# NAVARCH 583 – Adaptive Control
# Multiple Estimation Models for Adaptive Control
## Project Report

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# 1 Introduction

Transient response of adaptive systems is not predefined while their steady state behaviour can be predicted. Large errors in the initial parameter estimates can result in poor and oscillatory transient performance characteristics. This affects operational safety and reliability and can also result in failure in practical systems. Examples of such systems include flight control systems due to controller effector failures [1] and flexible transmission systems [2]. Thus, there is a need to improve the transient performance of adaptive systems, in terms of parametric, identification and tracking errors. To achieve this, concepts of multiple models, switching and tuning have been applied extensively in control theory. Back in the 1970's, multiple Kalman filter-based estimation models were studied to improve the accuracy of state estimates [3, 4]. For adaptive control, the method of switching was first introduced by Mårtennson [5]. In [6, 7], multiple adaptive identification models were set up to identify an LTI system for the case of model reference adaptive control. Switching technique based on minimization of a performance index at every instant was utilized. The results showed an improvement in the transient response in the presence of large parametric uncertainties. Therefore, the use of multiple estimation models has been shown to cope well with rapid changes in plant parameters, resulting in faster, more accurate transient response in adaptive systems. As part of our project, we conducted a study on the efficacy of multiple estimation models and investigated their use in adaptive systems for identification and control in order to improve transient performance. Furthermore, detailed comparative studies are presented between the different parameter estimation algorithms, switching methodologies and sensitivity to number of estimation models.

# 2 Adaptive Control using Multiple Models

We consider the Multiple Models, Switching and Tuning (MMST) methodology proposed by Narendra and Balakrishnan [7, 8] for continuous-time systems. This was extended to discrete-time systems by Narendra and Xiang [9]. The standard Model Reference Adaptive Controller

(MRAC) can suffer from poor transient performance, in parametric, identification and tracking errors. This is primarily due to a poor initial estimate of the parameter vector.

In the multiple model algorithm for discrete-time adaptive systems, a number of models (say $M$) with identical structures, but with different initial estimates of the plant parameter are initialized in the parameter space (ref Figure 1). Each of these is updated according to a parameter estimation algorithm such as projection algorithm or recursive least-squares [10] at each sample. At each sample, one model is chosen according to a predefined criterion, and the parameter estimates corresponding to that model are used for control design. The most common criterion used is a minimum identification error criterion. This model is then subsequently used for control design.

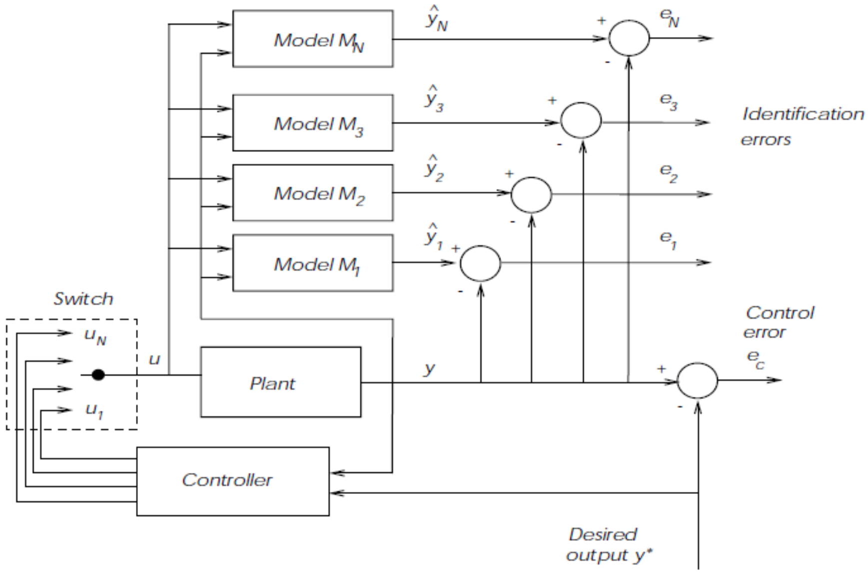



Figure 1: Multiple Models, Switching and Tuning Overview [11]

# 3 Design Approach

## 3.1 Parameter Estimation Algorithms

In many practical problems, some a priori knowledge is of use to design the adaptive laws. One would like to use such a priori information to search for estimates of the parameter in the parameter space. Such a procedure might speed up convergence and help reduce larger transients.

Consider an nth-order discrete-time plant of the form:

$$y_{k+d_0} = a_1 y_k + a_2 y_{k-1} + \ldots + a_n y_{k-n+1} + b_1 u_k + b_2 u_{k-1} + \ldots + b_n u_{k-m+1} \tag{1}$$

where, $a = [a_1,\ a_2,\ \ldots,\ a_n]^{\mathrm{T}}$ and, $b = [b_1,\ b_2,\ \ldots,\ b_m]^{\mathrm{T}}$ are unknown. We rely on the methods of on-line parameter estimation to estimate the unknown plant parameters. To do this, we first obtain an appropriate parameterization of the plant model of the form:

$$z_k = \theta^{*T}\phi_k$$

where $\theta^* = [b,\ a]^{\mathrm{T}}$ and $\phi_k = [u_k,\ u_{k-1},\ \ldots,\ u_{k-m+1},\ \ y_k,\ y_{k-1},\ \ldots,\ y_{k-n+1}]^T$. Next, we update our plant parameters $\theta_k$ using a parameter estimation algorithm. As part of our project, we have used the projection and recursive least squares algorithm.

The projection algorithm for estimating $\theta^*$ is given by:

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \frac{\phi_k}{1 + \phi_k^T\phi_k}\left[z_k - \hat{\theta}_k^T\phi_k\right]$$

Here, $\hat{\theta}$ is the parameter estimate for a single parameter estimate. The least square algorithm for estimating $\theta^*$ is given by:

$$\theta_k = \theta_{k-1} + P_k\phi_k\epsilon_k$$

$$P_k = \left(P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^{\mathrm{T}}P_{k-1}}{\lambda + \phi_k^{\mathrm{T}}P_{k-1}\phi_k}\right)/\lambda$$

where $\epsilon_k = z_k - \hat{\theta}_k^T\phi_k$. $P = \rho_0 I$ and $\lambda$ are design parameters.

We use the projection and least squares algorithm for $M$ estimation models and estimate $\hat{\theta}_{j,k}$, where $j = 1, \ldots, M$.

## 3.2 Switching Methodologies

At each sample, a model is picked based on a predefined switching criterion. The most common criterion used is the minimum instantaneous identification error. We have also explored two other switching criterion, minimum accumulated identification error and minimum weighted identification error. [11]

### 3.2.1 Minimum Instantaneous Identification Error

In this method, we pick the model $j^*$ that satisfies

$$j^* = \arg\min_{j=1,\ldots,M} |\hat{e}_j(k)| \tag{2}$$

where $\hat{e}_j(k)$ denotes the identification error corresponding to model $j$ at sample $k$.

### 3.2.2 Minimum Accumulated Identification Error

In this method, we pick the model $j^*$ that satisfies

$$j^* = \arg\min_{j=1,\ldots,M} \sum_{\tau=0}^{k} |\hat{e}_j(\tau)| \tag{3}$$

where $\hat{e}_j(\tau)$ denotes the identification error corresponding to model $j$ at sample $\tau$.

### 3.2.3 Minimum Weighted Identification Error

In this method, we pick the model $j^*$ that satisfies

$$j^* = \arg\min_{j=1,\ldots,M} \sum_{\tau=0}^{k} \rho^{k-\tau} |\hat{e}_j(\tau)| \tag{4}$$

where $\hat{e}_j(\tau)$ denotes the identification error corresponding to model $j$ at sample $\tau$, and $\rho \in [0,1]$ is a user defined memory weight.

## 3.3 Control Algorithm

For the model given in Equation 1, the predictive control to ensure $y_k$ tracks reference $r_k$ is given by:

$$u_k = \frac{1}{b_1} \left[r_{k+d_0} - b_2 u_{k-1} - \ldots - b_m u_{k-m+1} - a_1 y_k - \ldots - a_n y_{k-n+1}\right]$$

When the parameter $\theta = [b_1,\ b_2,\ \ldots,\ b_m,\ \ a_1,\ a_2,\ \ldots,\ a_n]^T$ is unknown, we use the parameter estimates $\hat{\theta}_{j,k},\ j = 1,\ldots,M$ from Section 3.1, and decide on the best estimate $\hat{\theta}_{j^*,k}$ using any methodology from Section 3.2 and finally design the input as:

$$u_k = \frac{1}{\hat{b}_{1,j^*,k}} \left[r_{k+d_0} - \hat{b}_{2,j^*,k} u_{k-1} - \ldots - \hat{b}_{m,j^*,k} u_{k-m+1} - \hat{a}_{1,j^*,k} y_k - \ldots - \hat{a}_{n,j^*,k} y_{k-n+1}\right]$$

# 4 Simulation Results

In this section, simulation results are presented for the methodologies discussed previously. A detailed comparison study between the different parameter estimation algorithms, switching methodologies and number of estimation models is also presented.

We consider a third-order, discrete-time plant:

$$y_{k+1} = a_1 y_k + a_2 y_{k-1} + a_3 y_{k-2} + b_1 u_k + b_2 u_{k-1} + b_3 u_{k-2} \tag{5}$$

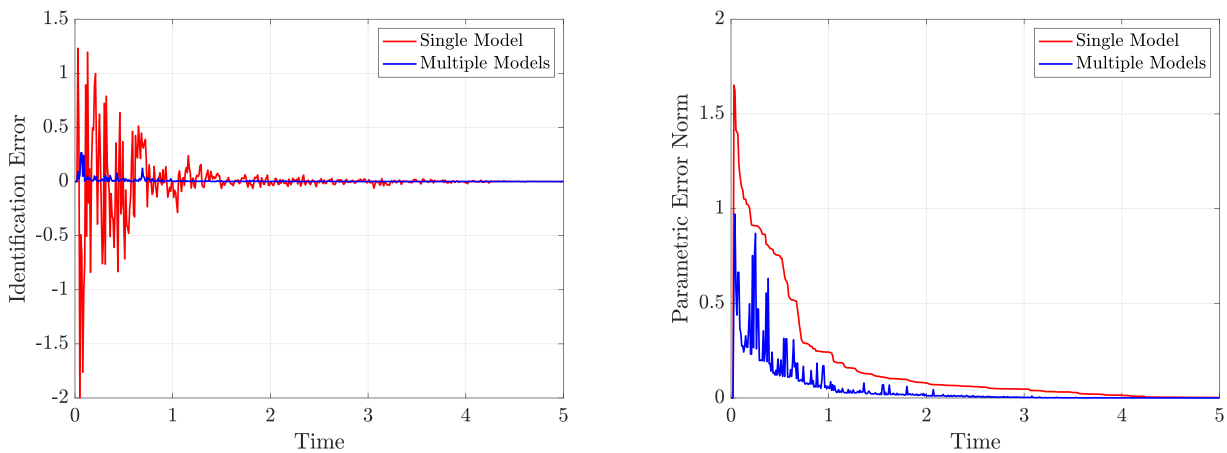
where, $a = [a_1,\ a_1,\ a_3]^{\mathrm{T}} = [1,\ 0.125,\ \ -0.3125]^{\mathrm{T}}$ and, $b = [b_1,\ b_1,\ b_3]^{\mathrm{T}} = [1.25,\ \ -0.6,\ 0.45]^{\mathrm{T}}$. The system is parameterized as,

$$z_k = \theta^{*T} \phi_k$$

where $\theta^* = [b,\ a]^{\mathrm{T}}$ and $\phi_k = [u_k,\ u_{k-1},\ u_{k-2},\ y_k,\ y_{k-1},\ y_{k-2}]^{\mathrm{T}}$.

In the next few subsections, we present results for the comparison of single versus multiple models, followed by analysis on sensitivity to change in number of estimation models for $M = 3, 10, 20$ and 50. The initial conditions for these estimation algorithms are chosen at random. Finally, we present results for different switching methodologies.

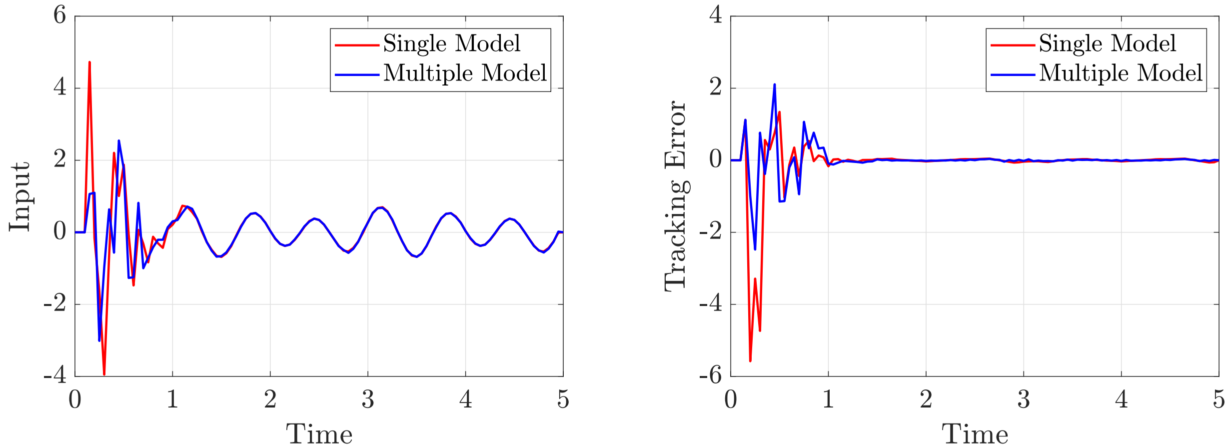
## 4.1 Single Model vs Multiple Model




(a) Identification Error (b) Parameter Error

Figure 2: Projection algorithm: random input excitation.

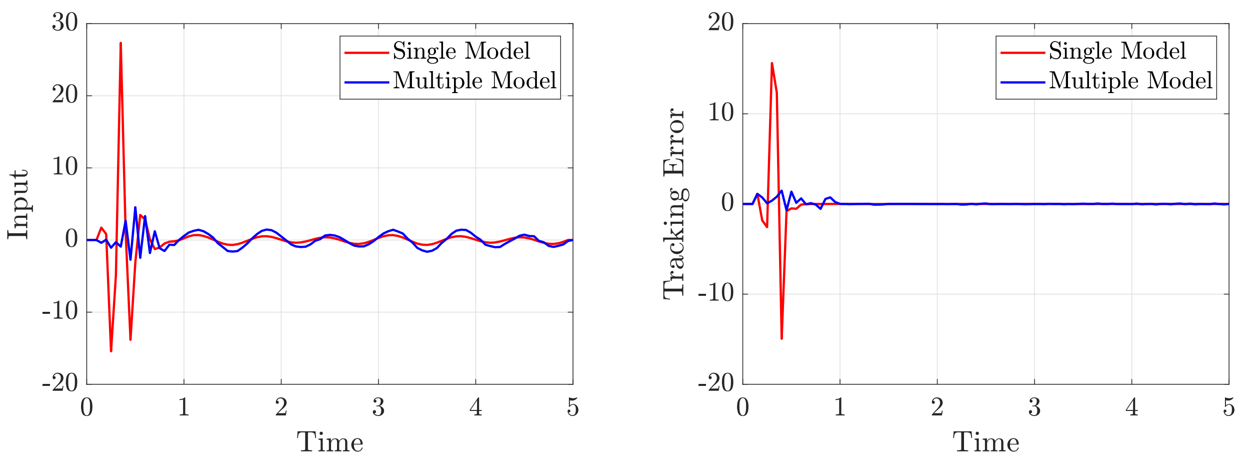
It can be seen from Fig 2 that using multiple model results in lower transients, faster identification and parametric error convergence when compared to the case with a single estimation model for random input excitation. Note that for this comparison we have used projection algorithm for parameter identification, $M = 10$ and minimum instantaneous identification error as the switching methodology.




(a) Control Input (b) Tracking Error

Figure 3: Projection algorithm: tracking a sinusoidal signal $r_k$.

In the next set of simulations, we track a reference signal: $r_k = \sin(2\pi t) + 2\cos(3\pi t)$. Figures 3 and 4 show the tracking and control energy results for projection and recursive least squares algorithm respectively.



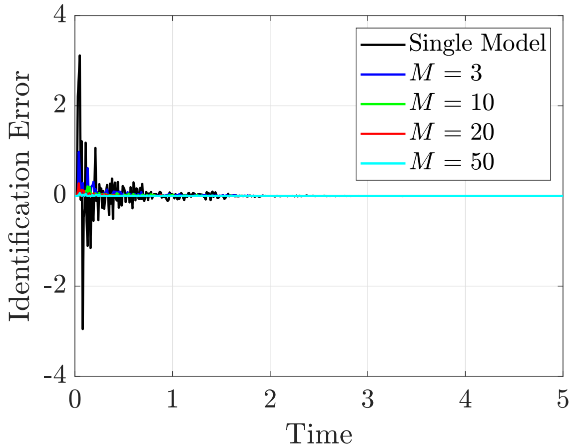

(a) Control Input (b) Tracking Error

Figure 4: Recursive least squares algorithm: tracking a sinusoidal signal $r_k$.

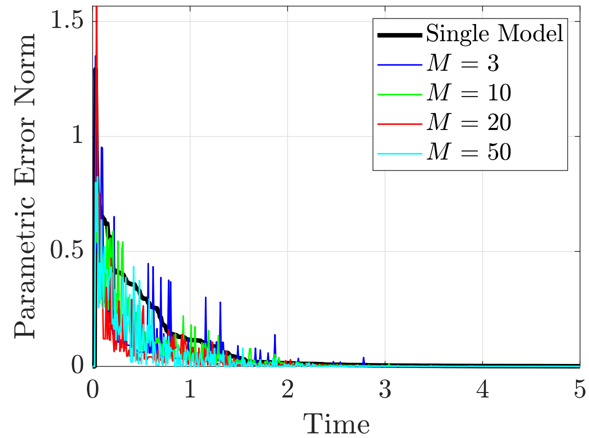
Again we see that for both estimation algorithms, multiple models give lower tracking error and use lesser control energy in the transient phase.

## 4.2 Comparison with different number of models

Fig. 5 shows the system identification results with random excitation $u_k$ for the projection algorithm with various number of multiple models, and Fig. 6 shows the same for recursive least square algorithm. The switching methodology used here is that of the minimum instantaneous identification error.



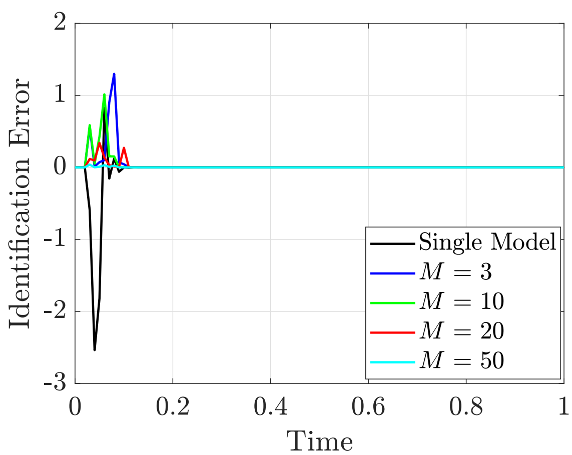

(a) Identification Error



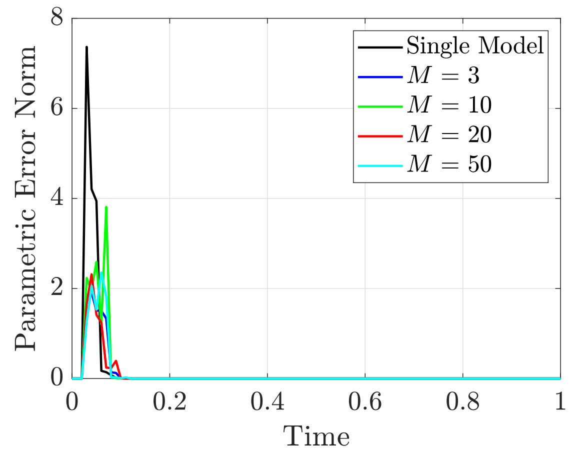

(b) Parametric Error

Figure 5: Projection algorithm: random excitation of $u_k$.

From figures 5 and 6, we can conclude that the increase in the number of estimation models generally results in better performance. However, the difference between the performance of $M = 20$ and $M = 50$ is very slight. It should be noted that each run with the same number of models results in a different output due to random initialization for all the simulation results. However, the general consistency of the results remain.



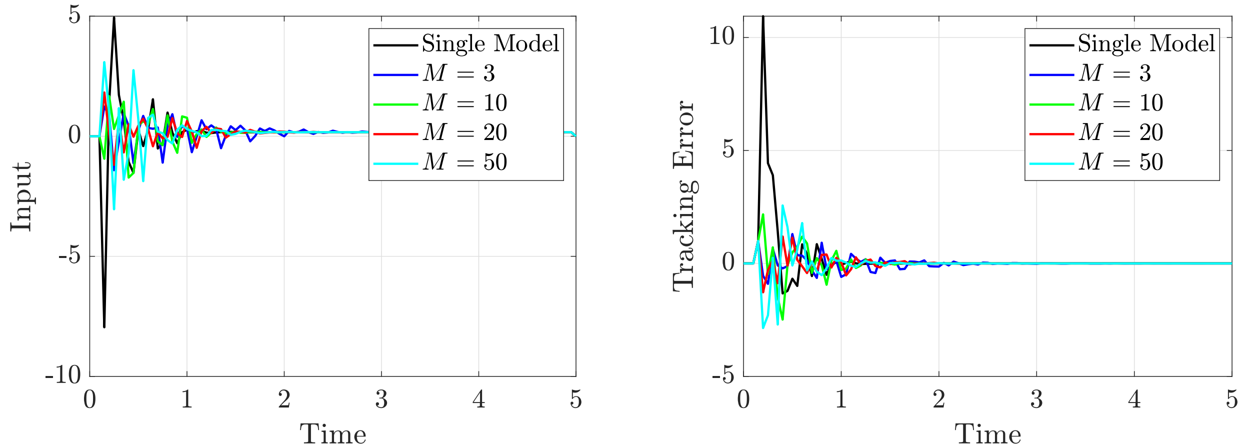

(a) Identification Error




(b) Parametric Error

Figure 6: Recursive least squares algorithm: random excitation of $u_k$.

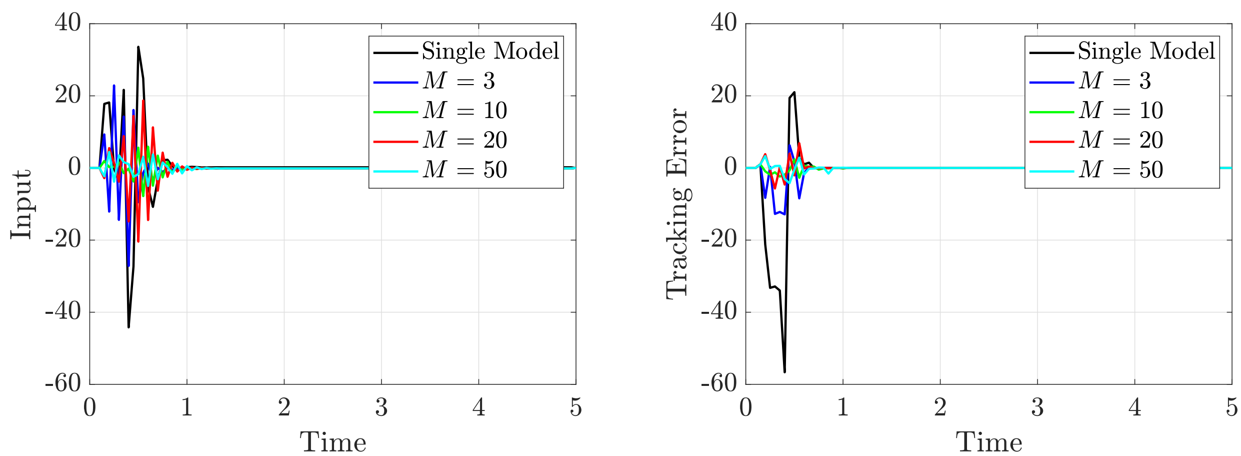
Fig. 7 shows the results for control where the goal is to track a unit step signal ($r_k$). Fig. 8 shows the same using recursive least squares algorithm.




(a) Control Input (b) Tracking Error

Figure 7: Projection algorithm: tracking a unit step signal $r_k$.

Overall, it can be concluded that the case with multiple estimation model results in faster convergence, lower transients and lower overall utilization of control energy to track the reference signal $r_k$. Increasing the number of models generally improved the performance until $M = 20$. Increasing beyond this did not improve the performance as seen in $M = 50$.

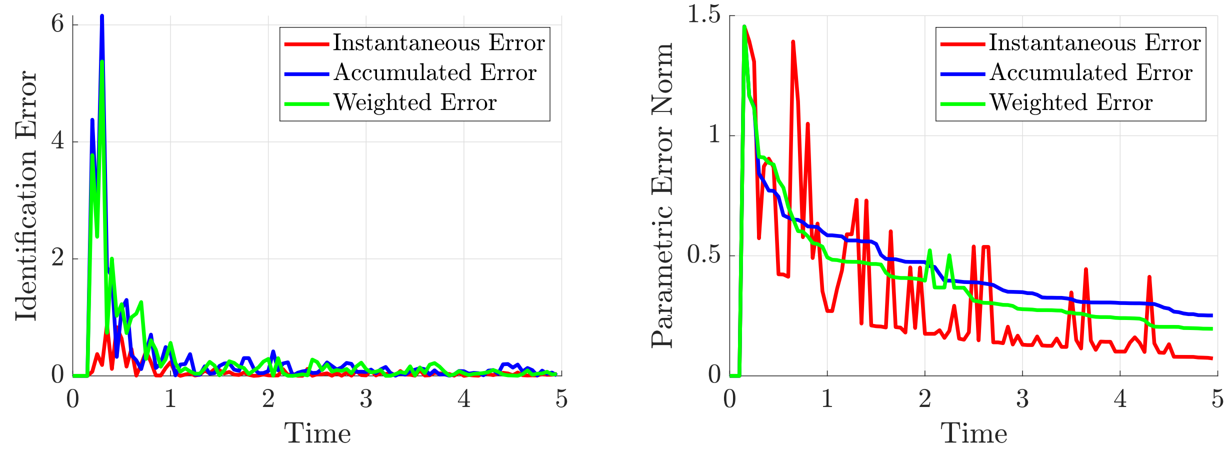



(a) Control Input (b) Tracking Error

Figure 8: Recursive least squares algorithm: tracking a unit step signal $r_k$.

## 4.3 Different Switching Methodologies

Three different switching methodologies as specified in Section 3.2 are implemented with projection algorithm for estimation and $M = 10$. For the minimum weighted identification error, the memory factor $\rho$ is taken as 0.9. Fig. 9 shows the identification and parameter error for random excitation input $u_k$. We can see that even though the identification error is lower for instantaneous error methodology, the parametric errors are monotonic for accumulated error.

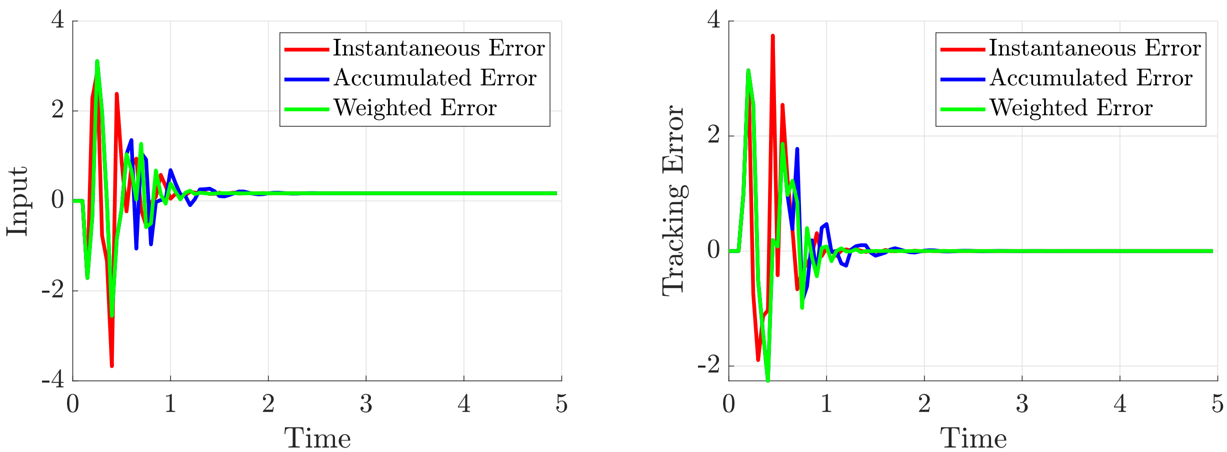



(a) Identification Error

(b) Parametric Error

Figure 9: Comparison of different switching methods: Projection algorithm, random input excitation

Fig. 10 shows the control input and tracking error for tracking a step input signal. Again, we can see that the transients are slightly lower for accumulated error.




(a) Control Input

(b) Tracking Error

Figure 10: Comparison of different switching methods: Projection algorithm, step input tracking

# 5 Conclusion

From the simulation studies presented in Section 4, we can conclude that the use of multiple estimation models result in lower transients and faster convergence when compared to the case of single model for both projection and recursive least squares algorithm. Compared to the projection algorithm, the recursive least squares algorithm resulted faster convergence at the cost of higher transients.

Increasing the number of estimation models improved the transient and convergence performance up to a certain threshold. It is important to remember that this comes at a cost of added computational complexity. Therefore the designer should consider the trade-off between the desired accuracy versus the computation budget of the system while choosing the number of estimation models.

Finally, we also compare different switching methodologies. Though the accumulated error methodology gives lower transients than using instantaneous error for switching, there are some cases when instantaneous error can be beneficial for lower parametric errors, especially when the range of the parameters is known to some extent. Weighted error methodology proves to be a good middle ground between these two, especially since you can tune the memory factor for the best performance.

The future directions for this project could include investigating the MMST methodology on continuous-time systems, trying a more sophisticated switching methodology and testing its effectiveness on a real-world system.

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