

Floating-point numbers

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Based on slides by Randal E. Bryant and David R. O'Hallaron.
Some material by Michael Kirkedal Tomsen.

Agenda

Why are numbers exciting?

Preliminaries: biased numbers

Floating-point arithmetic

- Background: Fractional binary numbers

- IEEE floating-point standard

- Examples and properties

- Rounding, addition, and multiplication

- Floating-point in C

Summary

Suppose Kerbal Space Program



The Deep Space Kraken¹

- Physics simulation of each rocket *part*.
- Forces from e.g. engines affect connected parts.
- If forces become too great, *boom*.



Players who travelled far from the launch site found their craft becoming increasingly fragile.

- By the end of the lecture you will understand why...
- ...and how it was fixed.

¹https://wiki.kerbalspaceprogram.com/wiki/Deep_Space_Kraken

Learning Objectives

A lot of this stuff can seem very dry.

...because it is.

Main Things You Should Understand At The End

- Non-uniform distribution of numbers.
- The consequences of roundoff.
- That it is surprisingly hard to do numbers well.

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Biased number representation

For *biased numbers*, raw bits are interpreted as unsigned, then a *bias* is subtracted.

Unsigned

$$\text{Bits2N}(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's complement

$$\text{TC2Int}(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

- Typically $b = 2^{w-1} - 1$
- **Examples for $w = 8, b = 127$**

Biased

$$\text{B2Int}(X) = \text{Bits2N}(x) - b$$

x	$\text{Bits2N}(x)$	$\text{B2Int}(x)$
$\langle 00000000 \rangle$		

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Integral binary numbers

We have seen that

$$10010101_2$$

is basically interpreted like

$$149_{10}$$

in particular “structure” is the same, just with 2 instead of 10.

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Can we do the same thing for fractional numbers?

$$1011.101_2$$

Fractional numbers

$$123.456 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 + 4 \cdot 10^{-1} + 5 \cdot 10^{-2} + 6 \cdot 10^{-3}$$

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Generally

$$a_{m-1} \cdots a_0 . a_{-1} \cdots a_{-n} = \sum_{i=-n}^{m-1} a_i \cdot 10^i$$

Fractional numbers

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Generally

$$a_{m-1} \cdots a_0.a_{-1} \cdots a_{-n} = \sum_{i=-n}^{m-1} a_i \cdot 10^i$$

Even more generally, for radix r

$$a_{m-1} \cdots a_0.a_{-1} \cdots a_{-n} = \sum_{i=-n}^{m-1} a_i \cdot r^i$$

Fractional binary numbers

Weight	2^{m-1}	2^{m-2}	\dots	4	2	1		$1/2$	$1/4$	$1/8$	\dots	2^{-n}
Digits	b_{m-1}	b_{m-2}	\dots	b_2	b_1	b_0		b_{-1}	b_{-2}	b_{-3}	\dots	b_{-n}

Representation

- Bits to the right of “binary point” represents fractional powers of 2.
- Represents rational number

$$\underbrace{b_{m-1} \dots b_0}_{\text{integral part}} . \underbrace{b_{-1} \dots b_{-n}}_{\text{fraction part}} = \sum_{i=-n}^{m-1} b_i \cdot 2^i$$

Examples of fractional binary numbers

Value	Representation
$5\frac{3}{4}$	101.11_2
$2\frac{7}{8}$	10.111_2
$1\frac{7}{16}$	1.0111_2

Observations

- Divide by 2 by logical shifting right.
- Multiply by 2 by shifting left.
- Numbers of form $0.111\dots$ are just below 1.0.
 - ▶ $1/2 + 1/4 + 1/8 + \dots 1/2^n + \dots \sim 1.0$.
 - ▶ Use notation $1.0 - \epsilon$.

Representable numbers

Limitation #1

- Can only represent fractional part of form $x/2^k$
- Other rational numbers have repeating binary representation

Value	Representation
$\frac{1}{3}$	$0.0101010101[01] \cdots_2$
$\frac{1}{5}$	$0.001100110011[0011] \cdots_2$
$\frac{1}{10}$	$0.0001100110011[0011] \cdots_2$

Limitation #2

- Just one setting of binary point within the w bits.
 - ▶ Limited range of numbers—very small values? Very large?

The fixed-point dilemma

Consider $w = 8$

1 bit for fraction

- Largest number: $1111111.1_2 = 127.5_{10}$
- Increment: $0000000.1_2 = 0.5_{10}$

7 bits for fraction

- Largest number: $1.1111111_2 = 1.9921875_{10}$
- Increment: $0.0000001_2 = 0.0078125_{10}$

4 bits for fraction

- Largest number: $1111.1111_2 = 15.9375_{10}$
- Increment: $0000.0001_2 = 0.0625_{10}$

Fixed-point has same absolute precision everywhere, but this means relative precision is worse for numbers close to 0!

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IEEE Floating-Point

IEEE Standard 754

- Established in 1985 as uniform standard for floating-point.
 - ▶ Many idiosyncratic formats before then.
- Supported by all major CPUs.

Driven by numerical concerns

- Nice standards for rounding, overflow, underflow.
- Hard to make fast in hardware.
 - ▶ Numerical analysts predominated over hardware designers in defining standard.
 - ▶ ... but (later) Turing Award winner William Kahan secretly knew that Intel had figured out how.
 - ▶ **Beware the wrath of Kahan!**
 - ▶ <http://people.eecs.berkeley.edu/~wkahan/>

Floating-Point Representation

Numerical form

$$(-1)^s \cdot m \cdot 2^e$$

- **Sign bit** s determines whether number negative or positive.
- **Significand** m normally a fractional value in range $[1, 2)$.
- **Exponent** e weights value by power of two.

Encoding

- Most significant bit is sign bit.
- E field encodes e (but is not equal to e).
- T field encodes m (but is not equal to m).

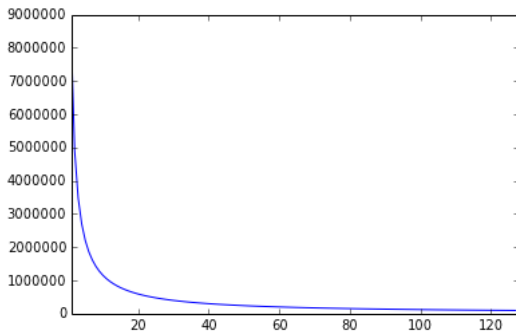


Why such a weird format?

The point is floating

- No fixed number of bits allocated to “fraction”.
- More bits close to 0, fewer bits for numbers with large magnitude.
- Symmetric around 0.

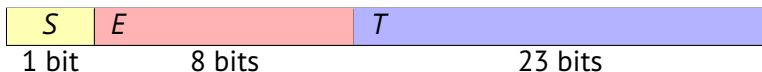
Density of floats



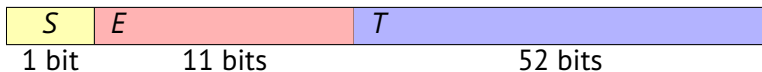
<https://stackoverflow.com/a/24179424/6131552>

Precision options

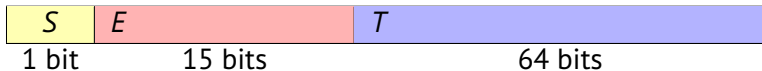
32-bit single precision: float



64-bit double precision: double



80-bit Extended precision (Intel only, never use): long double



Normalised values when $E \neq \langle 0 \dots 0 \rangle$ and $E \neq \langle 1 \dots 1 \rangle$

$$v = (-1)^s \cdot m \cdot 2^e$$



- Exponent encoded as *biased* value

$$e = \text{Bits2N}(E) - b$$

- ▶ $b = e_{\max}$.
 - ▶ Single precision: 127 ($E \in [-126, 127]$).
 - ▶ Double precision: 1023 ($E \in [-1022, 1023]$).

- Significand coded with implied leading 1:

$$m = 1.xxx \dots x_2 = 1 + \text{Bits2N}(\langle T \rangle) \cdot 2^{1-p}$$

- ▶ $xxx \dots x$: bits of T .
- ▶ Get extra leading bit for free.
- ▶ Precision
 - ▶ Single precision: $p = 24$.
 - ▶ Double precision: $p = 53$.
- ▶ Minimum value when $T = \langle 0000 \dots 0 \rangle$ ($m = 1$).
- ▶ Maximum value when $T = \langle 1111 \dots 1 \rangle$ ($m = 2 - \epsilon$).

Normalised encoding example

$$v = (-1)^s \cdot m \cdot 2^e$$

$$e = \text{Bits2N}(E) - b$$

Value: float $F = 15213.0$

$$\begin{aligned} 15213_{10} &= 11101101101101_2 \cdot 2^0 \\ &= 1.1101101101101_2 \cdot 2^{13} \end{aligned}$$

Significand

$$m = 1.1101101101101_2$$

$$T = \langle 110110110110100000000000 \rangle$$

Exponent

$$e = 13_{10}$$

$$b = 127_{10}$$

$$E = N2Bits(e + b) = \langle 10001100 \rangle$$

Result

0	10001100	110110110110100000000000
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Denormal values

$$v = (-1)^s \cdot m \cdot 2^e$$

$$e = 1 - b$$

Occur when $E = \langle 000 \dots 0 \rangle$.

- Exponent encoded as

$$e = 1 - b$$

- Significand coded with implied leading 0:

$$m = 0.xxx \dots x = \text{Bits2N}(\langle T \rangle) \cdot 2^{1-p}$$

- Cases

- ▶ $E = \langle 000 \dots 0 \rangle, T = \langle 000 \dots 0 \rangle$
 - ▶ Represents zero value.
 - ▶ Note distinct values $-0, +0$ – why do you think that is?
- ▶ $E = \langle 000 \dots 0 \rangle, T \neq \langle 000 \dots 0 \rangle$
 - ▶ Numbers closest to 0.0.
 - ▶ Called **subnormal numbers**.
 - ▶ Ensure that $x \neq y \Rightarrow x - y \neq 0$, i.e. avoid underflow.

Special values

Occur when $E = \langle 111 \dots 1 \rangle$.

When $E = \langle 111 \dots 1 \rangle, T = \langle 000 \dots 0 \rangle$

- Represents $\pm\infty$.
- Typically the result of *overflow*.
 - ▶ Overflow can be negative!
 - ▶ *Underflow* is when the result becomes zero due to rounding.
- Both positive and negative.
- Examples:

$$\frac{1}{0} = \frac{-1}{-0} = \infty \qquad \frac{1}{-0} = -\infty$$

When $E = \langle 111 \dots 1 \rangle, T \neq \langle 000 \dots 0 \rangle$

- Not A Number (NaN).
- Represents case when no numeric value can be determined.
- Examples:

$$\text{sqrt}(-1) \qquad \infty - \infty \qquad \infty \cdot 0$$

The floating-point number line

← very positive e very negative e → ← very negative e very positive e →

$-\infty$	-Normal	-Subnorm	-0	$+0$	+Subnorm	+Normal	$+\infty$
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NaN

NaN

Note that NaNs are unordered:

- NaN is different from everything *even other NaNs!*
 - ▶ $\text{NaN} == \text{NaN}$ is false.
 - ▶ Floating-point equality is not reflexive!
- $\text{NaN} > x$ and $\text{NaN} < x$ is false for all x .

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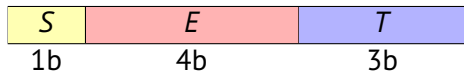
Floating-point in C

Summary

Play the game

<https://topps.diku.dk/compsys/floating-point.html>

Tiny 8-bit floating-point example



8-bit floating-point representation

- Sign bit is the most significant bit (leftmost).
- The next four bits are E with a bias of 7.
- The last three bits are T .

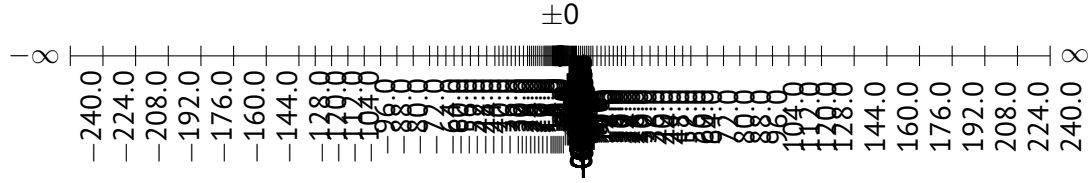
Same general form as IEEE Format

- Normalised, denormalised.
- Representation of 0, NaN, both infinities.

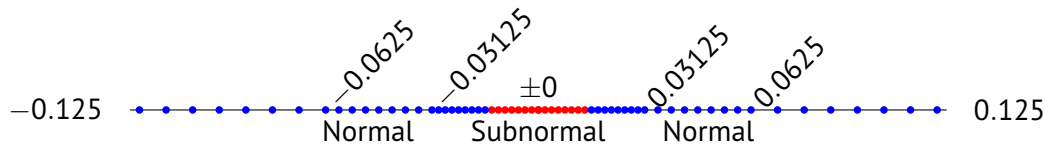
Dynamic range of positive numbers

	S	E	T	e	Value	
Denormalised	0	0000	000	-6	0	
	0	0000	001	-6	$1/8 \cdot 1/64 = 1/512$	closest to zero
	0	0000	010	-6	$2/8 \cdot 1/64 = 2/512$	
	...					
	0	0000	111	-6	$7/8 \cdot 1/64 = 7/512$	largest denorm
Normalised	0	0001	000	-6	$8/8 \cdot 1/64 = 8/512$	smallest norm
	0	0001	001	-6	$9/8 \cdot 1/64 = 9/512$	
	...					
	0	0110	110	-1	$14/8 \cdot 1/2 = 14/16$	Closest to 1
	0	0110	111	-1	$15/8 \cdot 1/2 = 15/16$	
	0	0111	000	0	$8/8 \cdot 1 = 1$	Closest to 1
	0	0111	001	0	$9/8 \cdot 9/8 = 1$	
	0	0111	010	0	$10/8 \cdot 10/8 = 1$	
	...					
	0	1110	110	7	$14/8 \cdot 128 = 224$	
	0	1110	111	7	$15/8 \cdot 128 = 240$	
	0	1111	000	N/A	∞	

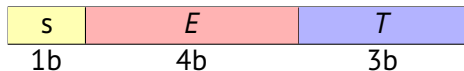
Distribution of values



Distribution of values (zooming in)



- Note how the distribution gets denser towards zero.
- Note the big gap there would be around 0 if we did not have subnormals.
- Each of the spans with same distance between neighbors corresponds to numbers with same E .



Useful properties of the IEEE encoding



- **Floating-point zero same as integer zero.**
 - ▶ All bits 0.
 - ▶ ...but negative zero is different.
- **Can almost compare floats with unsigned integer comparisons.**
 - ▶ Must first compare sign bit.
 - ▶ Must consider $-0 = 0$.
 - ▶ NaNs problematic:
 - ▶ Greater than any other value (because $E = \langle 111 \dots 1 \rangle$).
 - ▶ What should comparison yield?
 - ▶ Otherwise OK:
 - ▶ Normalised and denormalised compare as expected.
 - ▶ Infinities ordered properly relative to finities.

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Basic idea behind floating-point operations

$$x +_f y = \text{Round}(x + y)$$

$$x \times_f y = \text{Round}(x \times y)$$

- **Basic idea**

- ▶ First *compute exact result!*
- ▶ Then round it to fit into desired precision.
 - ▶ Overflow if exponent too large.
 - ▶ *Round to fit* into T .

Rounding and rounding modes

- There's more than one way to round a number, here to an integer.
- Round to nearest, with *tiebreaker* in case two numbers are equally close.

Tiebreaker	1.40	1.60	1.50	2.50	-1.50
Towards zero					

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Nearest even					

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Towards ∞	1	2	2	3	-1
Nearest even	1	2	2	2	-2

- “Round to nearest, ties to even” is the default rounding mode.

Closer look at *nearest even*

- **Default rounding mode**

- ▶ But can be changed dynamically.
 - ▶ `https://www.gnu.org/software/libc/manual/html_node/Rounding.html`
 - ▶ Never do this.
- ▶ All others are statistically biased.
 - ▶ Biased: Sum of set of positive numbers will consistently be over- or under-estimated.

- **Applying to other decimal places / bit positions**

- ▶ When exactly halfway between two possible values:
 - ▶ Round so that least significant digit is even.
- ▶ E.g. rounding to nearest hundredth:
 - ▶ 7.8949999:

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 - ▶ 7.8950000: 7.90
 - ▶ 7.8850000:

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 - ▶ 7.8950000: 7.90
 - ▶ 7.8850000: 7.88

Rounding binary numbers

- **Binary fractional numbers**

- ▶ “Even” when least significant bit is 0.
- ▶ “Half way” when bits to right of rounding position are $100 \cdots_2$.

- **Examples**

- ▶ Round to nearest $1/4$ (2 bits right of binary point).

Value	Binary	Rounded	Action	Rounded value
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Rounding binary numbers

- **Binary fractional numbers**

- ▶ “Even” when least significant bit is 0.
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Floating-point multiplication (assuming operands are numbers)

$$((-1)^{s_3} \cdot m_3 \cdot 2^{e_3}) = ((-1)^{s_1} \cdot m_1 \cdot 2^{e_1}) \cdot ((-1)^{s_2} \cdot m_2 \cdot 2^{e_2})$$

- **Exact result**

$$s_3 = s_1 \oplus s_2$$

$$m_3 = m_1 \cdot m_2$$

$$e_3 = e_1 + e_2$$

where \oplus is exclusive-or.

- **Fixing**

- ▶ If $m_3 \geq 2$, shift m_3 right and increment e_3 .
- ▶ If e_3 out of range, overflow to ∞ .
- ▶ Round m_3 to fit T precision.

- **Implementation**

- ▶ Biggest chore is multiplying significands.

Floating-point addition (assuming operands are numbers)

$$(-1)^{s_3} \cdot m_3 \cdot 2^{e_3} = ((-1)^{s_1} \cdot m_1 \cdot 2^{e_1}) + ((-1)^{s_2} \cdot m_2 \cdot 2^{e_2})$$

■ Approach

- ▶ Assume without loss of generality that $e_1 \geq e_2$.
- ▶ Rewrite smaller number such that its exponent matches e_1 :

$$((-1)^{s_3} \cdot m_3 \cdot 2^{e_3}) = ((-1)^{s_1} \cdot m_1 \cdot 2^{e_1}) + ((-1)^{s_2} \cdot m'_2 \cdot 2^{e_1})$$

■ Exact result

- ▶ Sign s_3 , significand m_3 :
 - ▶ Result of signed addition.

■ Fixing

- ▶ If $m_3 \geq 2$, shift m_3 right and increment e_3 .
- ▶ If $m_3 < 1$, shift m left k positions and decrement e_3 by k .
- ▶ If e_3 out of range, overflow to ∞ .
- ▶ Round m_3 to p bits.

$$\begin{array}{r} \leftarrow e_1 - e_2 \rightarrow \\ \boxed{-1^{s_1} \cdot m_1} \\ + \quad \boxed{-1^{s_2} \cdot m_2} \\ \hline \boxed{-1^{s_3} \cdot m_3} \end{array}$$

Example of floating-point addition with a 2-bit significand

$$\begin{aligned} & (-1.01 \cdot 2^2) + (1.1 \cdot 2^4) \\ = & (-1.01 \cdot 2^2) + (110.0 \cdot 2^2) && \text{Align exponents} \\ = & (-1.01 + 110.0) \cdot 2^2 && \text{Distributivity} \\ = & 100.11 \cdot 2^2 && \text{Add significands} \\ = & 1.0011 \cdot 2^4 && \text{Normalise} \\ = & 1.01 \cdot 2^4 && \text{Perform rounding} \end{aligned}$$

Algebraic properties of floating-point addition

- **Compared to those of Abelian Group**
 - ▶ Closed under addition?

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 - ▶ $(3.14 + 1e10) - 1e10 = 0$
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- ▶ 0 is additive identity?

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- ▶ $a \geq b \Rightarrow a + c \geq b + c?$

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■ Monotonicity

- ▶ $a \geq b \Rightarrow a + c \geq b + c$? **Almost**
 - ▶ Infinities and NaNs are the exception.

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- ▶ 1 is multiplicative identity? **Yes**
- ▶ Multiplication distributes over addition? **No**
 - ▶ Overflow and rounding again.
 - ▶ $1e20 * (1e20 - 1e20) = 0.0$
 - ▶ $1e20 * 1e20 - 1e20 * 1e20 = \text{NaN}$

Why are numbers exciting?

Preliminaries: biased numbers

Floating-point arithmetic

Background: Fractional binary numbers

IEEE floating-point standard

Examples and properties

Rounding, addition, and multiplication

Floating-point in C

Summary

Floating-point in C

- **C guarantees two types**

- ▶ `float`: 32-bit single precision.
- ▶ `double`: 64-bit single precision.

- **Conversions/casting**

- ▶ Casting between `int`, `float`, and `double` changes bit representation.
- ▶ `double/float to int`
 - ▶ Truncates fractional part.
 - ▶ Like rounding toward zero.
 - ▶ Not defined when out of range or NaN: generally sets to TMin.
- ▶ `int to double`
 - ▶ Exact conversion as long as `int` fits in 53 bits.
- ▶ `int to float`
 - ▶ Will round according to rounding mode.

Floating-point is exciting!



First “flight” of the Ariane 5 in 1996.

Floating-point is exciting!



First “flight” of the Ariane 5 in 1996.

- A `double` storing horizontal velocity of the rocket was converted to a 16-bit signed integer.
- The number was larger than 32767 so the conversion failed, causing an exception, crashing the guidance module.

Floating-point puzzles

For each of the following C expressions, either

- Argue that it is true for all argument values.
- Explain why it's not.

```
int    x = ...;  
float  f = ...;  
double d = ...;
```

Assume neither `d` nor `t`
is NaN.

Assume `int` is 32 bits.

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For each of the following C expressions, either

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- `(d+f)-d == f`

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Summary

Summary

- **IEEE floating-point has clear properties.**
 - ▶ But they may not match your intuition.
- **Represents numbers of the form $(-1)^s \cdot m \cdot 2^e$.**
- One can reason about operations independent of implementation.
 - ▶ Computed with perfect precision and then rounded.
 - ▶ But rounded after *every* “primitive” operation (e.g. addition, multiplication).
- **Not the same as \mathbb{Q}/\mathbb{R} arithmetic.**
 - ▶ Violates associativity and distributivity, mostly due to rounding.
 - ▶ Sometimes makes life difficult for heavy-duty numerical programming.
 - ▶ But carefully designed such that “naive” use mostly does what one expects.

Also try this tool: <https://evanw.github.io/float-toy/>

And read this: <https://moyix.blogspot.com/2022/09/someones-been-messing-with-my-subnormals.html>