a) (i) As $y = X\beta + \varepsilon$, we get

$$\hat{\beta} = Ay = A(X\beta + \varepsilon) = AX\beta + A\varepsilon$$

As A, X, and β are fixed and $E(\varepsilon) = 0$, we find

$$E(\hat{\beta}) = AX\beta + AE(\varepsilon) = AX\beta$$

$$\begin{aligned} var(\hat{\beta}) &= E\left((\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))'\right) \\ &= E\left(A\varepsilon(A\varepsilon)'\right) \\ &= E(A\varepsilon\varepsilon' A') \\ &= A\,E(\varepsilon\varepsilon')A' \\ &= A\sigma^2 I A' \\ &= \sigma^2 AA' \end{aligned}$$

(ii) Unbiased $E(\hat{\beta}) = \underbrace{AX\beta = \beta}_{AX = I}$ for all β

$AX = I$, or

$$DX = (A - A_0)X = AX - A_0X = I - (X'X)^{-1}X'X = I - I = 0$$

iii) $$\begin{aligned} AA' &= (D + A_0)(D + A_0)' \\ &= DD' + A_0D' + DA_0' + A_0A_0' \end{aligned}$$

$$DA_0' = DX(X'X)^{-1} = 0 \cdot (X'X)^{-1} = 0$$

$$A_0D' = (DA_0')' = 0$$

$$\begin{aligned} AA' &= DD' + A_0A_0' \\ &= DD' + (X'X)^{-1}X'X(X'X)^{-1} \\ &= DD' + (X'X)^{-1} \end{aligned}$$

b) As $var(b) = \sigma^2 (X'X)^{-1}$, we get

$$var(\hat\beta) \overset{(a-i)}{=} \sigma^2 AA' \overset{(a-iii)}{=} \sigma^2 (DD' + (X'X)^{-1})$$

$$= \sigma^2 DD' + \sigma^2 (X'X)^{-1} = \sigma^2 DD' + var(b)$$

c) $var(\hat\beta) - var(b) = \underbrace{\sigma^2 DD'}_{PSD}$ As $\sigma^2 > 0$

<u>prove that DD' is PSD</u>

D $(k \times n)$ matrix

D' $(n \times k)$ matrix

c $(k \times 1)$ vector

$d = D'c$ $(n \times 1)$ vector with components $d_1, \ldots, d_n$

$$c'DD'c = (D'c)'D'c = d'd = \sum_{i=1}^{n} d_i^2 \geq 0$$

Thus shows that DD' is PSD

d) Let c_j be the $(k \times 1)$ j-th unit vector, $c_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$c_j' (var(\hat\beta) - var(b)) c_j = \sigma^2 c_j' DD' c_j \geq 0$$

$c_j' var(\hat\beta) c_j \geq c_j' var(b) c_j$, or equivalently

$var(c_j' \hat\beta) \geq var(c_j' b)$

As $c_j' \hat\beta = \hat\beta_j$ and $c_j' b = b_j$, it follows that

$var(\hat\beta_j) \geq var(b_j)$ for every $j = 1, \ldots, k$