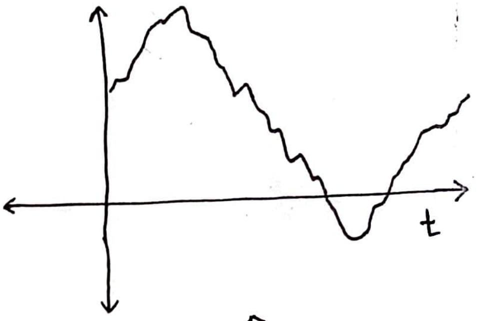
Spectral Analysis

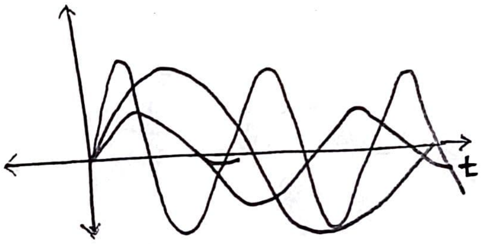
To write down a time series as sum of random sinusoidation sinnusoidals of different frequencies. The goal is to see the periodic nature and the dominating frequency.

This is for stationary time series. Also the ACVF is also studied as sum of deterministic waves. Those are called "Spectrial Analysis" or "Frequency Domain Analysis".

The studies we have done with ACVF's etc. is called "Time Domain Analysis".



$\updownarrow$



Def^n: Let $\{X_t\}_{t=-\infty}^{\infty}$ be a time series. $\{\gamma(h)\}_{h=-\infty}^{\infty}$ be the ACVF. Let $\sum_{h=-\infty}^{\infty}|\gamma(h)|<\infty$.

Then spectral density is defined as

$$f(\lambda)=\frac{1}{2\pi}\sum_{|h|<\infty} e^{-ih\lambda}\,\gamma(h)$$

Result: $f(\lambda)$ Spectral Density for $\gamma(h)$, which is absolutely summable, has the following properties,

(a) $f(\lambda)=f(-\lambda)$, $\forall\lambda$

(b) $f(\lambda)\geq 0$, $\forall\lambda$

(c) $\gamma(h)=\int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\,d\lambda$

→

$$f(\lambda)=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\,\gamma(h)$$

$$=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\cos(h\lambda)\gamma(h) \quad [\sin(h\lambda) \text{ part vanishes}]$$

So $f(\lambda)$ is even func, i.e. $f(\lambda)=f(-\lambda)$

Now shall show,

$$f(\lambda) \geq 0 \quad , \forall \lambda$$

Define $f_N(\lambda) = \frac{1}{2\pi N} E\left[\left|\sum_{|r|<N} e^{ir\lambda} X_r\right|^2\right]$

$$= \frac{1}{2\pi N} E\left[\sum_{|r|<N}\sum_{|s|<N} e^{ir\lambda} e^{-i\lambda s} X_r X_s\right]$$

$$= \frac{1}{2\pi N} \sum_{|r|<N}\sum_{|s|<N} e^{i\lambda(r-s)} E[X_r X_s]$$

$\cancel{\geq 0}$ ~~[Since $cov(X_r, X_s) = E[X_r X_s] \geq 0$]~~

$$= \frac{1}{2\pi N} \sum_{|r|<N}\sum_{|s|<N} e^{i\lambda(r-s)} \gamma(r-s)$$

$$= \frac{1}{2\pi} \sum_{|h|<N} \left(1 - \frac{|h|}{N}\right) \gamma(h) e^{-i\lambda h}$$

$$\to \frac{1}{2\pi} \sum_{|h|<\infty} e^{-i\lambda h} \gamma(h) \qquad \text{[by DCT]}$$

$f_N(\lambda) \geq 0 \Rightarrow f(\lambda) \geq 0$

To show, $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$

Now, $\int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda = \int e^{ih\lambda} \left(\frac{1}{2\pi} \sum_{|k|<\infty} e^{-ik\lambda} \gamma(k)\right) d\lambda$

$$= \sum_{|k|<\infty} \frac{1}{2\pi} \gamma(k) \int_{-\pi}^{\pi} e^{i\lambda(h-k)} d\lambda \quad \to \begin{cases} = 0 & \text{if } k \neq h \\ = 2\pi & \text{ow} \end{cases}$$

$$= \gamma(h)$$

Remark: If $\sum_{|h|<\infty} |\gamma(h)| < \infty$, then <u>Spectral Density</u> exists.

In other cases, Spectral Density may not exist.

Then we use the following definition:

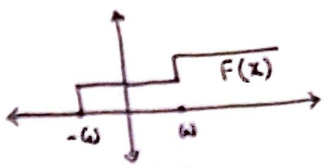
Def^n: $f(\lambda)$ is called Spectral Density for ACVF $\gamma(h)$ if

(a) $f(\lambda) \geq 0$, $\forall \lambda \in [-\pi, \pi]$

(b) $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$

Eg: $\gamma(h) = \cos(\omega h)$

$$= \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda)$$



pdf does not exist w.r.to measure $F(\cdot)$.

Remark: With $\sum_{|h|<\infty} |\gamma(h)| = \infty$ and with the above def^n of Spectral Density may not exist. Also if it exists, we may not have

$f(\lambda) = \frac{1}{2\pi} \sum_{|h|<\infty} e^{-ih\lambda} \gamma(h)$ [because the later may not be summable]

Result: If for any ACVF $\gamma(h)$ spectral density exists, then it is unique.

→ Let $f(\lambda)$ and $g(\lambda)$ be two spectral density.

$$\int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda = \gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} g(\lambda)\, d\lambda \quad , \forall h$$

$$\Rightarrow \int_{-\pi}^{\pi} e^{ih\lambda} (f(\lambda) - g(\lambda))\, d\lambda = 0$$

$$\Rightarrow f(\lambda) = g(\lambda)$$

To check for ACVF:

Theorem: Let $\sum_{|h|<\infty} |\gamma(h)| < \infty$, $\gamma : \mathbb{Z} \to \mathbb{R}$. Then $\gamma(h)$ is ACVF of some time series iff

(a) γ is even

(b) $\frac{1}{2\pi} \sum_{|h|<\infty} e^{-ih\lambda} \gamma(h) \geq 0$

Application:

Exc.

$$\gamma(h) = \begin{cases} 1 & \text{if } h=0 \\ \rho & \text{if } |h|=1 \\ 0 & \text{ow} \end{cases}$$

For which ρ, $\gamma(h)$ is ACVF?

$\rightarrow$

$$\frac{1}{2\pi} [\gamma(0) + 2\rho \cos\lambda] \geq 0$$

$$\Rightarrow 1 + 2\rho\cos\lambda \geq 0, \quad \forall \lambda$$

$$\Rightarrow |\rho| \leq 1/2$$

Exc. Let $\gamma(h) : \mathbb{Z} \to \mathbb{R}$ be defined as

$$\gamma(h) = \begin{cases} 1 & \text{if } h=0 \\ -0.5 & \text{if } |h|=1 \\ -0.25 & \text{if } |h|=2 \\ 0 & \text{ow} \end{cases}$$

$\rightarrow$

$$\left[1 - \cos\lambda - \frac{1}{2}\cos 2\lambda\right]$$

$$= \left[1 - \cos\lambda - \frac{1}{2}[2\cos^2\lambda - 1]\right]$$

$$= \left[\frac{1}{2} - \cos\lambda - \cos^2\lambda\right] < 0 \quad \text{for } \lambda = 0$$

$\therefore$ $\gamma(h)$ is not an ACVF

Spectral Analysis (Contd.)

For a time series $\{X_t\}_{t=-\infty}^{\infty}$ with ACVF $\{\gamma(h)\}_{h=0}^{\infty}$, $f(\lambda) : [-\pi,\pi] \to \mathbb{R}$ is called spectral density of $\{X_t\}$ if

(i) $f(\lambda) \geq 0 \quad \forall \lambda \in [0,\pi]$

(ii) $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$

Thm: For $f : \mathbb{R} \to \mathbb{R}$, $\exists$ an ACVF $\{\gamma(h)\}$ s.t. $f(\lambda)$ is spectral density for $\gamma(h)$ iff

(i) $f(\lambda) = f(-\lambda) \quad \forall \lambda$

(ii) $f(\lambda) \geq 0 \quad \forall \lambda$

(iii) $\int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty$

→ (Necessity)

$f(\lambda)$ is spectral density of $\{\gamma(h)\}$

$\gamma(h)$ is even

So $f(\lambda)$ is even i.e. $f(\lambda) = f(-\lambda)$

$\therefore f(\lambda) \geq 0 \quad \forall \lambda$ [Since $f(\lambda) \geq 0 \;\forall\; \lambda \geq 0$]

$\gamma(0) = \int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty$

(Sufficiency)

Suppose $f(\lambda) = f(-\lambda) \quad \forall \lambda > 0$

$f(\lambda) \geq 0$

$\int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty$

To show $\exists$ ACVF $\gamma(h)$ ~~whose~~ whose spectral density is $f(\lambda)$

Define, $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$ [well defined since absolutely integrable]

f is even function

$\therefore \gamma$ is also even

$\therefore \gamma(h) = \gamma(-h)$

Only thing to show γ is n.n.g.

Let, $[a_1\ a_2 \dots a_n] \in \mathbb{R}^n$

$$\sum_{r=1}^{n}\sum_{s=1}^{n} a_r a_s \gamma(r-s)$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} a_r a_s \int_{-\pi}^{\pi} e^{i(r-s)\lambda} f(\lambda)\, d\lambda$$

$$= \int_{-\pi}^{\pi} \left(\sum_{r=1}^{n} a_r e^{ir\lambda}\right)^2 f(\lambda)\, d\lambda \geq 0$$

$f(\lambda)$ is weight of the wave $e^{i\lambda h}$ in $\gamma(h)$ if it is high then the particular wave has high weight in $\gamma(h)$.

Thm. Let a func $\gamma(h): \mathbb{Z} \to \mathbb{R}$ have $\sum_{|h|<\infty} |\gamma(h)| < \infty$. Then $\gamma(h)$ is ACVF of some time series iff

(i) γ is even

(ii) $f(\lambda) = \frac{1}{2\pi} \sum_{|h|<\infty} e^{-ih\lambda} \gamma(h) \geq 0$.

→ (Necessity)

Let γ is ACVF

Then $\gamma(h)$ is even

and $f(\lambda) \geq 0$

(Sufficiency)

Suppose γ is even i.e. $\gamma(h) = \gamma(-h)$

and $f(\lambda) = \frac{1}{2\pi} \sum_{|h|<\infty} e^{-ih\lambda} \gamma(h) \geq 0$

So $f(\lambda)$ has following properties.

(i) $f(\lambda) = f(-\lambda)$ as $\gamma(h) = \gamma(-h)$

(ii) $f(\lambda) \geq 0$

(iii) $\int_{-\pi}^{\pi} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \left(\sum_{h=-\infty}^{\infty} e^{ih\lambda} \gamma(h)\right) f(\lambda)\, d\lambda$

$$= \sum_{h=-\infty}^{\infty} \gamma(h) \int_{-\pi}^{\pi} e^{ih\lambda}\, d\lambda = \gamma(0) < \infty$$

$\therefore f(\lambda)$ satisfies the properties of spectral density.

$\Rightarrow \exists$ an ACVF whose spectral density is $f(\lambda)$.

$$\int_{-\infty}^{\infty} e^{ih\lambda} f(\lambda)\, d\lambda$$

$$= \int_{-\pi}^{\pi} e^{ih\lambda} \frac{1}{2\pi}\left(\sum_{k=-\infty}^{\infty} e^{-ik\lambda} \gamma(k)\right) d\lambda$$

$$= \sum_{k=-\infty}^{\infty} \gamma(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(h-k)\lambda}\, d\lambda$$

$$= \gamma(h)$$

$\therefore$ By prev. thm. $f(\lambda)$ is spectral density and $\gamma(h)$ is ACVF.

Note: There may be some ACVF which do not have spectral density.

Eg: $X_t = A\cos(\omega t) + B\sin(\omega t)$; $A, B \overset{iid}{\sim} N(0,1)$

$$Cov(X_{t+h}, X_t) = \cos(\omega(t+h))\cos(\omega t) + \sin(\omega(t+h))\sin(\omega t)$$

$$= \cos(\omega h)$$

$\therefore \gamma(h) = \cos(\gamma h)$

This doesn't have spectral density.

$$\cos(\omega h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda) \quad \text{where} \quad F(t) = \begin{cases} 0 & \text{if } t < -\omega \\ 1/2 & \text{if } -\omega \le t < \omega \\ 1 & \text{if } t \geq \omega \end{cases}$$

$\cos(\omega h)$ is Fourier transform of $F(\lambda)$ measure

So spectral density doesn't exist.

Defn: $F(t) : [-\pi, \pi] \to [0, \infty)$ is called ~~as~~ a <u>generalised distribution</u> func if

(i) $F(-\pi) = 0$, $F(\pi) < \infty$

(ii) $F(\cdot)$ is non-decreasing

(iii) F is right continuous

Note: $\dfrac{F(t)}{F(\pi)}$ is a c.d.f on $[-\pi, \pi]$

Thm: For all $\gamma(h)$ ACVF, $\exists$ a generalised distn func F on $[-\pi, \pi]$ s.t. $\underbrace{\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda)}_{\text{spectral representation}}$.

$\Rightarrow$ CDF $G(\lambda) := \dfrac{F(\lambda)}{F(\pi)}$ gives spectral representation of

$P(h) = \dfrac{\gamma(h)}{\gamma(0)}$ ACF (Auto correlation function).

Note: If $G(\lambda)$ has a density, we get spectral density for $\gamma(h)$. And it is said to have <u>cont. spectra</u>.

If $G(\lambda)$ ~~has~~ a discrete prob. distn, the ACVF $\gamma(h)$ is said to have <u>discrete spectra</u>.

Note: Let $x_t = \sum$

ACVF of x_t is $\sum_{j=1}^{k} \sigma_j^2 \cos(\omega_j h)$.

These approximate each ACVF $\gamma(h)$

$x_t = \sum_{j=1}^{k} \left(A_j \cos(\omega_j t) + B_j \sin(\omega_j t)\right)$ well approximates each stationary time series.

Representation of Stationary Time Series U_t

$$U_t = \int_{-\pi}^{\pi} e^{it\lambda} \, dZ(\lambda) \quad \text{(proof not required)}$$

$Z(\lambda)$ is a random process

Increments of $Z(\lambda)$ $[d(Z(\lambda))]$ are independent.

As given

$$X_t = \sum_{j=1}^{k} \left(A_j \cos(\omega_j t) + B_j \sin(\omega_j t) \right)$$

ACVF of X_t,

$$\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(\omega_j h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda),$$

where $F(\lambda) = \sum_{j=1}^{k} \sigma_j^2 F_j(\lambda)$

and $F_j(t) = \begin{cases} 0 & \text{if } t < -\omega \\ 0.5 & \text{if } -\omega \le t < \omega \\ 1 & \text{if } \omega \le t \end{cases}$

Representation of X_t

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dZ(\lambda) \quad \text{where } dZ(\lambda) = \begin{cases} (A_j + iB_j)/2, & \text{if } \lambda = -\omega_j \\ (A_j - iB_j)/2, & \text{if } \lambda = \omega_j \\ 0 & , \text{ow} \end{cases}$$

$j = 1(1)k$

(Idea only! not so rigorous)

Note: ACVF $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda}\, dF(\lambda)$

$\gamma(h)$ is real if $\int_{[a,b]} dF(t) = \int_{[-b,-a]} dF(t)$ [i.e. F is symmetric]

Note: If σ_j is high, wave with freq^n ω_j is dominant.

$f(\lambda)$ high $\Rightarrow$ contribution of wave with freq^n λ is more.

$f(\lambda)$ low $\Rightarrow$ contribution of wave with freq^n λ is less.

$f(\lambda) = 0 \Rightarrow$ freq^n λ has no contribution.

$$\boxed{f(\lambda) = dF(\lambda)}$$

Eg: White noise.

$$X_t \sim WN(0, \sigma^2)$$

$$\gamma(h) = \begin{cases} 0 & \text{if } h \neq 0 \\ \sigma^2 & \text{if } h = 0 \end{cases}$$

$$\rightarrow \quad f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{\sigma^2}{2\pi} \quad \text{if } \lambda \in [-\pi, \pi]$$

$\therefore$ ALL freq^n are present equally

Eg: AR(1) Process $(|\phi| < 1)$

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

$$\gamma(h) = \phi^{|h|} \frac{\sigma^2}{1-\phi^2}$$

$$\rightarrow \quad f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)$$

$$= \frac{\sigma^2}{(1-\phi^2)2\pi} + \frac{1}{2\pi} \sum_{h=1}^{\infty} (\phi e^{-i\lambda})^h \frac{\sigma^2}{1-\phi^2} + \frac{1}{2\pi} \sum_{h=1}^{\infty} (\phi e^{i\lambda})^h \frac{\sigma^2}{1-\phi^2}$$

$$\cancel{= \frac{1}{2\pi} \left(\frac{\sigma^2}{1-\phi^2}\right) \left[1 + \sum_{h=1}^{\infty} \phi e^{-i\lambda}\right.}$$

$$= \frac{1}{2\pi} \frac{\sigma^2}{1-\phi^2} \left[1 + \frac{\phi e^{-i\lambda}}{1 - \phi e^{-i\lambda}} + \frac{\phi e^{i\lambda}}{1 - \phi e^{i\lambda}}\right]$$

$$= \frac{1}{2\pi} \frac{\sigma^2}{1-\phi^2} \frac{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda}) + \phi e^{-i\lambda}(1-\phi e^{i\lambda}) + \phi e^{i\lambda}(1-\phi e^{-i\lambda})}{(1-\phi e^{-i\lambda})(1-\phi e^{i\lambda})}$$

$$= \frac{1}{2\pi} \frac{\sigma^2}{1-\phi^2} \frac{1 - \cancel{\phi e^{ih\lambda}} + \cancel{\phi e^{ih\lambda}} - \phi^2}{1 - 2\phi \cos\lambda + \phi^2}$$

$$= \frac{\sigma^2}{2\pi} \frac{1}{1+\phi^2 - 2\phi\cos\lambda}$$