Lecture programme

Engineering Maths 1.2

Vector Algebra

Dr P. A. Sleigh

Email: P.A.Sleigh@leeds.ac.uk

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2 Scalars and Vectors

In engineering much of the work in both analysis and design involves forces. You will be familiar with forces in structural members in space frames, and know that a force acts in a given direction with a given magnitude. A force is a three dimensional quality and one of the most common examples of a vector. Two other common vector quantities are acceleration and velocity.

In this section of the module we will introduce some formal mathematical notation and rules for the manipulation (multiplication, addition etc.) of these vector quantities. In the early stages it will be easy to recognise the geometric meaning of vector addition and multiplication, but as the problems become more complex (notably, when they are three-dimensional) the formal rules become more important.

The general definitions of scalars and vectors are given below:

A **Scalar** quantity is one that is defined by a single number with appropriate units.
Some examples are length, area, volume, mass and time.

A **Vector** quantity is defined completely when we know both its magnitude (with units) and its direction of application.
Some examples are force, velocity and acceleration.

Two very simple (and common) examples demonstrating the differences between scalars and vectors are speed - a scalar, and velocity - a vector.

A speed of 10 km/h is a scalar quantity
A velocity of 10 km/h at 20 degrees is a vector quantity.

Examples:

What are these quantities, vectors or scalars?

a) A temperature of 100^{o} C
b) An acceleration of 9.8142 m/s towards earth
c) The weight of a 2 kg mass
d) The sum of £500.00
e) A North easterly wind of 20 knots

Answers: a) Scalar b) Vector c) Vector d) Scalar e) Vector

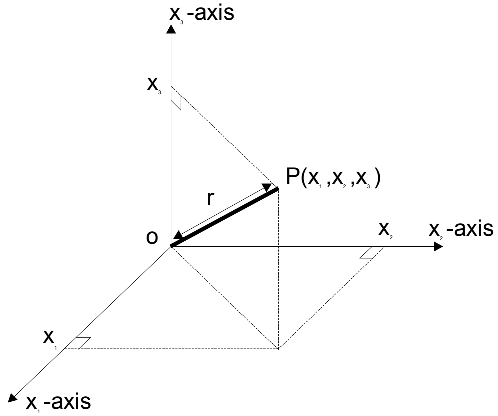
Note: To qualify as a vector the quantity must also satisfy some other basic rules of combination (addition, multiplication etc.) which we will see later in this course. For example, angular displacement is a quantity that has both direction and magnitude but it does NOT obey the addition rule - so it is NOT a vector

2.1 Cartesian Co-ordinates

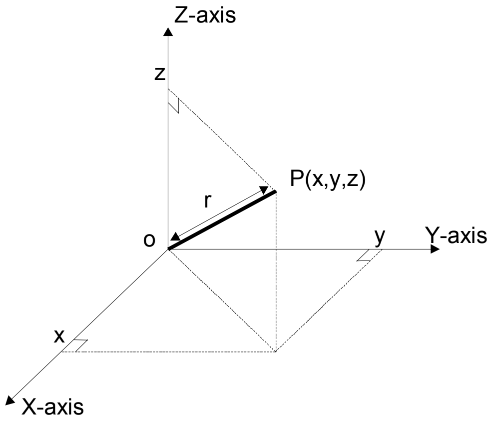
The theory of vectors is associated very closely with co-ordinate geometry so we shall start by introducing the co-ordinate system.

We use a rectangular Cartesian co-ordinate system with axes O_{xyz} or $O_{x_1 x_2 x_3}$, as shown below.

The position of any point, P say, is given by **co-ordinates**, or **components** (x,y,z) or (x_1,x_2,x_3).



x, y, z, axis system



x_1 , x_2 , x_3 axis system

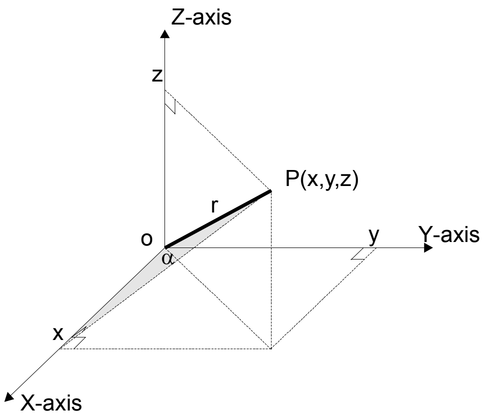
The order and direction of axes are assumed to be 'right handed'. This definition comes from imagining a right handed screw - if you turn the screw in the direction from axis O_x to O_y the screw advances along O_z . (and O_y to O_z along O_x, from O_z to O_x along O_y). This is convention and MUST be adhered to.

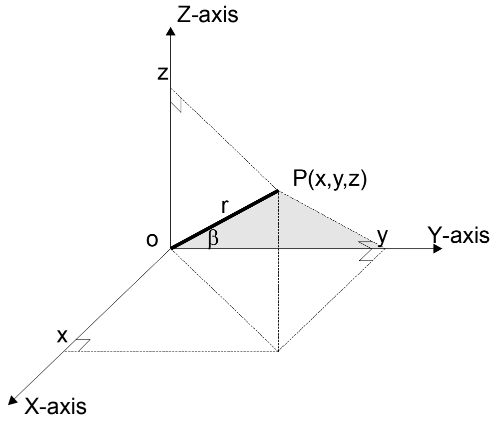
The distance from point O to point P, **r**, can be calculated using the Pythagoras theorem to give

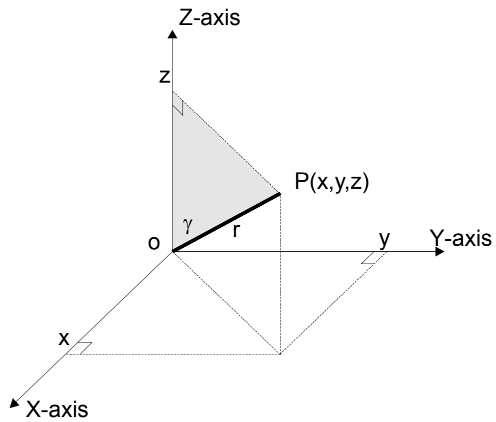
$$\mathbf{r} = \left(x^2 + y^2 + z^2\right)^{1/2}$$

2.2 Direction Cosines

We will see later that it is useful to know the angle the line OP makes with each axis. Fortunately this can be easily calculated. The shaded areas of the figures below each form right-angled triangles with the axes.







So, knowing the length and the appropriate co-ordinate, each angle can be calculated.

The angle $\angle POx$ is given by

$$l = \cos\alpha = \frac{x}{r}$$

The angle $\angle POy$ is given by

$$m = \cos\beta = \frac{y}{r}$$

The angle $\angle POz$ is given by

$$n = \cos\gamma = \frac{z}{r}$$

The cosines of these angles, α, β, γ are (often written) l, m, n and these are known as the **direction cosines** of the line OP.

It can easily be shown that

$$l^2 + m^2 + n^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2 + y^2 + z^2}{r^2} = 1$$

2.2.1 *Examples*

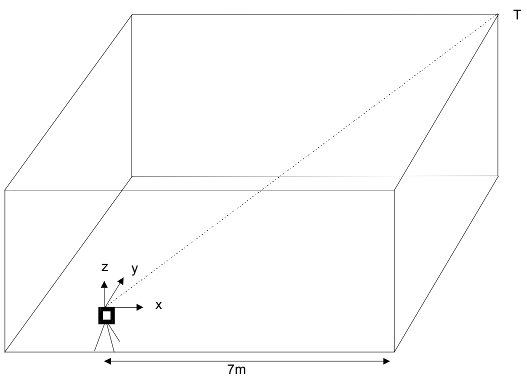
1) If P has co-ordinates (5, 4, -9). What is the length OP and its direction cosines.

$$OP^2 = 25 + 16 + 81$$
$$OP = \sqrt{122}$$

Direction cosines:

$$l = \frac{5}{\sqrt{122}}$$
$$m = \frac{4}{\sqrt{122}}$$
$$n = \frac{-9}{\sqrt{122}}$$

2) The position of the top corner of the lecture room was required. To find this a theodolite was set up as in the figure below. The distance to the side wall, along the x-axis is, 7m and the following angles were measured: TOx = 70^o and TOy = 50^o .
Calculate the co-ordinates of the corner relative to the position of he theodolite.



Calculate the direction cosines:

$$l = \frac{x}{r} = \cos\alpha = \cos 60 = 0.5$$
$$m = \frac{y}{r} = \cos\beta = \cos 50 = 0.6428$$

We need the third direction cosine.

Using $l^2 + m^2 + n^2 = 1$

$$m = \frac{z}{r} = \cos\gamma = \sqrt{1 - l^2 - m^2}$$
$$= 0.58035$$
$$\gamma = 54.52^\circ$$

The length OT, (**r**) can be calculated as we know the x co-ordinate, i.e. from l

$$l = \frac{x}{r} = 0.5$$
$$r = \frac{7.0}{0.5} = 14.0$$

From the other direction cosines

$$y = 0.6428 \times 14.0 = 9.0$$
$$z = 0.58035 \times 14.0 = 8.125$$

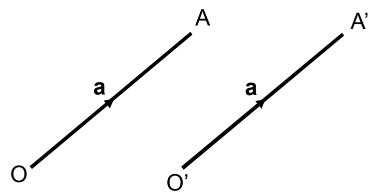
Co-ordinates of the corner are: $(x, y, z) = (7.0, 9.0, 8.125)$

2.3 A Geometric Representation

We represent vectors geometrically by line segments in space. The length of the line representing the magnitude and the direction of the line the direction of the vector.

From this definition the starting point is irrelevant.

What is the difference between these two vectors?



Their length is the same.

Their direction is the same.
(The arrows indicate the direction along the line)

These two vectors are equivalent (equal).

By the same argument, the two vectors are also equivalent to a vector of the same length and direction which starts at the origin. We can therefore use a co-ordinate notation of three numbers - as used previously for OP - to represent any vector (anywhere in three dimensional space).

We have shown a vector described in two ways: in terms of co-ordinates (x, y, z) and also represented geometrically by a line from an origin to this point. We say a dual definition can be used either a **geometrical** or a **co-ordinate (component)** one.

2.4 Vector notation found in text books

A vector can be written down in many ways. Some of the more common (and acceptable) ways you will come across are:

$$\underline{a} \qquad \overrightarrow{OA} \qquad \mathbf{a} \qquad (a_1, a_2, a_3)$$

2.5 Magnitude (or Modulus)

The magnitude, modulus or length of the vector $\mathbf{a}$ is written as $|\mathbf{a}|$ or $|\overrightarrow{OA}|$ and given in component form by

$$|\mathbf{a}| = \left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}$$

Geometrically this is the length of the line OA

2.6 The Unit Vector

A vector whose modulus is 1 is called a <u>unit vector</u>, sometimes written as $\hat{\mathbf{a}}$ and

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

<u>The unit vectors *i, j, k*</u>
The unit vectors in the co-ordinate directions are denoted *i, j,* and *k.*

$$i = (1,0,0), \quad j = (01,0) \quad \text{and} \quad k = (0,0,1)$$

Any vector $\mathbf{r}$ can be expressed in terms of its component x, y, z with respect to the unit vector (i,j,k) by

$$\mathbf{r} = xi + yj + zk$$

The notation (x,y,z) is interpreted as (xi, yj, zk).

2.6.1 Example:

A vector $\mathbf{a} = (3,-1,4) = 3i - j + 4k$

The modulus of $\mathbf{a} = |a| = \left(3^2, (-1)^2, 4^2\right)^{1/2} = \sqrt{26}$

So the unit vector in the direction of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{3}{\sqrt{26}} i - \frac{1}{\sqrt{26}} j + \frac{4}{\sqrt{26}} k\right)$$

2.7 Equal Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal if they have the same **magnitude** and **direction** i.e.

$$\mathbf{a} = \mathbf{b}$$

Using the component definition

$$\mathbf{a} = (a_1, a_2, a_3) \qquad \mathbf{b} = (b_1, b_2, b_3)$$

then

$$a_1 = b_1 \qquad a_2 = b_2 \qquad a_3 = b_3$$

2.8 Parallel Vectors

If λ is a scalar and $\mathbf{a} = \lambda\mathbf{b}$

then

if $\lambda>0$, the vector $\mathbf{a}$ is in the same direction as $\mathbf{b}$ and has magnitude $\lambda\mathbf{b}$

if $\lambda<0$, the vector $\mathbf{a}$ is in the opposite direction to $\mathbf{b}$ and has magnitude $\lambda\mathbf{b}$

In component form

$$a_1 = \lambda b_1 \qquad a_2 = \lambda b_2 \qquad a_3 = \lambda b_3$$

This can be summarised as the vectors $\mathbf{a}$ and $\mathbf{b}$ are

<u>Parallel</u> if $\lambda > 0$

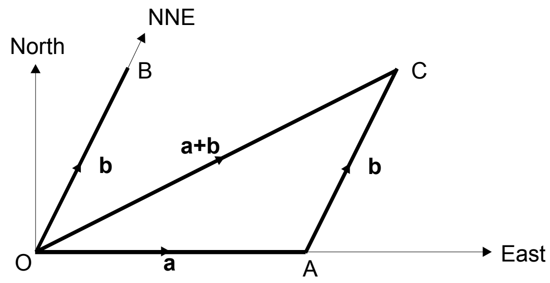
<u>Anti-parallel</u> if $\lambda < 0$.

3 Vector Addition

3.1 The boat problem

A boat steams at 4 knots due East for one hour. The tide is running North-North-East at 3 knots. Where will the boat be after one hour?

A diagram of the vectors involved looks like that below, where **a** represents the velocity of the boat and **b** represents the velocity of the tide.

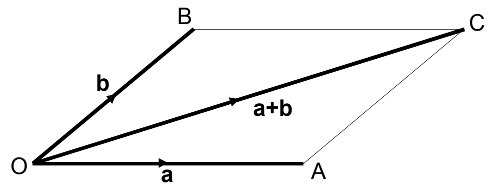


The net velocity of the boat is represented by the line OC which is the sum of **a** and **b**.

3.2 Geometric addition laws

This leads to the **parallelogram rule** for vector addition which may be written:

The sum, or resultant, of two vectors **a** and **b** is found by forming a parallelogram with **a** and **b** as two adjacent sides. The sum **a** + **b** is the vector represented by the diagonal of the parallelogram.

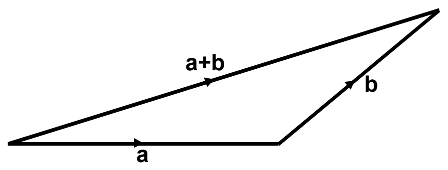


In component form

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1 \, , \, a_2 + b_2 \, , \, a_3 + b_3)$$

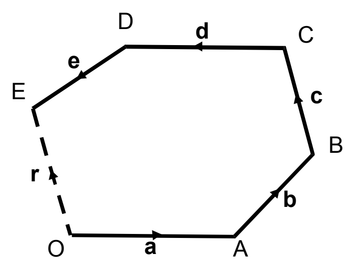
Because the vector **a** is the same as any other vector which is parallel and in the same direction, **a** or **b** could be "moved" in the figure above to form a triangle. This leads to the triangle law.

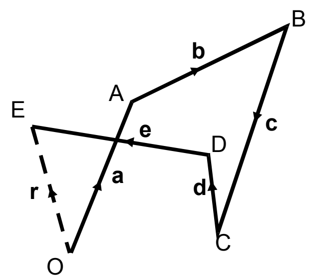
If two vectors **a** and **b** are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum is represented in magnitude and direction by the closing third side.



The triangle rule can be made more general to apply to any geometrical shape - or polygon. This then becomes the **polygon law**.

If from a point O, say, as in the figures below, lines are drawn to represent the vectors **a**, **b**, **c**, **d**, and **e**. Then the closing side represents the vector **r** and **r** is the sum of the vectors **a, b, c, d,** and **e**.





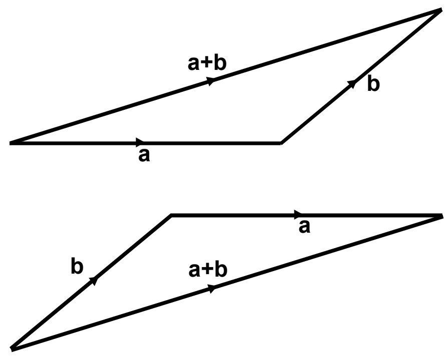
$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e}$$

3.3 Vector Laws for addition and subtraction

Commutative law

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

Order is NOT important



In component form

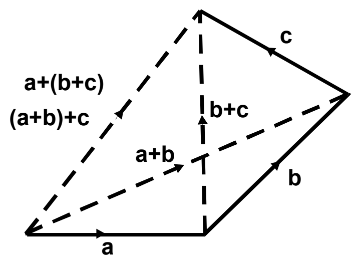
$$(a_1 + b_1 , a_2 + b_2 , a_3 + b_3) = (b_1 + a_1 , b_2 + a_2 , b_3 + a_3)$$

Associative law

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

This follows from the component definition and geometrically - from the triangle or polygon law.

In component form



$$(a_1 + b_1) + c_1 = a_1 + (b_1 + c_1)$$

$$(a_2 + b_2) + c_2 = a_2 + (b_2 + c_2)$$

$$(a_3 + b_3) + c_3 = a_3 + (b_3 + c_3)$$

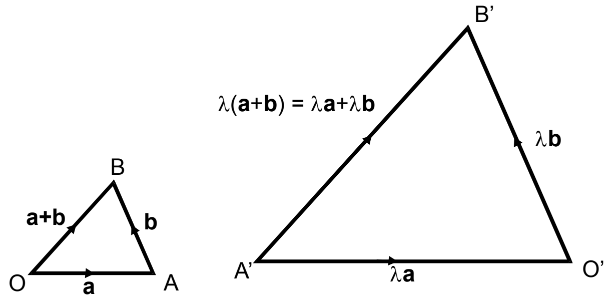
Distributive law

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$$

This follows from the component definition or geometrically from similar triangles

In component form

$$(\lambda(a_1 + b_1), \lambda(a_2 + b_2), \lambda(a_3 + b_3)) = (\lambda a_1 + \lambda b_1, \lambda a_2 + \lambda b_2, \lambda a_3 + \lambda b_3)$$



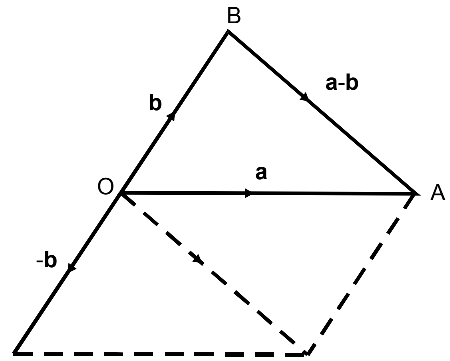
Subtraction

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

In component form:

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3)$$

Geometrically:



Note $\overrightarrow{BA} = \overrightarrow{\mathrm{OA}} - \overrightarrow{OB}$

This is an important result as it shows that the vector represented by the line joining two points is given by taking the vector giving the position of the first point from that for the second

$$\overrightarrow{BA} = \overrightarrow{\mathrm{OA}} - \overrightarrow{OB} = \overrightarrow{\mathrm{BO}} + \overrightarrow{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b}$$

3.4 Vector Addition Examples

(1)

$$\mathbf{a} = (2,1,0) \qquad \mathbf{b} = (-1,2,3) \quad \mathbf{c} = (1,2,1)$$
$$\mathbf{a} + \mathbf{b} = (1,3,3)$$
$$2\mathbf{a} - \mathbf{b} = (2\times 2 + 1,\ 2\times 1 - 2,\ 2\times 0 - 3) = (5,0,-3)$$
$$\mathbf{a} + \mathbf{b} - \mathbf{c} = (0,1,2)$$

Modulus of $\mathbf{c}$

$$|\mathbf{c}| = (1^2, 2^2, 1^2)^{1/2} = \sqrt{6}$$

The unit vector in the direction of $\mathbf{c}$

$$\hat{\mathbf{c}} = \frac{\mathbf{c}}{|\mathbf{c}|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

Not that the modulus of $\hat{\mathbf{c}} = |\hat{\mathbf{c}}| = 1$

2) From analysis of the forces in a new bridge it is calculated that the following forces are imposed at one of the bridge supports (in Newtons).

F_1 = (1,1,1)

F_2 is 6N and acts in the direction (1,2,-2)

F_3 is 10N and acts in the direction (1,2,-2)

In order to design the bridge support the resultant force is required. What is the force the support must impose on the bridge to reduce the resultant force to zero.

The first force is given in the usual vector form. The second two are in valid forms but need to be converted so that we can perform the usual vector addition.

First calculate the unit vectors in the direction of **F_2** and **F_3**

$$\hat{\mathbf{F}}_2 = \frac{\mathbf{F}_2}{|\mathbf{F}_2|} = \frac{(1,2,-2)}{\sqrt{1+4+4}} = \frac{1}{3}(1,2,-2)$$

$$\hat{\mathbf{F}}_3 = \frac{\mathbf{F}_3}{|\mathbf{F}_2|} = \frac{(3,-4,0)}{\sqrt{9+16+0}} = \frac{1}{5}(3,-4,0)$$

So the in vector for the two forces are

$$\mathbf{F}_2 = 6\left(\frac{1}{3},\frac{2}{3},\frac{-2}{3}\right) = (2,4,-4)$$

$$\mathbf{F}_2 = 10\left(\frac{3}{5},\frac{-4}{5},0\right) = (6,-8,0)$$

The resultant force **F** is found by vector addition

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_1 + \mathbf{F}_1 = (1,1,1) + (2,4,-4) + (6,-8,0) = (9,-3,-3)$$

The force the support must impose on the bridge is equal and in the opposite direction as the resultant i.e.

$$\text{Reaction force} = (-9,3,3)$$

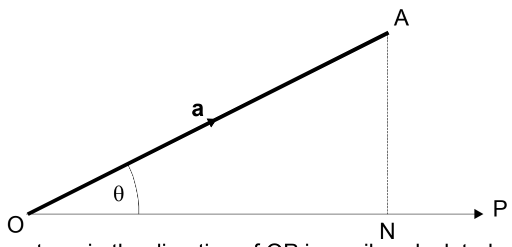
4 Vector Multiplication

We have seen how vectors can be added, another natural operation in maths is to multiply quantities together. The idea of multiplication of vectors becomes very useful in engineering. In particular we will show how to use them to calculate moments and work-done with great ease.

Two types of multiplication exist. They are called the **vector** and **scalar** products.

4.1 The Scalar Product

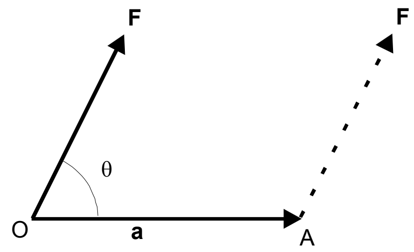
(also known as the **dot product** or the **inner product**)



The component of the vector **a** in the direction of OP is easily calculated as equivalent to the length ON by

$$ON = |\mathbf{a}|\cos\theta$$

Considering the constant force F in the figure below, which acts through the point O. If this force is moved along the line OA, along the vector **a**, then we can calculate the work done by the force. (Remember: work done is the component of the force in the direction of movement multiplied by distance moved by the point of application.)



The component of **F** along OA is $|\mathbf{F}|\cos\theta$. This force is moved the distance $|\mathbf{a}|$, giving,

$$\text{work done} = |\mathbf{F}||\mathbf{a}|\cos\theta$$

This is the **scalar product** of the two vectors **F** and **a** and can be seen as a geometric definition.

An equivalent component definition can be written.

Scalar product definition:

The scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is written with a dot $(\cdot)$ between the to vectors.
It is defined in component form as

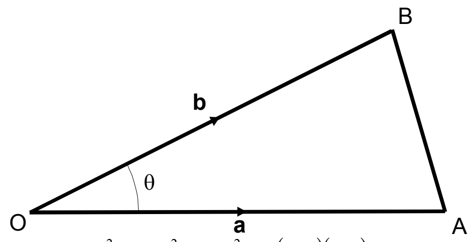
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

And in geometrical form

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}||\mathbf{a}|\cos\theta$$

Where θ is the angle between the two vectors and $0 \le \theta \le \pi$.

The two definitions can be proved to be equivalent by the cosine rule for a triangle,



$$AB^2 = OA^2 + OB^2 - 2(AO)(OB)\cos\theta$$

Which can be expanded to show that

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{b}||\mathbf{a}|\cos\theta$$

Three important points about a scalar product:

(1) The scalar product of two vectors gives a <u>number</u> (a scalar)

(2) The scalar product is a product of vectors
(it cannot be of two scalars nor a vector and a scalar)

(3) Use of the "*dot*" is *essential* to indicate that the calculation is a scalar product.

4.2 Rules for the scalar (or dot) product.

Commutative Law

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
$$|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{b}||\mathbf{a}|\cos\theta$$

Associative Law

You cannot have a scalar product of three vectors as 'dotting' the first two gives a scalar.

Distributive Law (for a scalar multiplier)

Brackets are multiplied out in the usual way.

$$\mathbf{a} \cdot (\lambda \mathbf{b}) = (\mathbf{a}\lambda) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$$

Distributive Law (for vector addition)

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Powers of vectors

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= a_1^2 + a_2^2 + a_3^2 \\ &= |\mathbf{a}||\mathbf{a}|\cos 0 \\ &= |\mathbf{a}|^2 \end{aligned}$$

No other powers of vectors are possible other than 2.
Note that for unit vectors *i, j, k*

$$i^2 = i \cdot i = 1 \qquad j^2 = j \cdot j = 1 \qquad k^2 = k \cdot k = 1$$

Perpendicular Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are <u>perpendicular</u> then $\theta = \frac{\pi}{2}$ and $\cos\theta = \cos\frac{\pi}{2} = 0$

Hence $\mathbf{a} \cdot \mathbf{b} = 0$

BUT $\mathbf{a} \cdot \mathbf{b} = 0$ does <u>not</u> imply that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, because $\mathbf{a}$ could be zero or $\mathbf{b}$ could be zero.

Note: Suppose $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ $\quad [\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0]$

This does <u>not</u> mean that $\mathbf{b} = \mathbf{c}$, since $\mathbf{a}$ could be zero or $\mathbf{a}$ perpendicular to $\mathbf{b}$-$\mathbf{c}$.

i.e. vectors cannot be cancelled in the same way as scalars.

For unit vectors which are perpendicular

$$i \cdot j = j \cdot k = k \cdot i = 0$$

Perpendicularity is important in engineering, for example pressure acts normal to a surface, so force per unit area is $p\hat{\mathbf{n}}$, where p is the pressure and $\hat{\mathbf{n}}$ the unit normal vector.

We often have to find a vector that is normal to another vector.

Component of a vector

The component of a vector, **F** say, in a given direction **a** is given by $|\mathbf{F}|\cos\theta$.

And we can write

$$\mathbf{F} \cdot \hat{\mathbf{a}} = |\mathbf{F}||\hat{\mathbf{a}}|\cos\theta = |\mathbf{F}|\cos\theta$$

to give the component of **F** in the direction on **a**.

4.3 Scalar Product Examples

(1) Find the angle between the two vectors $\mathbf{a}$ = (2, 0, 3) and $\mathbf{b}$ = (3, 2, 4)

Using $\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}||\mathbf{a}|\cos\theta$ and $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$\mathbf{a} \cdot \mathbf{b} = 6 + 0 + 12 = 18$$

$$|\mathbf{a}| = |(2,0,3)| = (4+0+9)^{1/2} = \sqrt{13}$$

$$|\mathbf{b}| = |(3,2,4)| = (9+4+16)^{1/2} = \sqrt{29}$$

$$18 = \sqrt{13}\sqrt{29}\cos\theta$$

$$\cos\theta = \frac{18}{\sqrt{13}\sqrt{29}} = 0.927$$

$$\theta = 22.02^\circ$$

(2) For $\mathbf{a}$ = (1, 2, -2), $\mathbf{b}$ = (0, 2, 2) $\mathbf{c}$ = (3, 2, 1) find

i) $\mathbf{a} \cdot \mathbf{c}$ ii) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$ iii) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

i) $\mathbf{a} \cdot \mathbf{c} = (3 + 4 - 2) = 5$

ii) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = (1,4,0) \cdot (3,2,1) = (3 + 8 + 0) = 11$

iii) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (0 + 4 - 4)(3,2,1) = 0(3,2,1) = (0,0,0)$

Note that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular since neither $\mathbf{a}$ or $\mathbf{b}$ are zero.

Remember that $\mathbf{a} \cdot \mathbf{b}$ is just a number (not a vector).

(3) Find the work done by the force $F = (3, 1, 5)$ in moving a particle from P to Q where position vectors of P and Q are (1, 2, 3) and (2, 4, 4) respectively.

$$\mathbf{r} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (1,2,1)$$

Work done by the force $F = \mathbf{F} \cdot \mathbf{r}$

$(3,1,5) \cdot (1,2,1) = 3 + 2 + 5 = 10$ units of work.

(4) Find the component of the vector $F = (2, -2, 3)$ in

(i) the i direction

(ii) the direction (3, 1, -2)

(i) The i direction is the vector (1,0,0) so the component in the i direction is

$$F \cdot (1,0,0) = (2,-2,3) \cdot (1,0,0) = 2$$

(iii) Is the vector (3, 1, -2) a unit vector?

$(9 + 1 + 4)^{1/2} = \sqrt{14}$ Therefore it is not a unit vector.

Unit vector = $\dfrac{(3,1,-2)}{\sqrt{14}}$

Component of F in direction (3, 1, -2) is

$$\mathbf{F} \cdot \frac{(3,1,-2)}{\sqrt{14}} = (2,-2,3) \cdot (3,1,-2)/\sqrt{14} = \frac{-2}{\sqrt{14}}$$

4.4 The Vector (or Cross) Product

The main practical use of the **vector** product is to calculate the moment of a force in three dimensions. It is of very limited use in two dimensions.

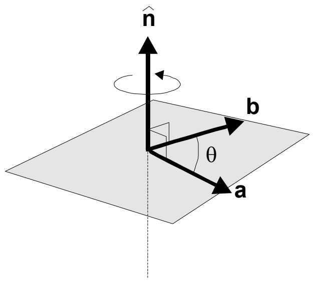
For two vectors $\mathbf{a}$ and $\mathbf{b}$, the vector product is defined as:

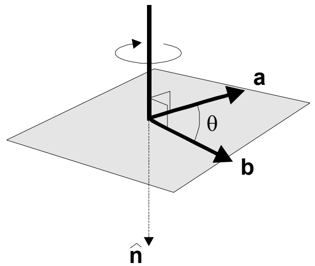
$$(\mathbf{a}\times\mathbf{b}) = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, ($0 \le \theta \le \pi$) and $\hat{\mathbf{n}}$ is the unit vector normal to both $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ are three vectors which form a right-handed set. (Remember how a right handed axes system was defined earlier.)

It is very important to realise that the result of a vector product is itself a vector.





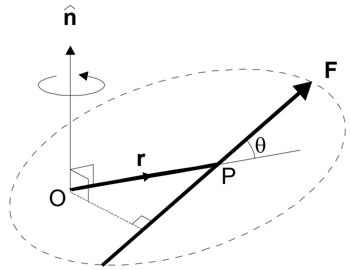
Note how from this definition order of multiplication matters:

The vector given by $(\mathbf{b}\times\mathbf{a})$ is in the opposite direction to $(\mathbf{a}\times\mathbf{b})$.

(This follows from the right-hand screw rule), so

$$(\mathbf{a}\times\mathbf{b}) = -(\mathbf{b}\times\mathbf{a})$$

4.5 The Moment of force, F.



If the force $\mathbf{F}$ passes through the point P and $\overrightarrow{OP} = \mathbf{r}$, then the moment of the force about O is defined as

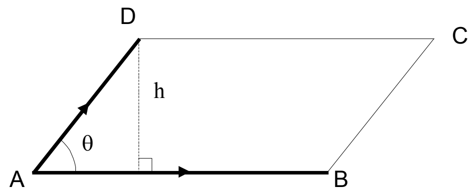
$$\mathbf{M} = \mathbf{r}\times\mathbf{F}$$

$\mathbf{M}$ is a vector in the direction of the normal $\hat{\mathbf{n}}$.

So, as with all vectors, moments add by the parallelogram rule.

The use of calculating moments by the vector (cross) product comes into itself when used in situations (particularly in three dimensions) which are very difficult to visualise.

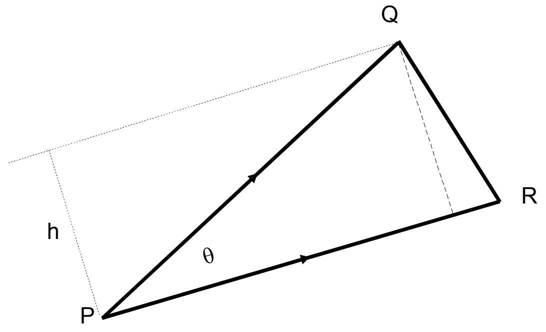
4.6 Area calculation



The area of the parallelogram ABCD is given by

$$\text{Area} = h\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\right|\sin\theta\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AD}\times\overrightarrow{AB}\right|$$

The area of the triangle PQR is given by



$$\begin{aligned}\text{Area} &= \frac{1}{2}\,\mathrm{h}\left|\overrightarrow{PR}\right| \\ &= \frac{1}{2}\left|\overrightarrow{PQ}\right|\sin\theta\left|\overrightarrow{PR}\right| \\ &= \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|\end{aligned}$$

4.7 Laws for the Vector (or Cross) products

Anti-Commutative

$$(\mathbf{a}\times\mathbf{b}) = -(\mathbf{b}\times\mathbf{a})$$

This follows from the definition that includes the 'right-handed axes', $\hat{\mathbf{n}}$ changes direction when the multiplication is reversed.

Non-associative

(The triple vector product)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

The vector $\mathbf{b}\times\mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ and the plane containing $\mathbf{b}$ and $\mathbf{c}$.

Similarly, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is in the plane of $\mathbf{a}$ and $\mathbf{b}$, so $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ are different vectors.

Brackets must always be used for more than two vectors in a vector product.

Distributive law with multiplication by a scalar

$$\mathbf{a}\times(\lambda\mathbf{b}) = (\lambda\mathbf{a})\times\mathbf{b}$$

Distributive law with addition

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$$

4.8 Parallel vectors.

From the definition of the vector product, if $\mathbf{a}$ and $\mathbf{b}$ are parallel, then

$$\theta = 0$$

and

$$\mathbf{a}\times\mathbf{b} = 0$$

So we can say that if $\mathbf{a}\times\mathbf{b} = 0$ then

$$\mathbf{a} = 0$$

or

$$\mathbf{b} = 0$$

or

$\mathbf{a}$ and $\mathbf{b}$ are parallel.

If $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$ it **<u>cannot</u>** be deduced that $\mathbf{b} = \mathbf{c}$. You must first show that $\mathbf{a} \neq 0$ and that $\mathbf{a}$ is not parallel to $\mathbf{b}$ - $\mathbf{c}$.

4.9 Cartesian (component) form.

For unit vectors i,j,k

$$i\times i = j\times j = k\times k = 0 \qquad (\text{as}\,\theta = 0, \sin\theta = 0)$$

Because of perpendiculatiry

$$i\times j = k, \quad j\times k = i, \quad k\times i = j$$

$$j\times i = -k, \quad k\times j = -i, \quad i\times k = -j$$

Component form of the vector product:

$$\mathbf{a} = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$$
$$\mathbf{b} = (b_1, b_2, b_3) = b_1 i + b_2 j + b_3 k$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1, a_2, a_3)\times(b_1, b_2, b_3) \\ &= (a_1 i + a_2 j + a_3 k)\times(b_1 i + b_2 j + b_3 k) \\ &= a_1 b_1 (i\times i) + a_1 b_2 (i\times j) + a_1 b_3 (i\times k) \\ &\quad + a_2 b_1 (j\times i) + a_2 b_2 (j\times j) + a_2 b_3 (j\times k) \\ &\quad + a_3 b_1 (k\times i) + a_3 b_2 (k\times j) + a_3 b_3 (k\times k) \\ &= a_1 b_2 k + a_1 b_3 (-j) + a_2 b_1 (-k) \\ &\quad + a_2 b_3 i + a_3 b_1 j + a_3 b_2 (-i)\end{aligned}$$

So, how do you remember this!!?

There are several ways, the determinant form is shown here

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a2b1)k$$

4.10 Examples

(1) Given the three vectors $\mathbf{a}$ = (3, 1, 1), $\mathbf{b}$ = (2, -1, 1), $\mathbf{c}$ = (0, 1, 2)

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - j\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= (2,-1,-5)$$

This vector is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} i & j & k \\ 2 & -1 & -5 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= i\begin{vmatrix} -1 & -5 \\ 1 & 2 \end{vmatrix} - j\begin{vmatrix} 2 & -5 \\ 0 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= (3,-4,-2)$$

This lies in the plane containing $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} &= 3\mathbf{b} - 1a \\ &= (6,-3,3) - (3,1,1) \\ &= (3,-4,2)\end{aligned}$$

This is the same as the solution for $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ above.

It can be shown that in general,

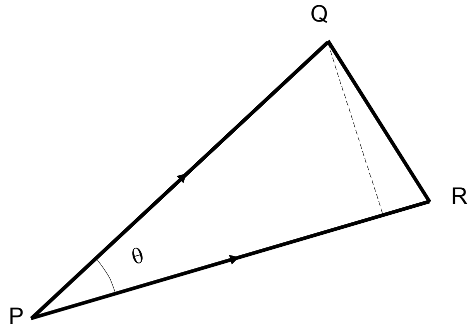
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

It can also be shown that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

(2) Find the area of the triangle having vertices at

P= (3, 2, 2), Q= (1, -1, 2), R = (2, 1, 1)



Area PQR = $\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$

$\overrightarrow{PQ} = (-2,-3,0)$ $\qquad$ $\overrightarrow{PR} = (-1,-1,-1)$

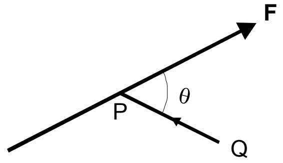
$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -2 & -3 & 0 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= i\begin{vmatrix} -3 & 0 \\ -1 & -1 \end{vmatrix} - j\begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix} + k\begin{vmatrix} -2 & -3 \\ -1 & -1 \end{vmatrix}$$

$$= (3,-2,-1)$$

Area = $\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{(9+4+1)} = \frac{\sqrt{14}}{2}$

(3) A force F of 6 units acts through the point P=(2,3,1) in the direction of the vector (4,1,2). Find the moment of the force about the point Q=(1,2,1).



The unit vector in the direction of the force is

$$\frac{4i+j+2k}{\sqrt{16+1+4}}=\left(\frac{4}{\sqrt{21}},\frac{1}{\sqrt{21}},\frac{2}{\sqrt{21}}\right)$$

So the force F, has components

$$F=6\left(\frac{4}{\sqrt{21}},\frac{1}{\sqrt{21}},\frac{2}{\sqrt{21}}\right)$$

The position vector of P relative to Q is

$$\overrightarrow{QP}=(1,1,0)$$

i.e. $\mathbf{r}$

So the moment of the force about Q is

$$\mathbf{M}=\overrightarrow{QP}\times F=\frac{6}{\sqrt{21}}\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 4 & 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 4 & 1 & 2 \end{vmatrix}=i\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}-j\begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix}+k\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}$$

$$=(2,-2,-3)$$

$$\mathbf{M}=\left(\frac{12}{\sqrt{21}},\frac{-12}{\sqrt{21}},\frac{-18}{\sqrt{21}}\right)$$

4.11 Triple products of Vectors

Definitions of the two **triple** products are

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is the <u>Triple Vector</u> product.

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is the <u>Triple Scalar</u> product.

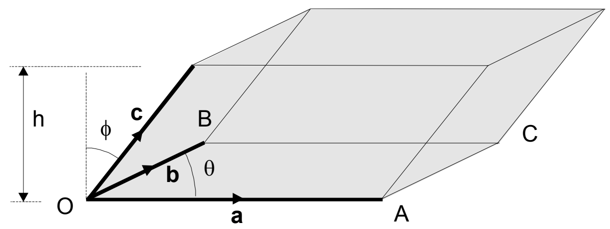
The triple <u>scalar</u> product has an interesting <u>geometrical</u> meaning:

We know that

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})&=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}\\ &=(\text{area of the parallelogram defined by }\mathbf{a}\text{ and }\mathbf{b})\end{aligned}$$

Thus

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}&=(\text{area of the parallelogram})\hat{\mathbf{n}}\cdot\mathbf{c}\\ &=(\text{area of the parallelogram})|\hat{\mathbf{n}}||\mathbf{c}|\cos\phi\end{aligned}$$



But $|\mathbf{c}|\cos\theta=h$ =height of the parallelepiped normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$. (ϕ is the angle between $\mathbf{c}$ and $\hat{\mathbf{n}}$).

So

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ = area of the parallelepiped defined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

It follows then that:

(a) If any two vectors are parallel $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$ (zero volume)

(b) If the three vectors a co-planar then $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$ (zero volume)

(c) If $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$ then either

i) $\mathbf{a}$ = 0, or

ii) $\mathbf{b}$ = 0 or

iii) $\mathbf{c}$ =0 or

iv) two of the vectors are parallel or

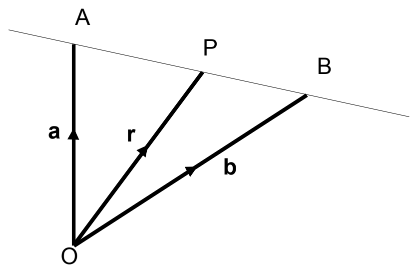
v) the three vectors are co-planar

(d) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ = the same volume.

5 Vector Equations of Lines and Planes

5.1 The Vector Equation of a Line

Consider the line which passes through the points A and B with position vectors **a** and **b** respectively and a point P, which has position vector **r,** and lies on this line as shown



$$\overrightarrow{OA}=\mathbf{a} \qquad \overrightarrow{OB}=\mathbf{b} \qquad \overrightarrow{OP}=\mathbf{r}$$

From this diagram we can see that

$$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$$
$$\mathbf{r}=\mathbf{a}+\overrightarrow{AP}$$

If t is some multiple of AB such that

$$\mathbf{r}=\mathbf{a}+t\overrightarrow{AB}$$

as $\mathbf{a}+\overrightarrow{AB}=\mathbf{b}$, then

$$\mathbf{r}=\mathbf{a}+t(\mathbf{b}-\mathbf{a})$$

So the equation of the line AB is:

$$\mathbf{r}=(1-t)\mathbf{a}+t\mathbf{b}$$

for $-\infty<t<\infty$.

In component form

$$\mathbf{r}=(x,y,z)$$
$$\mathbf{a}=(a_1,a_2,a_3)$$
$$\mathbf{b}=(b_1,b_2,b_3)$$

$$\begin{aligned}(x,y,z)&=(1-t)(a_1,a_2,a_3)+t(b_1,b_2,b_3)\\&=(a_1,a_2,a_3)+t[(b_1,b_2,b_3)-(a_1,a_2,a_3)]\end{aligned}$$

If we equate components e.g. those of x gives

$$x = (1-t)a_1 + tb_1$$
$$t = \frac{x-a_1}{b_1-a_1}$$

Similarly for y and z

$$t = \frac{y-a_2}{b_2-a_2} \qquad t = \frac{z-a_3}{b_3-a_3}$$

Hence the Cartesian equation of a line passing through the points (a_1,a_2,a_3) and (b_1,b_2,b_3).is

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3} = t$$

Note: For any equation in Cartesian form we can readily express the equation in vector form. e.g.

$$\frac{x}{3} = \frac{y-9}{-1} = \frac{z-2}{1} = t$$

In vector form

$$\mathbf{r} = \mathbf{a} + t(\mathbf{b}-\mathbf{a})$$
$$\mathbf{r} = (0,9,2) + t(3,-1,1)$$

5.2 Equation of line examples:

$$\mathbf{r} = (1-t)\mathbf{a} + t\mathbf{b} = \mathbf{a} + t(\mathbf{b}-\mathbf{a})$$

(1) Find the equation of the line $\mathbf{r_1}$ through the points (0,1,-2) and (3,4,3).

$$\begin{aligned}\mathbf{r} &= (1-t)\mathbf{a} + t\mathbf{b}\\ &= (0,1-t,-2+2t) + (3t,4t,3t)\\ &= (3t,1+3t,-2+5t)\end{aligned}$$

i.e. $x = 3t \qquad y = 1+3t \qquad z = -2+5t$

In Cartesian form

$$\frac{x-0}{3-0} = \frac{y-1}{3-1} = \frac{z+2}{3+2} = t$$
$$\frac{x-0}{3} = \frac{y-1}{2} = \frac{z+2}{5} = t$$

(2) Find the equation of the line $\mathbf{r_2}$ through the points (1,1,0) and (-3,-2,-7)

$$\begin{aligned}\mathbf{r} &= (1-s)\mathbf{a} + s\mathbf{b}\\ &= (1-s,1-s,0) + (-3s,-2s,-7s)\\ &= (1-4s,1-3s,-7s)\end{aligned}$$

In Cartesian form

$$\frac{x-1}{4} = \frac{y-1}{-3} = \frac{z}{-7} = s$$

(3) Do $\mathbf{r_1}$ and $\mathbf{r_2}$ intersect?

For the lines to intersect then t and s should be such that:

$$\begin{aligned}3t &= 1-4s\\ 1+3t &= 1-3s\\ -2+5s &= -7s\end{aligned}$$

are all satisfied.

Solving the first two simultaneously s = 1 and t = -1.

Substituting these values into the third gives

-7 = -7

i.e. the three equations are satisfied, so the two lines intersect.

Substituting the values for s or t into the respective Cartesian equations gives the point of intersection as (-3,-2,-7)

(4) Find the angle between the two lines which are in the same direction as the two vectors $\mathbf{c}$ = (3,3,5) and $\mathbf{d}$ = (-4,-3,-7)

$$\mathbf{c}\cdot\mathbf{d} = |\mathbf{c}||\mathbf{d}|\cos\theta$$
$$\mathbf{c}\cdot\mathbf{d} = -12-9-35 = -56$$

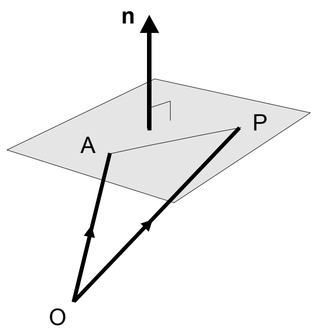
$$|\mathbf{c}| = (9+9+25)^{1/2} = \sqrt{43}$$
$$|\mathbf{d}| = (16+9+49)^{1/2} = \sqrt{74}$$

$$\cos\theta = \frac{-56}{\sqrt{43}\sqrt{74}} = -0.9927$$
$$\theta = 173.1^\circ$$

5.3 The Vector Equation of a Plane

We can use the fact that a line joining any two points in the plane is perpendicular to the normal to the plane.

i.e. $\mathbf{n}$ and $\overrightarrow{AP}$ are perpendicular.



The vector $\mathbf{n}$ is normal to the plane, $\mathbf{a}$ is the position vector of A, $\mathbf{r}$ is the position vector of P ($\mathbf{r}$ = (x, y, z)).

The vector

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \mathbf{a}$$

Now $\mathbf{r}$ - $\mathbf{a}$ is perpendicular to $\mathbf{n}$ if

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$

or

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

or

$$\mathbf{r} \cdot \mathbf{n} = p$$

The above two are the general form of the vector equation of a plane.

If we can take $\mathbf{n}$ = (α, β, γ) the equation in Cartesian form is

$$\alpha x + \beta y + \gamma z = p$$

since $\mathbf{r}$ = (x, y, z)

5.4 Examples involving both lines and planes

(1) Find the equation of the plane through the points

$$\mathbf{a} = (3,2,0) \qquad \mathbf{b} = (1,3,-1) \qquad \mathbf{c} = (0,-2,3)$$

$\mathbf{a}$ - $\mathbf{b}$ lies in the plane

$$\mathbf{a} - \mathbf{b} = (2, -1, 1)$$

$\mathbf{a}$ - $\mathbf{c}$ lies in the plane

$$\mathbf{a} - \mathbf{c} = (3, 4, -3)$$

A normal, $\mathbf{n}$, to the plane is

$$\begin{aligned}
\mathbf{n} &= (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} - \mathbf{c}) = (2,-1,1) \times (3,4,-3) \\
&= \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -3 \end{vmatrix} \\
&= i\begin{vmatrix} -1 & 1 \\ 4 & -3 \end{vmatrix} - j\begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \\
&= ((3-4), -(-6-3), (8+3)) \\
\mathbf{n} &= (-1, 9, 11)
\end{aligned}$$

Equation of the plane is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\begin{aligned}
\mathbf{r} \cdot (-1,9,11) &= (3,2,0) \cdot (-1,9,11) \\
&= -3 + 18 + 0 \\
&= 15
\end{aligned}$$

In component form this is

$$\begin{aligned}
(x, y, z) \cdot (-1,9,11) &= 15 \\
-x + 9y + 11z &= 15
\end{aligned}$$

Check

$$\begin{aligned}
a: &\quad -3 + 18 + 0 = 15 \\
b: &\quad -1 + 27 - 11 = 15 \\
c: &\quad 0 - 18 + 33 = 15
\end{aligned}$$

All are consistent.

(2) (a) Find the point where the plane $\mathbf{r}\cdot(1,2,1)=4$ (or $x+2y+z=4$) meets the line

$$\mathbf{r} = (2, 1, -1) + t(1, 1, 2) \text{ or } \left(\frac{x-2}{1}=\frac{y-1}{1}=\frac{z+1}{2}=t\right)$$

(b) Find the angle that the line makes with the plane.

[Remember the equations of a line

$$\mathbf{r}=\mathbf{a}+t(\mathbf{b}-\mathbf{a})$$

$$\frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}=\frac{z-a_3}{b_3-a_3}=t$$

]

The point of intersection must satisfy both the equation of the plane AND the equations of the line.

i.e (using the vector form)

$$\begin{aligned}[\mathbf{a}+t(\mathbf{b}-\mathbf{a})]\cdot(1,2,1)&=4\\(2+2-1)+t(1+2+2)&=4\\3+5t&=4\\t&=1/5\end{aligned}$$

Substituting into the equation of the line gives the point of intersection

$$\begin{aligned}\mathbf{r}&=\mathbf{a}+t(\mathbf{b}-\mathbf{a})\\&=(2,1,-1)+\frac{1}{5}(1,1,2)\\&=\left(\frac{11}{5},\frac{6}{5},-\frac{3}{5}\right)\end{aligned}$$

(b) Find the angle that the line makes with the plane.

The normal to the plane is (1, 2, 1) = $\mathbf{n}$.

A vector in the direction of the line is (1, 1, 2) = $\mathbf{b}$ - $\mathbf{a}$

$$(\mathbf{b}-\mathbf{a})\cdot\mathbf{n}=|\mathbf{b}-\mathbf{a}||\mathbf{n}|\cos\theta$$

$$(\mathbf{b}-\mathbf{a})\cdot\mathbf{n}=1+2+2=5$$

$$|\mathbf{b}-\mathbf{a}|=(1+1+4)^{1/2}=\sqrt{6}$$

$$|\mathbf{n}|=(1+4+1)^{1/2}=\sqrt{6}$$

$$\cos\theta=\frac{5}{6}=0.833$$

$$\theta=33.56°$$

(3) Find the perpendicular distance from the point P (2,-3,4) to the plane $x+2y+2z=13$

The equation of the plane in vector form is

$$\mathbf{r}\cdot(1,2,2)=13$$

A vector normal to the plane is

$$\mathbf{n}=(1,2,2)$$

Hence the equation of a line perpendicular to the plane and passing through P is

$$\begin{aligned}\mathbf{r}&=\mathbf{a}+t(\mathbf{b}-\mathbf{a})\\&=(2,-3,4)+t(1,2,2)\end{aligned}$$

This line meets the plane when

$$\mathbf{r}\cdot(1,2,2)=(2,-3,4)\cdot(1,2,2)+t(1,2,2)\cdot(1,2,2)=13$$

i.e.

$$\begin{aligned}(2-6+8)+t(1+4+4)&=13\\4+9t&=13\\t&=1\end{aligned}$$

Thus the line meets the plane at N with co-ordinates

$$(2,-3,4)+1(1,2,2)=(3,-1,6)$$

Hence the perpendicular distance PN

$$\left[(3-2^2)+(-1+3)^2+(6-4)^2\right]^{1/2}=3\text{ units}$$

(3) Find the equation of the line of intersection of the two planes given by:

$x+y+z=5$ and $4x+y+2z=15$

In vector from these planes can be written:

$$\mathbf{r}\cdot(1,1,1)=5$$
$$\mathbf{r}\cdot(4,1,2)=15$$

The line of intersection must be perpendicular to both the vectors (1,1,1) and (4,1,2).

Hence a vector in the direction of the line must be in the direction

(1,1,1)×(4,1,2) = (1,2,-3)

To complete the equation of the line we need to find any one point on the line. Choosing x=0 then, from the Cartesian equations of the two planes,

$y+z=5$

$y+2z=15$

Hence for $x=0$, $y=-5$ and $z=10$

And the point (0, -5, 10) is a point on the line.

The equation of the line can then be written $\mathbf{r} = (0, -5, 10) + t\,(1, 2, -3)$