Probability Theory:-

Sample space is the set of all possible outcomes of a random experiment.

It is usually denoted by Ω.

e.g. for toss a coin : $\Omega = \{H, T\}$

for roll a die : $\Omega = \{1, \dots, 6\}$

The element of the sample space which occures is the "outcome" of the experiment.

Subset of the sample space are 'events'.

e.g. Roll a die.

$A = \{$We see an even number$\}$.

$= \{2, 4, 6\}$

An event occures or doesn't occure depending on wheather or not the outcome of the experiment belongs to that event.

Classical Definition of Probability:-

For a event A.

$$P(A) = \frac{\# A}{\# \Omega} \quad [\# \text{ means cardinality}]$$

Example :- We roll a die twice. Find the probability that the sum of the two numbers obtained is odd.

$\Omega = \{(1,1), (1,2), \dots, (1,6), (2,1), (2,2), \dots, (2,6), \dots, (6,1), (6,2), \dots, (6,6)\}$

$\# \Omega = 36$

$A =$ the event the sum is odd.

$= \{ (1,2), \ldots, (2,1), \ldots, (3,2), \ldots, (6,1) \ldots \}$

$= \{ (i,j) ;\ 1 \le i,j \le 6 ;\ i+j = \text{odd} \}$

$\# A = 6 \times 3$ [For each of 6 choices of i there are 3 choices of j.]

$= 18.$

$\therefore P(A) = \dfrac{\# A}{\# \Omega} = \dfrac{18}{36} = \dfrac{1}{2}$

Observations:-

1. $0 \le P(E) \le 1$ for every event E
2. For disjoint events (mutually exclusive) A and B.

   $P(A \cup B) = P(A) + P(B).$

3. $P(\Omega) = 1.$

Theorems (Th):-

Th 1:- For every events A and B.

$P(A \cup B) = P(A) + P(B) - P(A \cap B).$

Proof:- $A \cup B = A \cup (B - A)$ $[B - A = B \cap A^c]$

$P(A \cup B) = P(A) + P(B - A)$

$A = (A \cap B) \cup (A - B)$

$$P(A) = P(A \cap B) + P(A - B) \Rightarrow P(A-B) = P(A) - P(A \cap B)$$

$$\begin{aligned} P(A \cup B) &= P(A \cap B) + P(A-B) + P(B-A) \\ &= P(A \cap B) + P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

<u>Th. 2</u> :- For every event A and B.

$$P(A \cup B) \leq P(A) + P(B).$$

<u>Th. 3</u> :- For events A and B

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

<u>proof</u> :- $P(A \cap B) = P(A) + P(B) - \underbrace{P(A \cup B)}_{\leq 1}.$

<u>Th. 4</u> :- $A_1, A_2, \ldots, A_n$ are disjoint events

$(A_i \cap A_j = \cancel{\neq}\ \emptyset \quad \forall\ 1 \leq i < j \leq n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

<u>proof</u> :- Known for $n = 2$

Induction hypothesis. Assume the result for $n = k$

We shall show it for $n = k+1$

Let, $A_1, A_2, \ldots, A_k, A_{k+1}$ be disjoint ~~set~~ events

$$\begin{aligned} P\left(\bigcup_{i=1}^{k+1} A_i\right) &= P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right) \\ &= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) \\ &= \left(\sum_{i=1}^{k} P(A_i)\right) + P(A_{k+1}) \\ &= \sum_{i=1}^{k+1} P(A_i) \end{aligned}$$

Th. 5. (Inclusion Exclusion Formula)

For events $A_1, \dots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j)$$

$$+ \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) - \dots\dots + (-1)^{n+1} P(A_1 \cap A_2 \cap \dots \cap A_n)$$

Proof:-

Result is known for $n=2$

Induction hypothesis ; Assume for $n=k$.

Show for $k+1$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)$$

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + \underbrace{P(A_{k+1})}_{*} - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$

Note that, $\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1} = \bigcup_{i=1}^{k} (A_i \cap A_{k+1})$

Let, $\omega \in$ LHS

i.e. $\omega \in A_{k+1}$ and $\omega \in \bigcup_{i=1}^{k} A_i$.

therefore ~~ω be~~ $\omega \in A_{i_0}$
for some $i_0 \in \{1, 2, \dots, k\}$.

Hence, $\omega \in A_{i_0} \cap A_{k+1}$.
and thus $\omega \in$ RHS.

Conversely,
Let, $\alpha \in$ RHS.

i.e. $\alpha \in A_{i_0} \cap A_{k+1}$ for some $i_0 \in \{1, 2, \dots, k\}$

i.e. $\alpha \in A_{i_0}$ and $\alpha \in A_{k+1}$

$\therefore$ $\alpha \in \bigcup_{i=1}^{k} A_i$ and $\alpha \in A_{k+1}$.

$\therefore$ $\alpha \in$ LHS.

Now,

$$P\left(\bigcup_{i=1}^{n}(A_i \cap A_{k+1})\right) = \sum_{i=1}^{k} P(A_i \cap A_{k+1})$$

$$- \sum_{1\le i<j\le k} (A_i \cap A_j \cap A_{k+1}) + \cdots + (-1)^{k} \sum_{i=1}^{k} P\left(\bigcap_{\substack{j=1\\ i\neq j}}^{k+1} A_j\right)$$

$$+ (-1)^{k+1} P(A_1 \cap A_2 \cap \cdots \cap A_{k+1})$$

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \underbrace{\sum_{i=1}^{k} P(A_i)}_{*} - \sum_{1\le i<j\le n} P(A_i \cap A_j)$$

$$+ \sum_{1\le i<j\le k} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{k+1} P(A_1 \cap A_2 \cap \cdots \cap A_k)$$

$$\therefore P\left(\bigcup_{i=1}^{k+1} A_i\right) = \underbrace{\sum_{i=1}^{k} P(A_i)} - \underline{\sum_{1\le i<j\le k} P(A_i \cap A_j)}$$

$$+ \sum_{1\le i<j\le k} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{k+1} P(A_1 \cap A_2 \cap \cdots \cap A_k)$$

$$+ \underbrace{P(A_{k+1})} - \underline{\sum_{i=1}^{k} P(A_i \cap A_{k+1})}$$

$$+ \sum_{i\le i<j\le k} (A_i \cap A_j \cap A_{k+1}) + \cdots + (-1)^{k+1} \sum_{i=1}^{k} P\left(\bigcap_{\substack{j=1\\ i\neq j}}^{k+1} A_j\right)$$

$$+ (-1)^{k+2} P(A_1 \cap A_2 \cap \cdots \cap A_{k+1}).$$

$$= \sum_{i=1}^{k+1} P(A_i) - \sum_{1\le i<j\le k+1} P(A_i \cap A_j)$$

$$+ \cdots + \sum (-1)^{k+1} P(\text{Intersection of events taking } k \text{ at a time})$$

$$+ (-1)^{k+2} P(A_1 \cap A_2 \cap \cdots \cap A_{k+1}).$$

This proves the identity for any natural number.

Exc. 1. An urn contains 10 balls, lebeled 1, 2, ..., 10. Three balls are drawn from the urn at random with replacement. Calculate the probability that the lebel on at least one ball does not exceed 5.

~~Let~~ $\Omega = \{(i,j,k) : 1 \le i,j,k \le 10\}$.

$\# \Omega = 10^3$.

A be the event ~~that~~ of interest. that the label of at least one ball is at most 5.

So, look at A^c = none of them is less or equal 5.

i.e. all of them are from 6 to 10.

$\# A^c = 5^3$

$\therefore \# A = \# \Omega - \# A^c = 10^3 - 5^3$.

$\therefore P(A) = \dfrac{\# A}{\# \Omega} = 1 - \dfrac{5^3}{10^3} = \dfrac{7}{8}$.

2. Now what if the balls are drawn without replacement?

~~$\Omega = \{(i,j,k) : 1 \le i \ne j, k \le 10\}$~~ ;

$\# \Omega = {}^{10}C_3$

$\# A^c = {}^{5}C_3$

$$P(A) = 1 - \frac{{}^5P_3}{{}^{10}P_3}$$

$$= 1 - \frac{5 \times 4 \times 3}{10 \times 9 \times 8}$$

$$= 1 - \frac{1}{12}$$

$$= \frac{11}{12}$$

Exc. Seven students are chosen at random from 61 class. What is the probability that they are born on 7 distinct days of the week?

Ans:- $\frac{7!}{7^7}$

Exc. 1. Two cards are chosen at random without replacement from a standard deck. Find the probability that the two cards have either the same suit or the same rank.

$$\frac{{}^4C_1 \cdot {}^{13}C_2}{{}^{52}C_2} + \frac{{}^{13}C_1 \cdot {}^4C_2}{{}^{52}C_2}$$ (those two are disjoint events.)

2. if the cards are drawn with replacement

$$P(A \cup B) = \underset{\text{same suit}}{P(A)} + \underset{\text{same rank.}}{P(B)} - P(A \cap B).$$

$$= 4 \cdot \left(\frac{1}{4}\right)^2 + 13 \cdot \left(\frac{1}{13}\right)^2 - 52 \cdot \left(\frac{1}{52}\right)^2$$

Exc. :- A number is chosen at random from the first thousand natural number. Find the probability that it is divisible by either 4 or 6.

$$\frac{250 + 166 - 83}{1000}$$

$6 \,)\, 99\cancel{8} \,(\, 166$
6
39
36
36

$12 \,)\, 996 \,(\, 83.$
96
36

Exc. A die is rolled 10 -times. What is the probability that all the possible odd number have occured at least once.?

~~Let~~ ~~that~~

$A = \{$'1' occurs in at least one roll$\}$.

$B = \{$'3' occurs in at least one roll$\}$.

$C = \{$'5' occurs in at least one roll$\}$.

$$P(A \cap B \cap C) = 1 - P(A^c \cup B^c \cup C^c).$$

$$= 1 - \Big[P(A^c) + P(B^c) + P(C^c)$$

$$- P(A^c \cap B^c) - P(A^c \cap C^c) - P(A^c \cap B^c)$$

$$+ P(A^c \cap B^c \cap C^c)\Big].$$

$$= 1 - \left[3 \times \frac{5^{10}}{6^{10}} - 3 \times \left(\frac{4}{6}\right)^{10} + \left(\frac{3}{6}\right)^{10}\right]$$

Exc A urn contains 5 red balls, 5 white balls and 10 black balls, all the 20 balls being distinct. 7 balls are chosen at random without replacement. Calculate the probability that at least one ball of each colour is chosen?

$$1 - \left[\frac{{}^{15}C_7}{{}^{20}C_7} + \frac{{}^{15}C_7}{{}^{20}C_7} + \frac{{}^{10}C_7}{{}^{20}C_7}\right]$$

$R = \{$ At least one red ball is drawn $\}$.

$W = \{$ At " " white " " " $\}$.

$B = \{$ At " " black " " " $\}$.

$$P(R \cap W \cap B) = 1 - P(R^c \cup W^c \cup B^c)$$

$$= 1 - \Big[P(R^c) + P(W^c) + P(B^c) - P(R^c \cap W^c) - P(W^c \cap B^c) - P(B^c \cap R^c) + P(R^c \cap B^c \cap W^c) \Big]$$

$$= 1 - \left[\frac{\binom{15}{7}}{\binom{20}{7}} + \frac{\binom{15}{7}}{\binom{20}{7}} + \frac{\binom{10}{7}}{\binom{20}{7}} - \frac{\binom{10}{7}}{\binom{20}{7}} \right]$$

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r balls $\quad \underbrace{\sqcup \; \cdots \; \sqcup}_{n \text{ boxes}}$.

n^r ways. (if the balls are ~~not~~ not identical)

if the balls are ~~not~~ identical.

$${}^{n+r-1}C_{n-1}.$$

$\longrightarrow$

Bose - Einstein Experiment:-

r "identical" balls and n ~~identical~~ distinct boxes.

$(n-1)$ partitions $\underbrace{| \ldots |}_{n-1}$.

r balls. $\underbrace{0 \ldots 0}_{r}$.

Permute all the partitions and the balls together.

Each permutation determine an unique way of distributing the balls into the boxes.

$$\# \text{ ways to distribute the balls} = \frac{(n+r-1)!}{(n-1)!\, r!}$$

$$= \binom{n+r-1}{r}.$$

Clearly,

$$\Omega = \{(k_1, \ldots, k_n) : k_i \in \mathbb{N} \cup \{0\},\ k_1 + \cdots + k_n = r\}.$$

$\# \Omega$ = the coefficient of x^r in the expansion of $(1+x+x^2+\cdots)^n$.

Define $f(x) = (1+x+x^2+\cdots)^n$, $|x|<1$.

$$= (1-x)^{-n}$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

coefficient of x^r in $f(x)$ equals

$$= \frac{n(n+1)\cdots(n+r-1)}{r!}$$

$$= \binom{n+r-1}{r}.$$

Exc:- Concider the Bose-Einstein-Experiment with n boxes and r balls, where $r \geq n$. Calculate the probability that each box has at least one ball.

$\#A$ = coefficient of x^r in $(x+x^2+\cdots)^n$.

$$f(x) = (x+x^2+\cdots)^n$$

$$= x^n(1-x)^{-n}.$$

$$=$$

Exc:- A die is rolled 5 times calculate the probability that the sum of the number observed, equals 15

$$\#\Omega = 6^5$$

$\# A$ = coefficient of x^{15} in $(x+\cdots+x^6)^5$

$$(x+\cdots+x^6)^5 = x^5 \cdot \left(\frac{x^6-1}{x-1}\right)^5$$

$$= x^5(x-1)^{-5}(x^6-1)^5$$

$$= x^5(1-x)^{-5}(1-x^6)^5$$

$$= x^5 \left(\sum_{i=0}^{5} \binom{5}{i} (-1)^i x^{6i}\right)\left(\sum_{j=0}^{\infty} \binom{4+j}{4} x^j\right)$$

$$6i + j + 5 = 15.$$

$$6i + j = 10.$$

$i=0 \quad j=10$

$i=1 \quad j=4.$

$$\# A = {}^5C_0 \cdot {}^{14}C_4 - {}^5C_1 \cdot {}^8C_4$$

$$P(A) = \frac{{}^{14}C_4 - 5 \cdot {}^8C_4}{6^5}$$

N.B. $x^5 \left(\sum_{i=0}^{5} \binom{5}{i} (-1)^i x^{6i}\right)\left(\sum_{j=0}^{\infty} \binom{4+j}{4} x^j\right)$

is not a infinite polynomial.

1. Toss a coin with chances of head 0.3.
2. Toss a fair coin till you get the first tails.

Digression:-

Defn:- A set E is called countably infini[te if] there exists a one-one and onto function from $\mathbb{N} \to E$

A set E is said to have cardinality $n \in \mathbb{N}$ if ~~∃~~ there exists a bijection from

$f \; \{1, 2, \ldots, n\} \rightarrow E$

A set is called "countable" if it is either finite ~~of~~ or is countably infinite

A probability assignment on Ω is a function $p : \Omega \xrightarrow{} [0,1]$.

such that $\sum_{n \geq 1} p(\omega_n) = 1$.

For event $A \subseteq \Omega$ the probability of A is find by $P(A) = \sum_{i,\, \omega_i \in A} p(\omega_i)$.

The tuple (Ω, P) is called a probability space. $= ??$

Example:-

Toss a coin with chances of head 0.3.

$\Omega = \{H, T\}$.

$p(H) = 0.3$

$p(T) = 0.7$.

subsets of Ω are $= \phi, \{H\}, \{T\}, \Omega$

$P(\phi) = 0$.

$P(\{H\}) = 0.3$. $\quad P(\Omega) = 1$.

$P(\{T\}) = 0.7$.

Example:- Toss the same coin twice. independently

$\Omega = \{HH, HT, TH, TT\}$.

Define $p : \Omega \to [0,1]$ by.

$$p\{HH\} = 0.09.$$

$$p(HT) = 0.21$$

$$p(TH) = 0.21.$$

$$p(TT) = 0.49.$$

Example:- Toss a coin, independently, with chances of Heads θ, untill you get the first Tails.

$$\Omega = \{T, HT, HHT, HHHT, \ldots\},$$

$$p(\underbrace{H\cdots H}_{n}T) = \theta^n(1-\theta), \quad n \geq 0.$$

$$P(\Omega) = \sum_{i=0}^{\infty} \theta^i(1-\theta).$$

$$= \frac{1-\theta}{1-\theta} = 1.$$

Exc. In the above experiment calculate the probability that you do not get Tails in the first 5 draws.

$$P(E) = \frac{\cancel{\frac{\theta^5 \cancel{(1-\theta)}}{(1-\theta)}}}{\cancel{\frac{(1-\theta)}{(1-\theta)}}}$$

The event in the question is.

$$E = \{HHHHHT,\ HHHHHHT, \ldots\}.$$

$$P(E) = \sum_{i=0}^{\infty} \theta^{5+i}(1-\theta)$$

$$= \theta^5$$

<u>Example</u>:- Consider an urn containing 10 red balls and 10 green balls. Suppose that 15 balls are chosen from the urn without replacement.

~~$\Omega = (10R, 5G), (9R, 6G), (8R, 7G)$~~,

Let us call the ball $R_1, R_2, \ldots, R_{10}$, $G_1, G_2, \ldots, G_{10}$.

$$\Omega = \{(x_1, x_2, \ldots, x_{15}) \cdot x_i \in \{R_1, \ldots, R_{10}, G_1, \ldots, G_{10}\}\ x_i \neq x_j\ \forall\ 1 \leq i < j \leq 15\}.$$

For every $\omega \in \Omega$

$$p(\omega) = \frac{1}{{}^{20}P_{15}}$$

<u>Remark</u> :- If a finite sample space ω has all outcomes equally likly that is.

$$p(\omega) = \frac{1}{\# \Omega}, \quad \omega \in \Omega$$

then, probability can be computed using the classical definition of Probability.

Example:- A fair coin is tossed. If you get heads, you roll a fair die independently. Otherwise, you pick a card from a standard deck at random. (independently)

$P(H\,1) = \frac{1}{12}$

$P(T \text{ Ace of Heart}) = \frac{1}{104}$.

Thm. Let (Ω, P) be a probability space.

1. $P(\Omega) = 1$.

2. If $A_1, A_2, \ldots$ are disjoint events, then

$$P\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} P(A_n).$$

$0 \leq P(A) \leq 1$ for all A.

Proof:-

1. If p is the probability assignment, then

$$P(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1.$$

2.

$$P\left(\bigcup_{n\geq 1} A_n\right) = \sum_{\omega \in \bigcup_{n\geq 1} A_n} p(\omega).$$

$$= \sum_{n\geq 1} \sum_{\omega \in A_n} p(\omega). \quad \left[\text{because } A_n\text{'s are disjoint and } p(\omega) \geq 0.\right]$$

$$= \sum_{n\geq 1} P(A_n).$$

3. Trivial.

Conditional Probability:-

Definition:- Let, A be an event with $P(A) > 0$: For any event B, the conditional probability of B given A defined as,

$$P(B/A) = P(A \cap B) / P(A).$$

Example:- A number is chosen at random from $\{0, 1, ..., 99\}$. Given that the ~~first~~ digit of the ten's place is 2. What is the conditional probability that the number is prime

Let, A be the event that the digit in the ten's place is 2

and B be the event that the number is prime.

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\{23, 29\})}{P(\{20, 21, ..., 29\})} = \frac{1}{5}.$$

Example:- Concider an urn containing 10 white balls and 20 black balls. Two balls are ~~selected~~ drawn at random without replacement

ⓐ What is the conditional probability that the 2nd ball is black given that the first one is white.

ⓑ Calculate the probability that the 2nd ball is black.

(a) $W1 = \{$The first ball drawn is white$\}$.

$B2 = \{$ " second " " " black$\}$

$$P(B2/W1) = \frac{P(B2 \cap W1)}{P(W1)} = \frac{{}^{10}C_1 \cdot {}^{20}C_1}{{}^{10}C_1 \cdot {}^{29}C_1} = \frac{20}{29}$$

(b) $$P(B2) = P(B2 \cap W1) + P(B2 \cap W1^c).$$

$$= P(B2/W1)P(W1) + P(B2/W1^c)P(W1^c).$$

$$= \frac{20}{29} \cdot \frac{10}{30} + \frac{19}{29} \cdot \frac{20}{30}$$

$$= \frac{20\cancel{(10+19)}}{\cancel{29} \cdot 30}$$

$$= \frac{20}{30}$$

$$= \frac{2}{3}.$$

Theorem: (Polya urn model).

Consider an urn containing W white balls and B black balls. Suppose that n balls $(1 \le n \le W+B)$ are drawn at random without replacement from the urn. Then,

$P($getting a black ball in the nth draw$)$

$$= \frac{B}{W+B}.$$

Proof:-

Let, $B_1 = \{$ The first draw is black $\}$.
$\vdots$
$B_n = \{$ The n-th " is black $\}$.

Let, the balls be labeled $w_1, \dots w_W, b_1, \dots b_B$.

$$\Omega = \{(i_1, i_2, \dots, i_n) : i_1, \dots, i_n \in \{w_1, \dots, w_W, b_1, \dots, b_B\}\}$$
$$1 \leq j < k \leq n$$

$$B_1 = \{ i \in \Omega \ . \ i_1 \in \{b_1, \dots, b_B\} \}.$$
$$B_n = \{ i \in \Omega \ , \ i_n \in \{b_1, \dots, b_B\} \}.$$

Define $f : B_1 \to B_n$ by
$$f((i_1, i_2, \dots, i_n)) = (i_2, \dots, i_n, i_1).$$
Clealy, f is a bijection and hence $\# B_1 = \# B_n$.

Since, every outcome in Ω is equally likely it follows that,
$$P(B_n) = P(B_1) = \frac{B}{W+B}.$$

Proof:- We want to proove the following statement For all choices of $W, B \in \mathbb{N}$ and $n \in \{1, \dots, W+B\}$ it holds that
$$P(\text{getting black in the nth draw}) = \frac{B}{W+B}.$$

The claim is clearly true for $n = 1$.

Inductive Hypothesis:- The claim is true for ~~n~~ $(n-1)$ where $1 \leq n-1 \leq W+B-1$.

P(getting black in the nth draw)

= P(black in nth draw/black in 1st draw). $\frac{B}{W+B}$

~~P(black in 1st draw)~~

+ P(black in nth draw/white in 1st draw). $\frac{W}{W+B}$

Using the induction hypothesis with B replaced B−1 we get,

P(Black in nth draw/black in first draw)

$$= \frac{B-1}{W+B-1}$$

Ily, replacing W by W−1 we get,

P(~~white~~ black in the nth draw/ white ~~black~~ in first draw)

$$= \frac{B}{W+B-1}$$

P(getting black in the nth draw)

$$= \frac{B-1}{W+B-1} \cdot \frac{B}{W+B} + \frac{B}{W+B-1} \cdot \frac{W}{W+B}$$

$$= \frac{B\cancel{(B-1+W)}}{\cancel{(W+B-1)}(W+B)}$$

$$= \frac{B}{W+B}$$

The proof follows from the principle of mathematical induction.

Example:- In the Polya urn scheme, calculate the probability that the first ball is white given that the 2nd ball is black.

$W_1 = \{ \text{1st white} \}$

$B_2 = \{ \text{2nd ball is black} \}$

$$P(W_1 / B_2) = P(W_1 \cap B_2) / P(B_2).$$

$$= \frac{P(B_2/W_1)\,P(W_1)}{P(B_2/W_1)\,P(W_1) + P(B_2/W_1^c)\,P(W_1^c)}$$

$$= \frac{\frac{10}{30} \cdot \frac{20}{29}}{\frac{20}{29} \cdot \frac{10}{30} + \frac{19}{29} \cdot \frac{20}{30}}$$

$$= \frac{10}{29}$$

Bayes Theorem:-

Let, $A_1, A_2, \ldots$ be disjoint events such that

$$\Omega = \bigcup_{n \geq 1} A_n.$$

Suppose that $P(A_n) > 0$ for every n. If B is an event with $P(B) > 0$, then,

$$P(A_i / B) = \frac{P(B/A_i)\,P(A_i)}{\sum_{n \geq 1} P(B/A_n)\,P(A_n)}, \forall i \geq 1.$$

Proof:- For a fixed i,

$$P(A_i / B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B/A_i)\,P(A_i)}{P(B)}$$

Note that, $P(B) = P(B \cap \Omega)$ →

$$= P\left(B \cap \left(\bigcup_{n \geq 1} A_n\right)\right)$$

$$= P\left(\bigcup_{n \geq 1} (A_n \cap B)\right)$$

$$= \sum_{n \geq 1} P(A_n \cap B) \quad \left[\because A_n\text{s are disjoint, so are } (A_n \cap B)\right]$$

$$= \sum_{n \geq 1} P(B/A_n)\, P(A_n).$$

$$\therefore P(A_i / B) = \frac{\cancel{P(A_i \cap B)}\; P(B/A_i)\, P(A_i)}{\sum_{n \geq 1} P(B/A_n)\, P(A_n)}.$$

Probability theory 1

Homework 1

2. $A_i = \{$The ith stick is black$\}$, $1 \leq i \leq 100$.

$$P\left(\bigcup_{i=1}^{100} A_i\right) = \sum_{i=1}^{100} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \cdots - P\left(\bigcap_{i=1}^{100} A_i\right).$$

$$= {}^{100}C_1 \cdot \frac{99!}{100!} - {}^{100}C_2 \cdot \frac{98!}{100!} + {}^{100}C_3 \frac{97!}{100!}$$

$$= \sum_{k=1}^{100} (-1)^{k+1} \binom{100}{k} \frac{(100-k)!}{100!}$$

$$= \sum_{k=1}^{100} (-1)^{k+1} \frac{\cancel{(100-k)!}}{\cancel{100!}} \cdot \frac{\cancel{100!}}{k!\,\cancel{(100-k)!}}$$

$$= \sum_{k=1}^{100} (-1)^{k+1} \frac{1}{k!}$$

5. K = { The student knew the correct answer}

C = { The student answered it correctly}

We have to calculate $P(K|C)$.

$$P(K/C) = \frac{P(C/K)\,P(K)}{P(C/K)\,P(K) + P(C/K^c)\,P(K^c)}$$

$$= \frac{1 \times 0.8}{1 \times 0.8 + 0.25 \times 0.2}$$

$$= \frac{80}{85}$$

$$= \frac{16}{17}.$$

8. Suppose that instead of stopping the experiment when all the balls in the box are of same colour, we decide to continue until all the $(a+b)$ balls have been drawn.

Therefore, the required probability equals

P(the last ball drawn is white in the new experiment)

$$= \frac{a}{a+b}$$

9. A = { All the tickets are numbered less than or equal to M }

$$P(A) = \frac{M^n}{N^n}$$

B = { All the tickets are numbered less than or equal to $M-1$ }

$$P(B) = \frac{(M-1)^n}{N^n}$$

Required event $= A - B \; (= A \cap B^c)$.

Required probability $= P(A - B)$.

$$= P(A) - P(B) \quad [\text{as } B \subseteq A]$$

$$= \frac{M^n - (M-1)^n}{N^n}$$

[**Exc:-** Show that if $B \subseteq A$

$P(A - B) = P(A) - P(B)$.]

(11) $$p_n = \frac{[n/k]}{n}, \quad n \geq 1.$$

$$\frac{n}{k} - 1 \leq \left\lfloor \frac{n}{k} \right\rfloor \leq \frac{n}{k}.$$

$$\therefore \frac{1}{n}\left(\frac{n}{k} - 1\right) \leq \frac{1}{n}\left[\frac{n}{k}\right] \leq \frac{1}{k}.$$

as $\lim_{n \to \infty} \frac{1}{n}\left(\frac{n}{k} - 1\right) = \frac{1}{k}$.

$$\therefore \lim_{n \to \infty} \frac{1}{n}\left\lfloor \frac{n}{k} \right\rfloor = \frac{1}{k}.$$

Independence

Let, A, B be two events. such that.

$$P(A) > 0.$$

$$P(B/A) = P(B)$$

$$\frac{P(A \cap B)}{P(A)} = P(B)$$

$$\Rightarrow P(A \cap B) = P(A)P(B).$$

Defi^n:- Events A and B are independent if $P(A \cap B) = P(A)P(B)$.

- Saying that "A is independent of B" is same as "B is independent of A".
- Independence should ~~not~~ never be confused with disjointness.

A, B are disjoin $A \cap B = \phi$

A, B are independent $P(A \cap B) = P(A)P(B)$.

Exc:- Suppose that A and B are both disjoint and independent, Show that atleast one of A and B are with probability 0.

Example:- A fair die is rolled twice.

A = { The first number is even }

B = { The sum of the two numbers is odd }

Then, A and B are independent.

What if the die is five faced? No.

Theorem If A and B are independent, then so are,

(i) A and B^c

(ii) A^c and B

(iii) A^c and B^c.

(i) $P(A \cap B^c) = P(A) - P(A \cap B)$

$= P(A) - P(A)P(B)$

$= P(A)(1 - P(B))$

$= P(A)P(B^c)$

(ii) $P(A^c \cap B) = P(B) - P(A \cap B)$

$= P(B) - P(A)P(B)$

$= P(B)(1 - P(A))$

$= P(A^c)P(B)$

(iii) $P(A^c \cap B^c) = P(A \cup B)^c$

$= 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$.

Examples:- A card is chosen at random from a standard deck. Define the events.

A = { The card is an Ace }

B = { The colour of the card is black }

C = { The card belongs to the suit hearts }

are they indep.

✓ (i) A & B

✓ (ii) A & C

✗ (iii) B & C

Def^n:- Events $A_1, \ldots, A_n$ are pairwise independent if for all $1 \le i \le j \le n$, A_i and A_j are independent.

Example:-

A fair coin is tossed twice.

$A = \{$ The first toss is a Heads $\}$

$B = \{$ The second toss is a Heads $\}$

$C = \{$ The out comes of the two tosses are same $\}$.

See, A, B & C are pairwise independent.

Example:-

Consider a disease which is prevalent in 1% of the population. A diagnostic test for the same disease gives the correct answer with probability 0.9 independently of the event that the subject being tested has the disease. If for a particular subject the test yields a positive result, what are the chances that the subject actually have the disease?

$$\frac{\frac{1}{100} \times \frac{9}{10}}{\frac{1}{100} \times \frac{9}{10} + \frac{99}{100} \times \frac{1}{10}}$$

$$= \frac{9}{\ }$$

D be the event that the subject have the disease

$C = \{$ The test result is correct $\}$.

$P = \{$ the test result is positive $\}$.

$$P(D/P) = \frac{P(P/D)\,P(D)}{P(P/D)\,P(D) + P(P/D^c)\,P}$$

$$= \frac{P(D \cap P)}{P(P)}$$

$$P(D \cap P) = P(D \cap C) = P(D)\,P(C) = (0.01)(0.9)$$
$$= 0.009.$$

$$P(P) = P(P \cap D) + P(P \cap D^c)$$
$$= 0.009 + P(C^c \cap D^c)$$
$$= 0.009 + 0.1 \times 0.99$$
$$= 0.009 + 0.099$$
$$= 0.108.$$

$$P(D/P) = \frac{0.009}{0.108} = \frac{9}{108} = \frac{1}{12}$$

What if ~~two~~

- the disease were more prevalent?
- the test outcomes was negative?
- also do it from first principal.

Exc :- Suppose that the events A_1, A_2, A_3 are pairwise independent and $P(A_i) = \frac{1}{2}$ for $\forall i = 1(1)3$. If $P(A_1 \cap A_2 \cap A_3) = \frac{1}{4}$ then show that P(An odd number of events from A_1, A_2, A_3 occur) $= 1$.

$$P(A_1 \cap A_2 \cap A_3^c)$$

$$= P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$$

$$= P(A_1)P(A_2) - P(A_1 \cap A_2 \cap A_3)$$

$$= \frac{1}{4} - \frac{1}{4}$$

$$= 0.$$

$$\Rightarrow P(A_1 \cap A_2^c \cap A_3) = P(A_1^c \cap A_2 \cap A_3) = 0.$$

$$P(A_1^c \cap A_2^c \cap A_3^c)$$

$$= P(A_1 \cup A_2 \cup A_3)^c$$

$$= 1 - P(A_1 \cup A_2 \cup A_3)$$

$$= 1 - P(A_1) - P(A_2) - P(A_3) + P(A_1)P(A_2) + P(A_2)P(A_3) + P(A_3)P(A_1) - P(A_1 \cap A_2 \cap A_3)$$

$$= 1 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0.$$

$$P(A_1 \cap A_2^c \cap A_3^c) = P(A_1) - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{4}$$

Defi^n:- Events $A_1, \dots, A_n$ are independent if for every choice of $k \in \{2, \dots, n\}$ and $1 \le i_1 < \dots < i_k \le n$ it holds that,

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \dots P(A_{i_k}).$$

(*) "Pair wise independence" is not the same as "independence".

Example:-

A = {be the event that first toss is heads}
B = {The second toss is tail}
C = {The two tosses are identical}.

(*) Converse is not true. The definition of independence implies pairwise independence.

Exc:- If $A_1, A_2, \dots, A_n$ are independent, show that $\forall\ k \in \{2, \dots, n\}$, $A_1, \dots, A_k$ are independent.

Theorem:- If $A_1, A_2, \dots, A_n$ are independent then for every $1 \le k \le n$, $A_1^c, \dots, A_k^c, A_{k+1}, \dots, A_n$ are independent.

Proof:- We first show for $k = 1$.

Fix $j \in \{2, \dots, n\}$. Define $B_1 = A_1^c$, $B_2 = A_2, \dots, B_n = A_n$.
$1 \le i_1 < \dots < i_j \le n$.

Want to show that,

$$P(B_{i_1} \cap \dots \cap B_{i_j}) = P(B_{i_1}) \dots P(B_{i_j}).$$

Nothing to show, if $i_1 > 1$. So assume without loss of generality $i_1 = 1$.

$$P(B_{i_1} \cap \cdots \cap B_{i_j}) = P(A_{i_1}^c \cap A_{i_2} \cap \cdots \cap A_{i_j}).$$

$$= P(A_{i_2} \cap \cdots \cap A_{i_j}) - P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j})$$

$$= P(A_{i_2}) P(A_{i_3}) \cdots P(A_{i_j})$$

$$- P(A_{i_1}) \cdot P(A_{j_2}) \cdots P(A_{i_j})$$

$$= P(A_{i_2}) P(A_{i_3}) \cdots P(A_{i_j}) (1 - P(A_{i_1}))$$

$$= P(A_{i_1})^c P(A_{i_2}) P(A_{i_3}) \cdots P(A_{i_j}).$$

Thus the claim holds for $k = 1$.

Exc :- Complete the proof.

Let, we assume that the result holds good for ~~not~~. $k = p$.

i.e.

$$P(A_1^c \cap A_2^c \cap \cdots \cap A_p^c \cap A_{p+1} \cap \cdots \cap A_n)$$

$$= P(A_1^c) P(A_2^c) \cdots P(A_p^c) P(A_{p+1}) \cdots P(A_n)$$

We have to prove it for $k = p+1$

Let, we define,

$B_1 = A_1^c$, $B_2 = A_2^c$, $\cdots$, $B_p = A_p^c$, $B_{p+1} = A_p^c$, $B_{p+2} = A_{p+2}$, $\cdots$ $B_n = A_n$

$$\therefore P(B_1 \cap B_2 \cap \cdots \cap B_n)$$

$$= P(A_1^c \cap A_2^c \cap \cdots \cap A_p^c \cap A_{p+1}^c \cap A_{p+2} \cap \cdots \cap A_n)$$

$$= P(A_1^c \cap A_2^c \cap \cdots \cap A_p^c \cap A_{p+2} \cap \cdots \cap A_n)$$

$$- P(A_1^c \cap A_2^c \cap \cdots \cap A_p^c \cap A_{p+1} \cap A_{p+2} \cap \cdots \cap A_n)$$

$$= P(A_1^c) P(A_2^c) \cdots P(A_p^c) P(A_{p+2}) \cdots P(A_n)$$

$$- P(A_1^c) P(A_2^c) \cdots P(A_p^c) P(A_{p+1}) P(A_{p+2}) \cdots P(A_n)$$

$$= P(A_1^c) P(A_2^c) \cdots P(A_p^c) P(A_{p+2}) \cdots P(A_n) (1 - P(A_{p+1}))$$

$$= P(A_1^c) P(A_2^c) \cdots P(A_p^c) P(A_{p+1}^c) P(A_{p+2}) \cdots P(A_n).$$

Hence prove completed.

1. a. Two numbers are selected at random without replacement from $\{1, \dots, N\}$. What's the probability that the first one is larger or equal to?

Ans:- $\frac{1}{2}$

b. What if the selection was with replacement?

$$\cancel{\frac{1 - \frac{{}^nC_1}{{}^nC_2}}{2}} = \cancel{\frac{1 - \frac{n}{\frac{n(n-1)}{2}}}{2}}$$

$$= \cancel{\tfrac{1}{2}\left(1 - \tfrac{2}{n-1}\right)}.$$

Ans:- $\frac{1}{2}\left(1 - \frac{1}{N}\right) + \frac{1}{N}$.

c. Answer 1(b) & 1(a) if 3 numbers are selected.

1. a. $\rightarrow \frac{1}{3}$

1. b. three numbers chosen from $\{1, \dots, N\}$ with replacement. $A = \{$First one is largest$\}$.

$$P(A) = \sum_{i=1}^{N} P(A / \text{first number is } i)\, P(\text{first number is } i)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{i^2}{N^2}$$ [For first number i we have i choices for 2nd one and i choices of 3rd one.]

$$= \frac{1}{N^3} \cdot \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{(N+1)(2N+1)}{6N^2}.$$

(d) If three numbers are chosen with replacement find the probability that the first one is not larger than the 2nd one and 2nd one is not larger than the 3rd one

Ans:- Define the events:

A = { First number is strictly larger than the 2nd one }

B = { 2nd number is strictly larger than the 3rd one }.

The event of interest is $(A \cup B)^c$

∴ Required probability

$$= 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - \frac{1}{2}\left(1 - \frac{1}{N}\right) - \frac{1}{2}\left(1 - \frac{1}{N}\right) + \frac{\binom{N}{3}}{N^3}$$

$$= \frac{1}{2N} + \frac{1}{2N} + \frac{(N-1)(N-2)}{6N^2}$$

$$= \frac{3N + 3N + N^2 - 3N + 2}{6N^2}$$

$$= \frac{N^2 + 3N + 2}{6N^2}$$

$$= \frac{(N+1)(N+2)}{6N^2}$$

2. Suppose that A and B are events with $0 < P(B) < 1$ and $P(A/B) = P(A/B^c)$.

Show that A and B are independent.

~~P(B/A)~~ ~~P(A/B)~~

Let, $P(A/B) = P(A/B^c) = p$.

$$P(A) = P(A/B)\,P(B) + P(A/B^c)\,P(B^c)$$

$$= p\,\big(P(B) + P(B^c)\big)$$

$$= p = P(A/B).$$

$\therefore$ A and B are independent.

③ Consider a knockout tournament with 8 teams. Assume that the outcomes of different matches are independent.

ⓐ If any of the two teams are equally likly to win in a particular match (and there is no draws) What is the probability that a particular team will ~~wo~~ win the tournament? Ans :- $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$.

ⓑ Suppose that the teams are ranked prior to the tournament from 1.(best) to 8 (worst). Assume that in a particular match, a higher ranked teams wins with probability 2/3. What is the probability that Team 1 wins the tournament? Ans:- $\frac{8}{27}$.

ⓒ Formats are

1 8   4 5   2 7   3 6

What is the probability that team 2 wins.

$\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{27}$.

$$\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{27}$$
$$+$$
$$\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{8}{27} \quad = \frac{12}{27}.$$

P(Team 2 wins)

= P(Team 2 wins the final / It reaches final) P(Team 2 reaches the final).

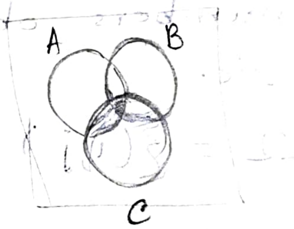
$= \frac{4}{9}$ [P(Team 2 wins the final / Final is b/w 1 and 2) P(1 reaches final)

+ P(team 2 wins the final / final is with one with lower than 2) P(lower than 2 team reaches final)]

$$= \frac{4}{9}\left[\frac{1}{3}\cdot\frac{4}{9} + \frac{2}{3}\cdot\frac{5}{9}\right]$$

$$= \frac{4}{9}\cdot\frac{14}{27}$$

$$= \frac{56}{81}.$$



A, B are independent events of positive probability.

??

$$P(C/A) = P(C\cap B/A) + P(C\cap B^c/A).$$

$$= \frac{P(A\cap B\cap C)}{P(A)} + \frac{P(A\cap C\cap B^c)}{P(A)}.$$

$$= \frac{P(A\cap B\cap C)\,P(B)}{P(A\cap B)} + \frac{P(A\cap B^c\cap C)}{P(A\cap B^c)}\,P(B^c)$$

$$= P(C|A\cap B)\,P(B) + P(C|A\cap B^c)\,P(B^c).$$

Random Variables:-

Def^n:- A random variable is a function from sample space to real line ($\mathbb{R}$).

For a subset $A \subseteq \mathbb{R}$

$$P(X \in A) = P(\underbrace{\{\omega \in \Omega : X(\omega) \in A\}}_{X^{-1}(A)}).$$

Ω is countable.

$\{X(\omega) : \omega \in \Omega\}$ is the set of possible values of X. The "distribution" of X is a ~~description~~ table containing the possible values along with the probabilities with which X takes those values.

Example:- Let, X denote the total of the numbers obtained in two rolls of a fair die.

$$\Omega = \{(i,j) : 1 \le i,j \le 6\}$$

$$X : \Omega \to \mathbb{R}$$

$$X((i,j)) = i + j.$$

Distribution of X

x	$P(X=x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	5/36.
8	4/36
9	3/36
10	2/36
11	1/36
12	~~[illegible]~~

Example:- A coin which shows Heads with probability p is tossed n-times. Let X be the number of heads.

$$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & ; x = 0(1)n. \\ 0 & ; \text{otherwise}. \end{cases}$$

This is also known as probability mass function (PMF).

Binomial Distribution:-

A random variable X is said to follow Binomial Distribution with parameters n and p ($n \in \mathbb{N}$, $p \in [0,1]$) if

(i) $$P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0(1)n. \\ 0 & , \text{otherwise}. \end{cases}$$

and we tell $X \sim \text{Bin}(n,p)$.

Exc:- Suppose $X \sim \text{Bin}(n,p)$. Let, $Y = n - X$.

Ans:- $Y \sim \text{Bin}(n, 1-p)$.

$$P(Y=k) = P(\{\omega : Y(\omega) = k\})$$

$$= P(\{\omega : X(\omega) = n-k\})$$

$$= P(X = n-k)$$

$$= \begin{cases} \binom{n}{n-k} p^{n-k} \cdot (1-p)^{n-(n-k)} & , k = 0(1)n \\ 0 & , \text{otherwise} \end{cases}$$

$$\overset{?}{=} \begin{cases} \binom{n}{k} (1-p)^k p^{n-k} & , k = 0(1)n. \\ 0 & , \text{otherwise}. \end{cases}$$

Example:- Consider a coin which tosses Heads with probability p. It is tossed until the first Heads. Let X denote the number of tosses needed.

$$P(X=k) = \begin{cases} q^{k-1}p & , \quad k=1,2,3,\ldots \\ 0 & , \quad \text{otherwise.} \end{cases}$$

This distribution is known as Geometric with parameter p for $p \in (0,1]$, abbreviated as Geo (p).

Example:- Consider an urn containing 10 black balls, 5 white balls and 5 red balls. Balls are drawn one by one from the urn with replacement untill one ball of each colour is drawn. Let, Z be the number of balls.

So, Z can take any natural number ≥ 3.

Fix $k \geq 2$

$P(Z>k)$

Define $NB = \{$no black ball in the first k draws$\}$

$NW = \{$no white ball in the first k draws$\}$

$NR = \{$no red ball in the first k draws$\}$

$$P(Z>k) = P(NB \cup NW \cup NR)$$

$$= \left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{3}{4}\right)^k - \left(\frac{1}{4}\right)^k - \left(\frac{1}{4}\right)^k - \left(\frac{1}{2}\right)^k$$

$$= 2\left[\left(\frac{3}{4}\right)^k - \left(\frac{1}{4}\right)^k\right]$$

$$P(Z=k) = P(Z>k-1) - P(Z>k).$$

$$= 2\left[\left(\frac{3}{4}\right)^{k-1} - \left(\frac{3}{4}\right)^{k} - \left(\frac{1}{4}\right)^{k-1} + \left(\frac{1}{4}\right)^{k}\right]$$

The distribution of Z is.

$$P(Z=k) = \begin{cases} 2\left[\left(\frac{3}{4}\right)^{k-1} - \left(\frac{3}{4}\right)^{k} - \left(\frac{1}{4}\right)^{k-1} + \left(\frac{1}{4}\right)^{k}\right] & , k = 3,4,5,\ldots \\ 0, \text{o.w.} \end{cases}$$

$$\sum_{k=3}^{\infty} P(Z=k) = \lim_{n\to\infty} \sum_{k=3}^{n} P(Z=k).$$

$$= \lim_{n\to\infty} 2\left[\left(\frac{3}{4}\right)^{2} - \left(\frac{3}{4}\right)^{n} - \left(\frac{1}{4}\right)^{2} + \left(\frac{1}{4}\right)^{n}\right]$$

$$= \lim_{n\to\infty} 2\left[\frac{1}{2} - \left(\frac{3}{4}\right)^{n} - \left(\frac{1}{4}\right)^{n}\right]$$

$$= 2\cdot\frac{1}{2}$$

$$= 1.$$

Example:- Consider an urn containing w white balls and b black balls. From the urn n balls $(1 \le n \le w+b)$ are chosen without replacement. Let Y denote the number of white balls chosen.

$$Y \le n \wedge w$$

$$n - Y \le b.$$

$$\underbrace{n-b \vee 0}_{\max} \le Y \le \underbrace{n \wedge w}_{\min}$$

$$P(Y=k) = \begin{cases} \dfrac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}} , & (n-b)\vee 0 \le k \le n \wedge w. \\ 0 & , \text{o.w.} \end{cases}$$

This is known as Hypergeometric Distribution with parameter w, b, n.

Example:- Suppose that n numbers are chosen from $\{1, \dots, N\}$ with replacement. Let X denote the largest number.

$$P(X=k) = \frac{k^n - (k-1)^n}{N^n}, \quad k = 1(1)N.$$

Suppose $n \leq N$, and the selection is without replacement.

$$P(X=k) = \begin{cases} \dfrac{{}^{k-1}C_{n-1}}{{}^{N}C_n}, & n \leq k \leq N. \\ 0, & \text{o.w.} \end{cases}$$

Def^n:- We say that "events A_n increase to A_∞" if $A_1 \subseteq A_2 \subseteq A_3, \dots$ and $A_\infty = \bigcup_{n=1}^{\infty} A_n$.

Theorem:- If $A_n \uparrow A_\infty$, the $P(A_n) \uparrow P(A_\infty)$.

Proof:-
Since, for every n,

$$A_n \subseteq A_{n+1}$$

it holds that, $P(A_n) \leq P(A_{n+1})$.

Thus, to complete the proof all that needs to be shown is $\lim_{n \to \infty} P(A_n) = P(A_\infty)$.

Define $B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$\vdots$

$B_n = A_n \setminus A_{n-1}, \; n \geq 2.$

Then, $B_1, B_2, \ldots$ are disjoint and ~~there~~ $\bigcup_{n=1}^{\infty} B_n = A_\infty$.

suppose that, $\omega \in B_i \cap B_j$ for some $1 \leq i < j$.

Since, $\omega \in B_i \subseteq A_i \subseteq A_{j-1}$ $[\because i < j]$

As, $B_j = A_j - A_{j-1}$

so, ω cannot belong to B_j. $(\Rightarrow \Leftarrow)$

$\because$ B_i's are disjoint.

Next, observe that $B_n \subseteq A_\infty$ for $n \geq 1$.

and hence, $\bigcup_{n=1}^{\infty} B_n \subseteq A_\infty$.

Conversely, suppose that $\omega \in A_\infty$.

That means $\{n : \omega \in A_n\}$ is non-empty.

Let $n_0 = \min \{n : \omega \in A_n\}$.

If $n_0 = 1$, then $\omega \in B_1$

otherwise $\omega \in A_{n_0} \setminus A_{n_0 - 1} = B_{n_0}$ $[\because \omega \notin A_{n_0 - 1}]$.

Thus. $\bigcup_{n=1}^{\infty} B_n = A_\infty$.

$$\therefore P(A) = \sum_{n=1}^{\infty} P(B_n).$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} P(B_k).$$

$$= \lim_{n \to \infty} \left[P(A_1) + \sum_{k=2}^{n} \left[P(A_k) - P(A_{k-1}) \right] \right]$$

$$= \lim_{n \to \infty} P(A_n).$$

Definition:- If the events A_n are such that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots \supseteq A_n$ and $A = \bigcap_{n=1}^{\infty} A_n$, then we say $A_n \downarrow A$.

If $A_n \downarrow A$ then $P(A_n) \downarrow P(A)$.

$\because$ $A_n \downarrow A$;
it holds that,

$$A_n^c \uparrow A^c.$$

By the previous theorem,

$$P(A_n^c) \uparrow P(A^c)$$

$$\Rightarrow 1 - P(A_n^c) \downarrow 1 - P(A^c).$$

$$\Rightarrow P(A_n) \downarrow P(A).$$

Definition:- Let, X be the random variable. The cumulative distribution function (CDF) of X is the function $F : \mathbb{R} \to [0,1]$ defined by.

$$F(x) = P(X \leq x), x \in \mathbb{R}.$$

Theorem:- The CDF F of a random variable X has the following properties.

1. F is non-decreasing.
2. $\lim_{x \to \infty} F(x) = 1$ (this means for a seq. x_n as $x_n \uparrow \infty$ $F(x_n$
3. $\lim_{x \to -\infty} F(x) = 0$. (this " " " x_n " $x_n \downarrow -\infty$ $F(x_n)$
4. F is right continuous.

Proof:- (i) Let $x \le y$.

Since $[X \le x] \subseteq [X \le y]$.

$$\Rightarrow P(X \le x) \le P(X \le y).$$

$$\Rightarrow F(x) \le F(y).$$

(ii) Let $x_n \uparrow \infty$

Define $A_n = [X \le x_n], n \ge 1$.

clearly $A_n \uparrow \Omega$.

$$\therefore P(A_n) \uparrow P(\Omega) = 1.$$

$$\Rightarrow F(x_n) \uparrow 1.$$

Since this holds for any $x_n \uparrow \infty$ we get

$$\lim_{x \to \infty} F(x) = 1.$$

(iii) Let, $x_n \downarrow -\infty$

Define, $A_n = [X \le x_n]$

Then $A_n \downarrow \phi$.

$$\therefore P(A_n) \downarrow P(\phi) = 0.$$

$$F(x_n) \to 0.$$

$$\therefore \lim_{x \to -\infty} F(x) = 0.$$

(iv) For $x \in \mathbb{R}$. Suppose that $x_n \downarrow x$.

Let, $A_n = [X \le x_n], n \ge 1$

Clearly $A_n \downarrow \bigcap_{n=1}^{\infty} A_n = [X \le x]$.

Therefore $P(A_n) \downarrow P(X \le x)$.

i.e. $F(x_n) \downarrow F(x)$. This prove rigt continuity of x.

Thm:- For every $x \in \mathbb{R}$,

$$F(x-) = \lim_{y \to x^-} F(y) = P(X < x)$$

i.e. For every $x_n \uparrow x$ such that $x_n < x$ it holds that $F(x_n) \to P(X < x)$.

Proof:- Let $\{x_n\}$ be a sequence such that $x_n \uparrow x$ and $x_n < x$ for all n. Let $A_n = [X \leq x_n]$, $n \geq 1$, Then

$$A_n \uparrow [X < x]$$

Therefore $F(x_n) = P(A_n) \to P(X < x)$, as desired

Corollary:- If F is the CDF of X, then

$P(X = x) = F(x) - F(x-)$, $x \in \mathbb{R}$

Corollary:- If F is a CDF of X, then $\forall\, x \in \mathbb{R}$; F is continuous at x iff

$$P(X = x) = 0.$$

Example:- Let $X \sim Geo(p)$.

CDF of X is,

$P(X \leq n)$ $\cancel{F(x) = P(X \leq x)}$ $=$ 1 − P(Head has not occured in first x tosses).

$\cancel{= \sum_{i=1}^{x} P(X = i)}$ $= 1 - q^n$.

$\therefore$ The CDF of X is

$$F(x) = \begin{cases} 1 - q^{\lfloor x \rfloor}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

Example:- A number is chosen at random from $\{1,\dots,N\}$ and denoted by X.

The CDF of X is.

$$F(x) = \begin{cases} 0, & x < 1 \\ \dfrac{\lfloor x \rfloor}{N}, & 1 \le x \le N. \\ 1, & x > N \end{cases}$$

This distribution is known as uniform distribution on $[1,N]$.

Example:- Suppose that n numbers are chosen with replacement from $\{1,\dots,N\}$. Denote their maximum by X.

Let $X = k$. $k \in \mathbb{N}$.

$$\therefore\ P(X \le k) = \frac{k^n}{N^n}$$

$\therefore$ CDF of X is.

$$F(x) = \begin{cases} 0, & x < 1. \\ \left(\dfrac{\lfloor x \rfloor}{N}\right)^n, & 1 \le x \le N. \\ 1, & x > N \end{cases}$$

In the above example Let Y denote the minimum of the n numbers.

$$P(Y \le i) = 1 - P(Y > i), \quad i = 1,\dots,N.$$

$$= 1 - P(Y \ge i+1).$$

$$= 1 - \left(\frac{N-i}{N}\right)^n.$$

The CDF of Y is.

$$F_Y(x) = \begin{cases} 0 & ; \quad x < 1. \\ 1-\left(\frac{N-[x]}{N}\right)^n, & 1 \le x \le N. \\ 1, & x > N. \end{cases}$$

Expectation:-

Def^n:- Suppose that x takes values $x_1, x_2, \ldots \infty$ with respective probabilities $p_1, p_2, \ldots$ Then X has an expectation if

$$\sum |x_n| p_n < \infty.$$

and in that case, the expectation is defined as.

$$E(X) = \sum_{n \ge 1} x_n p_n.$$

Example:- Let $X \sim Bin(n, p)$.

$$E(X) = \sum_{i=0}^{n} i \, P(X=i).$$

$$= \sum_{i=0}^{n} i. \binom{n}{i} p^i q^{n-i}.$$

$$= \sum_{i=0}^{n} i. \frac{n!}{(n-i)!\, i!} p^i q^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} p^{i-1} q^{(n-1)-(i-1)}$$

$$= np. (p+q)^{n-1}$$

$$= np \quad [\because p+q=1].$$

(3) Let, A be the event that no box gets more than two balls.

$\# A$ = coefficient of x^{10} in $(1+x+x^2)^7$.

$\# \Omega$ =

(4) Let, A be the event that first box is empty.
B " " " " second " " " .
C " " " " third " " " .

We want $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= 3 \times P(A) - 3 \times P(A \cap B) + 0$.

$= \dfrac{3 \times 2^{10} - 3}{3^{10}}$

(6) Let, A be the event that two rooks can take each other.

$\#(A) = \binom{64}{1} \times 14$.

$P(A) = \dfrac{64 \times 14}{{}^{64}P_2} = \dfrac{14}{63} = \dfrac{2}{9}$.

(7) A be the event that exactly one box is empty.

$P(A) = \dfrac{n(n-1)\binom{n}{2}(n-2)!}{n^n}$

10.

(b) Let, $A_i = \{ i \text{ has no pre-image} \}$.

Our event of interest is

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = 1 - P\left(\bigcup A_i\right).$$

$$= 1 - \sum P(A_i) + \sum P(A_i \cap A_j)$$

$$\cdots\cdots + (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right).$$

$$\cancel{= 1 - \frac{1}{n^m}\left[n(n-1)^m + n\right.}$$

$$= 1 - \left[\sum (-1)^{r+1} \cdot {}^{n}C_{r} \frac{(n-r)^m}{n^m}\right].$$

<u>back to Expectation</u>

<u>Example:-1</u> Suppose $X \sim Geo(p)$.

$$E(X) = 1 \cdot p + 2 \cdot qp + 3 \cdot q^2 p + \cdots$$

$$= (p + qp + q^2 p + \cdots) + (qp + q^2 p + \cdots)$$

$$+ (q^2 p + \cdots).$$

$$= p \frac{1}{1-q} + p \cdot \frac{q}{1-q} + p \cdot \frac{q^2}{1-q} + \cdots$$

$$= \frac{p}{1-q} \cdot (1 + q + q^2 + \cdots)$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

[Often "X-1" is the geometric random variabl

Example-2:- Let X be a number chosen at random from $\{1, \dots, N\}$.

$$E(X) = \frac{1}{N} \sum_{i=1}^{N} i = \frac{N+1}{2}.$$

Theorem:- Let, X be random variable taking value $\mathbb{N} \cup \{0\}$. If expectation of X exists then

$$E(X) = \sum_{n=0}^{\infty} P(X > n)$$

Assume that $E(X)$ exists.

$$\therefore E(X) = \sum_{n=0}^{\infty} n P(X=n).$$

$$= \sum_{n=0}^{\infty} P(X=n) \sum_{k=1}^{n} 1.$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{n} P(X=n).$$

$$= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P(X=n).$$

$$= \sum_{k=1}^{\infty} P(X \geq k).$$

$$= \sum_{k-1=0}^{\infty} P(X > k-1).$$

$$= \sum_{n=0}^{\infty} P(X > n) \text{ proved.}$$

Example:-3 $X \sim Geo(p)$.

$$E(X) = \sum_{k=0}^{\infty} P(X > n) = \sum_{k=0}^{\infty} q^n = \frac{1}{1-q} = \frac{1}{p}.$$

Example:-4 A urn contain 10 red, 5 black and 5 white balls. Balls are drawn at random with replacement. Let, X denote the minimum no. of draws to get balls of each colour. Calculate $E(X)$

$$P(X > n) = 2\left[\left(\frac{3}{4}\right)^n - \left(\frac{1}{4}\right)^n\right], \quad n = 1, 2, 3, \ldots$$

$$E(X) = \sum_{n=0}^{\infty} P(X > n).$$

$$= \cancel{2\left[1 - 1 + \frac{3}{4} - \frac{1}{4} + \right]} \; 2\sum_{n=0}^{\infty}\left[\left(\frac{3}{4}\right)^n - \left(\frac{1}{4}\right)^n\right]$$

$$= 2\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n - 2\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= \frac{2}{1-\frac{3}{4}} - \frac{2}{1-\frac{1}{4}} = 8 - \frac{8}{3} = \frac{16}{3}.$$

Example :- 5

From $\{1, \ldots, N\}$, n numbers are chosen with replacement. Calculate the expectation of ~~to~~ the maximum.

Let, X be the maximum.

$$P(X > k) = 1 - P(X \leq k).$$

$$= 1 - \left(\frac{k}{N}\right)^n$$

$$\therefore E(X) = \sum_{k=0}^{\infty} P(X > k).$$

$$= \sum_{k=0}^{N-1}\left[1 - \left(\frac{k}{N}\right)^n\right]$$

$$= N - N^{-n}\sum_{k=1}^{N-1} k^n.$$

Theorem:- Let X be a random variable defined of the probability space (Ω, P). If $E(X)$ exists then

$$E(X) = \sum_{\omega \in \Omega} X(\omega) P(\{\omega\}).$$

Proof:- Let $X(\Omega) = \{x_1, x_2, \dots\}$

by definition,

$$E(X) = \sum_{n \geq 1} x_n P(X = x_n).$$

$$= \sum_{n \geq 1} x_n P(\omega \in \Omega : X(\omega) = x_n).$$

$$= \sum_{n \geq 1} x_n P(X^{-1}\{x_n\}).$$

$$= \sum_{n \geq 1} x_n \sum_{\omega : X(\omega) = x_n} P(\{\omega\}).$$

$$= \sum_{n \geq 1} \sum_{\omega : X(\omega) = x_n} X(\omega) P(\{\omega\}).$$

$$= \sum_{\omega \in \Omega} X(\omega) P(\{\omega\}).$$

HW-2

4. (a) ~~$P(A_1 \cap A_2 \cap \dots \cap A_n)$~~ use induction.

(b) $P(A_1 | A_2 \cap \dots \cap A_n)$

~~$P(A_1 \cap A_n | A_2 \cap \dots \cap A_{n-1})$~~

$$= \frac{P(A_1 \cap A_2 \cap \dots \cap A_n)}{P(A_2 \cap \dots \cap A_n)}$$

$$= \frac{P(A_1 \cap A_n | A_2 \cap \dots \cap A_{n-1}) P(A_2 \cap \dots \cap A_{n-1})}{P(A_n | A_2 \cap \dots \cap A_{n-1}) P(A_2 \cap \dots \cap A_{n-1})}$$

Thm:- If X and Y are random variable with expectation defined on the same probability space. Then,

$$E(X+Y) = E(X) + E(Y).$$

Pf:-

$$E(X+Y) = \sum_{\omega \in \Omega} (X+Y)(\omega)\, P(\{\omega\})$$

$$= \sum_{\omega \in \Omega} X(\omega)\, P(\{\omega\}) + \sum Y(\omega)\, P(\{\omega\}).$$

$$= E(X) + E(Y).$$

e.g. Suppose a coin which tosses Heads with probability p is tossed n times. Let X denotes the number of Heads.

Define $Y_i = \begin{cases} 1, & \text{if the } i\text{th toss is Heads,} \\ 0, & \text{otherwise.} \end{cases}$ $i = 1(1)n.$

$\therefore X = Y_1 + Y_2 + \dots + Y_n.$

By the previous theorem,

$$E(X) = E(Y_1) + \dots + E(Y_n)$$

$$= p + \dots + p$$

$$= np.$$

Note:- A Random variable which takes values 1 and 0 with respect to probabilities p and $1-p$ is said to follow Bernoulli (p).

e.g. Suppose n balls are thrown at random into ~~one of~~ k boxes. Let X denote the number of empty boxes.

~~Define Y_i~~

Let, Y_i be the indicator of the event that the ith box is empty, $i = 1, \dots, k$.

$\therefore X = Y_1 + \dots + Y_k$

$\therefore E(X) = E(Y_1) + \dots + E(Y_k)$.

$P(Y_i = 1) = \left(\frac{k-1}{k}\right)^n$

$\therefore E(Y_i) = \left(\frac{k-1}{k}\right)^n$.

$\therefore E(X) = k\left(\frac{(k-1)}{k}\right)^n$.

<u>Thm</u>; Let X be a random variable having expectation then ~~for~~ $\forall \alpha \in \mathbb{R}$,

$$E(\alpha X) = \alpha E(X).$$

<u>Pf</u>:-

$$E(\alpha X) = \sum_{\omega \in \Omega} \alpha X(\omega) P(\{\omega\}).$$

$$= \alpha \sum_{\omega \in \Omega} X(\omega) P(\{\omega\}).$$

$$= \alpha E(X)$$

Exc:- Let n balls be thrown into n boxes and denote by X_n the proportion of empty boxes. Calculate $\lim_{n\to\infty} E(X_n)$.

Solⁿ:- For a fix n,
Let Y_i be the indicator of the ith box being empty, $i = 1, \dots, n$.

$$X_n = \frac{1}{n}\sum_{i=1}^{n} Y_i .$$

$$\therefore E(X_n) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i).$$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \frac{(n-1)^n}{n^n}$$

$$= \left(\frac{n-1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} E(X_n) = e^{-1}.$$

Defⁿ:- Let X be a random variable whose expectation is μ, Its variance is
$Var(X) = E[(X-\mu)^2]$
which is defined if the expectation on the right hand side is also defined.

Exc:- Prove that, $Var(X-a) = Var(X)$.

Exc:- $Var(X) = E(X^2) - (E(X))^2$

Suppose, $E(X) = \mu$.

$$Var(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

$$= E(X^2) - E(2\mu X) + E(\mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$[\because E(X) = \mu]$

$$= E(X^2) - (E(X))^2.$$

Corollary:- $E(X^2) \geq (E(X))^2$

Exc:- Show that $E(X^2) = E^2(X)$ iff X takes only one value. (X is a degenerate random variable).

Solⁿ:-

Assume $E(X^2) = E^2(X)$.

$\therefore Var(X) = 0.$

$\Rightarrow E[(X-\mu)^2] = 0 \quad [\because \mu = E(X)].$

$\therefore X - \mu = 0$

$P[(X-\mu)] = 1.$

~~X takes values $x_1, x_2, \ldots, x_n$. (non-~~

$\Rightarrow X = \mu.$

$\therefore (\Rightarrow)$ proved.

~~Exc: Show th~~

now, if $X = \mu$ (constant),

$\therefore E(X) = \mu$

$\therefore$ ~~$X-\mu$ is a random vari.~~

$E(X^2) = \mu^2.$

$\therefore Var(X) = E(X^2) - E^2(X).$

$= \mu^2 - (\mu)^2$

$= 0.$

$(\Leftarrow)$ proved.

Exc. Calculate the variable of all the standard distributions covered so far:

ⓐ Bin (n, p)

ⓑ Geo (p).

ⓒ Uniform $\{1, \ldots, N\}$

ⓒ
$$E(X^2) = \frac{1}{N} \sum_{i=1}^{N} i^2$$
$$= \frac{\cancel{N}(N+1)(2N+1)}{6\cancel{N}}$$
$$E(X) = \frac{1}{N} \sum_{i=1}^{N} i$$
$$= \frac{N+1}{2}$$

$$\therefore \; \mathrm{Var}(X) = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4}$$
$$= \frac{(N+1)(4N+2-3N-3)}{12}$$
$$= \frac{N^2-1}{12}$$

Defⁿ :- For random variables X and Y defined on the same probability space, their join PMF is a function $p : \mathbb{R} \times \mathbb{R} \to [0, 1]$ defined by,
$$p(x, y) = P(X = x, Y = y), \quad x, y \in \mathbb{R}.$$

e.g. Consider the urn containing 10 red, 5 black and 5 white balls. Two balls are drawn from the urn with replacement. Let X and Y denote the number of black balls and white balls respectively.

$y \backslash x$	0	1	2	$P(Y=y)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{9}{16}$
1	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{3}{8}$
2	$\frac{1}{16}$	0	0	$\frac{1}{16}$
	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$	1.

$$P(X=x, Y=y) = \frac{2}{x!\,y!\,(2-x-y)!} 4^{-x} \cdot 4^{-y} 2^{x+y}$$

<u>Exc:-</u> Let $p(x,y)$ be the joint PMF of X and Y. Show that,

$$P(X=x) = \sum_y p(x,y), \quad x \in \mathbb{R}.$$

$$P(Y=y) = \sum_x p(x,y), \quad y \in \mathbb{R}$$

<u>Defn:-</u> Random variables X and Y are indep. if

$$P(X=x, Y=y) = P(X=x)\,P(Y=y). \ \forall\, x, y \in \mathbb{R}.$$

e.g. A fair coin is tossed 15 times.

X = No. of Heads in the 1st 10 tosses.

Y = No. of Tails in the last 3 tosses.

The X and Y are independent.

<u>Thm:-</u> If X and Y are indep. random variables having expectations, then

$$E(X.Y) = E(X)E(Y).$$

<u>Pf:-</u> Let $x_1, x_2, \ldots$ be the possible values of X and $y_1, y_2, \ldots$ be the possible values of Y.

Using a previous result,

$$E(XY) = \sum_{\omega \in \Omega} (XY)(\omega)\, P(\{\omega\}).$$

$$= \sum_i \sum_j \sum_{\substack{\omega \in \Omega \\ X(\omega)=x_i,\, Y(\omega)=y_j}} (XY)(\omega)\; P(\{\omega\}).$$

$$= \sum_i \sum_j \sum_{\omega \in \Omega:\, X(\omega)=x_i,\, Y(\omega)=y_i} x_i y_j\; P(\{\omega\}).$$

$$= \sum_i \sum_j x_i y_j \sum_{\omega \in \Omega:\, X(\omega)=x_i.\, Y(\omega)=y_j} P(\{\omega\}).$$

$$= \sum_i \sum_j x_i y_j\; P(X=x_i, Y=y_j).$$

$$= \sum_i \sum_j x_i y_j\; P(X=x_i)\, P(Y=y_j).$$

$$= \left(\sum_i x_i\, P(X=x_i)\right)\left(\sum_j y_j\, P(Y=y_j)\right)$$

$$= E(X)E(Y).$$

<u>Thm</u>:- If X and Y are independent, then $P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B) \quad \forall\, A, B \subseteq \mathbb{R}$.

<u>Pf</u>:- Let, $x_1, x_2, \ldots$ be the possible values of X and $y_1, y_2, \ldots$ be " " " Y.

$$P(X \in A, Y \in B) = \sum_{i:\, x_i \in A} \sum_{j:\, y_j \in B} P(X=x_i, Y=y_j).$$

$$= \sum_i \sum_j P(X=x_i)\, P(Y=y_j)$$

$$= \left(\sum_i P(X=x_i)\right)\left(\sum_j P(Y=y_j)\right)$$

$$= P(X \in A)\, P(Y \in B).$$

Defn:- For random variable ~~x and y~~ X and Y, their covariance is defined as.

$$Cov(X,Y) = E\left[(X-E(X))(Y-E(Y))\right]$$

which is defined when expectation on RHS is defined.

Theorem :- ~~Pf~~

1. If X and Y have a covariance, then
$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

2. If X has a variance, then
$$Cov(X,X) = Var(X). \quad (EXC)$$

3. If X and Y both have variances, then
$$Var(X+Y) = Var(X) + Var(Y) + 2\,cov(X,Y).$$

4. If X and Y are independent then,
$$Cov(X,Y) = 0.$$

Proof:-

1. Denote $\mu_x = E(X)$, $\mu_y = E(Y)$

$$\begin{aligned}
Cov(X,Y) &= E\left[(X-\mu_x)(Y-\mu_y)\right] \\
&= E\left[XY - \mu_x Y - \mu_y X + \mu_x\mu_y\right] \\
&= E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x\mu_y \\
&= E(XY) - \mu_x\mu_y - \mu_x\mu_y + \mu_x\mu_y. \\
&= E(XY) - E(X)E(Y).
\end{aligned}$$

3.
$$\begin{aligned}
Var(X+Y) &= E\left[(X+Y)^2\right] - \left(E(X+Y)\right)^2 \\
&= E(X^2+Y^2+2XY) - \cancel{E(X+Y)}\left(E(X)+E(Y)\right)^2 \\
&= E(X^2)+E(Y^2)+2E(XY) \\
&\quad - E^2(X) - E^2(Y) - 2E(X)E(Y) \\
&= E(X^2) - E^2(X) + E(Y^2) - E^2(Y) + 2\left(E(XY) - E(\ldots\right.
\end{aligned}$$

$= Var(X) + Var(Y) - 2\,Cov(X, Y).$

4. If X & Y are independent.

$\therefore E(XY) = E(X)E(Y).$

$\Rightarrow E(XY) - E(X)E(Y) = 0.$

$\Rightarrow Cov(X, Y) = 0.$

Thm:-
Let $X_1, \dots, X_n$ are random variable each having Variance.

$\therefore Var(X_1 + \dots + X_n)$

~~Variance~~

$$= E\left[\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} X_i\right)\right]^2$$

$$= E\left[\sum_{i=1}^{n} (X_i - E(X_i))\right]^2$$

$$= E\left[\sum_{i=1}^{n} (X_i - E(X_i))^2 + 2\sum_{1\le i<j\le n} (X_i - E(X_i))(X_j - E(X_j))\right]$$

$$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i<j\le n} Cov(X_i, X_j).$$

Corollary:- If $X_1, \dots, X_n$ are random variable with variances such that $\forall\, i \neq j$ X_i and X_j are independent then

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i).$$

Exc:- Let, $X \sim Bin(n, p)$. Calculate $var(X)$.

Sol:- Take a coin which tosses Heads with probability p. Toss it n times.

Let, Y_i ~~be a~~ be a random variable such that, $Y_i = 1$ if ith toss is head $\forall\ 1 \le i \le n$.
$= 0$ otherwise.

Define
~~Clearly~~ $Y = \sum_{i=1}^{n} Y_i$. the $Y \sim Bin(n, p)$.

~~Now, $Var(Y_i) = \frac{1}{2}[$~~

$\because Var(X) = Var(Y)$

$= Var(Y_1 + \cdots + Y_n)$.

$= \sum_{i=1}^{n} Var(Y_i)$.

$Var(Y_i) = E(Y_i^2) - (E(Y_i))^2$

$= p - p^2$

$= p(1-p)$.

$\therefore Var(X) = np(1-p)$.

<u>Exc:-</u> Let n distinct ball be thrown into k distinct boxes. Denote the number of empty boxes by X and calculate Variance of X.

Sol: Let Y_i be the indicator of the event that the ith box is empty $i = 1(1)k$.

then $X = \sum_{i=1}^{k} Y_i$.

$\therefore Var(X) = \sum_{i=1}^{k} Var(Y_i) + 2 \sum_{1 \le i < j \le k} cov(Y_i, Y_j)$.

Now, $P(Y_i = 1) = \left(\frac{k-1}{k}\right)^n$

$P(Y_i = 0) = 1 - \left(\frac{k-1}{k}\right)^n$

$Var(Y_i) = \left(\frac{k-1}{k}\right)^n \left(1 - \left(\frac{k-1}{k}\right)^n\right)$

~~For~~ ~~$1 \le i \le j \le k$~~.

$\longrightarrow$

Fix $1 \le i < j \le k$

$\text{Cov}(Y_i, Y_j) = E(Y_i Y_j) - E(Y_i)E(Y_j)$.

$Y_i Y_j$ takes only 2 values 0 & 1.

$Y_i Y_j = 1 \Rightarrow Y_i = 1, Y_j = 1 \quad P(Y_i = 1, Y_j = 1) = \left(\frac{k-2}{k}\right)^n$.

$\therefore \text{Cov}(Y_i, Y_j) = \left(\frac{k-2}{k}\right)^n - \left(\frac{k-1}{k}\right)^{2n}$.

$$\therefore \text{Var}(X) = k\left(\frac{k-1}{k}\right)^n \left[1 - \left(\frac{k-1}{k}\right)^n\right] + k(k-1)\left[\left(\frac{k-2}{k}\right)^n - \left(\frac{k-1}{k}\right)^{2n}\right].$$

Exc:- Suppose a sample of size n is chosen from $\{1, \ldots, N\}$ without replacement. Fix a $k \le N$. Let X denote the number of chosen units are less than or equal to k. Calculate $E(X)$ and $\text{Var}(X)$.

Soln: Denote $X_i = \begin{cases} 1 & \text{if the } i\text{th unit is } \le k. \\ 0 & \text{otherwise.} \end{cases}$

$i = 1(1)n$.

clearly, $X = \sum_{i=1}^{n} X_i$.

For each i,

$P(X_i = 1) = \frac{k}{N}$

For $1 \le i < j \le n$, $P(X_i = 1, X_j = 1) = \frac{k(k-1)}{N(N-1)}$.

$$E(X_i) = 1 \times \frac{k}{N} + 0 \times \left(1-\frac{k}{N}\right) = \frac{k}{N}.$$

$$\therefore E(X) = \sum_{i=1}^{n} E(X_i) = \frac{nk}{N}.$$

$$Var(X) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \le i < j \le n} cov(X_i, X_j)$$

$$= n \cdot \frac{k}{N} \cdot \left(1-\frac{k}{N}\right) + n(n-1)\left[\frac{k(k-1)}{N(N-1)} - \frac{k^2}{N^2}\right].$$

Exc:- Suppose that X and Y are jointly distributed random variables. If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $E[f(X,Y)]$ exists, then show that

$$E[f(X,Y)] = \sum_i \sum_j f(x_i, y_j) \, P(X = x_i, Y = y_j).$$

where $x_1, x_2, \ldots$ are possible values of X and $y_1, y_2, \ldots$ are possible values of Y.

Soln. $$E[f(X,Y)] = \sum_{\omega \in \Omega} f(X(\omega), Y(\omega)) \, P(\{\omega\}).$$

$$= \sum_i \sum_j \sum_{\omega \in \Omega : X(\omega) = x_i, Y(\omega) = y_j} f(X(\omega), Y(\omega)) \, P(\{\omega\}).$$

$$= \sum_i \sum_j f(x_i, y_j) \sum_{\omega \in \Omega;\, X(\omega) = x_i, Y(\omega) = y_j} P(\{\omega\})$$

$$= \sum_i \sum_j f(x_i, y_j) \, P(X = x_i, Y = y_j).$$

$$\sum_{\omega \in \Omega} f(X(\omega)) \, P(\{\omega\})$$

$$= \sum_i \sum_{\omega : X(\omega) = x_i} f(X(\omega)) \, P(\{\omega\})$$

$$= \sum_i f(x_i) \sum_{\omega : X(\omega) = x_i} P(\{\omega\})$$

$$= \sum_i f(x_i) P(X = x_i)$$

Exc:- Let X and Y follow Bin (m, p) and Bin (n, p) independent of each other. Calculate the distribution of $X+Y$.

Sol:- Fix $k \in \{0, 1, \ldots, m+n\}$.

$$P(X+Y = k) = \sum_{i=0}^{k} P(X = i, Y = k-i)$$

$$= \sum_{i=0}^{k} P(X = i) \, P(Y = k-i). \quad [\text{independence}]$$

$$= \sum_{i=0}^{k} \binom{m}{i} p^i q^{m-i} \cdot \binom{n}{k-i} p^{k-i} q^{n-k+i}$$

[convention $\binom{n}{r} = 0$ if $r < 0$ or $r > n$]

$$= p^k q^{m+n-k} \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

$$= \binom{m+n}{k} p^k q^{m+n-k} \quad [\text{where } q = 1-p]$$

$\therefore (X+Y) \sim Bin(m+n, p)$.

Exc:-

Suppose that $X_n \sim Bin(n, p_n)$ for every $n \geq 1$.
Assume that $\lim_{n \to \infty} n p_n = \lambda \in (0, \infty)$.

Fix $k \in \mathbb{N} \cup \{0\}$.

$$P(X_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k}.$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot p_n^k (1-p_n)^{n-k}.$$

$$= \frac{np_n(np_n - p_n)\cdots(np_n - (k-1)p_n)}{k!} (1-p_n)^{n-k}$$

$$= \frac{1}{k!} \left(\prod_{i=0}^{k-1} (n-i)p_n \right) (1-p_n)^{n-k}.$$

$$= \frac{1}{k!} \underbrace{(1-p_n)^{-k}}_{\downarrow \, 1} \underbrace{\left(\prod_{i=0}^{k-1} (n-i)p_n \right)}_{\downarrow \, \lambda^k} \underbrace{(1-p_n)^{n}}_{\downarrow \, ??}$$

Recall that $\lim_{n \to \infty} n p_n = \lambda$. Fix $\varepsilon > 0$

$\exists$ $N \in \mathbb{N}$ such that.

$$|n p_n - \lambda| \leq \varepsilon \quad , \quad \forall \; n \geq N.$$

That is $\forall$ $n \geq N$,

$$\lambda - \varepsilon \leq n p_n \leq \lambda + \varepsilon.$$

$$\Rightarrow \frac{\lambda - \varepsilon}{n} \leq p_n \leq \frac{\lambda + \varepsilon}{n}.$$

$$\Rightarrow 1 - \frac{\lambda + \varepsilon}{n} \leq 1 - p_n \leq 1 - \frac{\lambda - \varepsilon}{n}.$$

if n is large enough so that $\frac{\lambda + \varepsilon}{n} < 1$,

then,

$$\left(1 - \frac{\lambda - \varepsilon}{n}\right)^n \geq (1 - p_n)^n \geq \left(1 - \frac{\lambda + \varepsilon}{n}\right)^n.$$

Letting $n \to \infty$.

$$e^{-(\lambda + \varepsilon)} \leq \liminf_{n \to \infty} (1 - p_n)^n \leq \limsup_{n \to \infty} (1 - p_n)^n \leq e^{-(\lambda - \varepsilon)}$$

Letting $\varepsilon \to 0$ we get,

$$e^{-\lambda} \leq \liminf_{n \to \infty} (1 - p_n)^n \leq \limsup_{n \to \infty} (1 - p_n)^n \leq e^{-\lambda}.$$

$$\therefore \lim_{n \to \infty} (1 - p_n)^n = e^{-\lambda}.$$

<u>Theorem:-</u> For $n \geq 1$, Let $X_n \sim \text{Bin}(n, p_n)$.

where, $\lim_{n \to \infty} n p_n = \lambda \in (0, \infty)$

For every $k \in \mathbb{N} \cup \{0\}$.

$$\lim_{n \to \infty} P(X_n = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Definition :- A ~~P~~ random variable X follows Poisson distribution with parameter λ if

$$P(X=k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & , \ k = 0,1,2,\ldots \\ 0 & , \ \text{otherwise.} \end{cases}$$

clearly $\sum_{k=0}^{\infty} P(X=k)$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda}\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right)$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= 1.$$

$$E(X) = \sum_{k=0}^{\infty} k\,P(X=k).$$

$$= 0 + \sum_{k=1}^{\infty} k \cancel{P} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \lambda \sum_{(k-1)=0}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot 1.$$

$$= \lambda.$$

If $\text{cov}(X,Y) = 0$ then $\not\Rightarrow$ X and Y independent.

Example:-

Let, X be a random variable having a variance

Suppose Z takes values 1 and -1, each with ~~poro~~ probability $\frac{1}{2}$, independently of X.

Define $Y = XZ$

Calculate,

$\text{Cov}(X,Y) = ?$

$E(Y) = E(XZ) = E(X)E(Z) = 0.$

$$\begin{aligned}\text{cov}(X,Y) &= E(XY) - E(X)E(Y)\\ &= E(X^2Z) - E(X)\,E(X)E(Z).\\ &= E(X^2)E(Z) - E^2(X)E(Z)\\ &= (E(X^2) - E^2(X))E(Z)\\ &= \text{Var}(X)E(Z)\\ &= 0.\end{aligned}$$

$\because$ X has variance.

$\exists\, x \neq 0$ such that $0 < P(X=x) < 1.$

$\therefore P(X = x, Y = x)$

$= P(X = x, Z = 1)$

$= \frac{1}{2} P(X = x).$

$P(Y=x) = P(X=x, Z=1) + P(X=-x, Z=-1).$

$= \frac{1}{2}[P(X=x) + P(X=-x)]$

If $P(X=x) + P(X=-x) < 1$, then

$P(Y=x) < \frac{1}{2}.$

If the random variable X we started with takes at least three distinct values with +ve probability then X and Y are dependent.

~~For the~~

e.g. $X \sim Bin(n,p)$ Z takes ± 1 w.p. $\frac{1}{2}$.

$$Y = XZ$$

$$Cov(X,Y) = ?$$

Example:- Let, X be uniform on $\{-10,-9,\dots,9,10\}$. Define $Y = X^2$. Calculate $Cov(X,Y) = ?$

$$E(XY) = E(X^3) = 0.$$

$$E(X) = 0.$$

$$\therefore E(XY) - E(X)E(Y) = 0.$$

$$\Rightarrow Cov(X,Y) = 0.$$

Theorem:- (Cauchy - Schwarz Inequality)

Let, X and Y are random variables. Such that $E(X^2)$ and $E(Y^2)$ are defined. The $E(XY)$ is defined and

$$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}.$$

Pf:- We enumerate the elements of the sample space Ω as $\{\omega_1, \omega_2, \dots\}$

In order to show that $E(XY)$ is defined it suffices to prove that

$$\sum_n |X(\omega_n) Y(\omega_n)| \, P(\{\omega_n\}) < \infty.$$

Fix $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$

$$\sum_{i=1}^{n} \left[|X(\omega_i)| - \lambda |Y(\omega_i)| \right]^2 P(\{\omega_i\}) \geq 0.$$

quadratic of λ. and the whole expression is greater than or equal to 0. $\therefore$ Discriminant ≤ 0.

$**(\rightarrow)^5$

Qn: Suppose A and B are independent events. Is any subset of A indep. of B?

Ans No.
$C = A \cap B$ $\quad C \subseteq A$. C is not indep of B.
$D = A \cap B^c$ $\quad D \subseteq A$. D is not indep of B.

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$
$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$
$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$
$$= P(A)[P(B) + P(C) - P(B \cap C)]$$
$$= P(A)P(B \cup C).$$

$B - C = B \cap C^c$. $\quad P(B)P(C^c)$.

$P(A \cap B \cap C^c) = P(A)P(B)P(C^c)$.

$$\frac{P(A^c \cap B) + P(A \cap B)}{P(B)}$$

HW-3

$$\frac{P(B)}{P(B)} = 1.$$

4. $\cancel{P(X = x)} = \cancel{\left(\frac{N-M}{N}\right)^x \cdot \frac{M}{N}}, \quad \cancel{x = 1},$

$$P(X = x) = \begin{cases} \dfrac{{}^{N-M}C_x}{{}^{N}C_{x+1}} \cdot M, & x = 1(1)(N-M) \\ 0, & \text{otherwise} \end{cases}$$

3.

$$E(x) \sum_{n=k}^{\infty} n \binom{n-1}{k-1} p^k (1-p)^{n-k}.$$

$$= \sum_{n=k}^{\infty} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} \quad [q=1-p]$$

$$= k \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k q^{n-k}.$$

Let $N = n+1$
$K = k+1$

$$\therefore E(x) = k \sum_{N=K}^{\infty} \binom{N-1}{K-1} p^{K-1} q^{N-K}$$

$$= \frac{k}{p} \sum_{N=K}^{\infty} \binom{N-1}{K-1} p^K q^{N-K}$$

Now, $\sum \binom{n-1}{k-1} p^k q^{n-k}.$

$$= p^k \left[1 + kq + \frac{k(k+1)}{2} q^2 + \frac{k(k+1)(k+2)}{3!} q^3 + \cdots\right]$$

$$= p^k (1-q)^{-k} = 1.$$

$$\therefore E(x) = \frac{k}{p}$$

This is just called Negative Binomial Distribution.

$X \sim$ Negative Bin(k, p)

4. $$P(X=x) = \begin{cases} \dfrac{\binom{N-M}{x} M}{\binom{N}{x+1}} & ; x = 1(1)N-M \\ 0 & ; \text{otherwise} \end{cases}$$

Let numbered the black balls $\{1, \ldots, N-M\}$

Let X_i be another ~~even~~ random variables. such that.

$$X_i = \begin{cases} 1, & \text{if the } i\text{th black is drawn before the first white ball.} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $X = \sum_{i=1}^{N-M} X_i$.

$$P(X_1 = 1) = \frac{\frac{N!}{(M+1)!} M!}{N!}$$

$$= \frac{1}{M+1}$$

[first consider black ball no. 1 and m white balls and treat them as same then permute the M white balls.]

$$\therefore E(X) = \sum_i E(X_i)$$

$$= \frac{N-M}{M+1};$$

Exc:- Calculate $Var(X)$.

~~6~~ 7 $$P(X=x) = \begin{cases} \left(\dfrac{n-1}{n}\right)^{x-1} \cdot \dfrac{1}{n} & ; x = 1, 2, 3, \ldots \\ 0 & ; \text{otherwise} \end{cases}$$

Clearly $X \sim Geo(p)$ where, $p = \frac{1}{n}$.

$$E(X) = \frac{1}{p} = n.$$

8. ~~1~~ X follows uniform distribution with parameter $\frac{1}{n}$.

$E(X) = \frac{n+1}{2}$.

HW-4

3. ~~$P(X=x) =$~~

$$P(X=x, Y=y) = (1-p)^{(x-1)+(y-1)} p^2$$

$$= (1-p)^{(x+y-2)} p^2$$

$$P(X=x) = (1-p)^{(x-1)} p$$

$$P(Y=y) = (1-p)^{y-1} p.$$

7.

$$P\left(X_n = \frac{k}{n}\right) = \left(\frac{n-k}{n}\right)^n - \left(\frac{n-k-1}{n}\right)^n.$$

$$E(X_n) = \sum_{k=0}^{n-1} \frac{k}{n} \left[\frac{(n-k)^n - (n-k-1)^n}{n^n}\right].$$

$$P\left(X_n \geq \frac{k}{n}\right) = \left(\frac{n-k}{n}\right)^n.$$

$$E(X_n) = \sum_{k=0}^{n-1} \frac{(n-k)^n}{n^n} = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^n.$$

$$\cancel{= \sum_{i=1}^{n} \frac{i^n}{n^n}}.$$

$$\cancel{\lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^n}{n^n}}.$$

$$\lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^n.$$

$$= \sum_{k=0}^{n-1}.$$

8. ~~Pof~~

Let, X be the random variable which denote number of empty boxes.

$$P(X \geq k) = \binom{2n-k-1}{n} \qquad k = 0, \ldots, n-1.$$

$$\frac{a+b}{2} \quad \underbrace{\quad}_{(a^2+b^2)} \qquad (a+b)^2 > a^2+b^2$$

$$\therefore E(X) = \sum_{k=0}^{n-1} P(X \geq k).$$

$$= \sum_{k=0}^{n-1} \binom{2n-k-1}{n}.$$

$$= \binom{2n-1}{n} + \binom{2n-2}{n} + \cdots + \binom{n}{n}.$$

10. Let, Y be another variable such that

$X = k + Y$.

Clearly Y takes value $0, 1, \ldots \infty$.

Now, $P(X = k+y) = P(Y=y) =$

** $(\longleftarrow)^5$

$$= \lambda^2 \sum_{i=1}^{n} Y^2(\omega_i)\, P(\{\omega_i\})$$

$$- 2\lambda \sum_{i=1}^{n} |X(\omega_i) Y(\omega_i)| P(\{\omega_i\})$$

$$+ \sum_{i=1}^{n} X^2(\omega_i)\, P(\{\omega_i\}).$$

Therefore.

$$\left(\sum_{i=1}^{n} |X(\omega_i) Y(\omega_i)| P(\{\omega_i\})\right)^2$$

$$\leq \left(\sum_{i=1}^{n} X^2(\omega_i) P(\{\omega_i\})\right)\left(\sum_{j=1}^{n} Y^2(\omega_j) P(\{\omega_j\})\right)$$

$$\leq E(X^2)\, E(Y^2).$$

~~For $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$~~

$\cancel{\sum_{i=1}^{n} \left[|X(\omega_i)| - \lambda \right.}$

Since this holds for every n.

$$\sum_{i \geq 1} |X(\omega_i) Y(\omega_i)| P(\{\omega_i\}) \leq \sqrt{E(X^2) E(Y^2)} < \infty$$

Thus $E(XY)$ exists.

For $\lambda \in \mathbb{R}$

$$E\left[(X - \lambda Y)^2\right] = E(X^2) - 2\lambda E(XY) + \lambda^2 E(Y^2) \geq 0.$$

$\forall \lambda.$

hence,

$$E^2(XY) \leq E(X^2)E(Y^2).$$

thus

$$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$$

Corollary

1. If $E(X^2)$ exists, the so does $E(X)$.

Let, say $Y=1$

then $E(Y)=1.$

$E(Y^2)=1.$

$$\therefore |E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$$

$$|E(X)| \leq \sqrt{E(X^2)}.$$

2. Suppose X and Y have variances. They $Cov(X,Y)$ is defined.

Definition :- Let, X & Y be non-degenerate random variables whose variances are defined. The corelation Coefficient b/w X & Y is defined as.

$$Cor(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}}.$$

Corollary - 3

$$|Cor(X,Y)| \leq 1.$$

Thm:- If $\operatorname{Corr}(X,Y) = 1$, then

$$\frac{X-\mu_X}{\sigma_X} = \frac{Y-\mu_Y}{\sigma_Y} \quad \text{almost surely,}$$

where $\mu_X = E(X)$; $\mu_Y = E(Y)$; $\sigma_x^2 = Var(X)$; $\sigma_Y^2 = Var(Y)$

and If $\operatorname{corr}(X,Y) = -1$, then.

$$\frac{X-\mu_X}{\sigma_X} = -\frac{Y-\mu_Y}{\sigma_Y} \quad \text{almost surely.}$$

Pf:- Suppose that $\operatorname{Corr}(X,Y) = 1$.

$$E\left[\left(\frac{X-\mu_X}{\sigma_X} - \frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]$$

$$= E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + \frac{(Y-\mu_X)^2}{\sigma_Y^2} - \frac{2(X-\mu_X)(Y-\mu_X)}{\sigma_X \sigma_Y}\right]$$

$$= 1 + 1 - 2\operatorname{Corr}(X,Y)$$

$$= 2 - 2\operatorname{corr}(X,Y)$$

$$= 0.$$

$\therefore$ Expectation of a nonnegative random variable is 0

$\therefore$ the random variable is 0.

$$\frac{X-\mu_X}{\sigma_X} = \frac{Y-\mu_Y}{\sigma_Y} \quad \text{almost surely.}$$

Exc:-

1. Show that $Corr(aX+b, cY+d) = sgn(ac)\, Corr(X,Y)$ such that $a, c \neq 0$, $b, d \in \mathbb{R}$.

2. Show that, $Var(aX+b) = a^2 Var(X)$.

3. $Cov(aX+b, cY+d) = ac\, cov(X,Y)$.

~~4. cov (Show that for constants $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$~~

Exc:- Let X and Y have variances σ_X^2 and σ_Y^2 respectivly. Show that $(\sigma_X - \sigma_Y)^2 \leq Var(X+Y) \leq (\sigma_X + \sigma_Y)^2$.

Soln:-

$$Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y).$$
$$\leq Var(X) + Var(Y) + 2|cov(X,Y)|$$
$$\leq Var(X) + Var(Y) + 2\sqrt{Var(X)Var(Y)}$$
$$\leq (\sigma_X + \sigma_Y)^2$$

$$Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y)$$
$$\geq Var(X) + Var(Y) - 2|cov(X,Y)|$$
$$\geq Var(X) + Var(Y) - 2\sqrt{Var(X)Var(Y)}$$
$$\geq (\sigma_X - \sigma_Y)^2$$

$$\Rightarrow (\sigma_X - \sigma_Y)^2 \leq Var(X+Y) \leq (\sigma_X + \sigma_Y)^2$$

E.xc:- If X, Y are independent then show that $Corr(X,Y) = 0$.

The converse is FALSE.

$E(X^2) = npq + np$

$E(X) = np$

$E(X^3) =$

- If $X \sim Bin(n,p)$.

$Y = X^2$.

then calculate $Corr(X,Y) > 0$.

Exc:- Let, A_1, A_2, A_3 be pairwise independent events each having probability ~~each~~ $\frac{1}{2}$. If $P(A_1 \cap A_2 \cap A_3) =$ then show that ~~P~~ almost surely an odd number of events among A_1, A_2 & A_3 occures.

Solⁿ:- For $i = 1, 2, 3$, define

$$X_i = 1_{A_i} = \begin{cases} 1, & \text{if } A_i \text{ occures.} \\ 0, & \text{if } A_i \text{ doesn't occure,} \end{cases}$$

All we need to show, is that

$X_1 + X_2 + X_3$ is odd. almost surely.

Then, $X_3 = \begin{cases} 1, & \text{if } X_1 = X_2 \\ 0, & \text{if } X_1 \neq X_2 \end{cases}$ almost surely.

In other words, suffices to show that,

$$X_3 = 1 - (X_1 - X_2)^2 \text{ . almost surely}$$

$\therefore$ Our task is to compute

$$\text{corr}\left(X_3, (X_1 - X_2)^2\right)$$

$$\text{Cov}\left(X_3, (X_1 - X_2)^2\right) = \text{Cov}\left(X_3, (X_1^2 + X_2^2 - 2X_1X_2)\right)$$

> Sub Exc:- Show that, For constant $\alpha_1, \dots, \alpha_m$, $\beta_1, \dots, \beta_n$ and random variable $Y_1, \dots, Y_m$, $Z_1, \dots, Z_n$
>
> $$\text{Cov}\left[\sum_{i=1}^{m} \alpha_i Y_i, \sum_{j=1}^{n} \beta_j Z_j\right] = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j \text{ cov}(Y_i, Z_j)$$

$$= \text{Cov}(X_3, X_1^2) + \text{Cov}(X_3, X_2^2) - 2\,\text{Cov}(X_3, X_1X_2)$$

$$= -2\,\text{cov}(X_3, X_1X_2)$$ [bcoz X_1 and X_3 are indep and so are X_2 and X_3]

$= -2\,[E[X_1X_2X_3] - E(X_3)E(X_1X_2)]$

$= -2\quad [P(A_1\cap A_2\cap A_3) - P(A_3)P(A_1\cap A_2)]$

$= -2\left[\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4}\right]$ [Bcoz A_1, A_2 are indep]

$\therefore$ $\text{Cov}(X_3, (X_1-X_2)^2) = -\frac{1}{4}$.

$\text{Var}(X_3) = \frac{1}{4}$.

Clearly $(X_1-X_2)^2 \sim \text{Ber}(1/2)$.

and hence $\text{Var}((X_1-X_2)^2) = \frac{1}{4}$.

$\therefore$ $\text{Corr}(X_3, (X_1-X_2)^2) = -1$.

$$\frac{X_3 - \frac{1}{2}}{\frac{1}{2}} = -\frac{(X_1-X_2)^2 - \frac{1}{2}}{\frac{1}{2}}$$ almost surely.

$\Rightarrow$ $X_3 = 1 - (X_1-X_2)^2$. almost surely.

as directed.

FACT:- $F: \mathbb{R} \to \mathbb{R}$

If for all $x_n \uparrow \infty$ it holds that

$F(x_n) \to a$, then

$\lim_{x\to\infty} F(x) = a$.

Proof:- Let if possible, $\exists$ $\varepsilon > 0$ such that

$\forall k$ $\exists$ $x > k$ with $|F(x) - a| > \varepsilon$.

$\exists$ $x_1 > 0$ s.t. $|F(x_1) - a| > \varepsilon$.

& $\exists$ also $x_2 > x_1 + 1$ s.t. $|F(x_2) - a| > \varepsilon$.

$\rightarrow$

Illy, $\exists\, x_n > x_{n-1} + 1$ s.t. $|F(x_n) - a| > \varepsilon$.

Clearly, $x_n > x_{n-1} > \dots > x_2 > x_1$.

Further more $x_n > n-1 \quad \forall n$.

Hence $x_n \uparrow \infty$

and Clearly,

$F(x_n) \not\to a$. Hence, $(\Rightarrow\Leftarrow)$

HW-4

14. $P(X<Y)$

$$= \sum_{y=0}^{\infty} P(X=y) \sum_{x=y+1}^{\infty} P(X=x).$$

$$= P(X=0)\,P(Y\geq 1) + P(X=1)\,P(Y\geq 2) + \dots + P(X=k)\,P(Y\geq k+1) + \dots$$

14. $P(X<Y)$

$$= \sum P(X=x)\,P(Y>x).$$

$$= \sum P(X=x)$$