Unscented Model Predictive Static Programming for Impulsive Systems

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1 Review of Unscented Optimal Control

Let a general nonlinear system be defined by the following continuous-time system dynamics:

$$\begin{aligned}
\dot{X} &= f(X, \theta, t), \qquad && for \;\; t \neq t_q \;\; where \;\; q = 1, \dots, n_q \\
X^+ &= g(X^-, U_q, \theta), && for \;\; t = t_q \\
Y &= h(X)
\end{aligned} \tag{1}$$

Here, $X \in \mathbb{R}^n$ denotes the state vector i.e., $X = [x_1 \; x_2 \; \dots \; x_n]^T$. $U \in \mathbb{R}^m$ is the impulse control input applied at t_q for a total of n_q instances. $Y \in \mathbb{R}^p$ is the output vector and $\theta \in \mathbb{R}^{n_p}$ denotes the system parameters. f is a nonlinear function of the state and system parameters. The impulse control is applied as per the nonlinear function g at t_q giving rise to the impulse state X^+.

In practical scenarios, the initial condition (X_0) as well as the parameters (θ) are often known with a degree of uncertainty associated with them. The known value of the initial conditions is taken to be the mean, $\mu_{X_0} \in \mathbb{R}^n$ whereas the associated uncertainty is taken as the covariance matrix, P_{X_0}. Similarly for the parameters, the certain or known value is taken as $\mu_\theta \in \mathbb{R}^{n_p}$ and the uncertain value is the covariance matrix, P_θ.

The objective of unscented optimal control is two-fold. First, it aims to design a control solution insensitive to the various uncertainties such that $Y \longrightarrow Y^*$, where Y^* is the desired output. The uncertain characteristics in the system are introduced in the form of sigma points which utilizes the mean and covariance matrix. Second, the unscented control is designed so as to minimize the variance of the sigma points at final time. This is achieved by combining the techniques of unscented transform with optimal control theory. Unscented transform is a mathematical function for estimating the effect of applying a given nonlinear transformation to a probability distribution characterized by a finite set of statistics. To obtain this transform, the set of sigma points are derived as follows:

For parameter:

$$\begin{aligned}
\theta_0 &= \mu_\theta \\
\theta_j &= \mu_\theta + (\sqrt{P_\theta})_j \qquad && j = 1, 2, \dots, n_p \\
\theta_j &= \mu_\theta - (\sqrt{P_\theta})_{j-n_p} && j = n_p + 1, \dots, 2n_p
\end{aligned} \tag{2}$$

For initial condition:

$$
\begin{aligned}
X_0 &= \mu_{X_0} \\
X_{0j} &= \mu_{X_0} + (\sqrt{P_{X_0}})_j \qquad j = 1, 2, \dots, n \\
X_{0j} &= \mu_{X_0} - (\sqrt{P_{X_0}})_{j-n} \quad j = n+1, \dots, 2n
\end{aligned} \tag{3}
$$

These sigma points could be used to redefine the original system as follows:

$$
\dot{\hat{X}} = \begin{bmatrix} f(\chi_1, t) \\ f(\chi_2, t) \\ \vdots \\ f(\chi_{n_\sigma}, t) \end{bmatrix} = f_0(\hat{X}, t) \quad \hat{X}^+ = \begin{bmatrix} g(\chi_1^-, U) \\ g(\chi_2^-, U) \\ \vdots \\ g(\chi_{n_\sigma}^-, U) \end{bmatrix} = g_0(\hat{X}^-, U) \tag{4}
$$

Here, $n_\sigma = 2n_p + 2n + 1$ is the total number of sigma points and χ represents a specific combination of each of these sigma points. $\dot{\hat{X}}$ represents the augmented system matrix and $\hat{X}^+$ denotes the augmented impulse state matrix.

2 Formulation of Unscented MPSP for Impulsive Systems

According to the philosophy of Model Predictive Static Programming (MPSP), the continuous-time dynamics given in Eq. 1 is written in discrete-time as:

$$
\hat{X}_{k+1} = F(\hat{X}_k) \tag{5}
$$

$$
\hat{X}_k^+ = g(\hat{X}_k^-, U_q) \tag{6}
$$

$$
\hat{Y}_k = H(\hat{X}_k) = \frac{1}{n_\sigma} \left[\sum_{i=1}^{n_\sigma} H(x_{1i}(k)) \cdots \sum_{i=1}^{n_\sigma} H(x_{ni}(k)) \right]^T \tag{7}
$$

Here, $k = 1, \dots, N$ represents all the time instants, whereas $q = 1, \dots, n_q$ denotes the impulse instants as stated previously. The q^{th} impulse is applied at the last node (n_d) of the q^{th} segment. The output at each instant, $\hat{Y}_k$ is given by the mean of the augmented state ($\hat{X}_k$) as given in Eq. 7. The output at the final instant N is:

$$
\hat{Y}_N = \frac{1}{n_\sigma} \left[\sum_{i=1}^{n_\sigma} H(x_{1i}(N)) \cdots \sum_{i=1}^{n_\sigma} H(x_{ni}(N)) \right]^T \tag{8}
$$

Starting with a guess control history applied at every impulse instant, the IMPSP technique is used to update the control trajectory at each step to achieve the desired objectives. The output error ($\Delta \hat{Y}_N = \hat{Y}_N^* - \hat{Y}_N$) at final time is represented by,

$$
\Delta \hat{Y}_N \cong d\hat{Y}_N = \left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N} \right] d\hat{X}_N \tag{9}
$$

This can be further expanded using $\hat{X}_{N-1}$ through Eq. 5 as:

$$
d\hat{Y}_N = \left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N} \right] \left[\frac{\partial F_{N_1}}{\partial \hat{X}_{N-1}} \right] d\hat{X}_{N-1}
$$

Similarly, this can be expanded till the last impulse from the end ($k = N - (n_d - 1)$),

$$
d\hat{Y}_N = \left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N} \right] \left[\frac{\partial F_{N-1}}{\partial \hat{X}_{N-1}} \right] \cdots \left[\frac{\partial F_{N-(n_d-1)}}{\partial \hat{X}^+_{N-(nd-1)}} \right] d\hat{X}^+_{N-(n_d-1)} \tag{10}
$$

Using Eq. 6, $d\hat{X}_k^+$ can be written as,

$$d\hat{X}_k^+ = \left[\frac{\partial g(\hat{X}_k^-, U_q)}{\partial \hat{X}_k^-}\right] d\hat{X}_k^- + \left[\frac{\partial g(\hat{X}_k^-, U_q)}{\partial U_q}\right] dU_q \tag{11}$$

Eq. 11 can be used to replace $d\hat{X}^+_{N-(n_d-1)}$ in Eq. 10,

$$d\hat{Y}_N = \left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N}\right]\left[\frac{\partial F_{N-1}}{\partial \hat{X}_{N-1}}\right]\cdots\left[\frac{\partial F_{N-(n_d-1)}}{\partial \hat{X}^+_{N-(nd-1)}}\right]\left[\left[\frac{\partial g_{N-(n_d-1)}}{\partial \hat{X}^-_{N-(n_d-1)}}\right] d\hat{X}^-_{N-(n_d-1)} + \left[\frac{\partial g_{N-(n_d-1)}}{\partial U_q}\right] dU_q\right] \tag{12}$$

From Eq. 5, $d\hat{X}^-_{N-(n_d-1)}$ can be expanded as a function $d\hat{X}_{N-(n_d-1)-1}$,

$$\begin{aligned} d\hat{Y}_N = &\left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N}\right]\left[\frac{\partial F_{N-1}}{\partial \hat{X}_{N-1}}\right]\cdots\left[\frac{\partial F_{N-(n_d-1)}}{\partial \hat{X}^+_{N-(nd-1)}}\right]\left[\frac{\partial g_{N-(n_d-1)}}{\partial \hat{X}^-_{N-(n_d-1)}}\right]\left[\frac{\partial F_{N-(n_d-1)-1}}{\partial \hat{X}_{N-(nd-1)-1}}\right] d\hat{X}_{N-(n_d-1)-1} \\ &+ \left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N}\right]\left[\frac{\partial F_{N-1}}{\partial \hat{X}_{N-1}}\right]\cdots\left[\frac{\partial F_{N-(n_d-1)}}{\partial \hat{X}^+_{N-(nd-1)}}\right]\left[\frac{\partial g_{N-(n_d-1)}}{\partial U_q}\right] dU_q \end{aligned} \tag{13}$$

This can similarly be expanded till the 1^{st} impulse where $k = N - n_q(n_d-1)$,

$$d\hat{X}^-_{N-n_q(n_d-1)} = \left[\frac{\partial F_{N-n_q(n_d-1)}}{\partial \hat{X}_{N-n_q(n_d-1)}}\right]\cdots\left[\frac{\partial F_1}{\partial \hat{X}_1}\right] d\hat{X}_1 \tag{14}$$

As the initial conditions are known to us, $d\hat{X}_1 = 0$, the final condensed equation in terms of the sensitivity matrix B can be written as:

$$d\hat{Y}_N = \sum_{q=1}^{n_q} B_q dU_q \tag{15}$$

The sensitivity matrix is calculated recursively as,

$$B_q^0 = \begin{cases} \left[\frac{\partial \hat{Y}_N}{\partial \hat{X}_N}\right]\left[\frac{\partial F_{N-1}}{\partial \hat{X}_{N-1}}\right]\cdots\left[\frac{\partial F_{N-(n_d-1)}}{\partial \hat{X}^+_{N-(nd-1)}}\right], & \text{for } q = n_q \\ B^0_{q+1}\left[\frac{\partial g_{N-(n_q-q)(n_d-1)}}{\partial \hat{X}^-_{N-(n_q-q)(n_d-1)}}\right]\left[\frac{\partial F_{N-(n_q-q)(n_d-1)-1}}{\partial \hat{X}_{N-(n_q-q)(n_d-1)-1}}\right]\cdots\left[\frac{\partial F_{N-(n_q-(q-1))(n_d-1)}}{\partial \hat{X}^+_{N-(n_q-(q-1))(n_d-1)}}\right], & \text{for } q = n_q-1,\ldots,2,1 \end{cases} \tag{16}$$

$$B_q = B_q^0\left[\frac{\partial g}{\partial U_q}\right] \qquad for \;\; q = 1,\ldots,n_q \tag{17}$$

The second objective is to minimize the trace of covariance matrix at final time. For this, a new variable which is taken to be the final time error between the achieved augmented state ($\hat{X}_N$) and mean of the augmented state ($\bar{X}_N$) is taken:

$$Z_N = \hat{X}_N - \bar{X}_N$$

Here, $\hat{X}_N = [\chi_1(N), \ldots, \chi_{n_\sigma}(N)]$, and $\bar{X}_N = \frac{1}{n_\sigma}\sum_{i=1}^{n_\sigma}\chi_i(N)$. The term $Z_N^T Q Z_N$ gives the covariance matrix trace of the augmented system at final time. Z_N can be expressed as,

$$Z_N = Z_N^* + \Delta Z_N = dZ_N = \left[\frac{\partial Z_N}{\partial X_N}\right] dX_N \tag{18}$$

Here, Z_N^* denoting the desired covariance at end time which is taken as zero. Following a similar procedure as above the variable Z_N can be represented in terms of the control input U with the help of sensitivity matrices as,

$$Z_N = \sum_{q=1}^{n_q} C_q dU_q \tag{19}$$

where C is given by,

$$C_q^0 = \begin{cases} \left[\frac{\partial Z_N}{\partial \hat{X}_N}\right]\left[\frac{\partial F_{N-1}}{\partial \hat{X}_{N-1}}\right]\cdots\left[\frac{\partial F_{N-(n_d-1)}}{\partial \hat{X}^+_{N-(nd-1)}}\right], & \text{for } q = n_q \\ C_{q+1}^0\left[\frac{\partial g_{N-(n_q-q)(n_d-1)}}{\partial \hat{X}^-_{N-(n_q-q)(n_d-1)}}\right]\left[\frac{\partial F_{N-(n_q-q)(n_d-1)-1}}{\partial \hat{X}_{N-(n_q-q)(n_d-1)-1}}\right]\cdots\left[\frac{\partial F_{N-(n_q-(q-1))(n_d-1)}}{\partial \hat{X}^+_{N-(n_q-(q-1))(n_d-1)}}\right], & \text{for } q = n_q-1,\dots,2,1 \end{cases} \tag{20}$$

$$C_q = C_q^0\left[\frac{\partial g}{\partial U_q}\right] \quad for \;\; q = 1,\dots,n_q \tag{21}$$

The cost function is selected such that it satisfies both the objectives discussed previously:

$$J = \frac{1}{2}\sum_{k=1}^{n_q}\big(U_k^i + dU_k\big)^T R_k\big(U_k^i + dU_k\big) + \frac{1}{2}(Z_N^T Q Z_N) \tag{22}$$

Writing Z_N using Eq. 19, we get,

$$J = \frac{1}{2}\sum_{k=1}^{n_q}\big(U_k^i + dU_k\big)^T R_k\big(U_k^i + dU_k\big) + \frac{1}{2}\bigg(\sum_{j=1}^{n_q}(C_j dU_j)\bigg)^T Q\bigg(\sum_{j=1}^{n_q}(C_j dU_j)\bigg) \tag{23}$$

Here, R_k and Q are the weighing matrices which are taken to be positive definite. The cost function in Eq. 23 is subjected to the equality function given in Eq. 15. This gives rise to an augmented cost function written using the lagrangian multiplier λ,

$$\tilde{J} = \frac{1}{2}\sum_{k=1}^{n_q}\big(U_k^i + dU_k\big)^T R_k\big(U_k^i + dU_k\big) + \frac{1}{2}\bigg(\sum_{j=1}^{n_q}(C_j dU_j)\bigg)^T Q\bigg(\sum_{j=1}^{n_q}(C_j dU_j)\bigg) + \lambda(d\hat{Y}_N - \sum_{q=1}^{n_q} B_q dU_q) \tag{24}$$

The augmented cost function, $\tilde{J}$ in Eq. 24 is differentiated with respect to the free variables dU_k and λ to obtain the necessary conditions of optimality,

$$\frac{d\tilde{J}}{dU_k} = 0 \;\;\; and \;\;\; \frac{d\tilde{J}}{d\lambda} = 0$$

The necessary conditions result in,

$$R_j(U_j^i + dU_j) - B_j^T\lambda + \sum_{q=1}^{n_q} C_j^T Q C_q dU_q = 0$$

$$R_j dU_j + \sum_{q=1}^{n_q} C_j^T Q C_q dU_q = B_j^T\lambda - R_j U_j^i \quad \forall j = 1,\dots,n_q \tag{25}$$

Eq. 25 can be expressed in a more compact form using the dirac delta function,

$$\sum_{q=1}^{n_q}\Big(C_j^T Q C_q + \delta_{qj} I_{m\times m} R_j\Big) dU_q = B_j^T\lambda - R_j U_j^i$$

$$\text{where,} \quad \delta_{qj} = \begin{cases} 1, & q = j \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

Further,

$$\sum_{q=1}^{n_q} \phi_q^j dU_q = B_j^T \lambda - R_j U_j^i \quad \forall j = 1, \dots, n_q$$

$$\text{where,} \quad \phi_q^j = \left(C_j^T Q C_q + \delta_{qj} I_{m\times m} R_j \right) \tag{27}$$

Eq. 28 can now be represented in the vector or matrix form ($[\cdot]$) as,

$$[\phi] dU = -[R]U^i + [B]^T \lambda \tag{28}$$

Here,

$$[\phi] = \begin{bmatrix} \phi_1^1 & \dots & \phi_{n_q}^1 \\ \vdots & \ddots & \vdots \\ \phi_1^{n_q} & \dots & \phi_{n_q}^{n_q} \end{bmatrix}, dU = \begin{bmatrix} dU_1 \\ \vdots \\ dU_{n_q} \end{bmatrix}, [R] = \begin{bmatrix} R_1^1 & 0 & 0 \\ 0 & \ddots & \vdots \\ 0 & 0 & R_{n_q} \end{bmatrix}, U^i = \begin{bmatrix} U_1^i \\ \vdots \\ U_{n_q}^i \end{bmatrix}, [B] = \begin{bmatrix} B_1^T \\ \vdots \\ B_{n_q}^T \end{bmatrix}$$

From Eq. 28, we can find the change in the control input (dU) as,

$$dU = -[\phi]^{-1}[R]U^i + [\phi]^{-1} B^T \lambda \tag{29}$$

Using Eq. 29 in Eq. 15 (in matrix notation) gives us,

$$d\hat{Y}_N = -[B][\phi]^{-1}[R]U^i + [B][\phi]^{-1}[B]^T \lambda \tag{30}$$

Solving for λ from the above equation,

$$\lambda = \left[[B][\phi]^{-1}[B]^T \right]^{-1} \left[d\hat{Y}_N + [B][\phi]^{-1}[R]U^i \right] \tag{31}$$

This can be substituted in Eq. 29 to obtain dU, which thereby gives us the updated control law,

$$U^{i+1} = dU + U^i \tag{32}$$

This iteration is carried out until the desired objective is obtained.

3 Simulation Results

This section discusses the simulation results obtained using the proposed algorithm.

3.1 Hare & Lynx Problem

A highly specialized prey-predator relationship between Snowshoe Hare and Canadian Lynx is seen in the North American forests. The records of pelts collected by the Hudson Bay Company for over 100 years reveals this unusually tight relationship between the two species. The data between the years 1900-1920 is shown in Fig. 1. From this, we can notice the distinct rise in the predator population following the rise in the prey population. This intertwined relationship is better visualized in Fig. 2. Thereby, we can use the simplified Lotka-Volterra model to predict the changes in the Hare and Lynx Population over time. The model is as follows:

$$\begin{aligned} \dot{H} &= \mu_1 H - r_{12} HL, && \text{at } t \neq t_q \\ \dot{L} &= -\mu_2 L + r_{21} LH, && \text{at } t \neq t_q \\ H^+ &= H^- + u_{Hq}, && \text{at } t = t_q \\ L^+ &= L^- + u_{Lq}, && \text{at } t = t_q \end{aligned} \tag{33}$$

Year	Hares ($\times$1000)	Lynx ($\times$1000)	Year	Hares ($\times$1000)	Lynx ($\times$1000)
1900	30	4	1911	40.3	8
1901	47.2	6.1	1912	57	12.3
1902	70.2	9.8	1913	76.6	19.5
1903	77.4	35.2	1914	52.3	45.7
1904	36.3	59.4	1915	19.5	51.1
1905	20.6	41.7	1916	11.2	29.7
1906	18.1	19	1917	7.6	15.8
1907	21.4	13	1918	14.6	9.7
1908	22	8.3	1919	16.2	10.1
1909	25.4	9.1	1920	24.7	8.6
1910	27.1	7.4			

Figure 1: Hudson Bay Pelt Collection Data

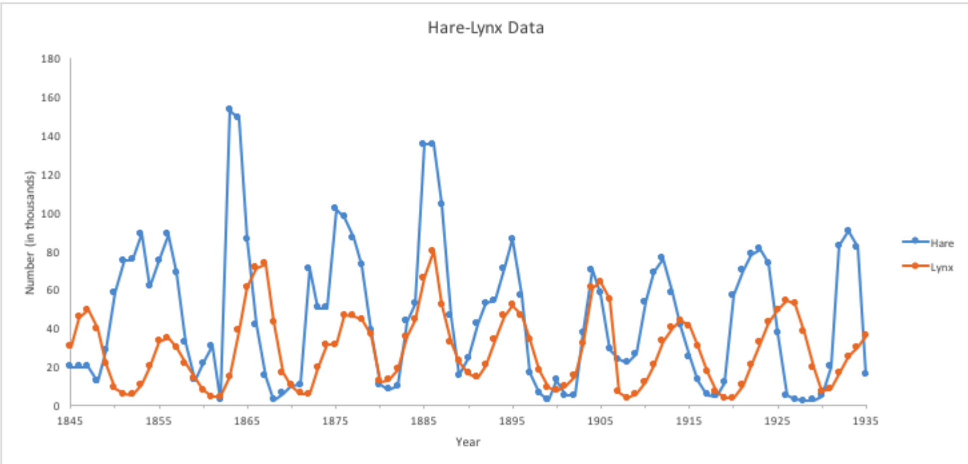


Figure 2: Hare and Lynx Population Cycle

Here, H and L denote the Hare and Lynx population respectively. q denotes the impulse instant ranging from $1, \ldots, n_q$. H^- and H^+ denote the Hare population just before and after the application of the impulse input u_{Hq}. Similarly, L^- and L^+ denote the Lynx population just before and after the application of the impulse input u_{Lq}. μ_1 is the growth rate of Hare and μ_2 is the death rate of Lynx. r_{12} and r_{21} are the interaction rates. The initial condition is taken as $(H(0), L(0)) = (30{,}4)$ (values for the year 1900). The equilibrium condition is taken to be the average population value between the periodic cycles lasting about 10.5 years[1] resulting in $\mu_1 \mu_2 = 0.358$. Thus, the equilibrium is:

$$(H_e, L_e) = (\frac{\mu_2}{r_{21}}, \frac{\mu_1}{r_{12}}) = (34.6, 22.1)$$

Using the lowest density of Lynx population between 1900-1901, it is assumed that the Hare population during this period satisfies the following relation,

$$\begin{aligned} H(t) &= H(0)e^{\mu_1 t} \\ 47.2 &= 30e^{\mu_1} \\ \mu_1 &= 0.453 \end{aligned}$$

μ_1 can now be used to calculate the values of other parameters which are tabulated in table 1.

$$\mu_2 = \frac{0.358}{\mu_1} = 0.790; \qquad r_{12} = \frac{\mu_1}{22.1} = 0.0205; \qquad r_{21} = \frac{\mu_1}{34.6} = 0.0229$$

[1] https://jmahaffy.sdsu.edu/courses/f17/math636/beamer/lotvol.pdf

Parameters	Values
μ_1	0.453
μ_2	0.790
r_{12}	0.0205
r_{21}	0.0229

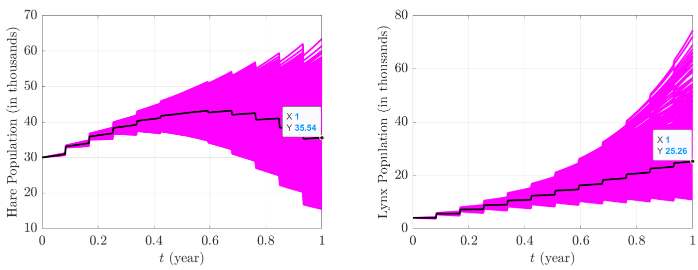
Table 1: Parameter values for Hare & Lynx

The impulse control is applied at an interval of every month excluding the last month, resulting in a total of 11 impulses. The RK4 integration constant is taken as 1/365. The total time duration is taken as one year. The objective is to achieve the desired equilibrium state $(H_e,\, L_e) = (34.6,\, 22.1)$. The problem is first solved using IMPSP by considering the nominal parameters and subjecting it to the following cost function:

$$J = \frac{1}{2}\int_{t=0}^{t_f} U^T R U \tag{34}$$

Fig. 3 shows the results for 1000 randomized Monte-Carlo like simulations. It can be noticed that the variance at final time is significant owing to the uncertainty in the interaction constants $(r_{12},\, r_{21})$ with $P_\theta = diag[0.001, 0.001]$. To reduce this variance, U-IMPSP algorithm is used. As explained in the previous sections, the sigma points are defined as:

$$\begin{aligned} \theta_0 &= \mu_\theta \\ \theta_i &= \mu_\theta + (\sqrt{P_\theta})_i \quad & i = 1, 2 \\ \theta_i &= \mu_\theta - (\sqrt{P_\theta})_i \quad & i = 3, 4 \end{aligned} \tag{35}$$



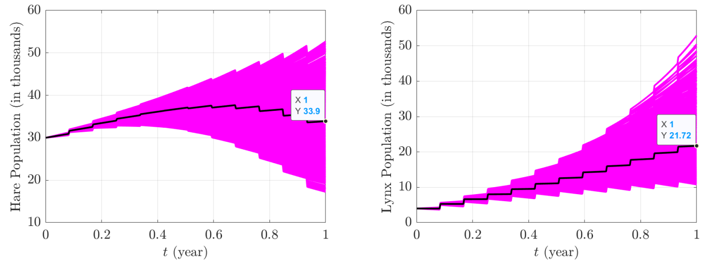
(a) Hare Population, H (b) Lynx Population, L

Figure 3: IMPSP applied to Hare & Lynx Problem

The cost function given in Eq. 34 is redefined to minimize the variance at final time:

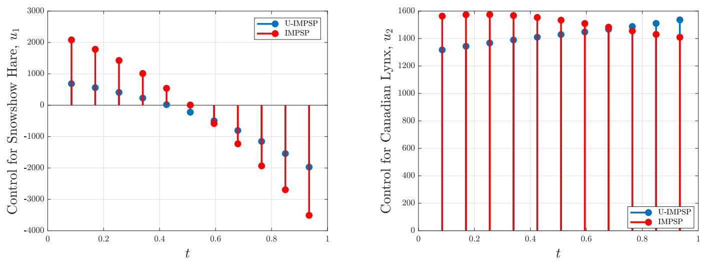
$$J = (X(N) - \bar{X}_N)^T Q (X(N) - \bar{X}_N)^T + \frac{1}{2}\int_{t=0}^{t_f} U^T R U \tag{36}$$

Here, $Q = I_2$ and $R = diag[0.1, 1]$. $X(N)$ is the augmented system at final time and $\bar{X}_N$ is the mean of the augmented system at final time. Figure 4 shows the result for the U-IMPSP algorithm. The H and L variance obtained using MPSP is 10.38^2 and 11.70^2 respectively which is higher compared to the variance obtained using U-IMPSP, i.e., 8.35^2 and 8.38^2 for H and L respectively. Figure 5 shows the comparison of the IMPSP and U-IMPSP control trajectories. It can also be noticed that U-IMPSP demands comparatively lower control effort.



(a) Hare Population, H (b) Lynx Population, L

Figure 4: U-IMPSP applied to Hare & Lynx Problem



(a) Control Input, u_1 (b) Control Input, u_2

Figure 5: Control applied to Hare & Lynx Problem

Note: The mean for the Lynx population as seen in Figs. 3b and 4b appears to be slightly skewed, this is due to the disparity between the points lying above (≈ 400) and below the mean (≈ 600) at final time.

4 Conclusion

An unscented MPSP design for impulsive systems was discussed in order to handle the uncertainties in the initial conditions and/or parameters. The control was designed to achieve the final time constraints as well as to minimize the final time variance for impulsive systems. An example prey-predator problem containing parametric uncertainties was discussed to verify the algorithm.