1)$,

the, $$E(X) = E\left(\frac{\sum_{i=1}^{m} Y_i^2/m}{\sum_{i=1}^{n} Z_i^2/n}\right)$$

$$= E\left(\frac{1}{m}\sum_{i=1}^{m} Y_i^2\right) E\left(\frac{n}{\sum_{i=1}^{n} Z_i^2}\right).$$

$$= E\left(\frac{n}{\sum_{i=1}^{n} Z_i^2}\right)$$

$$= \int_0^\infty \frac{n}{z} \frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma\frac{n}{2}} e^{-\frac{z}{2}} z^{\frac{n}{2}-1} dz.$$

$$= \frac{n}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \int_0^\infty e^{-\frac{z}{2}} z^{\frac{n}{2}-1-1} dz$$ (3b)

$$= \frac{n}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} \frac{\Gamma\frac{n}{2}-1}{\left(\frac{1}{2}\right)^{\frac{n}{2}-1}} \quad \text{if } \frac{n}{2}-2 > -1.$$
$$\Rightarrow n > 2$$

$$= \frac{n}{n-2}$$

$$\therefore E(X) = \begin{cases} \frac{n}{n-2} & \text{if } n \geq 3. \\ \infty & \text{if } n = 1, 2. \end{cases}$$

3. (c) $X \sim t_n$.

$$X \stackrel{d}{=} \frac{Y}{\sqrt{\frac{Z}{n}}}, \quad \begin{matrix} Y \sim N(0,1) \\ Z \sim \chi^2_n \end{matrix} \quad \text{indep}.$$

$$E(|X|) = E(|Y|) \cdot E\left(\sqrt{\frac{n}{Z}}\right).$$
$$= \sqrt{\frac{2}{\pi}} \; E\left(\sqrt{\frac{n}{Z}}\right).$$

$$E\left(\sqrt{\frac{n}{Z}}\right) =$$

4.

(a) $X^2 \sim Exp(1)$.

(a) ~~$\cdots$~~ $X \geq 0$,

for $x > 0$,

$$P(X \leq x) = P(X^2 \leq x^2) = 1 - e^{-x^2}.$$

Density of X is.

$$f(x) = 2xe^{-x^2} 1(x > 0)$$

(b) $X \stackrel{d}{=} -X$.

for $x > 0$.

$$P(X \leq x) = P(X \leq 0) + P(0 < X \leq x).$$

$$= P(X \le 0) + \frac{1}{2} P(|X| \le x).$$

$$= P(X \le 0) + \frac{1}{2} P(X^2 \le x^2).$$ (4b)

$$= \frac{1}{2} + \frac{1}{2}(1 - e^{-x^2}).$$

Density of X is

$$f(x) = |x| e^{-x^2}.$$

Thm:- Suppose that MGF of a random variable X is finite ~~a~~ in an open neighbourhood of 0. Then all the moments of X are finite.

Pf:- The hypothesis implies that for some t in $(0, \infty)$ $E[e^{tX}] < \infty$ and $E(e^{-tX}) < \infty$.

Then $$E(e^{t|X|}) \le E[e^{tX} + e^{-tX}] < \infty.$$

$$e^{t|X|} = \sum_{n=0}^{\infty} \frac{t^n |X|^n}{n!}$$

Now, all the terms in RHS is positive.

hence, for a fixed $n \in \mathbb{N}$.

$$|X|^n \le n!\, t^{-n} e^{t|X|}.$$

$$E(|X^n|) = E(|X|^n) \le n!\, t^{-n} E(e^{t|X|}) < \infty \quad \blacksquare$$

Exm:- Suppose that $X \sim \text{Pareto}(\alpha)$ for some $\alpha > 0$

$$M(t) = \int_1^\infty e^{tx} \alpha x^{-\alpha-1} dx.$$

$$< \infty \iff t \le 0.$$

Therefore, the previous theorem fails if 0 is not an interior point.

Thm:- Suppose that the MGF M of X is finite in $\pm t$ for some $0 < t < \infty$. Then M is finite on $[-t, t]$ and infinitely many times diff'ble on $(-t, t)$ and

$$\frac{d^n}{ds^n} M(s)\Big|_{s=0} = E(X^n) \quad , n \geq 1.$$

Proof:- It is already proved that $M(s) < \infty$ for $|s| < t$.

For the remaining part it suffices to show

$$M(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} E(X^n) \quad , |s| \leq t.$$

Fix $s \in [-t, t]$.

Define. $$Y_n = \sum_{k=0}^{n} \frac{s^k}{k!} X^k \quad , n \geq 0.$$

For any $\omega \in \Omega$,

$$Y_n(\omega) = \sum_{k=0}^{n} \frac{s^k}{k!} (X(\omega))^k = \sum_{k=0}^{n} \frac{(s X(\omega))^k}{k!}$$

$$\therefore \quad Y_n(\omega) \to e^{sX(\omega)} \quad \text{as } n \to \infty.$$

i.e., $Y_n \to e^{sX}$ a.s.

Next observe that

$$|Y_n| \leq \sum_{k=0}^{n} \frac{|sX|^k}{k!} \leq e^{|sX|}.$$

$\because$ $M(\pm s) < \infty$ it follows the $E(e^{|sX|}) < \infty$.

hence, DCT applies and implies that,

$$E\left[e^{sX}\right] = \lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} \frac{s^k}{k!} E(X^k)\right]$$

$$= \sum_{k=0}^{\infty} \frac{s^k}{k!} E(X^k).$$

In other words, $M(s)$ has the desired power series expansion $|s| \le t$.

Therefore M is infinitly many times diff'ble on $(-t,t)$ and $n \geq 1$,

$$\frac{d^n M(s)}{ds^n} = \sum_{k=n}^{\infty} \frac{1}{(k-n)!} s^{k-n} E(X^k), \quad |s| < t.$$

$$\therefore \left.\frac{d^n M(s)}{ds^n}\right|_{s=0} = E(X^k). \quad \blacksquare$$

Exm:- Suppose that $X_1, \dots, X_n$ are i.i.d. with density

$$f(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$

Let, $Y = X_1 + \cdots + X_n$. The MGF of Y is

$$M_Y(t) = \left(E\left(e^{tX_1}\right)\right)^n$$

$$= \begin{cases} \dfrac{1}{(1-t^2)^n}, & |t| < 1. \\ \infty, & |t| \geq 1. \end{cases}$$

for $|t| < 1$.

$$M_Y(t) = (1-t^2)^{-n} = 1 + nt^2 + \frac{n(n+1)}{2} t^4 + \frac{n(n+1)(n+2)}{3!} t^6 + \cdots$$

The previous theorem implies that for $|t|<1$,

$$M_Y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(Y^k).$$

Comparing coefficient we get,

$$E(Y^k) = \begin{cases} 0, & \text{if } k \text{ odd} \\ \frac{k!}{(k/2)!}\, n(n+1)\cdots(n+\frac{k}{2}-1) & \text{if } k \text{ even} \end{cases}$$

Example -

$X \sim$ std Normal

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx + t^2 - t^2)}\, dx$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2}\, dx.$$

$$= e^{t^2/2}.$$

$$e^{t^2/2} = 1 + \frac{t^2}{2} + \frac{1}{2!}\frac{t^4}{2^2} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{t^{2k}}{2^k}.$$

$E(X^k) = 0$ if k = odd.

$$\frac{t^{2k}}{2k!} E(X^k) = \frac{1}{k!}\frac{t^{2k}}{2^k}.$$

$$\therefore E(X^k) = \frac{2k!}{k!\, 2^k}.$$

$$\therefore E(X^k) = \begin{cases} \frac{k!}{(k/2)!\, 2^{k/2}} & \text{if } k = \text{even} \\ 0 & \text{if } k = \text{odd} \end{cases}$$

Example :- $X \sim Exp(\lambda)$. Then MGF of X is.

$$M(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & , t < \lambda. \\ \infty & , t \geq \lambda \end{cases}$$

for $|t| < \lambda$.

$$M(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots$$

$$E(X^n) = \frac{n!}{\lambda^n}, \quad n \geq 1.$$

For $t = -\lambda$.

$$M(t) = \frac{1}{2} \neq \sum_{k=0}^{\infty} \frac{(-\lambda)^n}{n!} E(X^n).$$

because the infinite sum diverges.

Example :- Let, Y be a R.V. with density.

$$f(y) = \frac{1}{2^{3/2}\sqrt{\pi}} \frac{e^{-\frac{1}{2}(\log|y|)^2}}{|y|} 1(y \neq 0), y \in \mathbb{R}.$$

$$\int_{-\infty}^{\infty} f(y)\,dy = 2\int_{0}^{\infty} f(y)\,dy. = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2} \frac{dy}{y}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

$$= 1.$$

$E(Y^n)$

$$= 2\int_0^\infty y^n \frac{1}{2^{3/2}\sqrt{\pi}} e^{-\frac{1}{2}(\log y)^2} \frac{dy}{y}$$

$$= \int_{-\infty}^{\infty} e^{nx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

$$= e^{n^2/2}.$$

For $t \neq 0$,

$$\int_{-\infty}^{\infty} e^{ty} f(y)\, dy = \int_{-\infty}^{\infty} \frac{1}{2^{3/2}\sqrt{\pi}} e^{ty - \frac{1}{2}(\log|y|)^2} \frac{dy}{|y|}.$$

Suppose $t > 0$, then $\exists\ y_0 > 0$ such that $ty > \frac{1}{2}(\log y)^2 \ \forall\ y \geq y_0$.

Hence, $$E(e^{tY}) \geq c\int_{y_0}^{\infty} e^{ty - \frac{1}{2}(\log y)^2} \frac{dy}{y}.$$

$$\geq c\int_{y_0}^{\infty} \frac{dy}{y} = \infty.$$

and, by symmetry, a similar argument works for $t < 0$.

Thus all moments of Y are finite, even though its MGF is infinite at every $t \neq 0$.

<u>Theorem</u>:- Suppose that X is a RV with all moments finite, and for some t the sum $\sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$ converges.

Then $E(e^{tX})$ is finite and equals the above.

Proof:- Since $M(0)=1$, we assume WLOG that $t \neq 0$. We prove the claim for $t>0$, the case $t<0$ being similar.

Suppose $t>0$; Finiteness of ~~the~~ ~~power~~ the $\sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$ implies that the radius of convergence of the power series, satisfies are

$R \geq t$.

That means $\forall s$ with $|s|<t$.

$\sum_{n=0}^{\infty} \frac{s^n}{n!} E(X^n)$ converges absolutely.

Hence $E(e^{|sX|}) < \infty$ for $|s|<t$.

By similar arguement as before, we get that.

$$M(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} E(X^n), \quad |s|<t. \ (*)$$

Since, $\sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$ converges, we get.

$$\lim_{s \uparrow t} \sum_{n=0}^{\infty} \frac{s^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n). \ (*)(*).$$

Now,

$$M(s) = E(e^{sX}) = E\left[e^{sX} 1(X \geq 0)\right] + E\left[e^{sX} 1(X<0)\right].$$

As $s \uparrow t$

$e^{sX} 1(X<0) \to e^{tX} 1(X<0)$, a.s.

Since, for $s \geq 0$ $0 \leq e^{sX} 1(X<0) \leq 1$.

BCT implies that $\lim_{s \uparrow t} E(e^{sX} 1(X<0)) = E(e^{tX} 1(X<0))$.

and,

$$e^{sX} 1(X \geq 0) \uparrow e^{tX} 1(X \geq 0) \quad \text{a.s.} \quad \text{as} \quad s \uparrow t.$$

$\therefore$ MCT implies.

$$\lim_{s \uparrow t} E\left[e^{sX} 1(X \geq 0)\right] = E\left[e^{tX} 1(X \geq 0)\right].$$

i.e. $\lim_{s \uparrow t} M(s) = M(t)$. $(***)$

So, $(*)$, $(**)$ & $(***)$ shows that,

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n). \quad \blacksquare$$

<u>FACT:-</u> If $\forall\ |s| < t$, the sum

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} E(X^n) \text{ converges absolutely.}$$

then $E\left(e^{|sX|}\right) < \infty$, $|s| < t$.

<u>Pf:-</u> for a fixed $s \in (-t, t)$.

$$\infty > \sum_{n=0}^{\infty} \left| \frac{s^n}{n!} E(X^n) \right|.$$

$$\geq \sum_{n=0}^{\infty} \left| \frac{s^{2n}}{(2n)!} E(X^{2n}) \right|$$

$$= \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!} E(X^{2n}).$$

Observe that as $n \to \infty$.

$$0 \leq \sum_{k=0}^{n} \frac{s^{2k} X^{2k}}{(2k)!} \uparrow \frac{1}{2}\left(e^{sX} + e^{-sX}\right) \text{ a.s.}$$

MCT implies that

$$E\left(\frac{e^{sX} + e^{-sX}}{2}\right) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{s^{2k}}{(2k)!} E(X^{2k}).$$

$$= \sum_{k=0}^{\infty} \frac{s^{2k} E(X^{2k})}{(2k)!} < \infty.$$

$\therefore E(e^{sX})$ is finite & $E(e^{-sX})$ is also finite.

Since, $e^{|sX|} \leq e^{sX} + e^{-sX}$, ~~we obtain~~

$$\therefore E(e^{|sX|}) < \infty.$$

<u>Thm</u>:- If X and Y are random variables such that for some $\varepsilon > 0$,

$$E[e^{sX}] = E[e^{sY}] < \infty, \quad |s| \leq \varepsilon;$$

then $X \overset{d}{=} Y$.

<u>Pf</u>:- Postponed for the time being.

<u>Exm</u>:- Suppose that X and Y are iid. Then $X \sim N(\mu, \sigma^2) \iff X+Y \sim N(2\mu, 2\sigma^2)$.

($\Rightarrow$) trivial

($\Leftarrow$)

$$E(e^{tZ}) = e^{2\mu t + \sigma^2 t^2}$$

and

$$\begin{aligned} E[e^{tZ}] &= E(e^{tX} \cdot e^{tY}) \\ &= E(e^{tX}) E(e^{tY}) \\ &= (E(e^{tX}))^2 \end{aligned}$$

$$\Rightarrow E(e^{tX}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad t \in \mathbb{R}.$$

$\because$ RHS is MGF of $N(\mu, \sigma^2)$.

$\therefore$ the previous theorem implies $X \sim N(\mu, \sigma^2)$.

Exc: Suppose that $X_1, \dots, X_n$ are i.i.d. Let $Y = X_1 + \dots + X_n$. Show that,

1. $Y \sim \text{Poisson}(\lambda) \Leftrightarrow X_1 \sim \text{Poisson}(\lambda/n)$,
2. $Y \sim \text{Bin}(n, p) \Leftrightarrow X_1 \sim \text{Ber}(p)$,
3. $Y \sim \text{Negative Bin}(nk, p) \Leftrightarrow X_1 \sim \text{Negative Bin}(k, p)$.
4. $Y \sim \text{Gamma}(\alpha, \lambda) \Longleftrightarrow X_1 \sim \text{Gamma}\left(\frac{\alpha}{n}, \lambda\right)$,
5. $Y \sim N(\mu, \sigma^2) \Leftrightarrow X_1 \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$.

Characteristic Functions:-

Defn:- Let, Z be a $\mathbb{C}$-valued R.V. Then $Z = X + iY$ where X, Y are $\mathbb{R}$-valued R.V. and $i = \sqrt{-1}$. If X and Y have finite expectations, then expectation of Z is defined as $E(Z) = E(X) + iE(Y)$.

Thm:- If Z_1 and Z_2 are complex valued R.V.s have expectation then

$$E(z_1 Z_1 + z_2 Z_2) = z_1 E(Z_1) + z_2 E(Z_2).$$

$$\forall z_1, z_2 \in \mathbb{C}.$$

Pf:- Exc:-

Thm:- For a $\mathbb{C}$ valued RV Z with expectation, it hold, that $|E(Z)| \leq E(|Z|)$.

$$\cancel{E(Y^2 + Y^2) \geq E^2\left(\sqrt{X^2+Y^2}\right) \geq 0}.$$

$$\cancel{E(X^2) + E(Y^2) \geq E^2\left(\sqrt{X^2+Y^2}\right)}.$$

Pf:- Define, $z = E(Z)$.

if $z = 0$ the claim is trivial.

Assume WLOG $z \neq 0$

Define $\quad Z' := z^{-1} Z$.

$\therefore$ The previous thm implies,

$$E(Z') = z^{-1} E(Z) = z^{-1} z = 1.$$

Therefore

$$1 = E\left[\mathrm{Re}\,(Z')\right] \leq E\,|\mathrm{Re}\,(Z')|$$

$$\leq E\,|Z'|.$$

$$= |z|^{-1} E\,|Z|.$$

$\Rightarrow \quad |z| \leq E\,|Z|.$

$\Rightarrow |E(Z)| \leq E\,|Z|.$

Def^n: For a Real valued R.V. X, its characteristic function (CHF) is a function $\phi : \mathbb{R} \to \mathbb{C}$ defined by

$$\phi(t) = E\left[e^{itX}\right] = E(\cos tX) + i\,E(\sin tX).$$

$\forall t \in \mathbb{R}$.

Exc:- ① Show that $|\phi(t)| \leq 1$ for all t.

② $\phi(0) = 1$.

HW-5

⑤ $F(X) \sim U(0,1)$.

Fix $y \in (0,1)$.

$$F^*(y) = \sup \{x \in \mathbb{R} : F(x) = y\}.$$

Since, $F(-\infty) = 0$ & $F(\infty) = 1$, $\exists\ x_0, x_1$ such that $F(x_0) \leq y \leq F(x_1)$. Hence,

$\{x \in \mathbb{R},\ F(x) = y\} \neq \phi$. otherwise

Furthermore, x_1 is an upper bound of $F^*(y)$.

Claim 1 $F(F^*(y)) = y$, for every $y \in (0,1)$.

Pf:- By defn. $\exists \{x_n\} \in \{x \in \mathbb{R}, F(x) = y\}$ such that $x_n \to F^*(y)$.

Continuity of F implies that,
$$F(F^*(y)) = \lim_{n\to\infty} F(x_n) = y.$$

~~For $y \in (0,1)$.~~

Claim 2 $\forall x \in \mathbb{R}$ and $y \in (0,1)$.
$$F(x) \le y \iff x \le F^*(y).$$

Pf. of claim:-
Suppose $F(x) \le y$. Then, either $F(x) = y$ (Case I) or $F(x) < y$ (Case II).

In case I, $x \in \{x' : F(x') \le y\}$. and $F^*(y)$ is sup of this set
$$\therefore x \le F^*(y).$$

In case II, $F(F^*(y)) = y$

~~$F(x) \le F(F^*(y))$~~

~~$x \le F^*(y)$~~.

Hence, $F^*(y) > x$.

Suppose that $x \le F^*(y)$. Then
$$\Rightarrow F(x) \le F(F^*(y)) = y$$ Hence '$\Leftarrow$' follows.
$\because F$ is non-decreasing

Now,
For fix $y \in (0,1)$,
$$P(F(X) \le y) = P(X \le F^*(y)) \text{ (Claim 2)}$$
$$= F(F^*(y)) = y \text{ [Claim 1]}$$

For $y=0$ & $y=1$ a limiting arguement works and gives us.

$$P(F(X)\leq y) = \begin{cases} 0 & , y\leq 0. \\ y & , 0<y\leq 1. \\ 1 & , y>1. \end{cases}$$

HW-7

① $X_n \to X$ a.s.

ⓐ If $\omega \in \Omega$ is such that

$$X_n(\omega) \to X(\omega),$$

then $\exists N$ s.t.

$$|X_n(\omega) - X(\omega)| \leq \varepsilon \quad \text{for } n\geq N.$$

and hence.

$$1(|X_n(\omega) - X(\omega)| > \varepsilon) = 0, \quad \text{for } n\geq N.$$

In other word:

$$P[\omega\in\Omega : 1(|X_n - X| > \varepsilon) \to 0] \geq P(X_n \to X) = 1.$$

ⓑ. $1(|X_n - X| > \varepsilon) \to 0$ a.s.

Since, $0 \leq 1(|X_n - X| > \varepsilon) \leq 1$, BCT applies

$$E(1(|X_n - X| > \varepsilon)) \to 0$$

$$\Rightarrow P(|X_n - X| > \varepsilon) \to 0.$$

(2) $X_n \to X$ a.s.

$$\Updownarrow$$

$P(|X_n - X| > \varepsilon \text{ for infinitely many } n) = 0, \ \varepsilon > 0.$

($\Rightarrow$)

$X_n \to X$ a.s.

Fix $\varepsilon > 0$. If $\omega \in \Omega$ is such that $X_n(\omega) \to X(\omega)$, then $\exists N$ depending on ω s.t. $|X_n(\omega) - X(\omega)| \leq \varepsilon$ for $n \geq N$.

such a $\omega \in [\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon \text{ for infinitely many } n]^c$

$$\Rightarrow [\omega \in \Omega : |X_n(\omega) - X_n| \leq \varepsilon] \subseteq [\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon \text{ for infinitely many } n]^c$$

$$\Rightarrow P(X_n \to X) \leq P(|X_n - X| \text{ for infinitely many } n)$$
$$\| $$
$$1.$$

($\Leftarrow$) $P(|X_n - X| \text{ for infinitely many } n) = 0$

$$\Rightarrow P(\limsup_{n\to\infty} |X_n - X| \leq \varepsilon) = 1.$$

$$\Rightarrow P(\limsup_{n\to\infty} |X_n - X| \leq 0) = P\left(\bigcap_{k=1}^{\infty} \limsup_{n\to\infty} |X_n - X| \leq \frac{1}{k}\right)$$

$$= 1.$$

Exc:- Show that, if ϕ is the CHF of X then,

ⓐ $\phi(0) = 1$.

ⓑ $|\phi(t)| \leq 1$.

ⓐ $\phi(0) = E[e^{i0X}] = 1$.

ⓑ $|\phi(t)| = |E(e^{itX})| \leq E|e^{itX}| = E(1) = 1$.

Exc:- Suppose that, $\phi(t) = 1$ for some $t \neq 0$, where ϕ is the CHF of X.

Show that X is a discrete R.V.

Sol^n:- $1 = \phi(t) = E[\cos(tX)] + i\,E[\sin(tX)]$.

i.e. $E[\cos(tX)] = 1$.

Since, $\cos(tX) \leq 1$. a.s.

$\Rightarrow \cos(tX) = 1$ a.s.

Therefore,

$P[tX = 2n\pi$ for some $n \in \mathbb{Z}] = 1$.

i.e.

$P[X \in \{\frac{2n\pi}{t} : n \in \mathbb{Z}\}] = 1$.

Furthermore, show that, ϕ has period $|t|$.

$E[\cos tX] = 1$ $E[\sin tX] = 0$.

$E[\cos(t+s)X] = E[\cos tX \cos sX - \sin tX \sin sX] = E[\cos sX]$

$\because \cos tX \cos sX = \cos sX$ a.s. & $\sin tX \sin sX = 0$ a.s.

Exc:- Suppose that $X \sim \text{Exp}(\lambda)$. Calculate the characteristic function of X.

Soln:- For $t \in \mathbb{R}$,

$$\mathbb{E}[e^{itX}] = \int_0^\infty e^{itx} \lambda e^{-\lambda x}\, dx.$$

$$= \lambda \int_0^\infty e^{(it-\lambda)x}\, dx.$$

$$= \frac{\lambda}{it-\lambda} e^{(it-\lambda)x} \Big|_0^\infty$$

$$= \frac{\lambda}{it-\lambda}(0-1).$$

$$= \frac{\lambda}{\lambda - it}$$

$$E[e^{itX}] = E[\cos tX] + i E[\sin tX].$$

$$E[\cos tX] = \int_0^\infty \cos tx\, \lambda e^{-\lambda x}\, dx.$$

Now,

$$\frac{\lambda e^{(it-\lambda)x}}{it-\lambda}$$

$$= \frac{\lambda(\cos tx + i \sin tx) e^{-\lambda x} \cdot (it+\lambda)}{-(t^2+\lambda^2)}$$

$$= \frac{\lambda e^{-\lambda x}(\lambda \cos tx - t \sin tx)}{-(\lambda^2+t^2)} + i(\ \ ).$$

$$= \frac{\lambda e^{-\lambda x}(t \sin tx - \lambda \cos tx)}{\lambda^2+t^2} + i(\ \ ).$$

$$\frac{d}{dx}(\ \ ) = \cos tx\, \lambda e^{-\lambda x}.$$ check!!

$$\therefore \; E[\cos tX] = \int_0^\infty \cos tx \, e^{-\lambda x} \lambda \, dx.$$

$$= \left. \frac{\lambda e^{-\lambda x}(t \sin tx - \lambda \cos tx)}{t^2+\lambda^2} \right|_0^\infty$$

$$= 0 - \left(- \frac{\lambda^2}{t^2+\lambda^2}\right).$$

$$= \frac{\lambda^2}{t^2+\lambda^2}.$$

(1) ~~Compltet~~

& $\;E(\sin tX) = \int_0^\infty \sin(tx) \, \lambda e^{-\lambda x} dx$

=

Exc:- Suppose, $X \sim Poi(\lambda)$. Calculate CHF of X.

Soln:-

For $t \in \mathbb{R}$,

$$\phi(t) = E(e^{itX})$$

$$= \sum_{n=0}^{\infty} e^{int} P(X=n).$$

$$= \sum_{n=0}^{\infty} e^{int} \cdot \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^{it})^n}{n!}$$

$$= \exp[\lambda(e^{it}-1)].$$

Show that, for any complex number $z \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

Hint:- $z = r(\cos\theta + i\sin\theta)$.

$$z^n = r^n (\cos\theta + i\sin\theta)^n.$$

$$= r^n (\cos n\theta + i\sin n\theta).$$

Thm:- CHF of a R.V. is uniformly cont. on $\mathbb{R}$.

Pf:- Let ϕ be the CHF of X.

For $t, h \in \mathbb{R}$.

$$|\phi(t+h) - \phi(t)| = |E[e^{i(t+h)X}] - E[e^{itX}]|.$$

$$= |E[e^{itX}(e^{ihX}-1)]|$$

$$\le E|e^{itX}(e^{ihX}-1)|.$$

$$= E|e^{itX}||e^{ihX}-1|$$

$$= E|e^{ihX}-1|.$$

Thm:- (DCT for complex valued RV) If Z_n, Z are $\mathbb{C}$ valued R.V. s.t.

$$Z_n \to Z \text{ a.s},$$

$$|Z_n| \le X \text{ for some } X \text{ such that } E(X) < \infty.$$

then, $E(Z_n) \to E(Z)$.

As $h \to 0$ $|e^{ihX} - 1| \to 0$ a.s.

& $|e^{ihX} - 1| \le |e^{ihX}| + 1 = 2$.

BCT implies that,

$$\lim_{h \to 0} E|e^{ihX} - 1| = 0.$$

$\therefore$ given $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$E|e^{ihX} - 1| \le \varepsilon \quad \forall \ |h| < \delta.$$

Thus for $t \in \mathbb{R}$ and $|h| < \delta$.

$$|\phi(t+h) - \phi(t)| < \varepsilon$$

Pf of DCT for $\mathbb{C}$-valued R.V. :-

$$Z_n \to Z \text{ a.s.}$$

then $Re(Z_n) \to Re(Z)$ a.s.

$$|Re(Z_n)| \le |Z_n| \le X$$

Applying DCT (for $\mathbb{R}$-valued RV) to $Re(Z_n)$, we get that,

$$E(Re(Z_n)) = E[Re(Z)].$$

A similar argument works for the $Im(Z_n)$ & $Im(Z)$. completes the proof.

Thm:- Suppose that for a R.V. X, its MGF $M(\cdot)$ is finite at $\pm t$ for some $t > 0$. Then, $\forall |s| \le t$

$$E\left[e^{isX}\right] = \sum_{n=0}^{\infty} \frac{(is)^n}{n!} E(X^n).$$

Pf: $\because$ $M(\pm t) < \infty$, the moments of X is finite.

Define for a fixed $s \in [-t, t]$

$$Z_n = \sum_{k=0}^{n} \frac{(is)^k X^k}{k!} \quad n \ge 0.$$

$$|Z_n| \le \sum_{k=0}^{n} \frac{|sX|^k}{k!} \le e^{|sX|}.$$

$$E\left[e^{|sX|}\right] \le E\left[e^{t|X|}\right] < \infty.$$

because $M(\pm t) < \infty$.

Since, $Z_n \to e^{isX}$ a.s., DCT yields that

$$E\left[e^{isX}\right] = \lim_{n\to\infty} E(Z_n).$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(is)^k}{k!} E(X^k)$$

$$= \sum_{k=0}^{\infty} \frac{(is)^k}{k!} E(X^k). \quad \blacksquare$$

Thm:- If the n-th moment of X is finite then its CHF ϕ is n times diff'ble &

$$\frac{d^n}{dt^n} \phi(t) = E\left((iX)^n e^{itX}\right).$$

Pf: Let, $n=1$.

Need to show, that for $t \in \mathbb{R}$,

$$\frac{\phi(t+h) - \phi(t)}{h} \to E\left[iXe^{itX}\right].$$

as $h \to 0$.

$$\text{L.H.S.} = \frac{1}{h} E\left[e^{i(t+h)X} - e^{itX}\right].$$

$$= E\left[e^{itX}\left(\frac{e^{ihX}-1}{h}\right)\right]$$

$$\frac{e^{ihX}-1}{h} = \frac{\cos(hX)-1}{h} + i \cdot \frac{\sin h(X)}{h}$$

MVT implies that, $\forall x \in \mathbb{R}$, and $h \neq 0$.

$$\frac{\cos hx - 1}{h} = \frac{\cos hx - \cos 0}{h-0} = -x \sin(\xi x).$$

for some $\xi \in (0,h)$.

$$\therefore \left|\frac{\cos(hx)-1}{h}\right| = |x \sin(\xi x)| \leq |x|.$$

Ily, $\left|\frac{\sin(hx)}{h}\right| \leq |x|$. for $x \in \mathbb{R}$ & $h \neq 0$.

$$\therefore \left|\frac{e^{ihX}-1}{h}\right| \leq \left|\frac{\cos(hX)-1}{h}\right| + \left|\frac{\sin(hX)}{h}\right|$$

$$\leq 2|X|.$$

Since, $E|X| < \infty$ & for a fixed $t \in \mathbb{R}$

$$\frac{e^{i(t+h)X} - e^{itX}}{h} \xrightarrow{a.s.} e^{itX} \cdot iX \quad \text{as } h \to 0.$$

DCT implies that,

$$\lim_{h \to 0} E\left[\frac{e^{i(t+h)X} - e^{itX}}{h}\right] = E\left[iX e^{itX}\right].$$

Or, equivalently,

$$\lim_{h\to 0} \frac{\phi(t+h)-\phi(t)}{h} = E\left[iX\, e^{itX}\right]$$

Thus; $\frac{d}{dt}\phi(t) = E\left[iX\, e^{itX}\right]$, $t \in \mathbb{R}$.

Assume that, the claim holds for n. Shall show for $n+1$.

$$\frac{d^n}{dt^n}\phi(t) = E\left[(iX)^n e^{itX}\right], \ t\in\mathbb{R}.$$

In ~~~~ view of the above, all that needs to be shown ~~that~~ is.

$$\lim_{h\to 0} \frac{1}{h}\left[E\left((iX)^n e^{i(t+h)X}\right) - E\left((iX)^n e^{itX}\right)\right]$$

$$= E\left[(iX)^{n+1} e^{itX}\right]. \qquad (*).$$

Note that the quantity in the limit in LHS of $(*)$ equals.

$$E\left[(iX)^n\left(e^{itX}\right)\left(\frac{e^{ihX}-1}{h}\right)\right]$$

Clearly, the R.V. inside Expectation converges ~~to~~ a.s to $(iX)^{n+1} e^{itX}$

furthermore,

$$\left|(iX)^n\left(e^{itX}\right)\left(\frac{e^{ihX}-1}{h}\right)\right| = |X|^n \cdot \left|\frac{e^{ihX}-1}{h}\right| \le |X|^n \cdot 2|X|$$
$$= 2|X|^{n+1}.$$

& $E\left(|X|^{n+1}\right) < \infty$ by (given)

DCT applies and implies that,

$$\lim_{h \to 0} E\left[(iX)^n \cdot e^{itX} \frac{e^{ihX} - 1}{h} \right] = E\left[(iX)^{n+1} e^{itX}\right]$$

i.e. $$\frac{d^{n+1}}{dx^{n+1}} \phi(t) = E\left[(ix)^{n+1} e^{itX}\right]$$ $\boxtimes$

Exc:- Give an ~~exm~~ example of a C^∞ function from $\mathbb{R}$ to $\mathbb{R}$ which does not admit a power series ~~od~~ around 0 in any neighbourhood.

Solⁿ: Let X be a RV with density.

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(\ln|x|)^2} \cdot \frac{1}{|x|} 1\cdot(x \neq 0), \ x \in \mathbb{R}.$$

If ϕ is CHF of X, then for any $t \in \mathbb{R}$.

$$\phi(t) = E[\cos tX] + i E[\sin tX].$$

$$= E[\cos tX] \quad \left[\because \sin(tX) \text{ is odd f}^n \text{ \& } X \text{ is symm}\right]$$

$$= \int_{-\infty}^{\infty} \cos(tx) f(x)\, dx.$$

$\therefore$ ϕ is a function from $\mathbb{R} \to \mathbb{R}$ which is C^∞ by the previous theorem, using the fact that all moments of X are finite.

Suppose that for some $t \neq 0$, ϕ has a power series expansion at t. ~~then~~ around 0, then

$$\phi(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \phi^n(0).$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} E[(iX)^n]$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E(X^n).$$

Since, the above sum converges, it is necessary that,

$$\frac{|t|^n \, |E(X^n)|}{n!} \to 0 \quad \text{as } n \to \infty.$$

$$\Rightarrow \quad \frac{1}{\limsup\limits_{n\to\infty} \left|\frac{E(X^n)}{n!}\right|^{1/n}} \geq |t|.$$

Hence, for every,

$s \in (-|t|, |t|)$;

the sum $\sum_{n=0}^{\infty} \frac{s^n E(X^n)}{n!}$ converges abs.

$\Rightarrow \quad E(e^{sX}) < \infty \quad (\Rightarrow \Leftarrow)$.

[because the MGF of X is infinite at every non-zero ~~no~~ point. (proved previously)].

Exc:- If X & Y are independent RV, and ϕ_X, ϕ_Y and ϕ_{X+Y} are respective CHF of X, Y and $X+Y$ then show that:

$$\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t), \quad t \in \mathbb{R}.$$

Exc. Calculate CHF of

1. Std. Normal.
2. Binomial (n, p).
3. Geometric (p).
4. χ^2_{2n}
5. Unif (a, b).

HW-7

4.

(a) If $X_n \xrightarrow{a.s.} X$ & $Y_n \xrightarrow{a.s.} Y$, then show

$$X_n + Y_n \xrightarrow{a.s.} X + Y.$$

$\omega \in \{\omega : X_n(\omega) \to X(\omega)\} \cap \{\omega : Y_n(\omega) \to Y(\omega)\}$.

then, $X_n(\omega) + Y_n(\omega) \to X(\omega) + Y(\omega)$.

Clearly, $P[X_n \to X \text{ and } Y_n \to Y] = 1$.

this completes the pf.

(b) If $X_n \xrightarrow{P} X$ & $Y_n \xrightarrow{P} Y$. the show

$$X_n + Y_n \to X + Y.$$

Fix $\varepsilon > 0$,

$$P\left[|(X_n + Y_n) - (X+Y)| > \varepsilon\right]$$

$\leq P\left[|X_n - X| > \frac{\varepsilon}{2}\right] + P\left[|Y_n - Y| > \frac{\varepsilon}{2}\right].$

The above is true because,

$\left[|X_n - X| > \frac{\varepsilon}{2}\right]^c \cap \left[|Y_n - Y| > \frac{\varepsilon}{2}\right]^c \subseteq \left[|(X_n+Y_n) - (X+Y)| > \varepsilon\right]^c$

$\therefore \lim_{n\to\infty} P\left[|X_n + Y_n - (X+Y)| > \varepsilon\right] = 0$

7.

(a) $X_n \xrightarrow{a.s.} X \Rightarrow f(X_n) \xrightarrow{a.s.} f(X).$

$P\left[f(X_n) \to f(X)\right] \geq P\left[X_n \to X\right] = 1.$

(b) Sol^n 1 $X_n \xrightarrow{P} X \Rightarrow f(X_n) \xrightarrow{P} f(X).$

Suffices to show, any subsequence of $f(X_n)$ has a further subsequence ~~com~~ converging to $f(X)$ a.s. ... $(*)$

Let, $1 \leq n_1 < n_2 < \cdots$

$X_{n_k} \xrightarrow{P} X$ as $k \to \infty$

hence, it has a subsequence $X_{n_{k_l}} \xrightarrow{a.s.} X$ as $l \to \infty$.

part (a) implies that,

$f(X_{n_{k_l}}) \to f(X)$ a.s.

Thus $(*)$ holds.

Sol^n 2:

Suppose f were unif cont $\mathbb{R}$.

unif cont. implies that $\exists\, \delta > 0$ s.t.

Fix $\varepsilon > 0$

$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon$ $P\left[|f(X_n) - f(X)| > \varepsilon\right]$

Hence,

$$P\left[|f(X_n) - f(X)| > \varepsilon\right] \le P\left[|X_n - X| > \delta\right].$$

$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

Suppose f is just cont.

fix $\varepsilon > 0$.

Find a, b s.t.

$$P\left[a \le X \le b\right] \ge 1 - \frac{\varepsilon}{2}.$$

$$P\left[|f(X_n) - f(X)| > \varepsilon\right]$$

$$\le P\left[|f(X_n) - f(X)| > \varepsilon,\ a \le X \le b\right] \cancel{\le P[X_n - X]}$$

$$+ P\left[X \notin [a, b]\right].$$

$$= P\left[|f(X_n) - f(X)| > \varepsilon,\ a \le x \le b\right]$$

Let, $\delta \in (0,1)$ be s.t. if

$x, y \in [a-1, b+1]$ ~~and~~ and

$$|x - y| \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon.$$

8 (c)

Exc:- Suppose $Z_n \xrightarrow{P} Z$ and $|Z_n| \leq X$ where $E(X) < \infty$. Then show that $E(Z_n) \to E(Z)$.

9. Z_n = Second Minimum of $X_1, \dots, X_n$.

(a) $P(Z_n \geq k)$.

$= P$ (Atmost one of $X_1, \dots, X_n$ is less than k).

$= P(X_1 \geq k)^n + n\, P(X_1 \leq k-1) \, P(X_1 \geq k)^{n-1}$.

$= q^{n(k-1)} + n(1-q^{k-1})(q^{(n-1)(k-1)})$.

(b) $\sum_{n=1}^{\infty} P(Z_n \neq 1)$

$= \sum_{n=1}^{\infty} P(Z_n \geq 2)$.

$= \sum_{n=1}^{\infty} (q^n + n(1-q)q^{n-1}) < \infty$. because $q<1$.

Borel - Cantelli implies that,

$P\{w \mid Z_n(w) = 1$ for all n large enough$\}$
$= 1$.

10. $\sup_{n \in \mathbb{N}} \operatorname{Var}(X_n) < \infty$.

$\operatorname{Cov}(X_i, X_j) \leq 0 \quad i \neq j$.

Show that, $\underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i))}_{Y_n} \xrightarrow{P} 0$.

$E(Y_n) = 0.$

$$P\left[|Y_n| > \varepsilon\right] \leq \frac{1}{\varepsilon^2} Var(Y_n).$$

$$= \frac{1}{\varepsilon^2 n^2} Var\left(\sum_{i=1}^{n} (X_i - E(X_i))\right).$$

$$= \frac{1}{n^2\varepsilon^2} Var\left(\sum_{i=1}^{n} X_i\right).$$

$$= \frac{1}{n^2\varepsilon^2}\left[\sum_{i=1}^{n} Var(X_i) + 2\underbrace{\sum_{1\leq i<j\leq n} Cov(X_i, X_j)}_{\leq 0}\right]$$

$$\leq \frac{1}{n^2\varepsilon^2} \sum_{i=1}^{n} Var(X_i).$$

$$\leq \frac{\cancel{n}}{n^{\cancel{2}}\varepsilon^2} \sup_{n\in\mathbb{N}} Var(X_i)$$

$$= \frac{1}{n\varepsilon} \sup_{n\in\mathbb{N}} Var(X_i). \xrightarrow{P} 0$$

so as $n \to \infty$!!

Exc:- If X and Y are independent random variable, then show that,

$$E\left[e^{it(X+Y)}\right] = E\left[e^{itX}\right] + E\left[e^{itY}\right].$$

Soln:- ~~$E\left[e^{it(X+Y)}\right]$~~

~~$= E[\cos$~~

$$E\left[e^{itX}\right] \cdot E\left[e^{itY}\right].$$

$$= \left(E[\cos tX] + i\, E[\sin tX]\right)\left(E[\cos tY] + i\, E[\sin tY]\right)$$

$$= \left(E[\cos tX]\,E[\cos tY] - E[\sin tX]\,E[\sin tY]\right)$$
$$+ i\left[E[\sin tX]\,E[\cos tY] + E[\cos tX]\,E[\sin tY]\right]$$

$$= E\left[\cos tX \cos tY - \sin tX \sin tY\right]$$
$$+ i\,E\left[\sin tX \cos tY + \cos tX \sin tY\right]$$

$$= E\left[\cos t(X+Y)\right] + i\,E\left[\sin t(X+Y)\right]$$

$$= E\left[e^{it(X+Y)}\right]$$

Exc:- Calculate CHF of Std. Normal.

Soln:- $X \sim N(0,1)$.

$E\left[e^{tX}\right] < \infty$ for all t.

Then for all $t \in \mathbb{R}$.

$$E\left[e^{itX}\right] = \sum_{n=0}^{\infty} \frac{(it)^n E(X^n)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} E(X^{2n}).$$

$$= \sum_{n=0}^{\infty} \frac{(-t^2)^n}{(2n)!} \cdot \frac{(2n)!}{2^n n!}.$$

$$= \sum_{n=0}^{\infty} \left(-\frac{t^2}{2}\right)^n / n!$$

$$= e^{-t^2/2}.$$

Exc: Calculate CHF of Gamma$(n,1)$.

Sol^n:- Suppose $X \sim$ Gamma $(n,1)$.

Then $X = Y_1 + \cdots + Y_n$.

where $Y_1, \ldots, Y_n$ are i.i.d. from Exp(1).

$$E[e^{itX}] = \left(E[e^{itY_1}]\right)^n = \frac{1}{(1-it)^n}, \ t \in \mathbb{R}.$$

$$= (1-it)^{-n}.$$

Theorem (Uniqueness Thm):-

If X has CHF ϕ then

$$P(a \le X \le b) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)\, dt.$$

$\forall$ $a < b$ such that $P(X \in \{a, b\}) = 0$.

Corollary:- If X and Y have the same CHF (they agree at every $t \in \mathbb{R}$), then $X \overset{d}{=} Y$.

Pf:- Exc.

FACT (Tonelli)

1. For any $f: \mathbb{R}^2 \to [0, \infty)$, it holds that,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y,x)\, dy\, dx.$$

2. If X is a real-valued RV and $f: \mathbb{R}^2 \to [0, \infty)$

then $$E\left[\int_{-\infty}^{\infty} f(X,t)\, dt\right] = \int_{-\infty}^{\infty} E[f(X,t)]\, dt.$$

FACT:- (Fubini)

1. If $f:\mathbb{R}^2 \to \mathbb{C}$ is such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|\,dx\,dy < \infty \text{, then}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx.$$

2. If $f:\mathbb{R}^2 \to \mathbb{C}$ and a R.V. X are such that

$$\int_{-\infty}^{\infty} E|f(X,t)|\,dt < \infty \text{ then.}$$

$$E\left[\int_{-\infty}^{\infty} f(X,t)\,dt\right] = \int_{-\infty}^{\infty} E\left[f(X,t)\right]dt.$$

Proof of Uniqueness Thm:-

(Fix $a<b$).

$$\int_{-T}^{T} \frac{e^{ita}-e^{itb}}{it}\,\phi(t)\,dt$$

Step 1

Claim: $\forall t$, $\left|\frac{e^{ita}-e^{itb}}{it}\right| \le b-a.$ and hence the integral make sense for $T>0$.

Pf: $$\left|\frac{e^{ita}-e^{itb}}{it}\right| = \left|\int_a^b e^{-itx}\,dx\right|$$
$$\le \int_a^b |e^{-itx}|\,dx$$
$$= b-a.$$

Step 2:-

Denote:- $$I_T = \int_{-T}^{T} \frac{e^{-ita}-e^{-itb}}{it}\,\phi(t)\,dt$$

$\longrightarrow$

$$= \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} E\left(e^{itX}\right) dt$$

$$= \int_{-T}^{T} \frac{e^{it(X-a)} - e^{it(X-b)}}{it} dt$$

Claim: 2 $\int_{-T}^{T} E \left| \frac{e^{it(X-a)} - e^{it(X-b)}}{it} \right| dt < \infty.$

Pf: $\left| \frac{e^{it(X-a)} - e^{it(X-b)}}{it} \right| = \left| \frac{e^{-ita} - e^{-itb}}{it} \right|.$

$\leq b-a.$

$\therefore$ the integral in the above claim is at most $2T(b-a)$.

Claim: 2 & Fubini together imply that,

$$I_T = E\left[\int_{-T}^{T} \left[\frac{e^{it(X-a)} - e^{it(X-b)}}{it}\right] dt\right].$$

Step 3:- $\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it} dt$ for a fixed $x \in \mathbb{R}$.

$$= 2\left[\operatorname{sgn}(x-a)\, S(T|x-a|) - \operatorname{sgn}(x-b), |f(x-b)\right)$$

$$LHS = \int_{-T}^{T} \frac{\cos t(x-a) - \cos t(x-b)}{it} dt$$

$$+ \int_{-T}^{T} \frac{\sin t(x-a) - \sin(t(x-b))}{t} dt.$$

$\Rightarrow$ Since, the integrand of the first integral is odd the ~~integ~~ integral vanishes. Therefore,

$$LHS = \int_{-T}^{T} \cancel{\operatorname{sgn}(x-a)} \frac{\sin(t(x-a))}{t} - \int_{-T}^{T} \frac{\sin(t(y-a))}{t}\, dt$$

$$= 2\,[\operatorname{sgn}(x-a)\, S(T|x-a|) \cancel{- \operatorname{sgn}(x-b)\, S(T|x-b|)} - \operatorname{sgn}(x-b)\, S(T|x-b|)].$$

$$\therefore I_T = 2E\left[\operatorname{sgn}(X-a)\, S(T|X-a|) - \operatorname{sgn}(X-b)\, S(T|X-y|)\right] \quad (***)$$

Step 4

$$\lim_{T\to\infty} = 2\alpha\, E\left[\operatorname{sgn}(X-a) - \operatorname{sgn}(X-a)\right].$$

where, $\alpha = \lim_{t\to\infty} S(t)$

~~it follows that~~

Pf: Since, S is a continuous function on $[0,\infty)$. whose limit as $t\to\infty$ exists and is finite, it follows that $\sup_{t\geq 0} |S(t)| < \infty$

Applying BCT to $(***)$ · completes the proof of step 4.

For $x \in \mathbb{R}$.

$$\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b) = \begin{cases} 2, & a<x<b. \\ 1, & x = a,b \\ 0, & \text{o.w.} \end{cases}$$

$\therefore$ Step 4. implies that,

$$\lim_{T\to\infty} I_T = 2\alpha\, E\left[2 \cdot 1(a<x<b) + 1(X\in\{a,b\})\right].$$

$= 4\alpha P(a < X < b) + 2\alpha P(X \in \{a,b\})$.

In particular if $P(X \in \{a,b\}) = 0$ then

$$\lim_{T\to\infty} J_T = 4\alpha P(a \leq X \leq b).$$

Corollary:- $\lim_{t\to\infty} \int_0^t \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

HW-7

(11) $Z_1, Z_2, \ldots$ are independent

$$\sup_{n\geq 1} Var(Z_n) < \infty.$$

Define $Y_i = Z_i - E(Z_i)$, $i \geq 1$.

Y_i^+, Y_i^-

$$X = X^+ - X^-.$$

$$Var(X) = Var(X^+) + Var(X^-) - 2Cov(X^+X^-)$$

$$Cov(X^+, X^-) = E(X^+X^-) - E(X^+)E(X^-).$$

$$= -E(X^+)E(X^-).$$

$$\therefore Var(X) = Var(X^+) + Var(X^-) + 2E(X^+)E(X^-).$$

$$\therefore \sup_{n\geq 1} Var(Y_n^+) = \sigma^2 < \infty, \quad \sup_{n\geq 1} Var(Y_n^-) < \infty.$$

$$S_n^+ = Y_1^+ + \cdots + Y_n^+, \quad n \geq 1.$$

$$P\left[\left|\frac{S_n^+}{n} - E\left(\frac{S_n^+}{n}\right)\right| > \varepsilon\right] \leq \frac{1}{\varepsilon^2} Var\left(\frac{S_n^+}{n}\right)$$

$$= \frac{1}{n^2\varepsilon^2} Var(S_n^+).$$

$$\leq \frac{n\sigma^2}{n^2\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2}$$

$\frac{S_{n^2}^+}{n^2} - E$ .. Later.

11. $Z_1, Z_2, \dots$ are independent, non-negative R.V. with $E(Z_n) = \mu$, $n \geq 1$.

$$\sup_{n\geq 1} Var(Z_n) < \infty.$$

Then show that, $\frac{1}{n}\sum_{i=1}^{n} Z_i \to \mu$ a.s.

Soln: Let, $S_n = Z_1 + Z_2 + Z_3 + \dots + Z_n$.

For $\varepsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2 n^2} Var(S_n).$$

$$\leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\frac{S_{n^2}}{n^2} \to \mu \text{ a.s.}, \; n \to \infty.$$

for, $n^2 \leq k \leq (n+1)^2$.

$$\frac{S_{n^2}}{(n+1)^2} \leq \frac{S_k}{k} \leq \frac{S_{(n+1)^2}}{n^2}$$

$$\underbrace{\frac{n^2}{(n+1)^2}}_{\downarrow\, 1} \underbrace{\frac{S_{n^2}}{n^2}}_{\downarrow\, \mu} \leq \frac{S_k}{k} \leq \underbrace{\frac{S_{(n+1)^2}}{(n+1)^2}}_{\downarrow\, \mu} \underbrace{\frac{(n+1)^2}{n^2}}_{\downarrow\, 1}$$

$$\therefore \frac{S_k}{k} \to \mu \text{ . a.s.}$$

12. $X_n \to \theta$ a.s.

$Y_n \overset{d}{=} X_n$.

$P(|Y_n - \theta| > \varepsilon) = P(|X_n - \theta| > \varepsilon) \longrightarrow 0$ as $n \to \infty$

13. (a) $p > \frac{1}{2}$

$S_n = X_1 + \cdots + X_n$

where X_i's are iid. and take values 1 and -1

w.p p and $1-p$, respectively.

SLLN implies that.

$$\frac{S_n}{n} \xrightarrow{a.s.} E(X_1) = 2p-1 > 0 \quad [\because p > \tfrac{1}{2}].$$

Hence, $S_n \to \infty$ a.s.

14. (a)

(b) Let, $Y_n = Z_n - E(Z_n)$.

$$P(|Y_n| > \varepsilon) \leq \frac{1}{\varepsilon^2} Var(Z_n) \to 0.$$

$\Rightarrow Y_n \xrightarrow{P} 0$

& $E(Z_n) \to \theta$ (given).

$\Rightarrow Y_n + E(Z_n) \xrightarrow{P} \theta$

$\Rightarrow Z_n \xrightarrow{P} \theta$

(c) Similar to b and Borel - Cantelli.

16. $Y_n = \dfrac{X_n}{np_n}$

$E(Y_n) = 1$.

$Var(Y_n) = \dfrac{1-p_n}{np_n} \to 0 \quad \Rightarrow Y_n \to 1$.

$P(X_n = 0) = (1-p_n)^n$.

After stage $np_n \le \varepsilon$.

$$(1-p_n)^n \le \left(1-\frac{\varepsilon}{n}\right)^n \to e^{-\varepsilon} \quad \text{as } n \to \infty.$$

$\liminf\limits_{n\to\infty} P(X_n = 0) \geq e^{-\varepsilon}$ for every $\varepsilon \in (0,1)$.

Letting $\varepsilon \downarrow 0$,

$\liminf \geq 1$.

$\limsup \leq 1$.

17. Since, $X_n \xrightarrow{P} X$, there is a subsequence $\{X_{n_k}\}$ of (X_n) such that,

$$X_{n_k} \to X \text{ a.s.}, \quad k \to \infty.$$

Since, X_n is monotonic., for every ω

$\lim\limits_{n\to\infty} X_n(\omega)$ exists.

Let, $Y(\omega) = \lim\limits_{n\to\infty} X_n(\omega)$.

~~Comp~~ : $X \doteq Y$ a.s.

therefore, $X_n \to X$ a.s.

19. Let, $X_1, X_2, \ldots$ be i.i.d. Exp(1).

M_n = median of $X_1, \ldots, X_{2n-1}$.

(a) Show $P(M_n \le \ln 2) = \frac{1}{2}$

$$P(M_n \le \ln 2) = P(\text{at least } n \text{ of } X_1, \dots, X_{2n-1} \text{ are } \le \ln 2).$$

$$= \sum_{j=n}^{2n-1} P(\text{exactly } j \text{ many of } X_1, \dots, X_{2n-1} \text{ are } \le \ln 2)$$

$$= \sum_{j=n}^{2n-1} \binom{2n-1}{j} P(X_1 \le \ln 2)^j P(X_1 > \ln 2)^{2n-1-j}.$$

$$= \sum_{j=n}^{2n-1} \binom{2n-1}{j} \left(\frac{1}{2}\right)^{2n-1}$$

$$= \left(\frac{1}{2}\right)^{2n} \sum_{j=n}^{2n-1} \left[\binom{2n-1}{j} + \binom{2n-1}{2n-1-j}\right]$$

$$= \left(\frac{1}{2}\right)^{2n} \sum_{k=0}^{2n-1} \binom{2n-1}{k}$$

$$= \left(\frac{1}{2}\right)^{2n} \cdot 2^{2n-1}.$$

$$= \frac{1}{2}.$$

(b) Show that $M_n \xrightarrow{P} \ln 2$.

For $\varepsilon > 0$.

$$P(M_n \le \ln 2 - \varepsilon).$$

$$= P\Big(\underbrace{\sum_{j=1}^{2n-1} 1(X_j \le \ln 2 - \varepsilon)}_{Z_n} \ge n\Big).$$

$$Z_n \sim \text{Bin}(2n-1, p).$$

$$p = P(X_1 \le \ln 2 - \varepsilon) = 1 - \frac{1}{2} e^{\varepsilon} < \frac{1}{2}.$$

$$P(M_n \le \ln 2 - \varepsilon) = P\left(\frac{Z_n}{2n-1} \ge \frac{n}{2n-1}\right).$$