TIME SERIES

Seq^n of obs^n indexed w.r.t. time [finite / infinite seq^n]

Eg1 Wheat production (in 1000 tons) in India $\{X_t\}$ in the year $t = 1970, 1971, \ldots, 2005$

Eg2 $\left\{\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right\}_{t=1901}^{2000}$

X_t = avg. rainfall in Kolkata in the year t

Y_t = avg. rainfall in Joypur in the year t

Eg3 $\{x_t\}_{0 \leq t \leq 1 \text{ hr.}}$

x_t in amp. is the current flow at particular point $\frac{t}{2}$ in the electric circuit, $0 \leq t \leq 1$ hr.

Eg4. $\{x_t\}$, t = 6 a.m., 7 a.m., ..., 5 p.m. (in a day)

x_t is the body temp. of a patient at time t.

Note: Sometime seq^n of random obs^n represent time series $x_1, x_2, \ldots, x_n$. Sometimes seq^n of underlying ~~dist~~ random variable is called time-series.

Theoratical Examples:

1) IID $(0, \sigma^2)$ seq^n

$X_1, X_2, \ldots, X_n, \ldots$ are iid r.v.s with mean = 0, variance = σ^2

2) White noise $(0, \sigma^2)$ [WN$(0, \sigma^2)$]

$\{X_t\}_t$ s.t. $Var(X_t) = \sigma^2$ $E(X_t) = 0$

$Cov(X_t, X_s) = 0$ $\forall s \neq t$

Note: IID$(0, \sigma^2)$ is WN$(0, \sigma^2)$

But not the reverse.

3) Random Walk

$X_1, X_2, \dots, X_n \overset{iid}{\sim}$ mean = 0, var = σ^2

$S_n = X_1 + X_2 + \dots + X_n$

$\{S_t\}_t$ is called a Random Walk.

$Cov(S_t, S_u) = \sigma^2 (\min(t,u))$

4) MA(1) moving average of order 1

$\{X_t\}$

$X_t = \theta Z_{t-1} + Z_t$, $\theta \in \mathbb{R}$, $Z_t \overset{iid}{\sim} WN(0,\sigma^2)$

$E(X_t) = 0$ doesn't depend on t

$Var(X_t) = (\theta^2+1)\sigma^2$ doesn't depend on t

$Cov(X_t, X_{t+1}) = \dots = Cov(X_2, X_1) = \theta\sigma^2$

Weakly Stationary Time Series

Variance - Covariance func, Mean func.

$\{X_t\}$ is a time series

$E(X_t) = \mu_t$

$Cov(X_t, X_s) = \gamma(t,s)$

Properties of $\gamma(t,s)$:

1. $\gamma(t,t) \geq 0$, $\forall t \in \mathbb{Z}$.
2. $\gamma(t,s) = \gamma(s,t)$
3. $|\gamma(t,s)| \leq \sqrt{\gamma(t,t) \cdot \gamma(s,s)}$
4. $\underset{\sim}{a_n} = (a_1, \dots, a_n) \in \mathbb{R}^n$

   $\underset{\sim}{a_n}' \Gamma_n \underset{\sim}{a_n} \geq 0$ where $\Gamma_n = ((\gamma(i,j)))^n$

- If a bivariate function satisfies the above cond^n then it is variance - covariance func of some time - series. (Exc)

Consider

$$\Sigma = ((\gamma(t,s)))_{t,s=-N}^{N}$$

Def^n: $\{X_t\}$ is a gaussian process if each finite subsets of coordinates of it is Multi-variate Normals.

Weekly Stationary Process:

Def^n: $\{X_t\}$ is Weakly Stationary Process if,

$$E(X_t) = \mu_t = \mu \text{ (doesn't depend on } t)$$

$$Cov(X_t, X_s) = \gamma(t,s) = \gamma^*(|t-s|)$$

i.e.

$$E(X_t) = \mu$$

$$Cov(X_t, X_{t+h}) = Cov(X_1, X_{1+h}) = \gamma^*(h)$$

- If $(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_k+h})$ $\forall h, k$ then the process is Strongly Stationary.

Auto Covariance func:

Def^n: If variance-covariance func of the time series $\{X_t\}$ is such that,

$$\gamma(t, t+h) = \gamma^*(h), \quad \forall t \quad \text{(depends only on } h)$$

then $\{X_t\}_t$ is weakly stationary (provided mean func is const) and $\gamma(h)$ is called ACVF of $\{X_t\}_t$

Eg 1: IID $(0, \sigma^2)$

$$\gamma(h) = \sigma^2 \quad \text{if } h \neq 0$$
$$= 0 \quad \text{ow}$$

Eg 2: WN $(0, \sigma^2)$

$$\gamma(h) = \sigma^2 \quad \text{if } h = 0$$
$$= 0 \quad \text{if } h \neq 0$$

weakly stationary

Eg 3: Random Walk.

$$\gamma(X_t, X_s) = \sigma^2 (\min(t, s))$$

not weakly stationary

Eg 4: MA(1) process

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } |h| = 1 \\ 0 & \text{ow} \end{cases}$$

Exc. It is given $\gamma(h) = \begin{cases} 1 & \text{if } h = 0 \\ \rho & \text{if } |h| = 1 \\ 0 & \text{ow} \end{cases}$

For what values of ρ $\gamma(h)$ is ACVF ?

(Soln): For $|\rho| < 1/2$, $\exists$ a MA(1) process

Properties of ACVF:

1. $\gamma(0) \geq 0$.
2. $|\gamma(h)| \leq \gamma(0)$
3. $\gamma(-h) = \gamma(h)$
4. $\forall \; \underset{\sim}{a_n} = (a_1, \ldots, a_n) \in \mathbb{R}^n$

$$\underset{\sim}{a_n}' \Gamma_n \underset{\sim}{a_n} \geq 0 \quad \text{where } \Gamma_n = ((\gamma(|i-j|)))^n$$

Exc. If some function $\gamma(h): \mathbb{Z} \to \mathbb{R}$ satisfies the 4 properties of ACVF, then $\exists$ a weakly stationary time series $\{X_t\}$ whose ACVF is $\gamma(h)$.

Auto correlation Function ($\rho(h)$)

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$ if $\gamma(0) > 0$ where $\gamma(h)$ corresponds to ~~a~~ some weakly stationary time series

Properties:

1. $\rho(0) = 1$
2. $|\rho(h)| \leq 1$
3. $\rho(-h) = \rho(h)$
4. ~~[illegible]~~ PD property

Exc. If some function $\rho(h) : \mathbb{Z} \to \mathbb{R}$ satisfies the above 4 properties, then $\exists$ a weakly stationary time series whose ACF is this.

Goals:

1> infer about the parameters of the process.

2> to predict future values of the time series when past few/ all values are known.

3> to understand the model of the time series.

4> to control the process generating obs^n of the series.

AR(1) Process:

$\{X_t\}_t$ is called AR(1) if it is stationary solution to

$$X_t = \phi X_{t-1} + Z_t \quad \text{with } |\phi| < 1, \phi \in \mathbb{R}$$
$$Z_t \sim WN(0, \sigma^2)$$
$$Cov(Z_t, X_s) = 0, \ \forall s < t$$

Note: Stationary sol^n always exists.
It is unique. (to be proved)

Start with $X_0 \overset{d}{=} V$ (r.v.)

$X_1 = \phi V + Z_1$ and carry on to form a time series.

But in most cases these are not stationary.

ACVF of $X(t) \sim AR(1)$:

$$\gamma(0) = Var(X_t) = Var(\phi X_{t-1} + Z_t)$$
$$= \phi^2 Var(X_{t-1}) + \sigma^2$$
$$\Rightarrow \gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

$$\gamma(1) = Cov(X_t, X_{t-1}) = Cov(\phi X_{t-1} + Z_t, X_{t-1})$$
$$= \phi \gamma(0) + 0$$
$$= \frac{\phi \sigma^2}{1-\phi^2}$$

$$\gamma(h) = Cov(X_t, X_{t-h}) = \phi^{|h|} \gamma(0)$$

$$\therefore \gamma(h) = \phi^{|h|} \frac{\sigma^2}{1-\phi^2}$$

$$X_t = \phi X_{t-1} + Z_t = \phi^{h} X_{t-h} + \sum_{m=0}^{h-1} Z_{t-m} \phi^{m}$$

$$\therefore E\left[X_t - \sum_{m=0}^{h-1} Z_{t-m} \phi^{m}\right]^2 = \phi^{2j}(\gamma(0)+c)$$

$$\to 0 \text{ as } j \to \infty \quad [\text{since } |\phi|<1]$$

$$X_t \overset{L_2}{=} \sum_{k=0}^{\infty} \phi^k Z_{t-k} \quad (L_2 \text{ convergence})$$

$\therefore L_1$ convergence is immediate.

$$E\left[\sum_{k=0}^{j-1} |\phi|^k |Z_{t-k}|\right] \leq \sum_{k=0}^{j-1} |\phi|^k \sigma = \sigma \frac{1-|\phi|^j}{1-|\phi|} < \infty$$

$$\therefore E\left[\sum_{k=0}^{\infty} |\phi|^k |Z_{t-k}|\right] < \infty$$

$$\therefore \sum_{k=0}^{\infty} |\phi|^k |Z_{t-k}| < \infty \quad \text{a.s.}$$

$$\therefore \sum_{k=0}^{\infty} \phi^k Z_{t-k} \text{ is absolutely convergent a.s.}$$

$$\boxed{\therefore X_t = \sum_{k=0}^{\infty} \phi^k Z_{t-k} \quad \text{a.s.}}$$

~~alternative proof using operator~~

Note: $\{Y_t\}_{-\infty}^{\infty}$ and $BY_t = Y_{t-1}$, $B^h Y_t = Y_{t-h}$

B = backward shift operator

alternate proof using operators

$$X_t = \phi X_{t-1} + Z_t, \quad |\phi|<1, \quad \{Z_t\}_{-\infty}^{\infty} \sim WN(0,\sigma^2)$$

$$= \phi B X_t + Z_t$$

$$\Rightarrow (1-\phi B) X_t = Z_t \Rightarrow X_t = (1-\phi B)^{-1} Z_t$$

$$= (1+\phi B + \phi^2 B^2 + \cdots) Z_t$$

$$\therefore X_t = \sum_{k=0}^{\infty} \phi^k Z_{t-k}$$

- $$X_t = \phi X_{t-1} + Z_t \quad , \quad |\phi| > 1$$
$$Z_t \sim WN(0, \sigma^2)$$

$$X_{t+1} = \phi X_t + Z_{t+1}$$

$$\Rightarrow X_t = \frac{1}{\phi} X_{t+1} - \frac{1}{\phi} Z_{t+1}$$

$$= \frac{1}{\phi^h} X_{t+h} - \sum_{m=1}^{h} \frac{1}{\phi^m} Z_{t+m}$$

$$\therefore X_t = - \sum_{m=1}^{\infty} \frac{1}{\phi^m} Z_{t+m} \quad \text{a.s.}$$ (similar calculation as $|\phi| < 1$)

alternative proof using operator

$$(1 - \phi B) X_t = Z_t$$

$$\Rightarrow X_t = \frac{1}{1 - \phi B} Z_t = - \frac{\phi^{-1} B^{-1}}{1 - \phi^{-1} B^{-1}} Z_t$$

$$= - \phi^{-1} B^{-1} (1 + \phi^{-1} B^{-1} + \cdots\cdots) Z_t$$

$$= - \sum_{k=1}^{\infty} \frac{1}{\phi^k} Z_{t+k}$$

- $$X_t = X_{t-1} + Z_t = X_{t-h} + \sum_{n=0}^{h-1} Z_{t-n}$$

$$\therefore (X_t - X_{t-h}) = \sum_{n=0}^{h-1} Z_{t-n}$$ ← happens if X_t is stationary

$$Var(X_t - X_{t-h}) \leq 4\gamma(0)$$

but $Var\left(\sum_{n=0}^{h-1} Z_{t-n}\right) \longrightarrow \infty$ as $h \to \infty$

$\therefore$ x_t is not stationary.

$\therefore$ $|\phi| \neq 1$