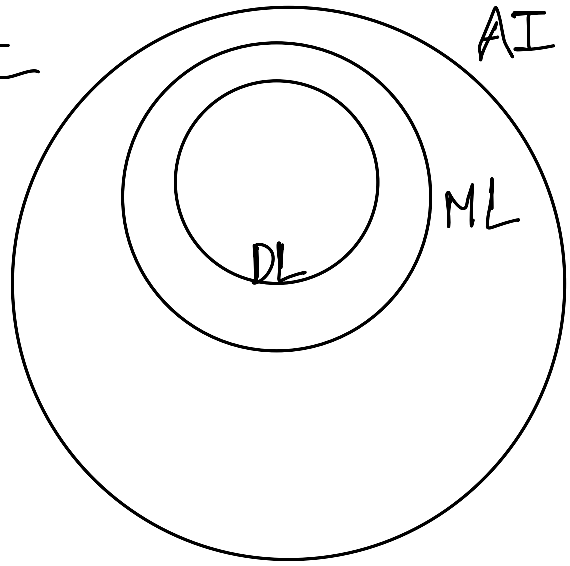
Diff. b/w AI, ML, DL



Notation

Vector : $v \in \mathbb{R}^d$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}$$

$$v^T = [v_1 \; v_2 \; \cdots \; v_d]$$

Matrix : $A \in \mathbb{R}^{m \times n}$

$$A = m\left\{ \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{n} \right.$$

m row-vectors of n dimension

or

n col vectors of m dim.

Vector-vector operations

• Inner product / dot product (Diff. concepts but same for this course)

$x, y \in \mathbb{R}^d$

$$\sum_{i=1}^{d} x_i y_i \quad \text{or} \quad x^T y$$

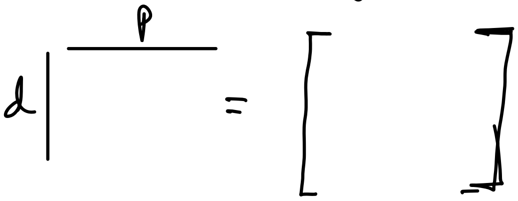
= • (scalar)

(2 vectors)

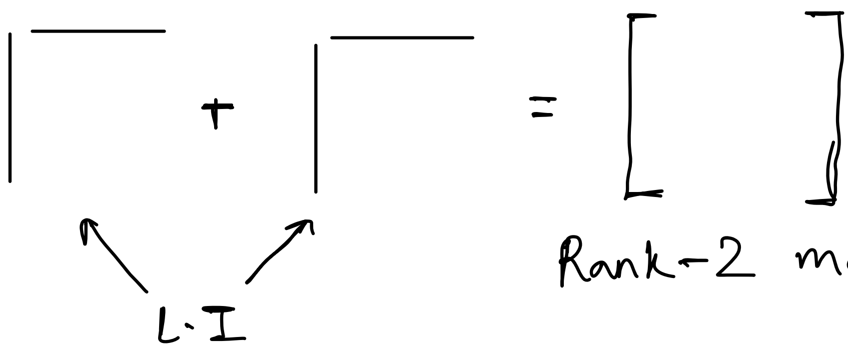
- Outer Product

$x \in \mathbb{R}^d, \; y \in \mathbb{R}^p$

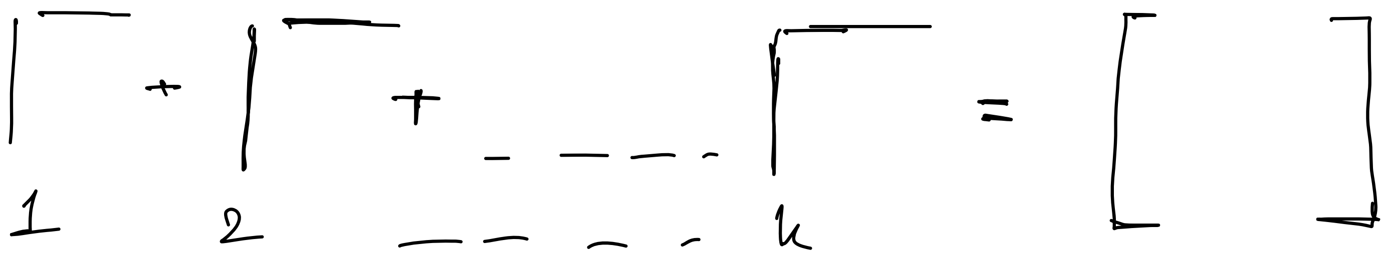
$xy^T \qquad yx^T$



Rank-1 matrix

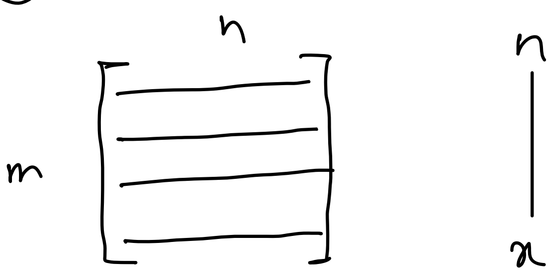


Rank-2 matrix



Rank $\leq \min(d, p, k)$

Matrix-vector operations



$A \in \mathbb{R}^{m \times n} \qquad x \in \mathbb{R}^n$

$Ax = \begin{bmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{bmatrix} m$

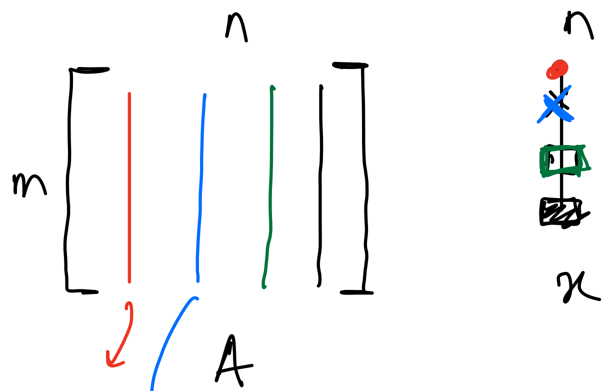
$Ax \in \mathbb{R}^m$

Interpretation

I

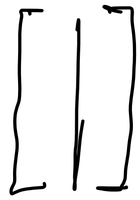
m. inner products of n-dim vectors.

Scaling means multiplying each element of that vector with another element.



Scale this by •

Scale this by x

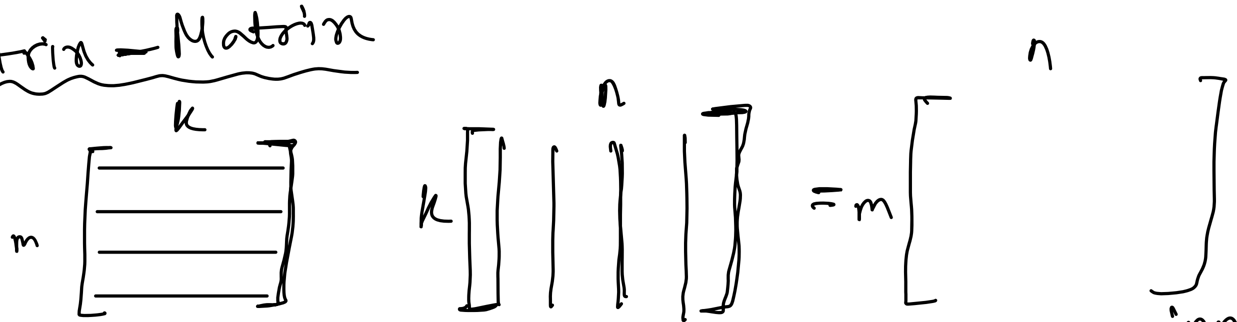
$Ax =$ 

Interpretation

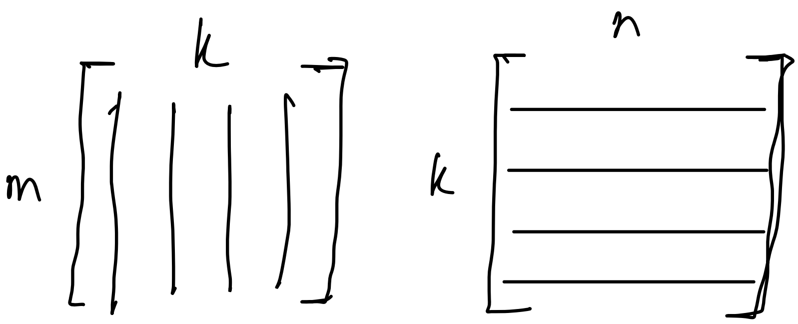
||

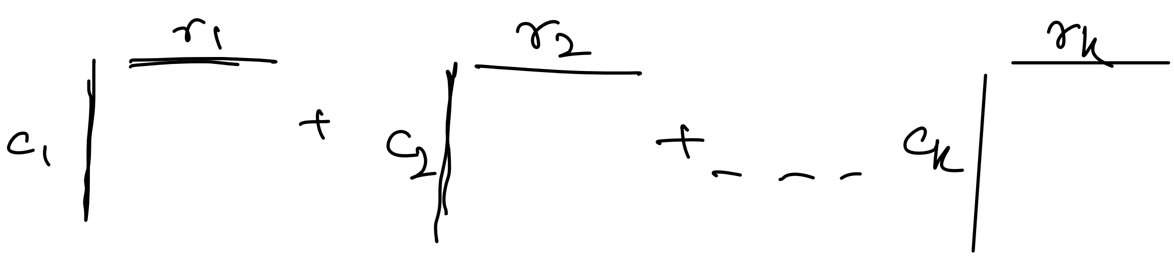
sum of all the 'scaled' m-dim. vectors

Matrix - Matrix

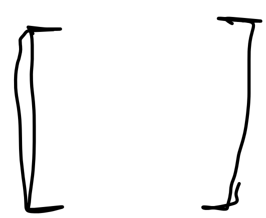


inner-product





outer-product

= 

* Both results will be same.

Application in ML

① Represent Data

$$X = \begin{array}{c} H_1 \\ H_2 \\ \vdots \\ H_n \end{array} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad \text{→ features}$$

design matrix

$$y = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{array}{l} \text{→ price of } H_1 \\ \vdots \\ \text{→ price of } H_n. \end{array}$$

value vector.

② Covariance Matrices

$$S \in \mathbb{R}^{d \times d}$$

③ Calculus :

Gradients – Vector

Hessians – Matrix (Symmetric)

Jacobians – Matrix

④ Kernel Method

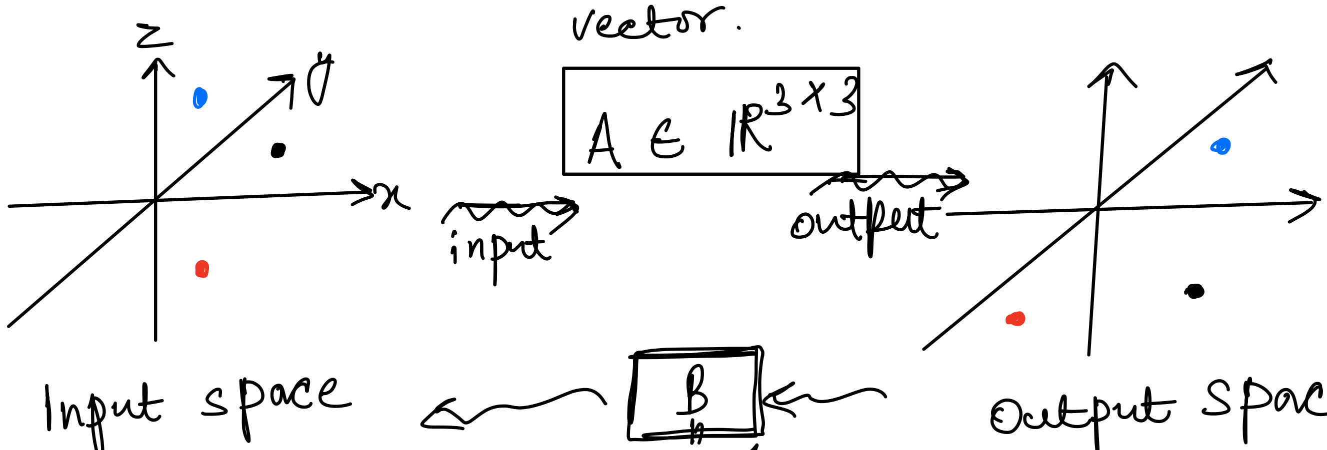
Geometrical Interpretation

$A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^{n}$

$A(x) \in \mathbb{R}^{m}$

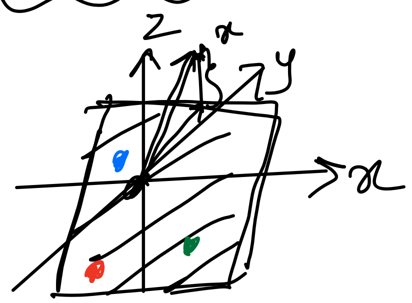
↳ Interpret this as func^n which takes vector as input and outputs a vector.



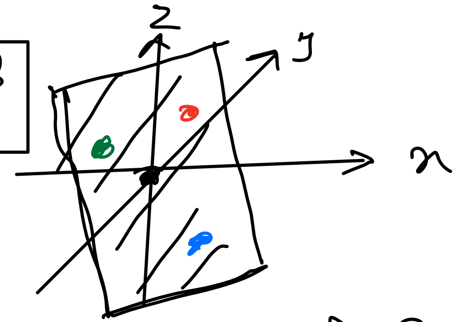
FULL RANK

- If A is full-rank then B exists.
- A is full-rank $\iff$ one to one mapping b/w I/O.

RANK DEFICIENT (Rank-2)



$$\boxed{A \in \mathbb{R}^{3\times 3}}$$



Rank-2 interpretation $\Rightarrow$ $\exists$ a 2-D subspace in input & output space which have one to one I/O mapping.

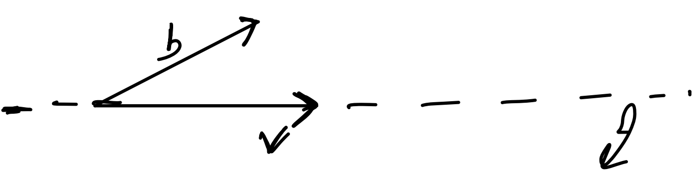
What about points outside this subspace?

Decomposition of this vector.

$$x = \text{Proj}(x;\ \text{Row space}) + \text{Proj}(x;\ \text{null.space})$$

$$A(x) = A(x_R + x_N)$$
$$= A(x_R) + \underbrace{A(x_N)}_{=0}$$
$$= A(x_R)$$

Projection



Projection matrix (V)

$$= \left[\frac{VV^T}{V^TV}\right] b$$

$$\frac{VV^T}{V^TV} b = \left(\frac{V}{\|V\|}\right)\left(\frac{V}{\|V\|}\right)^T b$$

$$= [\tilde{V}\,\tilde{V}^T]\,b$$

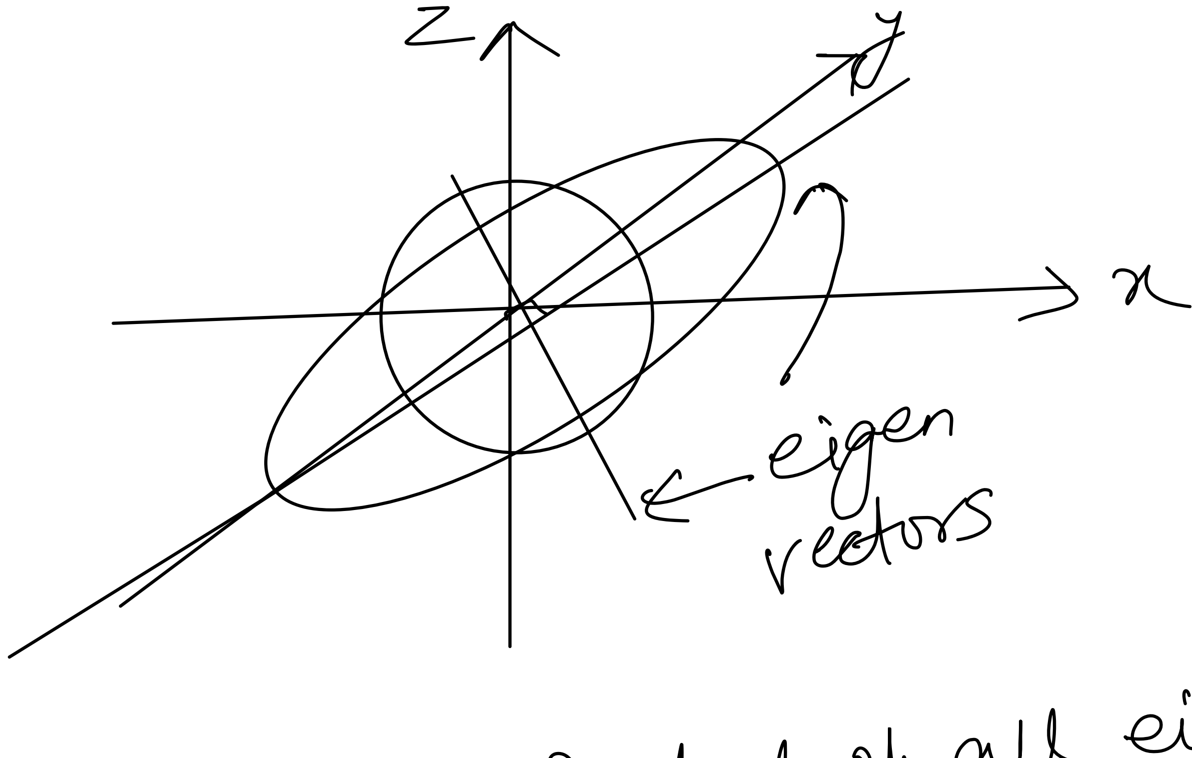
$$= \tilde{V}\,\underbrace{(\tilde{V}^T b)}_{\text{length}}$$

$$X = \begin{bmatrix} | & | & | & | \end{bmatrix}$$

set of L.I vectors

$X(X^TX)^{-1}X^T \rightarrow$ projection matrix.

$A \in \mathbb{R}^{3\times 3} \rightarrow$ symmetric



Determinant = Product of all eigen values

$$= \frac{\text{Vol. of output shape}}{\text{Vol. of input shape}}$$

For Non-full rank, atleast one dim. will be 0.

Thus, det. $= \frac{0}{x} = 0$.

<u>Spectrum</u>: collection of Eigen values

<u>Spectral Theorem</u>

$A \in \mathbb{R}^{d \times d}$, $A = A^T$

→ Real valued eigen values
→ orthonormal eigen vectors.

Hessians, Covariance Matrix, Kernel

square & symmetric.

Quadratic Forms

$A \in \mathbb{R}^{d \times d}$, $x \in \mathbb{R}^d$

$x^T A x$ — Quadratic Form

$$x^T B x = x^T A x \quad | \quad B = B^T$$

$$B = \frac{1}{2} A + \frac{1}{2} A^T$$

Definitiveness

$x^T A x > 0 \quad \forall x \neq 0 \rightarrow$ A is positive definite

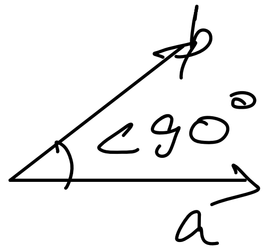
$\geq 0 \quad \forall x \neq 0 \rightarrow$ A is positive semi definite

< 0 " " negative definite

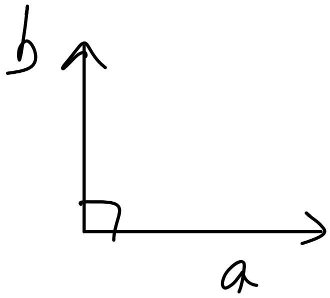
≤ 0 " " " semi-def.

$\geq, <$ " " indefinite

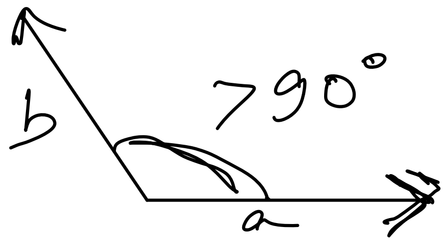
$a^T b$



> 0



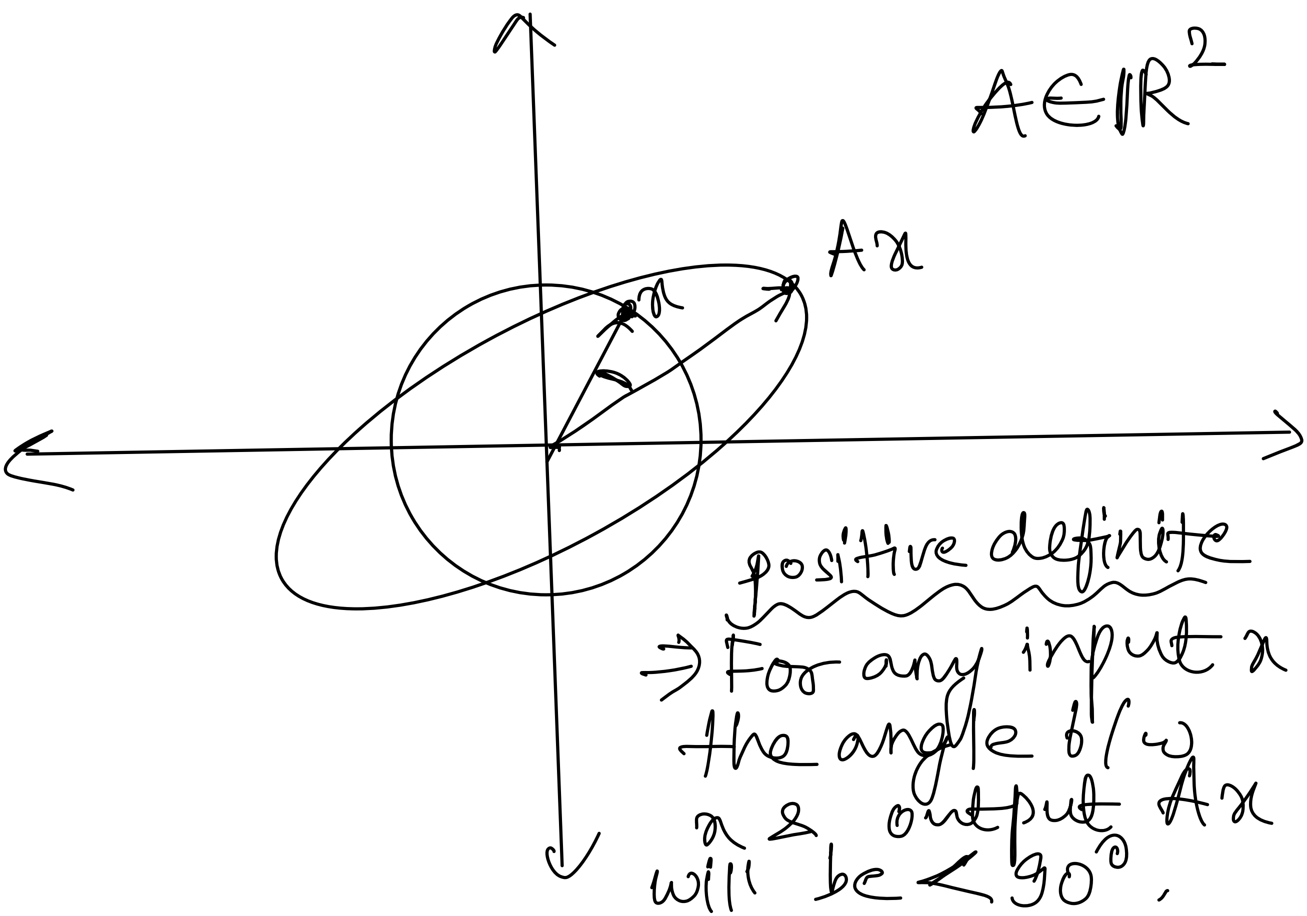
$= 0$



< 0

$\underbrace{(x)^T}_{\text{input}} \underbrace{Ax}_{\text{output}}$ → dot product of input & output

$A \in \mathbb{R}^2$



positive definite

⇒ For any input x the angle b/w x & output Ax will be $< 90°$.

For a P.D matrix

$\Rightarrow$ All eigenvalues > 0

P.S.D

$\rightarrow$ " " ≥ 0

N.D

$\rightarrow$ " " < 0

N.S.D

$\rightarrow$ " " ≤ 0

I.D

$\rightarrow$ " " $>, < 0$

Link b/w definitiveness & spectrum.

Decomposition of Matrices

- Singular value Decomposition
- Eigen value Decomposition.
- Cholesky Decomposition.

	A	Decomposition
SVD	Any	$A = USV^T$
EVD	square	$A = UDU^{-1}$

$$A(x) = U(S(V^T(x)))$$

Decomposed func^{n}s/ matrices

$U, V \Rightarrow$ orthonormal matrices

$S \rightarrow$ Diagonal

$D \rightarrow$ Diagonal

For EVD

$U \rightarrow$ set of eigen vectors

$D \rightarrow$ set of eigen values

$U \begin{bmatrix} | & | & | \end{bmatrix} \quad \begin{bmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{bmatrix} D$

A	Step 1	Step 2	Step 3
SVD	V^T (Rotation 1)	Scaling along axis (Real valued)	U (Rotation 2)
EVD	U^{-1} (Rotation)	Scaling along axes (Rotation if e.v is complex)	U (inverse of step 1)

A arbitrary → S.V.D

square → S.V.D & E.V.D may not be identical

square & symmetric → S.V.D & E.V.D same

Matrix Calculus

func^{n}s	Eg.	value	first der.	Sec. der.
$f: \mathbb{R} \to \mathbb{R}$	x^2	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
$f: \mathbb{R}^d \to \mathbb{R}$	loss func^n	$\mathbb{R}$	$\mathbb{R}^d$ [Gradient]	$\mathbb{R}^{d \times d}$ [Hessian] S^d (symmetric with dim. d)
$f: \mathbb{R}^d \to \mathbb{R}^p$	N.N layer	$\mathbb{R}^p$	$\mathbb{R}^{d \times p}$ [Jacobian]	$\mathbb{R}^{d \times p \times p}$ [Higher order Tensor]

Gradient $\to$ direc^n of steepest ascent

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_d} f(x) \end{bmatrix}$$

$$\nabla_x f(x) = \nabla_x f(x_1, x_2, \ldots, x_d)$$

$f: \mathbb{R}^d \to \mathbb{R}$

A vector

For matrices,

$$\nabla_A f(A) = \begin{bmatrix} \frac{\partial}{\partial a_{11}} f(A) & - & - & - \\ - & & - & \\ - & - & - & \frac{\partial}{\partial a_{mn}} f(A) \end{bmatrix}$$

$f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$

$f: \mathbb{R}^d \rightarrow R$

$\nabla_x^2 f(x)$

$$\nabla_x^2 f(x) = \begin{bmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f(x) & \cdots & \frac{\partial^2}{\partial x_1 \partial x_d} f(x) \\ \vdots & & \\ \frac{\partial^2}{\partial x_d \partial x_1} f(x) & \cdots & \frac{\partial^2}{\partial x_d \partial x_d} f(x) \end{bmatrix}$$

Eg: b is some constant.

$\nabla_x (b^T x)$

$$= \begin{bmatrix} \vdots \\ \frac{\partial}{\partial x_i} (b^T x) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \frac{\partial}{\partial x_i} (b_1 x_1 + \dots b_i x_i + \dots b_d x_d) \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_d \end{bmatrix} = b$$

Product Rule

$$\nabla_x x^T A x = \nabla_x x^T A x + \nabla_x x^T A x$$

$$= Ax + A^T x$$

$$= x(A + A^T)$$

$$= 2Ax \rightarrow \text{if } A \text{ is symm.}$$

$$\nabla_A \log|A| = A^{-1}$$

$$\frac{d}{dx} \log(x) = x^{-1}$$