Replacing $\underline{\beta} = \hat{\underline{\beta}}_H$

$$\|\underline{y} - X\hat{\beta}_H\|^2 = \|\underline{y} - X\hat{\underline{\beta}}\|^2 + \|X(\hat{\underline{\beta}} - \hat{\underline{\beta}}_H)\|^2$$

$$\hat{\underline{Y}}_H = X\hat{\underline{\beta}}_H \quad , \quad \hat{\underline{Y}} = X\hat{\underline{\beta}}.$$

<u>Centered covariate:</u>

So far the model was $\underline{Y} = X\underline{\beta} + \underline{\epsilon}$.

Suppose we center the x data and reparameterized model

$$Y_i = \alpha_0 + \beta_1(x_{i1} - \bar{x}_1) + \cdots + \beta_{p-1}(x_{i,p-1} - \bar{x}_{p-1}) + \epsilon_i$$

$$i = 1, \ldots, n$$

Where, $\alpha_0 = \beta_0 + \beta_1\bar{x}_1 + \cdots + \beta_{p-1}\bar{x}_{p-1}$

and, $\bar{x}_j = \sum_{i=1}^{n} x_{ij} / n$

Model now, $\underline{Y} = X_c\underline{\alpha} + \underline{\epsilon}$.

$$\alpha = (\alpha_0, \beta_1, \ldots, \beta_{p-1})' = (\alpha_0 ; \beta_c')$$

$$X_c = (1_n, \tilde{X}) \quad , \quad \tilde{X}_{n \times \overline{p-1}} = ((\tilde{x}_{ij})) \, ,$$

$$\tilde{x}_{ij} = x_{ij} - \bar{x}_j$$

The transformation between $\underline{\alpha}$ and $\underline{\beta}$ is 1-1, the least sq estimate of β_c remains the same.

$$\hat{\alpha} = (X_c'X_c)^{-1} X_c'\underline{Y}$$

$$= \begin{pmatrix} n & \underline{0}' \\ \underline{0} & \tilde{X}'\tilde{X} \end{pmatrix}^{-1} \begin{pmatrix} 1_n'\underline{Y} \\ \tilde{X}'\underline{Y} \end{pmatrix}$$

$\cancel{X_c'X_c} = \cancel{(1_n, \tilde{X})' (1_n, \tilde{X})}$

$[\underline{1_n}'\tilde{X} = \underline{0}']$

$$= \begin{pmatrix} n^{-1} & 0' \\ 0 & (\tilde{X}'\tilde{X})^{-1} \end{pmatrix} \begin{pmatrix} 1_n' Y \\ \tilde{X}' Y \end{pmatrix}$$

$$= \begin{pmatrix} \bar{Y} \\ (\tilde{X}'\tilde{X})^{-1} \tilde{X}' Y \end{pmatrix}$$

$$\therefore \begin{cases} \hat{\alpha}_0 = \bar{Y} \\ \hat{\beta}_c = (\tilde{X}'\tilde{X})^{-1} \tilde{X}' Y \end{cases}$$

$$\mathcal{C}(X_c) = \mathcal{C}(X)$$

Here, X_c & X have the same projection matrices.

Projection matrix

$$P = X_c (X_c' X_c)^{-1} X_c'$$

$$= (1_n, \tilde{X}) \begin{pmatrix} n & 0' \\ 0 & \tilde{X}'\tilde{X} \end{pmatrix}^{-1} (1_n, \tilde{X})'$$

$$= \frac{1}{n} 1_n 1_n' + \tilde{X} (\tilde{X}'\tilde{X})^{-1} \tilde{X}'$$

Let, $\underline{x}_i$: ith reduced row [omitting the first element from the ith row of X].

$S_{xx} \longrightarrow$ sample covariance matrix

$$S_{xx} = \frac{1}{(n-1)} \tilde{X}'\tilde{X} \quad , \quad \bar{\underline{x}} = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i$$

Then the ith diagonal element of P

$$P_{ii} = n^{-1} + (\underline{x}_i - \bar{\underline{x}})' S_{xx}^{-1} (\underline{x}_i - \bar{\underline{x}}) \frac{1}{(n-1)}$$

$MD_i = (x_i - \bar{x})' S_{xx}^{-1} (x_i - \bar{x})$ is the Mahalanobis distance between the ith reduced row of X and the average reduced row.

$$p_{ii} = \frac{1}{n} + \frac{1}{n-1} MD_i$$

→ a measure of how far away x_i is from the center of the x-data.

$$RSS = \underline{Y}' (I - P) \underline{Y}$$

$$= Y' \left(I_n - \frac{1}{n} 1_n 1_n' - \tilde{X} (\tilde{X}'\tilde{X})^{-1} \tilde{X}' \right) Y$$

$$= Y' \left(I_n - \frac{1}{n} 1_n 1_n' \right) Y - Y' \tilde{P} Y$$

$$= \sum_i (Y_i - \bar{Y})^2 - \underline{Y}' \tilde{P} \underline{Y} \quad , \quad \tilde{P} = \tilde{X} (\tilde{X}'\tilde{X})^{-1} \tilde{X}'$$

Q1 For a general regression model with X full rank, show that $\sum_{i=1}^{n} \hat{Y}_i(Y_i-\hat{Y}_i)=0$

$$\hat{\underline{Y}}(\underline{Y}-\hat{\underline{Y}}) = \hat{\underline{Y}}(\underline{Y}-P\underline{Y})$$

$$= \hat{\underline{Y}}(I-P)\underline{Y}$$

$$= P$$

$$\boxed{\hat{\underline{Y}}=P\underline{Y}}$$

$$\hat{\underline{Y}}'(\underline{Y}-P\underline{Y})$$

$$=(P\underline{Y})'(\underline{Y}-P\underline{Y})$$

$$=\underline{Y}'P'(\underline{Y}-P\underline{Y})$$

$$=\underline{Y}'P'\underline{Y}-\underline{Y}'P\underline{Y}$$

$$=0.$$

$$e_i = Y_i-\hat{Y}_i$$

$$\sum_{i=1}^{n} \hat{Y}_i e_i = 0$$

Q2: Suppose that explanatory variable are selected as $x_{ij}=k_j w_{ij}$ for all i,j

P.T. $\hat{\underline{Y}}$ remains unchanged under the change of scale.

Q3: Consider the simple regression model with two exp. variables x_1 & x_2 ~~linear parameters β_0 & β_1~~. Find the condition $\ni$ LSE of β_0 and β_1 are uncorrelated when n obs, write down $Var(\hat{\beta}_1)$ in terms of the sample correlation coeff of (x_i, y_i) $i=1,2,\ldots,n$)

Q.1

$$\sum_{i=1}^{n} \hat{y}_i (y_i - \hat{y}_i)$$

$$= \hat{\underline{y}} (\underline{y} - P\underline{y}) \qquad \left[\hat{y} = X\hat{\beta} = X(X'X)^{-1}X'y = Py\right]$$

$$= (P\underline{y})' (\underline{y} - P\underline{y})$$

$$= \underline{y}' P' (\underline{y} - P\underline{y})$$

$$= \underline{y}' P' \underline{y} - \underline{y}' P \underline{y} = 0 \qquad [P \text{ is idempotent and sym}]$$

Q.2

$$x_{ij} = k_j w_{ij} \quad \forall\, i, j$$

$$\hat{y}_{int} = X\hat{\beta} = X(X'X)^{-1}X'\underline{y}$$

$$\hat{y}_{new} = (XK)\,\hat{\beta}_{new} = (XK)\left[(XK)'(XK)\right]^{-1}(XK)'\underline{y}$$

$$= (XK)(XK)^{-1}(K'X')^{-1}(XK)'\underline{y}$$

$$= XK\,K^{-1}X^{-1}(X')^{-1}(K')^{-1}K'X'\underline{y}$$

$$= X(X'X)^{-1}X'y$$

$$= X\hat{\beta}$$

Where, $XK = (x_{ij}) = \underbrace{\begin{bmatrix} w_{10} & w_{11} & \cdots & w_{1p-1} \\ w_{20} & w_{21} & \cdots & w_{2p-1} \\ \vdots & & & \\ w_{n0} & w_{n1} & \cdots & w_{np-1} \end{bmatrix}_{n\times p}}_{X} \underbrace{\begin{bmatrix} k_0 & & & 0 \\ & k_1 & & \\ & & \ddots & \\ 0 & & & k_{p-1} \end{bmatrix}_{p\times p}}_{K}$

Q.3. Consider the regression model with two explanatory variable x_1 & x_2 with no intercept term. Find $Var(\hat{\beta}_1)$ in term of the sample correlation coefficient of x_1 & x_2 based on obsn (x_{1i}, x_{2i}), $i = 1, 2, \dots n$.

Scaling exp variable.

Suppose we scale the column of $\tilde{X}$ so that they have unit length.

Let, $s_j^2 = \sum_{i=1}^{n} \tilde{x}_{ij}^2$, $\tilde{x}_{ij} = x_{ij} - \bar{x}_j$,

$\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}$,

Define, $x_{ij}^* = \frac{\tilde{x}_{ij}}{s_j}$

They would become now.

$y_i = \alpha_0 + \gamma_1 x_{i1}^* + \gamma_2 x_{i2}^* + \cdots + \gamma_{p-1} x_{i,p-1}^* + \epsilon_i$

$i = 1, 2, \ldots, n.$

$\gamma_j = \beta_j s_j$

The trasformation is still 1-1. So

$\hat{\gamma}_j = \hat{\beta}_j s_j$, $\hat{\alpha}_0 = \bar{Y}$

If $X^* = ((x_{ij}^*))$, $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{p-1})'$

$$\underline{\hat{\gamma}} = (X^{*\prime} X^*)^{-1} X^{*\prime} Y$$

$$= R_{xx}^{-1} X^{*\prime} \underline{Y}$$

Where R_{xx} is the correlation matrix of x-variable.

$$\underset{(p-1)\times(p-1)}{R_{xx}} = \begin{bmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1,p-1} \\ r_{21} & 1 & & & r_{2,p-1} \\ \vdots & \vdots & & & \vdots \\ r_{p-1,1} & r_{p-1,2} & \cdots & & 1 \end{bmatrix}$$

$r_{jk} = \sum_i (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k) \Big/ (s_j s_k)$: sample correlation of jth and kth explanatory variables.

$$X^* = \left(x^{*(1)}, \ldots, x^{*(p-1)}\right)$$

Then, $r_{jk} = x^{*(j)'} x^{*(k)}$

<u>Example:</u>

$\underline{p=3}$ (two variable model)

$$R_{xx} = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}, \quad r = \underline{x}^{*(1)'} \underline{x}^{*(1)}$$

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^{-1} \begin{pmatrix} x^{*(1)'} \\ x^{*(2)'} \end{pmatrix} \underline{y}$$

$$= \frac{1}{1-r^2} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \begin{pmatrix} x^{*(1)'} \underline{y} \\ x^{*(2)'} \underline{y} \end{pmatrix}$$

$$\hat{\gamma}_1 = \frac{1}{1-r^2} \left(\underline{x}^{*(1)'} \underline{Y} - r \underline{x}^{*(2)'} \underline{y} \right)$$

$$\hat{\beta}_1 = \frac{\hat{\gamma}_1}{s_1}$$

$$\hat{\gamma}_2 =$$

$$\hat{\beta}_2 =$$

$$\tilde{X} = X^* \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} = X^* S_2, \quad S_2 = \begin{pmatrix} s_1 & 0 \\ 0 & s_1 \end{pmatrix}$$

$$P = \frac{1}{n} \underline{1}_n \underline{1}_n' + \tilde{X} (\tilde{X}'\tilde{X})^{-1} \tilde{X}'$$

$$= \frac{1}{n} \underline{1}_n \underline{1}_n' + X^* S_2 \left(S_2' X^{*'} X^* S_2 \right)^{-1} S_2 X^{*'}$$

$$= \frac{1}{n} \underline{1}_n \underline{1}_n' + X^* \left(X^{*'} X^* \right)^{-1} X^{*'}$$

$$= \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}' + \frac{1}{(1-r^2)} \left(\underset{\sim}{x}^{*(1)} \; \underset{\sim}{x}^{*(2)} \right) \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \left(x^{*(1)}, x^{*(2)} \right)'$$

Maximum Likelihood Estimation

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad , i = 1, 2, \dots, n \qquad , \epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$

y_i's are indp $N(\beta_0 + \beta_1 x_i, \sigma^2)$, $i = 1, 2, \dots, n$.

The likelihood,

$$L\left(\underset{i=1,2,\dots n}{x_i, y_i}, \beta_0, \beta_1, \sigma^2\right)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left[-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right]$$

$$\ln L\left(\beta_0, \beta_1, \sigma^2, \underset{\sim}{x}, \underset{\sim}{y}\right)$$

$$= -\frac{n}{2}\log 2\pi - \frac{n}{2}\ln \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$

Normal eq^{n}s:

$$\left.\begin{aligned}
\frac{\partial \ln L}{\partial \beta_0} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) = 0 \\
\frac{\partial \ln L}{\partial \beta_1} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) x_i = 0 \\
\frac{\partial \ln L}{\partial \sigma^2} &= -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 = 0
\end{aligned}\right\} (*)$$

Sol^n of (*) MLE: $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\sigma}^2$

$$\tilde{\beta}_0 = \bar{y} - \tilde{\beta}_1 \bar{x} \quad , \tilde{\beta}_1 = \frac{s_{xy}}{s_{xx}} ,$$

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{\beta}_0 - \tilde{\beta}_1 x_i)^2$$

$$\hat{\sigma}^2 = \frac{RSS}{n-2},$$

$$\tilde{\sigma}^2 = \frac{RSS}{n}$$

$$\Rightarrow \quad \tilde{\sigma}^2 = \frac{n-2}{n}\hat{\sigma}^2$$

$\tilde{B}_0, \tilde{B}_1 \rightarrow$ unbiased whereas $\tilde{\sigma}^2$ in biased but

Hessian matrix $\rightarrow$ matrix of second order partial derivate

[but $\tilde{\sigma}^2$ is asymptotically unbiased]

In matrix notation ~~of~~ considering multiple regression set up

$$\underset{\sim}{y} \sim N_n(X\underline{\beta}, \sigma^2 I_n), \; X_{n\times p} \text{ ranp } p$$

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \left(\frac{1}{\sigma^2 2\pi}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}(Y - X\underline{\beta})'(\underline{Y} - X\underline{\beta})\right]$$

$$= L(\underline{\beta}, \sigma^2)$$

but $l(\underline{\beta}, \sigma^2) = \ln L(\underline{\beta}, \sigma^2)$, $v = \sigma^2$

$$= l(\underset{\sim}{\beta}, v)$$

$$= -\frac{n}{2}\log v - \frac{1}{2v}\|\underline{Y} - X\underline{\beta}\|^2$$

$$\frac{\partial l}{\partial \underline{\beta}} = -\frac{1}{2v}(-2X'\underline{Y} + 2X'X\underline{\beta}) = 0$$

$$\frac{\partial l}{\partial v} = -\frac{n}{2v} + \frac{1}{2v^2}(\underline{Y} - X\underline{\beta})'(\underline{Y} - X\underline{\beta})$$

For every $v > 0$, the LSE of $\underline{\beta}$ ~~xxx~~ maximum $l(\beta, v)$

$\tilde{\beta} = (X'X)^{-1} X' \underline{Y} = \hat{\underline{\beta}}$.

$L(\underline{\beta}, v) \leq L(\hat{\underline{\beta}}, v) \quad \forall\, v > 0$

Maximization of $L(\hat{\beta}, v)$ or

w.r.t v

$$\frac{\partial l}{\partial v} = 0 \Rightarrow \hat{v} = \frac{1}{n} \| \underline{Y} - X \hat{\underline{\beta}} \|^2$$

$$l(\hat{\beta}, \hat{v}) - l(\hat{\beta}, v) = -\frac{n}{2} \ln \hat{v} - \frac{1}{2\hat{v}} \| \underline{Y} - X \hat{\underline{\beta}} \|^2$$

$$+ \frac{n}{2} \ln v + \frac{1}{2v} \| \underline{Y} - X \hat{\underline{\beta}} \|^2$$

$$= -\frac{n}{2} \log \frac{\hat{v}}{v} - \frac{1}{2\hat{v}} n \hat{v}$$

$$= -\frac{n}{2} \left[\log \frac{\hat{v}}{v} + 1 - \frac{\hat{v}}{v} \right] \geq 0$$

$x \leq e^{x-1}$ for $x \geq 0 \Rightarrow \log x \leq x - 1$ for $x \geq 0$

$$l(\hat{\underline{\beta}}, \hat{v}) \geq l(\hat{\underline{\beta}}, v) \quad \forall\, v > 0$$

with equility iff $\underline{\beta} = \hat{\underline{\beta}}$ & $v = \hat{v}$.

So, $\hat{\underline{\beta}}$ & $\hat{v} = \hat{\sigma}^2$ are the MLEs of β and σ^2.

$$L(\hat{\beta}, \hat{v}) = (2\pi \hat{\sigma}^2)^{-n/2} e^{-n/2}$$

Fisher information matrix

$$I = -E\left[\frac{\partial^2 l}{\partial \underline{\theta}\, \partial \underline{\theta}'}\right]$$

Where, $\underline{\theta} = (\beta', v)'$

$$\frac{\partial^2 l}{\partial \underline{\beta}\, \partial \underline{\beta}'} = -\frac{1}{v}(X'X) \quad ; \quad \frac{\partial^2 l}{\partial \underline{\beta}\, \partial v} = \frac{1}{2v^2}\left(-2X'\underline{y} + 2X'X\underline{\beta}\right)$$

$$\frac{\partial^2 l}{\partial v^2} = \frac{n}{2v^2} - \frac{1}{v^3}\|Y - X\beta\|^2$$

Note: $$\frac{\|\underline{y} - X\underline{\beta}\|^2}{v} = \frac{\underline{\epsilon}'\underline{\epsilon}}{\sigma^2} \sim \chi^2_n$$

$$E(\underline{\epsilon}'\underline{\epsilon}) = n\sigma^2 = nv$$

Replacing $\underline{y}$ by $\underline{Y}$ and taking expectations.

$$I = -\begin{pmatrix} -\frac{1}{v}(X'X) & 0 \\ 0 & -\frac{n}{2v^2} \end{pmatrix}$$

$$I^{-1} = \begin{pmatrix} v(X'X)^{-1} & 0 \\ 0 & 2v^2/n \end{pmatrix} = \begin{pmatrix} \sigma^2 (X'X)^{-1} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

$$\frac{(n-p)s^2}{\sigma^2} \sim \chi^2_{n-p}$$

So, $$\text{var}\left[\frac{(n-p)s^2}{v}\right] = 2(n-p)$$

$$\text{Var}(s^2) = \frac{2v^2}{(n-p)} = \frac{2\sigma^4}{n-p} \approx \frac{2\sigma^4}{n}$$

$\frac{2\sigma^4}{n(1-p/n)}$, the p/n is ~~to~~ close to the Large $\underline{n}$.

Hypothesis Testing:

Model: $G: \; y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{p-1} x_{ip-1} + \epsilon_i$

or, $\underline{y} = X\underline{\beta} + \underline{\epsilon}$

Where p is large, we are sometime interested including whether the some of the β_i's can be taken to be zero.

To test $\beta_s = \beta_{s+1} = \cdots = \beta_{p-1} = 0$

So, the model under hypothesis

$$H: \quad y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{s-1} x_{is-1} + \epsilon_i$$

$$= X_s \underline{\beta}^s + \epsilon$$

X_s -- first s column of X.

The hypothesis can be explained by

$A\underline{\beta} = \underline{c}$ where

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & & 1 \end{bmatrix}_{p-s \times p}$$

$$\underset{(p-s)\times 1}{\underline{c}} = (0, \cdots, 0)'$$

General rank : $A\underline{\beta} = \underline{c}$, $A_{q \times p}$ matrix of rank q.

F test: $A\underline{\beta} = \underline{c}$

$A\hat{\underline{\beta}} - \underline{c}$

$(A\hat{\underline{\beta}} - \underline{c})' \left[Var(A\hat{\underline{\beta}}) \right]^{-1} (A\hat{\underline{\beta}} - \underline{c})$, $\cancel{Var(A\hat{\beta})}\ \sigma^2$

$$\mathrm{Var}(A\hat{\beta}) = \sigma^2 A (X'X)^{-1} A'$$

Estimate σ^2 by $s^2 = \dfrac{RSS}{n-p}$

$$(A\hat{\underline{\beta}} - \underline{c})' \left[A(X'X)^{-1}A'\right]^{-1} (A\hat{\underline{\beta}} - \underline{c}) \Big/ s^2$$

<u>F-test :</u> to test $A\beta = c$

$$A\hat{\underline{\beta}} - \underline{c}$$

$$(A\hat{\underline{\beta}} - \underline{c})' \left[A(X'X)^{-1}A'\right]^{-1} (A\hat{\underline{\beta}} - \underline{c}) \Big/ \sigma^2$$

Estimate σ^2 by $s^2 = \dfrac{RSS}{n-p}$.

$$RSS = \| \underline{Y} - X\hat{\underline{\beta}} \|^2 - \| \underline{Y} - \hat{\underline{Y}} \|^2 = (n-p)s^2$$

$$RSS_H = \| \underline{Y} - X\hat{\underline{\beta}}_H \|^2 = \| \underline{Y} - \hat{Y}_H \|^2$$ (Min of $\underline{\epsilon}'\underline{\epsilon}$ sub to the $A\underline{\beta} = \underline{c}$)

$$\hat{\beta}_H = \hat{\beta} - (X'X)^{-1}A'\left[A(X'X)^{-1}A\right]^{-1}(A\hat{\beta} - c)$$

<u>Theorem:</u>

(i) $$RSS_H - RSS = \| \hat{Y} - \hat{Y}_H \|^2 = (A\hat{\beta} - \underline{c})' \left[A(X'X)^{-1}A'\right]^{-1} (A\hat{\beta} - \underline{c})$$

(ii) $$E(RSS_H - RSS) = \sigma^2 q + (A\beta - \underline{c})' \left[A(X'X)^{-1}A'\right]^{-1} (A\underline{\beta} - \underline{c})$$

(iii) When H is true

$$F = \frac{(RSS_H - RSS)/q}{(RSS)/(n-p)} = \frac{(A\underline{\beta} - \underline{c})' \left[A(X'X)^{-1}A'\right]^{-1} (A\hat{\underline{\beta}} - \underline{c})}{qs}$$

$$\sim F_{q, n-p}$$

(iv) when $\underline{c} = 0$, F can be expressed

$$F = \frac{n-p}{q} \cdot \frac{\underline{y}'(P - P_H)\underline{y}}{\underline{y}'(I_n - P)\underline{y}}$$

Where (i) P_H is symmetric & ~~identical~~ idempotent and

$$P_H P = P P_H = P_H$$

(iii) & (iv), normality assumption is required.

__Proof:__

(i) We already have seen

$$\|\underline{y} - X\hat{\underline{\beta}}_H\|^2 = \|\underline{y} - X\hat{\underline{\beta}}\|^2 + \|X(\hat{\underline{\beta}} - \hat{\underline{\beta}}_H)\|^2$$

$$\Rightarrow \|\underline{y} - X\hat{\underline{\beta}}_H\|^2 = \|\underline{y} - \hat{\underline{y}}\|^2 + \|\hat{\underline{y}} - \hat{\underline{y}}_H\|^2$$

$$\Rightarrow RSS_H = RSS + \|\hat{\underline{y}} - \hat{\underline{y}}_H\|^2$$

$$\Rightarrow RSS_H - RSS = \|\hat{\underline{y}} - \hat{\underline{y}}_H\|^2 = \|X(\hat{\underline{\beta}} - \hat{\underline{\beta}}_H)\|^2$$

$$\Rightarrow (\hat{\underline{\beta}} - \hat{\underline{\beta}}_H)' X'X (\hat{\underline{\beta}} - \hat{\underline{\beta}}_H)$$

Replacing $\hat{\underline{\beta}}_H = \hat{\underline{\beta}} + (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(\underline{c} - A\hat{\underline{\beta}})$

$$RSS_H - RSS = (A\hat{\underline{\beta}} - \underline{c})'[A(X'X)^{-1}A']^{-1}A(X'X)^{-1}X'X(X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\underline{\beta}} - \underline{c})$$

$$= (A\hat{\underline{\beta}} - \underline{c})'[A(X'X)^{-1}A']^{-1}(A\hat{\underline{\beta}} - \underline{c})$$

~~(ii)~~

(ii) $A_{q\times p}$ matrix of rank q

$$V(A\hat{\beta}) = \sigma^2 A (X'X)^{-1} A'$$

$\underset{\sim}{y} \sim N_n(X\underset{\sim}{\beta}, \sigma^2(X'X))$

$\Rightarrow \hat{\beta} \sim N_p(\underset{\sim}{\beta}, \sigma^2(X'X)^{-1})$

So, $A\hat{\beta} \sim N_q(A\beta, \sigma^2 A(X'X)^{-1}A')$

$E(RSS_H - RSS)$

$= E\left((A\hat{\beta} - \underset{\sim}{c})' [A(X'X)^{-1}A']^{-1} (A\hat{\underset{\sim}{\beta}} - \underset{\sim}{c})\right)$

$= E(\underset{\sim}{z}' W^{-1} \underset{\sim}{z})$

$= tr(W^{-1} \sigma^2 W) + (A\underset{\sim}{\beta} - \underset{\sim}{c})' W^{-1} (A\underset{\sim}{\beta} - \underset{\sim}{c})$

$E(\underset{\sim}{z}) = A\underset{\sim}{\beta} - \underset{\sim}{c}$

$V(\underset{\sim}{z}) = \sigma^2 W$

$E(X'AX) = tr(A\Sigma) + \mu' A \mu$

$= \sigma^2 q + (A\underset{\sim}{\beta} - \underset{\sim}{c})' [A(X'X)^{-1}A']^{-1} (A\underset{\sim}{\beta} - \underset{\sim}{c})$

(iii) $RSS_H - RSS = (A\hat{\underset{\sim}{\beta}} - \underset{\sim}{c})' [A(X'X)^{-1}A']^{-1} (A\hat{\underset{\sim}{\beta}} - \underset{\sim}{c})$

- is a continuous f^n of $\hat{\beta}$.

$RSS = (n-p)s^2$ is a f^n of s^2.

using $\hat{\beta}$ & s^2 are independent.

$\Rightarrow$ $RSS_H - RSS$ is indpt of RSS.

where H is true.

so, $A\underset{\sim}{\beta} = \underset{\sim}{c}$

$A\hat{\underset{\sim}{\beta}} \sim N_q(\underset{\sim}{c}, \sigma^2 A(X'X)^{-1}A')$

So, $$\frac{RSS_H - RSS}{\sigma^2} = (A\hat{\beta} - \underset{\sim}{c})' [Var(A\hat{\beta})]^{-1} (A\hat{\underset{\sim}{\beta}} - \underset{\sim}{c})$$

$$\frac{RSS}{\sigma^2} \sim \chi^2_{n-p}$$

So, when H is true

$$F = \frac{[RSS_H - RSS]/\sigma^2 q}{RSS/\sigma^2(n-p)} \text{ is of the form } \frac{\chi^2_q/q}{\chi^2_{n-p}/(n-p)}$$

Therefore, $F \sim F_{q,n-p}$ where H is true.

(iv) $\underline{c} = \underline{0}$,

So, $\hat{\beta}_H = \hat{\beta} - (X'X)^{-1}A'\left[A.(X'X)^{-1}A'\right]^{-1} A\hat{\beta}$

$$\hat{Y}_H = X\hat{\beta}_H = X\hat{\beta} - X(X'X)^{-1}A'\left(A(X'X)^{-1}A'\right)^{-1}A\hat{\beta}$$

$$= X(X'X)^{-1}X'\underline{Y} - X(X'X)^{-1}A'\left[A'(X'X)^{-1}A'\right]^{-1}A(X'X)^{-1}X'\underline{Y}$$

$$= \left\{X(X'X)^{-1}X' - X(X'X)^{-1}A'\left[A(X'X)^{-1}A'\right]^{-1}A(X'X)^{-1}X'\right\}Y$$

$$= \{P - P_1\}Y$$

$$= P_H \underline{Y}$$

$P_1' = P_1$, So, P_1 & P_H are symmetric.

$$P_1^2 = X(X'X)^{-1}\underbrace{A'\left[A(X'X)^{-1}A'\right]^{-1}A(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I_p}A'\left[A(X'X)^{-1}A'\right]^{-1}}_{I_q}A(X'X)^{-1}X'$$

$$= X(X'X)^{-1}A'\left[A(X'X)^{-1}A'\right]^{-1}A(X'X)^{-1}X'$$

$P_1P = P_1$, $\quad$ ||ly $PP_1 = P_1$

$$P_H^2 = (P-P_1)^2 = P^2 + P_1^2 - 2PP_1$$
$$= P - P_1 - P_1 + P_1 = P_H$$

$P_H P = (P - P_1) P = P - P_1 = P_H$

$P P_H = P_H$

$RSS_H = \| \underline{Y} - X\hat{\underline{\beta}}_H \|^2 = (\underline{Y} - \beta_H \underline{Y})' (\underline{Y} - \beta_H \underline{Y})$

$= Y'(I - P_H)^2 \underline{Y}$

$= \underline{Y}'(I - P_H)\underline{Y}$

So, $RSS_H - RSS = Y'(I - P_H)Y - \underline{Y}'(I - P)\underline{Y}$

$= \underline{Y}'(P - P_H)\underline{Y}$

$$F = \frac{(RSS_H - RSS)/q}{RSS/(n-p)} = \frac{n-p}{q} \cdot \frac{\underline{Y}'(P - P_H)Y}{Y'(I-P)\underline{Y}}$$

Write $S_H^2 = (RSS_H - RSS)/q$

$E(S_H^2) = \sigma^2 + \frac{1}{q}(A\beta - \underline{c})' [A(X'X)^{-1}A']^{-1}(A\underline{\beta} - \underline{c})$

$= \sigma^2 + \delta$ $\quad [\delta \geq 0$, since, $A(X'X)^{-1}A'$ is p.d.$]$.

$E(s^2) = \sigma^2$, $\quad \left(s^2 = \frac{RSS}{(n-p)}\right.$ & $RSS = (n-p)s^2 \left.\right)$

When H is true, $(A\underline{\beta} = \underline{c})$, $\delta = 0$ and S_H^2 & s^2 are both unbiased estimator for σ^2 and, $F = \frac{S_H^2}{s^2} \approx 1$

When H is false, $E(S_H^2) > E(s^2)$

$E(F) = E\left(\frac{S_H^2}{s^2}\right) = E(S_H^2) \cdot E\left(\frac{1}{s^2}\right) > \frac{E(S_H^2)}{E(s^2)} > 1$ (H is f[illegible]

H is rejected if F is sufficiently large.

[Let, X be non-ve non-degenerate r.v. If the expectation exists, then $E(X^{-1}) > [E(X)]^{-1}$]

[Jensen's ineq.]

$\frac{1}{x}$ is convex in $(0, \infty)$

<u>Likelihood Ratio Test</u>: (LR Test)

<u>Model:</u> $G: \underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$, X is $n \times p$ of rank p.

$\underset{\sim}{\epsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 I_n)$

To test $H: A\underset{\sim}{\beta} = \underset{\sim}{c}$, $A_{q\times p}$ of rank q.

likelihood f^n

$$L(\beta, \sigma^2) = \left(\frac{1}{\sigma^2 2\pi}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2} \|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|^2\right]$$

The MLE of $\underset{\sim}{\beta}$ & σ^2 are

$$\hat{\underset{\sim}{\beta}} = (X'X)^{-1} X'Y, \quad \hat{\sigma}^2 = \frac{1}{n} \|\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}\|^2$$

The max^m value of the likelihood f^n

$$L(\hat{\underset{\sim}{\beta}}, \hat{\sigma}^2) = (2\pi\hat{\sigma}^2)^{-n/2} e^{-n/2}$$

<u>Now,</u> to find the MLEs when H is true.

- needs Lagranse's multiplier approach

$$g = \log L(\beta, \sigma^2) + (\underset{\sim}{\beta}'A' - c')\underset{\sim}{\lambda}$$

$$= \text{Const} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\|\underset{\sim}{Y} - X\underset{\sim}{\beta}\|^2 + (\underset{\sim}{\beta}'A' - \underset{\sim}{c})\underset{\sim}{\lambda}$$

Using similar calculation as LSEs

$$\hat{\beta}_H = \hat{\beta} + (X'X)^{-1} A' \left[A (X'X)^{-1} A' \right]^{-1} (\underline{c} - A\underline{\hat{\beta}})$$

$$\& \ \hat{\sigma}_H^2 = \frac{1}{n} \| Y - X\hat{\beta}_H \|$$

So, $$L(\hat{\beta}_H, \hat{\sigma}_H)^2 = (2\pi \hat{\sigma}_H^2)^{-n/2} e^{-n/2}$$

The LR test statistic

$$\Lambda = \frac{L(\hat{\beta}, \hat{\sigma}_H)}{L(\hat{\beta}, \hat{\sigma}^2)}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_H^2} \right)^{n/2}$$

Reject H if Λ is too small.

$$F = \left(\frac{n-p}{q} \right) \left(\Lambda^{-2/n} - 1 \right) \sim F_{q, n-p}$$

F & LR test statistic:

Consider the MLEs of σ^2 incase of restricted and uncorrelated case.

$$\hat{\sigma}_H = \frac{RSS_H}{n} \quad , \quad \hat{\sigma}^2 = \frac{RSS}{n}$$

So,
$$F = \frac{(RSS_H - RSS)/q}{RSS/(n-p)}$$

$$= \frac{n-p}{q} \cdot \frac{\hat{\sigma}_H^2 - \hat{\sigma}^2}{\hat{\sigma}^2}$$

$$= \frac{n-p}{q}\left(\frac{\hat{\sigma}_H^2}{\hat{\sigma}^2} - 1\right)$$

$$= \frac{n-p}{q}\left(\Lambda^{-2/n} - 1\right)$$

Example:

$$y_1 = \alpha_1 + \epsilon_1$$
$$y_2 = 2\alpha_2 - \alpha_1 + \epsilon_2$$
$$y_3 = \alpha_1 + 2\alpha_2 + \epsilon_3$$

$\underset{\sim}{\epsilon} \sim N_3(0, \sigma^2 I_n)$ | To test $\alpha_1 = \alpha_2$

Sol^n:

$A\underline{\beta} = \underline{c}$ $\qquad A = (1, -1)$

$c = 0$

$$Y = X\beta + \epsilon = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \underline{\epsilon}$$

$\hat{\beta} = (X'X)^{-1}X'\underline{Y}$; $\quad \hat{\alpha}_1 = \dfrac{y_1 - \cancel{2}y_2 + y_3}{3}$

$\hat{\alpha}_2 = \dfrac{\cancel{3}y_3 + y_2}{4}$

$$A(X'X)^{-1}A' = \cancel{11/36}$$

$$F = \frac{(A\hat{\beta})' \left[A(X'X)^{-1}A'\right]^{-1} (A\hat{\beta})}{q s^2}$$

$$(X'X) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

$$A(X'X)^{-1}A' = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 0 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 3 & -8 \\ 3 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$\cancel{= 11}$$

$$= 11/24$$

$$s^2 = \frac{RSS}{n-p} = RSS =$$

$$[\; p = 2;\quad q = 1,\quad n = 3 \;]$$

$$F = \frac{(\hat{\alpha}_1 - \hat{\alpha}_2)^2}{\left(\frac{11}{24}\right) s^2} \sim F_{1,1}$$

Method : 2

Let, $\alpha_1 = \alpha_2 = \alpha$

$$\underset{\alpha}{\text{Min}} \quad (Y_1-\alpha)^2 + (Y_2-\alpha)^2 + (Y_3-3\alpha)^2 \longrightarrow \hat{\alpha}_H$$

$$\hookrightarrow \quad RSS_H = (Y_1-\hat{\alpha}_H)^2 + (Y_2-\hat{\alpha}_H)^2 + (Y_3-3\hat{\alpha}_H)^2$$

$$F = \frac{RSS_H - RSS}{RSS}$$

Example:

$U_1, U_2, \ldots U_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2)$ || sampled

$V_1, V_2, \ldots V_{n_2} \sim N(\mu_2, \sigma^2)$ independently.

To derive test statistic ~~for~~ for $H_0: \mu_1 = \mu_2$

$U_i = \mu_1 + \epsilon_i \qquad i = 1, 2, \ldots, n_1$

$V_j = \mu_2 + \epsilon_{n_1+j} \qquad j = 1, \ldots, n_2$

In matrix notation:

$$\underset{\sim}{Y} = (U_1, U_2, \ldots, U_{n_1}, V_1, \ldots, V_{n_2})'$$

$$X = \begin{array}{c} \\ n_1\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \\ 1 & 0 \\ 0 & 1 \\ \vdots & \\ 0 & 1 \end{bmatrix}_{n_1+n_2 \times 2} ; \quad \underset{\sim}{\beta} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

(the last $0, 1$ rows are marked n_2)

$$; \epsilon' = (\epsilon_1, \epsilon_2 \ldots, \epsilon_{n_1}, \epsilon_{n_1+1}, \ldots \epsilon_{n_1+n_2})$$

$$\underset{\sim}{\epsilon} \sim N_{n_1+n_2}\left(\underset{\sim}{0}, \sigma^2 I_{n_1+n_2}\right) \qquad n = n_1 + n_2$$

$H_0: \mu_1 = \mu_2 \implies A\underset{\sim}{\beta} = \underset{\sim}{0} \;;\; A = (1 \;\; -1)$

So, $p = 2,\; q = 1$

$$(X'X) = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} ; \quad \hat{\beta} = \begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/n_1 & 0 \\ 0 & 1/n_2 \end{pmatrix} \begin{pmatrix} \sum U_i \\ \sum V_j \end{pmatrix}$$

$$= \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix}$$

$$A\hat{\beta} = \hat{\mu}_1 - \hat{\mu}_2 = \bar{U} - \bar{V}$$

$$RSS = \underline{Y}'\underline{Y} - \hat{\underline{\beta}}'(X'X)\hat{\underline{\beta}}$$

$$= \sum_{i=1}^{n_1} (U_i - \bar{U})^2 + \sum_{j=1}^{n_2} (V_j - \bar{V})^2$$

$$A(X'X)^{-1}A' = \frac{1}{n_1} + \frac{1}{n_2}$$

$$F = \frac{(\bar{U} - \bar{V})^2}{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \sim F_{1, n_1+n_2-2} = t^2_{n_1+n_2-2}$$

$$s^2 = \frac{RSS}{n-p} = \frac{RSS}{n_1+n_2-2}$$

Example:

General model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{p-1} x_{ip-1} + \epsilon_i \quad , i = 1, 2, \ldots n$$

$H: \beta_j = c$, where $j > 0 \Rightarrow \underline{a}'\underline{\beta} = c$,

$$a = (0, 0, \ldots, 0, \underset{(j+1)}{1}, 0, 0, \ldots 0)$$

Write $(x'x)^{-1} = \begin{bmatrix} l & \underline{m}' \\ \underline{m} & D \end{bmatrix}$; $l \to$ scalar

$a'(x'x)^{-1}\underline{a} = d_{jj} \to j$ th diagonal element of D.

$= (j+1)$ th diagonal element of $(x'x)^{-1}$

& $\underline{a}'\hat{\beta} - c = \hat{\beta}_j - c$

$$F = \frac{(\hat{\beta}_j - c)^2}{s^2 d_{jj}} \sim F_{1, n-p} \, , \quad \text{under } H.$$

$$(A\hat{\beta} - c)'\left[A(x'x)^{-1}A'\right](A\hat{\underline{\beta}} - \underline{c})$$

$$= t^2_{n-p}$$

[Inversion of partioned matrix (symmetric)

$$A^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} + B_{12} B_{22}^{-1} B_{21} & -B_{12} B_{22}^{-1} \\ -B_{22}^{-1} B_{21} & B_{22}^{-1} \end{pmatrix}$$

$$= \begin{bmatrix} C_{11}^{-1} & -C_{11}^{-1} C_{12} \\ -C_{21} C_{11}^{-1} & A_{22}^{-1} + C_{21} C_{11}^{-1} C_{12} \end{bmatrix}$$

$B_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$, $B_{12} = A_{11}^{-1} A_{12}$

$B_{21} = A_{21} A_{11}^{-1}$, $C_{12} = A_{12} A_{22}^{-1}$

$C_{21} = A_{22}^{-1} A_{21}$ $\qquad C_{11} = A_{11} - A_{12} A_{22}^{-1} A_{21}$

[See CR Rao Linear Estimation]

Write $X = (\underline{1}_n, X)$

then $(X'X) = \begin{pmatrix} n & n\bar{\underline{x}}' \\ n\bar{\underline{x}} & X_1'X_1 \end{pmatrix}$; $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots \bar{x}_{p-1})'$

$$(X'X)^{-1} = \begin{bmatrix} \frac{1}{n} + \bar{\underline{x}}' V^{-1} \bar{\underline{x}} & -\bar{\underline{x}}' V^{-1} \\ -V^{-1}\bar{\underline{x}} & V^{-1} \end{bmatrix}$$

$$V = (v_{jk}) = X_1'X_1 - n\bar{\underline{x}}\bar{\underline{x}}'$$

$$v_{jk} = \sum_i x_{ij} x_{ik} - n \bar{x}_j \bar{x}_k$$

$$= \sum_i (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k)$$

D → inverse of V

~~Suppose~~ $\cancel{\underline{a}'\underline{\beta} = \underline{c}, \quad q=1}$

Example: $H: \underline{a}'\underline{\beta} = c$, then $q = 1$

$$\text{Var}(a'\hat{\beta}) = a' \text{Var}(\hat{\beta}) a = \sigma^2 a'(X'X)^{-1} a$$

$$F = \frac{(\underline{a}'\hat{\beta} - c)^2}{s^2 \, \underline{a}'(X'X)^{-1} \underline{a}} \sim F_{1, n-p} \text{ under } H$$

$$\underline{a}'\hat{\beta} \sim N(\underline{a}'\underline{\beta}, \sigma^2 \underline{a}'(X'X)^{-1}\underline{a})$$

$$U = \frac{\underline{a}'\hat{\underline{\beta}} - \underline{a}'\underline{\beta}}{\sigma^2 \{\underline{a}'(X'X)^{-1}\underline{a}\}^{1/2}} \sim N(0,1)$$

$$V = \frac{(n-p)s^2}{\sigma^2} \sim \chi^2_{n-p}$$

$$T = \frac{U}{\sqrt{V/(n-p)}} \sim t_{n-p}$$

The Straight line.

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad , i = 1, 2, \dots, n$$

To test $H: \beta_1 = c$ $\quad X = (\underline{1}_n, \underline{x})$

$$(X'X) = \begin{pmatrix} n & n\bar{x} \\ n\bar{x} & \sum x_i^2 \end{pmatrix} \qquad (X'X)^{-1} = \frac{1}{\sum(x_i-\bar{x})^2} \begin{pmatrix} \frac{1}{n}\sum x_i^2 & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$$

$$\underline{X}'\underline{Y} = \begin{pmatrix} \sum Y_i \\ \sum x_i Y_i \end{pmatrix}, \quad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_1 = \frac{\sum(x_i - \bar{x}) Y_i}{\sum(x_i - \bar{x})^2}$$

$$= \frac{\sum(x_i - \bar{x})(Y_i - \bar{Y})}{\sum(x_i - \bar{x})^2}$$

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

$$= \hat{\beta}_0 + \hat{\beta}_1 (x_i - \bar{x})$$

$$F = \frac{(\hat{\beta}_1 - c)^2}{s^2 / \sum(x_i - \bar{x})^2}$$

$$(n-2)s^2 = \sum_i (Y_i - \hat{Y})^2 = \sum\left[Y_i - \bar{Y} - \hat{\beta}_1 (x_i - \bar{x})\right]^2$$

$$= \sum(Y_i - \bar{Y})^2 - \hat{\beta}_1^2 \sum(x_i - \bar{x})^2$$

$$= \sum(Y_i - \bar{Y})^2 - \sum(\hat{Y}_i - \bar{Y})^2$$

$$\Rightarrow \sum(Y_i - \bar{Y})^2 = \sum(Y_i - \hat{Y}_i)^2 + \sum(\hat{Y}_i - \bar{Y})^2$$

$$= \sum(Y_i - \hat{Y}_i)^2 + r^2 \sum(Y_i - \bar{Y})^2$$

$$r^2 = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = \frac{\hat{\beta}_1^2 \sum(x_i - \bar{x})^2}{\sum(Y_i - \bar{Y})^2}$$

$$= \frac{\left\{\sum(Y_i - \bar{Y})(x_i - \bar{x})\right\}^2}{\sum(Y_i - \bar{Y})^2 \sum(x_i - \bar{x})^2} \rightarrow \text{sq of the sample corr}^n \text{ between } Y \text{ and } x.$$

$$RSS = \sum (y_i - \hat{y}_i)^2 = (1-r^2) \sum (y_i - \bar{y})^2$$

When $c=0$, the F statistic can also be written in terms of r^2.

$$(n-2)s^2 = (1-r^2) \sum (y_i - \bar{y})^2$$

$$= \frac{\hat{\beta}_1^2 \sum (x_i - \bar{x})^2 (n-2)}{(1-r^2) \sum (y_i - \bar{y})^2}$$

$$= \frac{r^2}{1-r^2} (n-2)$$

Multiple Correlation Coefficient :

$$RSS = (1-r^2) \sum (y_i - \bar{y})^2$$

For simple linear Regression

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i = \bar{y} + \hat{\beta}_1 (x_i - \bar{x})$$

$$r = \frac{\hat{\beta}_1 \sum (y_i - \bar{y})(x_i - \bar{x})}{\hat{\beta}_1 \left[\sum (y_i - \bar{y})^2 \sum (x_i - \bar{x})^2 \right]^{1/2}}$$

$$= \frac{\sum (y_i - \bar{y})(\hat{y}_i - \bar{y})}{\left\{ \sum (y_i - \bar{y})^2 \sum (\hat{y}_i - \bar{y})^2 \right\}^{1/2}}$$

$\rightarrow$ Corr^n coeff of the pair $(y_i, \hat{y}_i)$ if $\bar{y} = \bar{\hat{y}}$

$\bar{\hat{y}}$ ↳ mean of $\hat{y}_i$

$$\sum (y_i - \hat{y}_i) = 0$$

So the mean of $\hat{y}_i$, say $\bar{\hat{y}}$ is same as $\bar{y}$

Define the sample multiple correlation coefficient R is the correlation coefficient of the pair $(Y_i, \hat{Y}_i)$, the quantity R^2 is known as coefficient of determination

Theorem:

(1) $$\sum_i (Y_i - \bar{Y})^2 = \sum_i (Y_i - \hat{Y}_i)^2 + \sum_i (\hat{Y}_i - \bar{Y})^2$$

(2) $$R^2 = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = 1 - \frac{RSS}{\sum(Y_i - \bar{Y})^2}$$

Proof:

$$\hat{\underline{Y}} = X\hat{\underline{\beta}} = P\underline{Y}$$

$$\hat{\underline{Y}}'\hat{\underline{Y}} = \underline{Y}'P^2\underline{Y} = \underline{Y}'P\underline{Y} = \underline{Y}'\hat{\underline{Y}}$$

Using the first normal equation corresponding to β_0

$$\sum(Y_i - \hat{Y}_i) = 0$$

$$\sum_i (Y_i - \bar{Y})^2 = \sum(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2$$

$$= \sum(Y_i - \hat{Y}_i)^2 + \sum(\hat{Y}_i - \bar{Y})^2 + 2\underbrace{\sum(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y})}_{2\sum(Y_i - \hat{Y})\hat{Y}_i = 0}$$

$$= \sum(Y_i - \hat{Y}_i)^2 + \sum(\hat{Y}_i - \bar{Y})^2$$

$$\sum(Y_i - \bar{Y})(\hat{Y}_i - \bar{\hat{Y}}) = \sum(Y_i - \bar{Y})(\hat{Y}_i - \bar{Y})$$

$$= \sum(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})(\hat{Y}_i - \bar{Y})$$

$$= \sum_i (\hat{Y}_i - \bar{Y})^2$$

$$R^2 = \frac{\left[\sum(Y_i - \bar{Y})(\hat{Y}_i - \bar{\hat{Y}})\right]^2}{\sum(Y_i - \bar{Y})^2 \sum(\hat{Y}_i - \bar{\hat{Y}})^2} = \frac{\left[\sum(\hat{Y}_i - \bar{Y})^2\right]^2}{\sum(Y_i - \bar{Y})^2 \sum(\hat{Y}_i - \bar{Y})^2}$$

$$R^2 = \frac{\sum(\hat{y}_i - \bar{y})^2}{\sum(y_i - \bar{y})^2}$$

$$= \frac{\sum(y_i - \bar{y})^2 - RSS}{\sum(y_i - \bar{y})^2} = 1 - \frac{RSS}{\sum(y_i - \bar{y})^2}$$

$$\Rightarrow RSS = (1 - R^2) \sum (y_i - \bar{y})^2$$

if $y_i = \hat{y}_i \ \forall i$,

$R^2 = 1$, perfect fit.

otherwise $R^2 < 1$

$SST = \sum(y_i - \bar{y})^2$ [RSS of

$RSS = \sum(y_i - \hat{y}_i)^2$ [for general regression model]

$$R^2 = \frac{SST - RSS}{SST} = \frac{\frac{SST}{n} - \frac{RSS}{n}}{SST/n}$$

Adjusted R^2

$$R_{adj}^2 = R_a^2 = \frac{\frac{SST}{n-1} - \frac{RSS}{n-p}}{\frac{SST}{n-1}}$$

$$= 1 - \left(\frac{n-1}{n-p}\right)(1 - R^2)$$

Predictive R^2: leave-one-out cross-validation

$SST = \sum(y_i - \bar{y})^2$

$$\bar{y}_{(i)} = \frac{1}{(n-1)} \sum_{j \neq i}^{n} y_j$$