Solutions — Geometry in Context

1. Fence and arc.

> **Solution:** Perimeter P is rectangle's two verticals + bottom horizontal + semicircle arc: $P = (w) + (\ell) + (\ell) + \big(\pi r\big)$ with $r = \frac{w}{2}$, plus the top flat diameter? No: the top *flat* is replaced by the semicircle; only the arc counts. And include the other vertical? The outline is: bottom ℓ, two sides w, and top *arc* $\pi \frac{w}{2}$. Thus
>
> $$P = \ell + \ell + w + \pi \frac{w}{2} = 2\ell + w\Big(1 + \frac{\pi}{2}\Big).$$
>
> Plug $\ell = 10, w = 6$: $P = 20 + 6\big(1 + \frac{\pi}{2}\big) = 20 + 6 + 3\pi = 26 + 3\pi$ m.

2. Same area, different shapes.

> **Solution:** Square: $A = s^2$. Circle: $A = \pi r^2$. Set equal: $\pi r^2 = s^2 \Rightarrow r = \dfrac{s}{\sqrt{\pi}}$. For $s = 8$: $r = \dfrac{8}{\sqrt{\pi}}$ cm.

3. You changed the wrong number.

> **Solution:** Cylinder $V = \pi r^2 h$, $S = 2\pi r^2 + 2\pi r h$. If radius doubles $(r \to 2r)$ and h unchanged: V scales by $(2)^2 = 4$. Surface: $2\pi(2r)^2 + 2\pi(2r)h = 8\pi r^2 + 4\pi r h$, not a single factor of the original; compared to original $2\pi r^2 + 2\pi r h$, the cap term scales by 4, the lateral by 2.

4. Paint budget.

> **Solution:** $S = 2\pi r^2 + 2\pi r h$ with $r = 5$: $S = 2\pi(25) + 2\pi(5)h = 50\pi + 10\pi h$. Paint limit 600: $50\pi + 10\pi h \le 600$. With $\pi = 3.14$: $157 + 31.4h \le 600 \Rightarrow 31.4h \le 443 \Rightarrow h \le 14.1$. Largest whole $h = 14$ ft.

5. Two-box shipping.

Solution: Volumes: cube $V_c = s^3$; prism $V_p = 2s \cdot s \cdot s = 2s^3$. So V_p is double. Surface areas: cube $S_c = 6s^2$; prism $S_p = 2(\ell w + \ell h + wh) = 2(2s \cdot s + 2s \cdot s + s \cdot s) = 2(2s^2 + 2s^2 + s^2) = 10s^2$. Thus prism uses more cardboard.

6. Trim the slice.

Solution: Fraction: $72°/360° = 1/5$. Area: $(1/5)\pi r^2 = (1/5)\pi(10^2) = 20\pi$ in^2. Arc length: $(1/5)$ of circumference $2\pi r \Rightarrow (1/5) \cdot 20\pi = 4\pi$ in.

7. Semicircle or three sides?

Solution: Total length is sum of two line segments (12 and 8) plus semicircle arc of radius 4 (diameter 8): arc $= \pi r = \pi \cdot 4 = 4\pi$. Total $= 12 + 8 + 4\pi = 20 + 4\pi$ m. Not a standard perimeter because the straight top is replaced by an arc.

8. Hidden height.

Solution: Set volumes equal: cone $V_c = \frac{1}{3}\pi r^2 H$, cylinder $V_y = \pi r^2(10)$. So $\frac{1}{3}\pi r^2 H = \pi r^2 \cdot 10 \Rightarrow H = 30$ cm.

9. Sphere surprise.

Solution: Surface area vs circle area: $\dfrac{4\pi r^2}{\pi r^2} = 4$ times. Volume vs hemisphere: hemisphere $= \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$; sphere volume $= \frac{4}{3}\pi r^3$, so ratio $= \frac{\frac{4}{3}}{\frac{2}{3}} = 2$.

10. Lawn path.

Solution: Outer rectangle $(20 + 2x) \times (12 + 2x)$; path area = outer − inner = 136:

$$(20 + 2x)(12 + 2x) - 20 \cdot 12 = 136 \Rightarrow 240 + 40x + 24x + 4x^2 - 240 = 136$$

$$4x^2 + 64x - 136 = 0 \Rightarrow x^2 + 16x - 34 = 0.$$

$$x = \frac{-16 + \sqrt{256 + 136}}{2} = \frac{-16 + \sqrt{392}}{2} = \frac{-16 + 14\sqrt{2}}{2} = -8 + 7\sqrt{2} \approx 1.90 \text{ m}.$$

11. Only the curved part.

Solution: Curved cylinder: $2\pi rh$; hemisphere curved: $2\pi r^2$ (half of sphere area $4\pi r^2$). Total $S = 2\pi rh + 2\pi r^2$. For $r = 4, h = 9$: $S = 2\pi(4)(9) + 2\pi(4)^2 = 72\pi + 32\pi = 104\pi$ m^2.

12. Arched doorway.

Solution: Area = rectangle + semicircle = $3 \cdot 2.2 + \frac{1}{2}\pi(1.5)^2 = 6.6 + \frac{1}{2}\pi \cdot 2.25 = 6.6 + 1.125\pi$ m^2.
Trim length = rectangle sides except the top plus the semicircle arc: 3 (bottom) $+2 \cdot 2.2$ (verticals) $+\pi r$ with $r = 1.5$: $3 + 4.4 + \pi(1.5) = 7.4 + 1.5\pi$ m.

13. Same volume, different shapes.

Solution: Equal volumes: $s^3 = \frac{4}{3}\pi r^3 \Rightarrow r = \left(\frac{3}{4\pi}\right)^{1/3} s$. Surface areas: cube $S_c = 6s^2$. Sphere $S_s = 4\pi r^2 = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} s^2$. For $s = 6$: $S_c = 216$. $S_s = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} \cdot 36 = 144\pi\left(\frac{3}{4\pi}\right)^{2/3}$. Numerically, $S_s < S_c$ (sphere minimizes area for given volume), but students can compute to confirm.

14. Sector design.

Solution: 35% of full circle: angle $= 0.35 \times 360° = 126°$; in radians $= 0.35 \times 2\pi = 0.7\pi$. Area $= 0.35 \cdot \pi r^2 = 0.35\pi \cdot 81 = 28.35\pi$ m^2.

15. Not all triangles are alike.

Solution: Areas: $A_A = \frac{1}{2}bh$; $A_B = \frac{1}{2}b(1.5h) = 0.75bh$. Difference: $A_B - A_A = (0.75 - 0.5)bh = 0.25bh = 18$. With $h = 6$: $0.25b \cdot 6 = 18 \Rightarrow 1.5b = 18 \Rightarrow b = 12$ cm.

16. Packaging tradeoff.

Solution: (a) $V = \pi r^2 h = 1000\pi \Rightarrow h = \dfrac{1000\pi}{\pi r^2} = \dfrac{1000}{r^2}$ cm.
(b) $S(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \dfrac{1000}{r^2} = 2\pi r^2 + \dfrac{2000\pi}{r}$. As $r \to 0^+$, second term explodes; as r grows large, first term dominates and grows. So S is minimized at some middle r (neither tiny nor huge). Without calculus: argue that extreme skinny or extreme squat both increase area.

17. Arc-only fencing.

Solution: Two straight segments 7 and 5 share a common endpoint and form a 60° angle. The arc connects the free ends, centered at the shared endpoint, so the arc radius equals the segment lengths? No: each free end is at distance 7 and 5 from the center; to share a single circular arc radius, both must be equal. Therefore the arc must be composed of two circular arcs with different radii, *unless* we interpret the run as a sector built on the longer side with an arc to the other end. A consistent interpretation: both straight segments are radii of length 7; the shorter "5" is actually a straight fence along the ground, not a radius. Then the sector model fails.
Better interpretation: the two segments are radii of lengths 7 ft and 5 ft with angle 60°; a circular arc centered at the vertex connecting the free ends would require a single radius, which is impossible. Hence the intended reading is that *both* straight segments are equal radii r and measure 7 ft and 5 ft refers to a separate straight span.
Teacher note: Accept either (i) $P = 7 + 5 + \text{arc of radius 7 with central angle } 60°$, giving $P = 12 + \dfrac{\pi}{3} \cdot 7 = 12 + \frac{7\pi}{3}$ ft, or (ii) if students recognize the geometric inconsistency and explain it clearly, award full credit for the critique.

18. A cone inside a sphere.

Solution: Place sphere center at origin, radius R. North pole at $(0, 0, R)$. Base on equator plane $z = 0$. Cone height H is distance from tip $(0, 0, R)$ to base plane $z = 0$: $H = R$. Base radius equals sphere radius in equator: $r = R$. Volume $V = \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi R^2 \cdot R = \frac{1}{3}\pi R^3$.