DGP: $y = X_1\beta_1 + X_2\beta_2 + \varepsilon \longrightarrow b_1, b_2$

restricted model: $y = X_1\beta_1 + \tilde{\varepsilon} \longrightarrow b_R$

a) $b_1 = b_R - Pb_2$

$$
\begin{aligned}
Var(b_1) &= Var(b_R - Pb_2) \\
&= Var(b_R) + Var(Pb_2) - 2Cov(b_R, Pb_2) \\
&= Var(b_R) + Var(Pb_2) \\
&= Var(b_R) + PVar(b_2)P' \\
Var(b_R) &= Var(b_1) - PVar(b_2)P'
\end{aligned}
$$

b)

$$
\begin{aligned}
MSE(b) &= E((b-\beta)(b-\beta)') \\
&= E((b-\beta)(b'-\beta')) \\
&= E(bb' - b\beta' - \beta b' + \beta\beta') \\
&= E(bb') - E(b)\beta' - \beta E(b') + \beta\beta' \\
Var(b) &= E((b-E(b))(b-E(b))') \\
&= E((b-E(b))(b'-E(b)')) \\
&= E(bb' - bE(b)' - E(b)b' + E(b)E(b)') \\
&= E(bb') - E(b)E(b)' - E(b)E(b') + E(b)E(b)' \\
&= E(bb') - E(b)E(b)' \\
E(bb') &= Var(b) + E(b)E(b)' \\
MSE(b) &= Var(b) + E(b)E(b)' - E(b)\beta' - \beta E(b') + \beta\beta' \\
&= Var(b) + (E(b)-\beta)(E(b)-\beta)' \\
&= Var(b) + E(b-\beta)E(b-\beta)'
\end{aligned}
$$

c) $MSE(b_1) = Var(b_1) + E(b_1 - \beta_1)E(b_1 - \beta_1)'$

$= Var(b_1)$

$MSE(b_R) = Var(b_R) + E(b_R - \beta_1)E(b_R - \beta_1)'$

$MSE(b_1) - MSE(b_R) = Var(b_1) - Var(b_R) - E(b_R - \beta_1)E(b_R - \beta_1)'$

$= Var(b_1) - Var(b_R) - (E(b_R) - \beta_1)(E(b_R) - \beta_1)'$

$Var(b_1) = Var(b_R) + P\,Var(b_2)P'$, $b_R = b_1 + Pb_2$

$MSE(b_1) - MSE(b_R) = Var(b_R) + P\,Var(b_2)P' - Var(b_R)$

$- (E(b_1 + Pb_2) - \beta_1)(E(b_1 + Pb_2) - \beta_1)'$

$= P\,Var(b_2)P' - (\beta_1 + P\beta_2 - \beta_1)(\beta_1 + P\beta_2 - \beta_1)'$

$= P\,Var(b_2)P' - P\beta_2\beta_2'P'$

$= P(Var(b_2) - \beta_2\beta_2')P'$

d) The restricted estimator is better

$MSE(b_R) < MSE(b_1)$ $\quad MSE(b_1) - MSE(b_R) > 0$

1) $\beta_2 = 0$, $\quad P\,Var(b_2)P' > 0$

second group of regressors does not matter

2) $\beta_2 \neq 0$, in case $Var(b_2) - \beta_2\beta_2'$ PSD

variance b_2 ~~small~~ big compared to its influence