Test Exercise - 5

It is given that $\Pr[resp_i = 1] = \dfrac{\exp(\beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 (\frac{age_i}{10})^2)}{1 + \exp(\beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 (\frac{age_i}{10})^2)}$

for $i = 1, \dots, 925$

(a) We need to show that $\dfrac{\partial \Pr[resp_i = 1]}{\partial age_i} + \dfrac{\partial \Pr[resp_i = 0]}{\partial age_i} = 0$

For notational simplicity, let

$$\alpha = \beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 \left(\frac{age_i}{10}\right)^2$$

so, $\Pr[resp_i = 1] = \dfrac{e^\alpha}{1 + e^\alpha}$

$\Pr[resp_i = 0] = \dfrac{1}{1 + e^\alpha}$

$$\frac{\partial \Pr[resp_i = 1]}{\partial age_i} = \frac{\partial}{\partial age_i}\left(e^\alpha \cdot (1 + e^\alpha)^{-1}\right)$$

$$= \frac{\partial \alpha}{\partial age_i} \cdot e^\alpha \cdot (1 + e^\alpha)^{-1} - \frac{\partial \alpha}{\partial age_i} \cdot e^\alpha \cdot (1 + e^\alpha)^{-2} \cdot e^\alpha$$

$$\left[\because \frac{\partial (AB)}{\partial x} = \frac{\partial A}{\partial x} \cdot B + \frac{\partial B}{\partial x} \cdot A\right.$$

$$= \frac{\partial \alpha}{\partial age_i} \cdot e^\alpha (1 + e^\alpha)^{-2} \left\{(1 + e^\alpha) - e^\alpha\right\}$$

$$= \frac{\partial \alpha}{\partial age_i} \cdot e^\alpha \cdot (1 + e^\alpha)^{-2} \quad \text{—— } (1)$$

Now, $\dfrac{\partial \Pr[resp_i = 0]}{\partial age_i} = \dfrac{\partial}{\partial age_i}\left((1 + e^\alpha)^{-1}\right)$

$$= -\frac{\partial \alpha}{\partial age_i} \cdot e^\alpha \cdot (1 + e^\alpha)^{-2} \quad \text{—— } (2)$$

So, $\dfrac{\partial \Pr[resp_i = 1]}{\partial age_i} + \dfrac{\partial \Pr[resp_i = 0]}{\partial age_i} = (1) + (2)$

$$= \frac{\partial \alpha}{\partial age_i} \cdot e^\alpha \cdot (1 + e^\alpha)^{-2} - \frac{\partial \alpha}{\partial age_i} \cdot e^\alpha \cdot (1 + e^\alpha)^{-2}$$

$$= 0.$$

(b) $resp_i^{new} = -resp_i + 1$

$$\text{Odds ratio} = \frac{Pr[resp_i = 1]}{Pr[resp_i = 0]}.$$

Since, positive response is now defined to be equal to zero and negative response to be equal to 1,

So, $$Pr[resp_i^{new} = 1] = \frac{1}{1+e^{\alpha}}$$

$$Pr[resp_i^{new} = 0] = \frac{e^{\alpha}}{1+e^{\alpha}}$$

$$\text{odds ratio} = \frac{Pr[resp_i^{new} = 1]}{Pr[resp_i^{new} = 0]}$$

$$= \frac{\frac{1}{1+e^{\alpha}}}{\frac{e^{\alpha}}{1+e^{\alpha}}}$$

$$= \frac{1}{e^{\alpha}}$$

$$= e^{-\alpha}$$

Hence, we can observe from the odds ratio that the sign of all ~~pra~~ parameters in α get reversed.

Here, $\alpha = \beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 \left(\frac{age_i}{10}\right)^2$

(c) To allow age value to be different for males and females, we could add an interaction term between 'age' and 'male' dummy variable in the logit specification.

So, a term like ' $age_i \times male_i$ ' could be added to the logit specification to allow the gender variable to have an interaction effect on 'age'.

e.g.

$$Pr[resp_i = 1] = \frac{\exp\left(\beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 \left(\frac{age_i}{10}\right)^2 + \beta_5 age_i \times male_i\right)}{1 + \exp\left(\beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 \left(\frac{age_i}{10}\right)^2 + \beta_5 age_i \times male_i\right)}$$

So,

$$\text{Odds Ratio} = \frac{Pr[resp_i = 1]}{Pr[resp_i = 0]} = \exp\left(\beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 \left(\frac{age_i}{10}\right)^2 + \beta_5 age_i \cdot male_i\right)$$

$$\frac{\partial(\text{Odds Ratio})}{\partial age_i} = \left(\beta_3 + \frac{\beta_4 age_i}{50} + \beta_5 male_i\right) \times \exp\left(\beta_0 + \beta_1 male_i + \beta_2 active_i + \beta_3 age_i + \beta_4 \left(\frac{age_i}{10}\right)^2 + \beta_5 age_i \cdot male_i\right)$$

For maximum value of age,

$$\frac{\partial(\text{Odds Ratio})}{\partial age_i} = 0 \Rightarrow \beta_3 + \beta_4 \frac{age_{max}}{50} + \beta_5 male_i = 0$$

$$\Rightarrow age_{max} = \frac{50}{\beta_4} \cdot \left(-\beta_5 male_i - \beta_3\right)$$

So, 'age_i' depends on the gender. (the dummy variable '$male_i$').