AR(1)

Eq^n: $X_t = \phi X_{t-1} + Z_t$, $\{Z_t\} \sim WN(0,\sigma^2)$, indep.

For $|\phi|<1$, only stationary sol^n is

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

For $|\phi|>1$, only stationary sol^n is

$$X_t = \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$$

For $|\phi| = 1$, $\nexists$ any stationary sol^n

Def^n (Linear Process)

$\{Z_t\}_{-\infty}^{\infty} \sim WN(0,\sigma^2)$

$X_t = \sum_{j=-\infty}^{\infty} a_j Z_{t+j}$ is called a Linear Process if $\sum_{j=-\infty}^{\infty} |a_j| < \infty$

Def^n (Moving Average Process)

$\{Z_t\}_{-\infty}^{\infty} \sim WN(0,\sigma^2)$

$X_t = \sum_{j=0}^{\infty} a_j Z_{t-j}$ is called Moving Average Process if $\sum_{j=0}^{\infty} |a_j| < \infty$

$X_t \sim$ MA Process depends only on present and past values of Z_s's $(s \leq t)$. It is Cause-Effect dependence. That's why MA process is also called Causal Process.

$Cov(X_t, Z_s) = 0$ if $s > t$.

So it is also called Future Uncorrelated Process.

Def^n (Non-Causal Process):

- $Z_t \sim WN(0, \sigma^2)$

$$X_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j}$$ is called Non-Causal Process if

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty$$ and $cov(X_t, Z_s) \neq 0$ for some $s > t$

It is also called Future Correlated.

Note: For AR(1) process eq^n

$$X_t = \phi X_{t-1} + Z_t$$

- $|\phi| > 1 \Rightarrow$ unique non-causal process
- $|\phi| < 1 \Rightarrow$ unique causal process

Def^n (Strict Stationarity)

If $\{X_t\}_{-\infty}^{\infty}$ is a time series with property

$$(X_t, X_{t-1}, \ldots, X_{t-k}) \stackrel{d}{=} (X_s, X_{s-1}, \ldots, X_{s-k}) \quad \forall\, t, s \in \mathbb{Z} \text{ and } k \geq 0$$

it is called strictly stationary time series.

- $\{X_t\}$ strictly stationary

$\Rightarrow X_0, X_1, \ldots$ have same cdf.

Note: Strictly ~~sta~~ Stationary ~~Dist~~ Process are also Weakly Stationary given that Mean and Variance func^c exists.

Note: Weakly stationary process may not be Strictly Stationary.

Eg: $Z_t \sim WN(0, \sigma^2)$ indep. and $Z_t \sim F_t$.

Filter:

$\{X_t\}_{-\infty}^{\infty}$ is strictly stationary time series.

$$Y_t = g(X_t, X_{t-1}, \dots, X_{t-k})$$

$$g : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$$

Note that $\{Y_t\}_{-\infty}^{\infty}$ is also a strictly stationary time series.

g is called the Filter.

- Similarly, we can define, for Weakly Stationary Process.

- Let $\{X_t\}_{-\infty}^{\infty}$ be a weakly stationary time series.

We want to get other weakly stationary time series.

We should consider Linear filter.

$\{X_t\}_{-\infty}^{\infty}$ weakly stationary

$$\Rightarrow Y_t = \sum_{j=0}^{k} a_j X_{t-j} \text{ is also weakly stationary.}$$

MA(q)

$$Z_t \sim WN(0, \sigma^2)$$

$$X_t = \sum_{i=1}^{q} \theta_i Z_{t-i} + Z_t$$ is MA(q) Process $(q \geq 1)$

It is obtained from ~~weakly~~ stationary Process using Linear Filter.

Def^n (q-correlated process)

A stationary process is called q-correlated if

$Cov(X_t, X_{t-k}) = 0 \quad \forall\, k > q, \forall t$

Note: MA(q) process is q-correlated.

Result: If a stationary process $\{X_t\}$ is q-correlated, then $\exists$ a white noise $Z_t \sim WN(0, \sigma^2)$ such that $\{X_t\}$ is MA(q) using $\{Z_t\}$

Def^n (q-dependent process)

A strictly stationary time series is called q-dependent, if X_t and X_s are independent $\forall\, |t-s| > q$.

Note: Let $\{Z_t\}_{-\infty}^{\infty}$ iid $(0, \sigma^2)$

$Y_t = g(Z_t, Z_{t-1}, \dots, Z_{t-q})$ is also strictly stationary

as g is filter.

Y_t is q-dependent.

$\{Z_t\}_{-\infty}^{\infty} \overset{iid}{\sim} N(0,\sigma^2)$

a, b, c are constants.

Which of the following are stationary?
If yes, find ACVF
Is it strictly stationary?

(a) $X_t = a + bZ_{t-1} + cZ_{t-2}$

(b) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$

(c) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

→ (a) $E(X_t) = a$

$Var(X_t) = \sigma^2(b^2+c^2)$

$Cov(X_t, X_s) = Cov(a+bZ_{t-1}+cZ_{t-2},\ a+bZ_{s-1}+cZ_{s-2})$

$= \cancel{b^2 Var(Z_{t-1}) + c^2 Var(Z_{t-2})}$

$= \cancel{(b^2+c^2)\sigma^2}$

$$= \begin{cases} 0 & \text{if } |t-s| \geq 2 \\ bc\sigma^2 & \text{if } |t-s| = 1 \\ (b^2+c^2)\sigma^2 & \text{if } |t-s| = 0 \end{cases}$$

Strongly stationary since linear filter.

(b) $E(X_t) = 0$

$Var(X_t) = \sigma^2$

$Cov(X_t, X_{t+h}) = Cov(Z_1 \cos(t+h) + Z_2 \sin(t+h),\ Z_1 \cos t + Z_2 \cos t)$

$= \sigma^2 \cos ch$

$X_t \sim N(0,\sigma^2)$

strongly stationary

(c) $E(X_t) = 0$

$Var(X_t) = \sigma^2$

$$Cov(X_t, X_{t+h}) = Cov\left(Z_t\cos(ct) + Z_{t-1}\sin(ct), Z_{t+h}\cos(c(t+h)) + Z_{t+h-1}\sin(c(t+h))\right)$$

$$= \begin{cases} \sigma^2, & h=0 \\ \sigma^2(\sin(c(t+h))\cos(ct)), & h=1 \\ \sigma^2(\sin(ct)\cos(c(t+h))), & h=-1 \\ 0, & ow \end{cases}$$

not stationary.

d) $X_t = a + bZ_t$

e) $X_t = Z_0\cos(ct)$

$E(Z_t Z_{t-1}^2 Z_{t-2})$

f) $X_t = Z_t Z_{t-1}$

→ (f) $E(X_t) = 0$

$Var(X_t) = E(Z_t^2)E(Z_{t-1})^2 - E(Z_t)^2 E(Z_{t-1})^2$

$= \sigma^4$

$$Cov(X_t, X_{t+h}) = \begin{cases} \sigma^4 & \text{if } h=0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

$\therefore$ Stationary

Since filter, it is strongly stationary.

- $X_t \sim MA(2)$

$X_t = Z_t + \theta Z_{t-2}$, where $Z_t \sim WN(0,1)$, uncorrelated

(a) Find ACVF and ACF when $\theta = 0.8$

(b) Compute $Var\left(\frac{X_1 + \cdots + X_4}{4}\right)$ when $\theta = 0.8$

(c) Repeat (b) for $\theta = -0.8$

→ $\gamma(h) = Cov(X_t, X_{t+h}) = \begin{cases} (1+\theta^2) & \text{if } h=0 \\ \theta & \text{if } |h|=2 \\ 0 & \text{ow} \end{cases}$

$\rho(h) = \begin{cases} 1 & \text{if } h=0 \\ \frac{\theta}{1+\theta^2} & \text{if } |h|=2 \\ 0 & \text{ow} \end{cases}$

$Var\left(\frac{X_1 + \cdots + X_4}{4}\right)$

$= \frac{1}{16}\left[Var(X_1)\,4 + \cancel{Var(X_1)}\; 4\,Cov(X_1, X_3)\right]$

$= \frac{1}{4}\left[Var(X_1) + Cov(X_1, X_3)\right]$

$= \frac{1}{4}\left[1+\theta^2+\theta\right]$

- $X_1, \dots, X_n, X_{n+1}$ time series

We want to predict X_{n+1} by a func of $X_1, \dots, X_n$

i.e. $f(X_1, \dots, X_n)$ s.t. expected square error is minimised.

Find best such predictor.

$$E[(X_{n+1} - f(X_1, \dots, X_n))^2]$$

$$E(X-c)^2$$
$$= E(X - E(X) + E(X) - c)^2$$
$$= Var(X) + (E(X) - c)^2$$
$$+ 0 \quad 0$$

$$= \cancel{E[(X_{n+1} - E(X_{n+1}) f(X_1, \dots, X_n)}$$

$$= E\left[E\left[(X_{n+1} - f(X_1, \dots, X_n))^2 \mid X_1, \dots, X_n\right]\right]$$

$$= E\left[Var(X_{n+1} \mid X_1, \dots, X_n) + \left(E[X_{n+1} \mid X_1, \dots, X_n] - f(X_1, \dots, X_n)\right)^2\right]$$

minimized iff

$$E[X_{n+1} \mid X_1, \dots, X_n] = f(X_1, \dots, X_n)$$

<u>Note</u>: It is difficult to get conditional expectation even in computer. So we use BLUE, which depends only on mean and var-covariance ~~fun~~ func.

We can estimate mean and var-covariance func.

<u>Q</u> Find the set of values of ρ for which $\rho(h)$ is ACF for some stationary time series.

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0 \\ \rho & \text{if } |h| = 1 \\ 0 & \text{o/w.} \end{cases}$$

$$\rightarrow \quad \Sigma_n = \begin{bmatrix} 1 & \rho & 0 & 0 & \cdots & 0 \\ \rho & 1 & \rho & 0 & \cdots & 0 \\ 0 & \rho & 1 & \rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & & & & \cdots & 1 \end{bmatrix}$$

$$a_n \Sigma_n a_n' \geq 0$$

$$b_n \Sigma_n b_n' \geq 0$$

$$a_n = \underset{\sim}{1}$$

$$b_n = (1, -1, 1, -1, \dots)$$

$$\Rightarrow n + 2(n-1)\rho \geq 0 \quad \Big| \quad n - 2(n-1)\rho \geq 0$$

$$\Rightarrow \rho \geq -\frac{n}{2(n-1)} \quad \Big| \quad \Rightarrow \rho \leq \frac{n}{2(n-1)}$$

$$\therefore |\rho| \leq \frac{n}{2(n-1)} \quad \forall n$$

$$\Rightarrow |\rho| \leq \frac{1}{2}$$

If $|\rho| \leq 1/2$ to show $\exists\ X_t$ s.t. $\rho(h)$ is ACF of X_t.

$\rightarrow$ X_t has to be MA(1) (if it exists)

$$X_t = \theta Z_{t-1} + Z_t$$

$$\rho(1) = \frac{\theta}{1+\theta^2} = \rho \cancel{\Rightarrow \frac{1}{\rho} = \theta + \frac{1}{\theta} \Rightarrow \frac{1}{\rho}}$$

$$\Rightarrow \rho + \rho\theta^2 - \theta = 0$$

Sol^n for θ in $\mathbb{R}$ exists iff $\quad 1 - 4\rho^2 \geq 0 \Rightarrow |\rho| \leq \frac{1}{2}$

$\therefore \exists$ a time series with $\rho(h)$ as ACF.

- $\gamma : \mathbb{Z} \to \mathbb{R}$ is given s.t.
  - (i) $\gamma(0) \geq 0$
  - (ii) $\gamma(h) = \gamma(-h)$
  - (iii) $\gamma(h) \leq \gamma(0) \ \forall h$
  - (iv) $\forall a_1, \ldots, a_n \in \mathbb{R}, \ \sum_{i,j}^{n} a_i a_j \gamma(i-j) \geq 0$

Then $\exists$ a ~~st~~ stationary time series X_t with ACVF, $\gamma(\cdot)$.

→ Define $D_n = ((\gamma(i-j)))_{i,j=1}^{n}$

$$\underset{\sim}{X}_n \sim N_n(\underset{\sim}{0}, D_n)$$

$$D_{n+1} = ((\gamma(i-j)))_{i,j=1}^{n+1}$$

$\exists \ \underset{\sim}{Y}_{n+1} \sim N_{n+1}(\underset{\sim}{0}, D_{n+1})$ s.t.

$$\underset{\sim}{Y}_{n+1} = \begin{pmatrix} \underset{\sim}{Y}_n \\ V \end{pmatrix} \quad \text{and } \underset{\sim}{Y}_n \overset{d}{=} \underset{\sim}{X}_n$$

$\therefore \exists \ U \overset{d}{=} V$ s.t. $\begin{pmatrix} \underset{\sim}{X}_n \\ U \end{pmatrix} \sim N_{n+1}(\underset{\sim}{0}, D_{n+1})$

$$\begin{pmatrix} \underset{\sim}{X}_n \\ U \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{n+1} \end{pmatrix}$$

$\therefore Z_1, Z_2, \ldots, Z_n, Z_{n+1}, \ldots$ time series obtained inductive.