Method of Estimation / Elimination of Trend Component

Method 3: (Method of Differencing / Box-Jenkin's Method)

B = backward shift operator

$\nabla = I - B$

$y_t = m_t + \varepsilon_t$ where $m_t = c_0 + c_1 t + \cdots + c_k t^k$

$\nabla y_t = y_t - y_{t-1}$

∇m_t = $(k-1)$-th degree polynomial

$$\nabla^k y_t = \nabla^k m_t + \nabla^k \varepsilon_t$$

$$= k!\, c_k + \nabla^k \varepsilon_t$$

Since ε_t is assumed to be stationary, and ∇^k is a linear filter, $\nabla^k \varepsilon_t$ is also stationary.

Observed $\nabla^k \varepsilon_t$ is used to predict $\nabla^k \varepsilon_{t+h}$

From this $\hat{\varepsilon}_{t+h}$ is obtained.

Using this $\hat{m}_{t+h}$ is estimated which is used to predict y_{t+h}

Seasonal Component:

$s_1, s_2, \ldots, s_d$ are seasonal effects ($d = 12$ for months)

$s_1 = s_{1+kd}$, $\forall k \geq 1$

$s_k = s_{k+ld}$, $\forall k = 1(1)d$, $\forall l \geq 1$

$\hat{m}_t$ = estimated trend component

Consider the trend eliminated series $(y_t - \hat{m}_t)$

Let,

$$\omega_k := \frac{1}{\ell}\sum_{j=0}^{\ell}\left(y_{k+jd} - \hat{m}_{k+jd}\right) \quad , k = 1(1)d$$

$\sum_{k=1}^{d}\omega_k$ may not be equal to 0.

We calculate seasonal components by

$$\hat{s}_k = \omega_k - \frac{1}{d}\sum_{j=1}^{d}\omega_j \quad , \quad k = 1(1)d$$

$\hat{m}_t$ were estimated earlier.

Now, $y_t = m_t + s_t + \varepsilon_t$

We re estimate m_t from $(y_t - \hat{s}_t)$ and denote it $\hat{\hat{m}}_t$

$\therefore$ Estimated random component, $\hat{\varepsilon}_t = y_t - \hat{\hat{m}}_t - \hat{s}_t$

Again $\hat{\varepsilon}_t$ is assumed to be stationary.

From this we again predict $\hat{\varepsilon}_{n+h}$

~~We estimate $\hat{\hat{m}}_{t+h}$ from here~~

~~and continue so on~~. We have $\hat{\hat{m}}_{t+h}$ is known and so is $\hat{s}_{t+h}$. Using all these y_{t+h} is predicted.

Method 4: (Estimation of Seasonal Component by Differencing)

$$B y_t = y_{t-1}$$

$$B_d y_t = y_{t-d}$$

$$\nabla_d y_t = (I - B_d) y_t = y_t - y_{t-d}$$ [seasonal component eleminated ; trend may be there]

$$\nabla^k(\nabla_d y_t) = \nabla^k y_t - \nabla^k y_{t-d}$$

■ Show that, a linear filter $\{a_j\}$ passes an arbitrary polynomial of degree k without distortion, i.e.

$$m_t = \sum_{j=0}^{\infty} a_j m_{t-j} \quad , \forall \text{ k-th degree polynomial}$$

$$m_t = c_0 + c_1 t + \dots + c_k t^k$$

iff $$\sum_j a_j = 1$$

and $$\sum_j j^r a_j = 0 \quad , r = 1, \dots, k$$

→ Take, $m_t = t^k$

$$(t)^k = \sum_{j=0}^{\infty} a_j (t-j)^k$$

Comparing coeff. both side we have

$$\sum_{j=0}^{\infty} a_j = 1$$

and

$$\sum_{j=0}^{\infty} a_j \binom{n}{r} (-j)^{n-r} = 0 \quad , r = 1(1)k$$

$$\Leftrightarrow \sum_{j=0} j^r a_j = 0 \quad , r = 1(1)k$$

■ Show that the filter with coeff. $[a_{-2}, a_{-1}, a_0, a_1, a_2] = \frac{1}{9}[-1, 4, 3, 4, -1]$ passes third degree polynomial and elemenates seasonal components of period $3 = 3$.

$$m_t = -\frac{1}{9} m_{t-2} + \frac{4}{9} m_{t-1} + \frac{3}{9} m_t + \frac{4}{9} m_{t+1} - \frac{1}{9} m_{t+2}$$

ETS,

$$\sum a_j = 1$$

$$\sum j^r a_j = 0 \quad , r = 1,2,3$$

$\sum a_j = 1$ is clear.

$$\sum j a_j = \frac{2-4+4-2}{9} = 0$$

$$\sum j^2 a_j = \frac{-4+4+4-4}{9} = 0$$

$$\sum j^3 a_j = \frac{8-4+4-8}{9} = 0$$

$$s_t = (P, Q, -P-Q, P, Q, -P-Q\}, \ldots$$

$$\therefore \frac{1}{9}[-P + 4Q - 3P - 3Q + 4P - Q]$$

$$= 0$$

$\therefore$ seasonal variation is eleminated using this filter.

Find a filter of the form $1 + \alpha B + \beta B^2 + \gamma B^3$ (i.e. find α, β, γ) that passes linear ~~filters~~ trends without distortion and that eleminates arbitrary seasonal component of period 2.

→ Take, $m_t = a + bt$

So by the condn,

$$(1 + \alpha B + \beta B^2 + \gamma B^3)(a+bt) = a+bt$$

$$\cancel{\Rightarrow (1+\alpha+\beta+\gamma)a + b(t+\alpha t+\beta}$$

$$\Rightarrow a(1+\alpha+\beta+\gamma) + b(t + \alpha(t-1) + \beta(t-2) + \gamma(t-3)) = a+bt$$

$\therefore \quad 1+\alpha+\beta+\gamma = 1$

$bt(1+\alpha+\beta+\gamma) - b(\alpha+2\beta+3\gamma) = bt$

$\therefore \quad \alpha+\beta+\gamma = 0 \quad \ldots\ldots \langle i \rangle$

$\alpha+2\beta+3\gamma = 0 \quad \ldots\ldots \langle ii \rangle$

Now, the seasonal component $s_t = (P, -P)$

$\therefore \quad (1+\alpha B+\beta B^2+\gamma B^3)\, s_t = 0$

$\Rightarrow s_t + \alpha s_{t-1} + \beta s_{t-2} + \gamma s_{t-3} = 0$

$\Rightarrow P(1+\beta) - (\gamma+\alpha)P = 0$

$\Rightarrow 1+\beta-\gamma-\alpha = 0 \quad \ldots \langle iii \rangle$

Solving $\langle i \rangle$, $\langle ii \rangle$ and $\langle iii \rangle$ we have values for α, β, γ.

Exc. Show that $\exists$ some stationary time series $\{X_t\}$ whose ACVF is

(a) $\eta(t) = 1$, $\forall t = 0, \pm 1, \pm 2, \ldots$

(b) $\eta(t) = (-1)^t$, $\forall t = 0, \pm 1, \pm 2, \ldots$

(c) $\eta(h) = 1 + \cos\left(\frac{\pi h}{2}\right) + 2\cos\left(\frac{\pi h}{4}\right)$, $\forall h = 0, \pm 1, \pm 2, \ldots$

(d) $\eta(t) = \begin{cases} 1, & \text{if } t = 0 \\ 0.4, & \text{if } |t| = 1 \\ 0, & \text{ow} \end{cases}$ $\rightarrow$ MA (1) process

$\rightarrow$ Suppose X be a r.v. with finite variance.

$\therefore X_t = X$, $\forall t$ has ACVF of the form (a)

$X_t = (-1)^t X$, $\forall t$ has ACVF of the form (b)

$\sigma^2(1+\theta^2) = 1$

$\sigma^2 \theta = 0.4$

$\therefore \frac{\theta}{1+\theta^2} = 0.4$

$\Rightarrow 5\theta = 2 + 2\theta^2$

$\Rightarrow 2\theta^2 - 4\theta - \theta + 2 = 0$

$\Rightarrow (\theta - 2)(2\theta - 1) = 0$

Since $|\theta| < 1$, $\theta = \frac{1}{2}$

$\therefore X_t = \frac{1}{2} X_{t-1} + Z_t$

Consider, $X_t = A\cos(\omega t) + B\sin(\omega t)$, $t = 0, \pm 1, \pm 2, \dots$

$\boxed{A, B \overset{iid}{\sim} N(0,1)}$

Find mean and ACVF.

Hence show that $\cos(\omega h) = \eta(h)$ is non-neg. definite.

→ $E(X_t) = 0$

$Var(X_t) = 1$

$$Cov(X_t, X_{t+h}) = Cov\left(A\cos(\omega t) + B\sin(\omega t), A\cos(\omega(t+h)) + B\sin(\omega(t+h))\right)$$

$$= \cos(\omega(t+h))\cos(\omega t) \cdot Var(A) + \sin(\omega(t+h))\sin(\omega t) \cdot Var(B)$$

$$= \cos(\omega h)$$

$\therefore \gamma(h) = \cos(\omega h)$

$\therefore \cos(\omega h)$ is non-neg. definite.

Sol^n of (c):

Let ~~[illegible]~~

$X_1, X_2, X_3, X_4, X_5 \overset{iid}{\sim} N(0,1)$

Define $U_t = X_1 \quad \forall t$

$V_t = X_2\cos(\frac{\pi}{2}t) + X_3\sin(\frac{\pi}{2}t), \quad \forall t$

$W_t = X_4\cos(\frac{\pi}{4}t) + X_5\sin(\frac{\pi}{4}t), \quad \forall t$

$Z_t = U_t + V_t + 2W_t$ has ACVF of the form (c)

$\{\omega_t\} \sim WN(0, \sigma_\omega^2)$

$\{Z_t\} \sim WN(0, \sigma_Z^2)$

$cov(\omega_t, Z_s) = 0, \ \forall t, s$

$Y_t := X_t + \omega_t$ where $X_t \sim AR(1)$

$X_t - \phi X_{t-1} = Z_t \ , \ |\phi| < 1$

(i) Show Y_t is stationary and find its ACVF.

(ii) Show that $Y_t - \phi Y_{t-1}$ is 1-correlated. Hence ~~[illegible]~~ MA(1).

(iii) Show that Y_t is ARMA(1,1)

Find its parameters in terms of $\phi, \sigma_\omega^2, \sigma_Z^2$

→ Let $E(X_t) = \mu$

$\therefore E(Y_t) = \mu + 0 = \mu$

~~[illegible]~~ $\gamma_1(h)$ be the ACVF of X_t

$\therefore \gamma_1(h) = \phi^{|h|} \gamma_1(0) = \phi^{|h|} \dfrac{\sigma_Z^2}{1-\phi^2}$

$\therefore Cov(Y_t, Y_{t+h}) = \gamma_1(h) + Cov(\omega_t, \omega_{t+h})$

$\Rightarrow \gamma(h) = \gamma_1(h)$

$\therefore Y_t$ is stationary.

Now, $V_t := Y_t - \phi Y_{t-1} = Z_t + \omega_t - \phi\omega_{t-1}$

$Cov(V_t, V_{t+h}) = Cov(Z_t + \omega_t - \phi\omega_{t-1}, Z_{t+h} + \omega_{t+h} - \phi\omega_{t+h-1})$

$$= \begin{cases} -\phi\sigma_\omega^2 & \text{if } |h| = 1 \\ 0 & \text{ow} \end{cases}$$

$y_t - \phi y_{t-1} = P_t + \theta P_{t-1}$ [Since U_t is MA(1)]

$\therefore$ y_t is ARMA(1,1)

$$\cancel{(1+\theta^2)\sigma^2 = \gamma(0) = \frac{\sigma_z^2}{1-\phi^2}}$$

$$(1+\theta^2)\sigma^2 = \sigma_z^2 + (1+\phi^2)\sigma_\omega^2 \quad \cdots \text{①}$$

$$\mathrm{Cov}(V_t, V_{t-1}) = -\phi\sigma_\omega^2$$

$$\Rightarrow \theta\sigma^2 = -\phi\sigma_\omega^2 \quad \cdots\cdots \text{②}$$

From ① and ② we have

$$\frac{\theta}{1+\theta^2} = \frac{-\phi\sigma_\omega^2}{\sigma_z^2 + (1+\phi^2)\sigma_\omega^2}$$

$$\Rightarrow \theta\left(\sigma_z^2 + (1+\phi^2)\sigma_\omega^2\right) + \theta^2\left(\phi\sigma_\omega^2\right) + \left(\phi\sigma_\omega^2\right) = 0$$

Discriminant ≥ 0

$\therefore$ $\hat{\theta}$ be a sol^n

$$\therefore \hat{\sigma}^2 = -\frac{\phi}{\hat{\theta}}\sigma_\omega^2$$

Prediction Problem in Time ~~Seo~~ Series

$X_1, X_2, \ldots, X_n$ is a time series (stationary / non-stationary)

Best predictor to minimize expected square error loss is $E(X_{n+h} \mid X_1, \ldots, X_n)$, $h > 0$.

But it is difficult to handle and cumbersome to compute

So we use Best Linear Predictor.

It only requires Variance-Covariance function and mean function.

Linear Predictor:

$$P_n(X_{n+h}) = a_0 + a_1 X_n + a_2 X_{n-1} + \cdots + a_n X_1$$

We want to estimate the values of $a_0, a_1, \ldots, a_n$ by minimising

$$E = E\left((X_{n+h} - a_0 - a_1 X_n - a_2 X_{n-1} - \cdots - a_n X_1)^2\right) \quad \text{over } a_i\text{'s}.$$

Suppose are considering Stationary Time Series with ACVF $\gamma(h)$ and mean μ

Normal Equation

$$\frac{\partial E}{\partial a_i} = 0, \quad i = 0, 1, 2, \ldots, n$$

with $i = 0$, $E[X_{n+h} - a_0 - a_1 X_n - \cdots - a_n X_1] = 0$ — ①

with $i = j$, $E[(X_{n+h} - a_0 - a_1 X_n - \cdots - a_n X_1) X_{n-j+1}] = 0$, $j = 1(1)n$

From ①,

$$\mu - a_0 = \mu \sum_{i=1}^{n} a_i$$

$$\Rightarrow a_0 = \mu\left(1 - \sum_{i=1}^{n} a_i\right)$$

$$\Rightarrow \mu = \frac{a_0}{1 - \sum_{i=1}^{n} a_i}$$

From the rest of the normal eq^n,

$$\Gamma_n \underset{\sim}{a}_n = \underset{\sim}{\gamma}(h) \quad ; \text{ where } \Gamma_n = ((\gamma(i-j)))_{i,j=1}^{n}$$

$$\gamma_n(h) = (\gamma(h), \gamma(h+1), \dots, \gamma(h+n-1))$$

$$\underset{\sim}{a}_n' = (a_1, \dots, a_n)$$

So $\quad P_n(X_{n+h}) = \mu + \sum_{i=1}^{n} \hat{a}_i (X_{n-i+1} - \mu)$.

Properties of the Predictor

1) $E[\text{error}] = 0$, i.e. $E[X_{n+h} - P_n(X_{n+h})] = 0$

2) Cov (error, predictor) = 0 i.e. $E[(X_{n+h} - P_n(X_{n+h}))\, X_{n-i+1}] = 0$

3) $E(\text{error}^2) = \gamma(0) - 2\underset{\sim}{a}_n' \gamma_n(h) + \underset{\sim}{a}_n' \Gamma_n \underset{\sim}{a}_n$

$= \gamma(0) - \underset{\sim}{a}_n' \gamma_n(h)$ [as $\underset{\sim}{a}_n$ satisfies $\Gamma_n \underset{\sim}{a}_n = \gamma_n(h)$]

4) $P_n(X_{n+h})$ is unique, whatever be solution to normal eq^n.

(Γ_n may be singular)

→ Let the two values of predictor $P_n(X_{n+h})$ and $P_n'(X_{n+h})$

We shall show, $E(\tilde{Z}^2) = 0$ where $\tilde{Z} = Z_1 - Z_2$

$$E[\tilde{Z}^2] = E\left[\left((X_{n+h} - P_n(X_{n+h})) - (X_{n+h} - P_n'(X_{n+h}))\right)\tilde{Z}\right]$$

$= 0$ [Since Z is uncorrelated with errors]

$\Rightarrow Z_1 = Z_2$ a.s.

Eg. (AR(1) Process)

$X_n = \phi X_{n-1} + Z_n$, $Z_n \sim WN(0, \sigma^2)$

$Cov(Z_m, X_n) = 0$, $\forall\, m > n$

$|\phi| < 1$

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

$$\gamma(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2}$$

$$P_n(X_{n+h}) = \sum_{i=1}^{n} a_i X_{n-i+1}$$

$\mu = 0$

$\Gamma_n \underset{\sim}{a_n} = \gamma_n(1)$

$$\Gamma_n = \frac{\sigma^2}{1-\phi^2} \begin{bmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{n-1} \\ \phi & 1 & \phi & & \\ \phi^2 & \phi & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \cdots & 1 \end{bmatrix}$$

$\underset{\sim}{a_n}\ \cancel{\gamma_n(1)} = (\phi, 0, \ldots, 0)$

$$P_n(X_{n+1}) = \sum_{i=1}^{n} a_i X_{n-i+1} = \phi X_n$$

$E[\text{error}^2] \doteq E[Z_{n+1}^2] = \sigma^2$

$E[Z_{n+1} X_m] = 0 \quad \forall\ m \leq n$

This also implies $\quad$ error $= Z_{n+1}$

$P_n(X_{n+1}) = \phi X_n$

General Prediction Problem

Y is predicted on the basis of 1 and $W = (W_n, W_{n-1}, \ldots, W_1)$

by linear predictor $P(Y|W)$

$P(Y|W) = \mu_Y + \underset{\sim}{a_n}'(W - \mu_W)$ $\boxed{\text{to get } \underset{\sim}{a_n}}$

$\underset{\sim}{a_n}'$ is solution to

$\Gamma_n \underset{\sim}{a_n} = \gamma_n \quad$ where $\quad \Gamma_n = \left(\left(\text{cov}(W_{n-i+1}, W_{n-j+1})\right)\right)_{i,j=1}^{n}$

$\gamma_n' = \text{cov}(Y, W)$

$P(Y|W) = \mu_Y + \underset{\sim}{a_n}'(W - \mu_W)$

Properties:

1) $E[Y - P(Y|W)] = 0$

2) $E[(Y - P(Y|W))\,W] = 0$

3) $P(\alpha Y_1 + \beta Y_2 + \gamma \mid W) = \alpha P(Y_1|W) + \beta P(Y_2|W) + \gamma$

4) $P\left(\sum_{i=1}^{k} \alpha_i W_i \,\middle|\, W\right) = \sum_{i=1}^{k} \alpha_i W_i$

5) $P(y|\omega) = E(y)$ if $Cov(y, \omega) = 0$

6) $P(P(y|v,\omega)|\omega) = P(y|\omega)$

$\rightarrow$

$$\left.\begin{aligned} E(y - P(y|\omega, v)) &= 0 \\ E(y - P(y|\omega)) &= 0 \end{aligned}\right\} \Rightarrow E(P(y|\omega) - P(y|\omega, v)) = 0$$

$$\left.\begin{aligned} E((y - P(y|v,\omega)).\omega) &= 0 \\ E((y - P(y|\omega))\omega) &= 0 \end{aligned}\right\} \Rightarrow E((P(y|\omega) - P(y|v,\omega))\omega) = 0$$

$\therefore$ 6 follows.

Eg: Let $\{X_t\}$ is AR(1)

X_2 value is missing

To estimate X_2 value in term of known X_1 and X_3 value

$W = (X_1, X_3)$

$\underset{\sim}{a_n}' = (a_1, a_2)$

$\gamma_n = \frac{\sigma^2}{1-\phi^2}(\phi, \phi)$

$$\Gamma_n = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} 1 & \phi^2 \\ \phi^2 & 1 \end{pmatrix}$$

Now,

$$\frac{\sigma^2}{1-\phi^2}\begin{pmatrix} 1 & \phi^2 \\ \phi^2 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \phi \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & \phi^2 \\ \phi^2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \phi \\ \phi \end{pmatrix} = \frac{1}{1-\phi^4}\begin{pmatrix} 1 & -\phi^2 \\ -\phi^2 & 1 \end{pmatrix}\begin{pmatrix} \phi \\ \phi \end{pmatrix}$$

$$= \frac{1}{1-\phi^4}\begin{pmatrix} \phi - \phi^3 \\ \phi - \phi^3 \end{pmatrix}$$

$$= \frac{1}{1+\phi^2}\begin{pmatrix} \phi \\ \phi \end{pmatrix}$$

$$\text{predictor} = \frac{\phi}{1+\phi^2}(X_1+X_3)$$

$$E(\text{error}^2) = \operatorname{Var}(X_2) - \underset{\sim}{a_n}' \gamma_n = \frac{\sigma^2}{1-\phi^2} - \frac{\sigma^2}{1-\phi^2}\,\frac{1}{1+\phi^2}\,2\phi^2$$