Adaptive importance sampling in least-squares Monte Carlo algorithms for backward stochastic differential equations

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Abstract

We design an importance sampling scheme for backward stochastic differential equations (BSDEs) that minimizes the conditional variance occurring in least-squares Monte Carlo (LSMC) algorithms. The Radon-Nikodym derivative depends on the solution of BSDE, and therefore it is computed adaptively within the LSMC procedure. To allow robust error estimates w.r.t. the unknown change of measure, we properly randomize the initial value of the forward process. We introduce novel methods to analyze the error: firstly, we establish norm stability results due to the random initialization; secondly, we develop refined concentration-of-measure techniques to capture the variance of reduction. Our theoretical results are supported by numerical experiments.

Keywords: Backward stochastic differential equations, empirical regressions, importance sampling

MSC Classification: 49L20, 60H07, 62Jxx, 65C30, 93E24

1 Introduction

Importance sampling can be important for accelerating the convergence of Monte-Carlo approximation. To name a few examples and references, it has applications in numerical integration [26, 19, 20] and in rare event simulation [9, 25, 7]. The idea is to direct the simulations

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to important regions of space through an appropriate change of measure. In this work, we propose a fully implementable least-squares Monte-Carlo (LSMC, a.k.a. Regression Monte-Carlo or Empirical Regression) scheme with importance sampling for Backward Stochastic Differential Equations (BSDEs). BSDEs are usually associated to stochastic control problems [21, 12]. In the Markovian case with fixed terminal time $T > 0$, the BSDE takes the form

$$\mathcal{Y}_t = g(X_T) + \int_t^T f(s, X_s, \mathcal{Y}_s, \mathcal{Z}_s)\mathrm{d}s - \int_t^T \mathcal{Z}_s \mathrm{d}W_s \tag{1}$$

where the unknown processes are $(\mathcal{Y}, \mathcal{Z})$ and X is a given forward (d-dimensional) SDE driven by the (q-dimensional) Brownian motion W :

$$X_t = x_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s. \tag{2}$$

In the recent years, there have been many contributions to their numerical approximations (see references in [6, 8, 10, 16, 15]). Loosely speaking, the current work extends [15] by incorporating an importance sampling that minimizes the variance of the LSMC algorithm for computing the value functions y and z – defined as $\mathcal{Y}_t = y(t, X_t)$ and $\mathcal{Z}_t = z(t, X_t)$ – using a suitable Dynamic Programming Equation (DPE for short).

The work closest to ours is the one by Bender and Moseler [5] where they propose a importance sampling method for BSDEs for a general (but known) Radon-Nikodym derivative. In our setting, the change of measure is not given but sought within the algorithm. Indeed, the optimal Radon-Nikodym derivative depends on the solution $(\mathcal{Y}, \mathcal{Z})$ – see Proposition 2.1 – therefore it too must be approximated. In fact, the change of probability measure implies that traditional simulation and error analysis techniques cannot be applied to establish convergence of the approximations of y and z. In [15], the simulations of X for the LSMC scheme were generated from a fixed point at time 0. The error was analyzed using the $\mathbf{L}_2$-norm related to the law of simulations. Propagation of error terms due to dynamic programming were treated using BSDE estimates and an extension of Gronwall's inequality. In importance sampling, the simulations used in LSMC scheme have a modified drift because of the change of probability measure. The drift term depends on the solution of the BSDE or its approximation. This creates two novel difficulties. Firstly, one cannot generate the simulations starting from a fixed point at time 0. Indeed, the drift changes depend on the BSDE solution, which is computed recursively backwards in time, and therefore not available for simulation from time zero. To overcome this issue, we initialize the simulations at each time point i using a generic random variable. However, in doing so, in general we lose the ability to treat the DPE error with the usual Gronwall technique, see Remark 5.1. To retrieve the algorithm convergence, the initializing random variable is required to satisfy the Uniform Sub-Exponential Sandwiching (USES) property, see Proposition 3.1. This allows the distribution of simulations to have equivalent $\mathbf{L}_2$-norms, under different changes of measure and at every time-point. Secondly, the error due to the approximation of the Radon-Nikodym derivative must be treated.

Another significant contribution of this work is in the field of nonparametric statistics and it is crucial for capturing the effect of variance reduction in our scheme. In [15], the error analysis makes use of classical regression theory [17, Theorems 11.1 and 11.3]. In this theory, error estimates are not sufficiently tight to observe impact of variance reduction. In particular, uniform concentration-of-measure techniques are used, which only observe the diameter of the approximation space and cannot observe low variance. We make a non-trivial continuation of the recent work of [4] in order to improve concentration-of-measure techniques in the case of regression problems, so that we can recover the variance reduction effect. These results are novel, to the best of our knowledge, and may bring insights in problems beyond the immediate concern of BSDE approximation.

Overview. In Section 2.1, we identify the optimal importance sampling probability measure starting from the continuous time BSDE (1). In Section 2.2, we introduce the discrete-time approximation of (1) in the form of the importance sampling DPEs. We also state the assumptions on the data g and f, and summarize some key properties of the resulting discrete time BSDE. In Section 3, we define USES and give nontrivial, fully implementable examples (Proposition 3.3) relevant to practical problems. In Section 4, we detail the regression scheme (LSMC) on piecewise constant basis functions in a general setting (unrelated to BSDEs). We provide explicit nonparametric error estimates which do not require uniform concentration-of-measure techniques (Theorem 4.1). In Section 5, we apply the regression method of Section 4 in order to approximate the conditional expectations in the importance sampling DPEs. Explicit error estimates for this fully implementable scheme are derived. We conclude the paper in Section 6 with numerical examples that illustrate the performance of the scheme.

Model restrictions. Due to the novel nature of this scheme, we make simplified assumptions on the BSDE model and on the numerical method, with the aim of highlighting the main ideas rather than technical results. In particular, we assume that the function $f(t, x, y, z) \equiv f(t, x, y)$ is independent of the z component. Of course considering such an f is a restriction for applications, but nonetheless it still serves a significant interest since it allows to handle reaction-diffusion equations [27] and nonlinear valuations in finance [11]. We discuss more serious reasons for this simplification in Subsection 2.3, in the hope to motivate further investigation.

2 Derivation of the importance sampling Dynamic Programming Equations

2.1 Derivation of optimal Radon-Nikodym derivative

We are concerned with BSDEs driven by a q-dimensional Brownian motion W, supported by a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ with the usual conditions on the filtration. The solution $(\mathcal{Y}, \mathcal{Z})$ is progressively measurable, takes values in $\mathbb{R} \times (\mathbb{R}^q)^\top$, and satisfies

$$\mathcal{Y}_t = \xi + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathrm{d}s - \int_t^T \mathcal{Z}_s \mathrm{d}W_s \tag{3}$$

with the process $\mathcal{Y}$ being continuous; the operator $^\top$ denotes the vector transpose, i.e. $\mathcal{Z}$ is a row vector. Here, the BSDE is not necessarily of Markovian type.

Before introducing our numerical scheme, we first identify the optimal change of measure. This is inspired by the continuous time equation (3). Assume for discussion that the solution $(\mathcal{Y}, \mathcal{Z})$ of (3) is unique in $\mathbf{L}_2$-spaces, so that the above solutions satisfy $\mathbb{E}\left[\sup_{t\le T} |\mathcal{Y}_t|^2 + \int_0^T |\mathcal{Z}_t|^2 \mathrm{d}t\right] < +\infty$; this is valid under fairly general conditions on the data ξ and f [21, 12]. We start from the representation of $\mathcal{Y}$ as conditional expectation under $\mathbb{P}$:

$$\mathcal{Y}_t = \mathbb{E}_{\mathbb{P}}\left[\xi + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathrm{d}s \mid \mathcal{F}_t\right].$$

Let h be a $(\mathbb{R}^q)^\top$-valued progressively measurable process, such that one can define the process $(W_t^{(h)})_{0\le t\le T}$ and the $\mathbb{P}$-martingale $(1/\mathcal{L}_t^{(h)})_{0\le t\le T}$ by

$$W_t^{(h)} := W_t - \int_0^t h_r^\top \mathrm{d}r, \qquad \mathcal{L}_t^{(h)} := e^{-\int_0^t h_r \mathrm{d}W_r + \frac{1}{2}\int_0^t |h_r|^2 \mathrm{d}r} = e^{-\int_0^t h_r \mathrm{d}W_r^{(h)} - \frac{1}{2}\int_0^t |h_r|^2 \mathrm{d}r}, \tag{4}$$
$$\mathcal{L}_{t,s}^{(h)} := \frac{\mathcal{L}_s^{(h)}}{\mathcal{L}_t^{(h)}} \quad \text{for } 0 \le t \le s \le T.$$

Then we can set a new equivalent measure $\mathbb{Q}^h|_{\mathcal{F}_t} = [\mathcal{L}_t^{(h)}]^{-1}\mathbb{P}|_{\mathcal{F}_t}$ by the Girsanov theorem, and under $\mathbb{Q}^h$, $W^{(h)}$ is a Brownian motion. Moreover, we can express $\mathcal{Y}$ under the new measure

$$\mathcal{Y}_t = \frac{\mathbb{E}_{\mathbb{Q}^h}\left[\xi \mathcal{L}_T^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathcal{L}_s^{(h)}\mathrm{d}s \ \mid \mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}^h}\left[\mathcal{L}_T^{(h)} \mid \mathcal{F}_t\right]} = \mathbb{E}_{\mathbb{Q}^h}\left[\xi \mathcal{L}_{t,T}^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathcal{L}_{t,s}^{(h)}\mathrm{d}s \ \mid \mathcal{F}_t\right]. \tag{5}$$

To obtain variance reduction in the Monte-Carlo based algorithm, the $\mathcal{F}_t$-conditional variance under $\mathbb{Q}^h$ of $S(t,h) := \xi \mathcal{L}_{t,T}^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathcal{L}_{t,s}^{(h)}\mathrm{d}s$ has to be small; we see this later in Corollary 4.2. Under the assumptions

$(\mathbf{H_a})$ there exists $\varepsilon > 0$ such that $\mathbb{P}(\forall t \in [0,T] : \mathcal{Y}_t \ge \varepsilon) = 1$,

$(\mathbf{H_b})$ $\int_0^\cdot \mathcal{Z}_s \mathrm{d}W_s$ is a BMO martingale,

we determine an optimal Radon-Nikodym derivative in the following proposition.

Proposition 2.1. *Assume the existence of a unique solution* $(\mathcal{Y}, \mathcal{Z})$ *to* (3) *which satisfies* $\mathbb{E}(\sup_{t\le T} |\mathcal{Y}_t|^2 + \int_0^T |\mathcal{Z}_t|^2 \mathrm{d}t) < +\infty$ *and* $(\mathbf{H_a})$–$(\mathbf{H_b})$*. Then the equivalent probability measure* $\mathbb{Q}^h$ *such that the random variable* $S(t,h)$ *has zero* $\mathcal{F}_t$*-conditional variance in* (5) *is given by the drift* $h_s = \mathcal{Z}_s/\mathcal{Y}_s$*.*

The above result extends [24] to BSDEs.

Remark 2.2. The assumptions $(\mathbf{H_a})$–$(\mathbf{H_b})$ can be satisfied under quite general conditions on the data ξ and f. In fact, even if $(\mathbf{H_a})$ is not immediately satisfied, a trivial transformation may permit the application of Proposition 2.1: if $\mathcal{Y}_t \ge c$ for any $t \in [0,T]$ a.s. for some possibly non-positive constant $c \in \mathbb{R}$, the shifted BSDE $(\mathcal{Y} - c + 1, \mathcal{Z})$ associated to data $\xi - c + 1$ and $f(t, y + c - 1, z)$ satisfies the assumption $(\mathbf{H_a})$ with $\varepsilon = 1$; likewise, if $\mathcal{Y}_t \le c$, one can apply similar arguments with an additional sign flip of $\mathcal{Y}$. The BMO-condition $(\mathbf{H_b})$ is satisfied in many situations, in particular when the terminal condition ξ is bounded [3].

Proof of Proposition 2.1. Let h be a progressively measurable process such that $1/\mathcal{L}^{(h)}$ is an exponential martingale under $\mathbb{P}$. By Itô's formula applied to $\mathcal{Y}_\cdot \mathcal{L}_\cdot^{(h)}$ between on $[t,T]$ combined with (3), we readily obtain

$$
\begin{aligned}
S(t,h) &= (\mathcal{L}_t^{(h)})^{-1} \left(\mathcal{Y}_T \mathcal{L}_T^{(h)} + \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_s^{(h)} \mathrm{d}s \right) \\
&= \mathcal{Y}_t + (\mathcal{L}_t^{(h)})^{-1} \int_t^T \left(\mathcal{L}_s^{(h)} [-f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathrm{d}s + \mathcal{Z}_s \mathrm{d}W_s] + \mathcal{Y}_s \mathcal{L}_s^{(h)} (-h_s) \mathrm{d}W_s^{(h)} - \mathcal{L}_s^{(h)} h_s \cdot \mathcal{Z}_s \mathrm{d}s \right) \\
&\quad + (\mathcal{L}_t^{(h)})^{-1} \int_t^T f(s, \mathcal{Y}_s, \mathcal{Z}_s) \mathcal{L}_s^{(h)} \mathrm{d}s \\
&= \mathcal{Y}_t + (\mathcal{L}_t^{(h)})^{-1} \int_t^T \mathcal{L}_s^{(h)} (\mathcal{Z}_s - \mathcal{Y}_s h_s) \mathrm{d}W_s^{(h)}.
\end{aligned}
$$

Therefore, if we can set $h_s = \mathcal{Z}_s/\mathcal{Y}_s$ and ensure that $1/\mathcal{L}^{(h)}$ is an exponential martingale under $\mathbb{P}$, then $S(t,h)$ has zero $\mathcal{F}_t$-conditional variance under $\mathbb{Q}^h$; this follows readily from $(\mathbf{H_a})$–$(\mathbf{H_b})$. $\square$

2.2 Assumptions and discrete-time scheme

The resolution of $(\mathcal{Y}, \mathcal{Z})$ usually requires the time-discretization of (3). A possible approach is to use the Malliavin integration-by-parts formula to represent $\mathcal{Z}$ [22], which leads to the Malliavin weights DPE [15, 28] (MWDP) given by

$$
\text{for } i \in \{0, \dots, N-1\}, \quad \begin{cases} Y_i &= \mathbb{E}_{\mathbb{P},i}\left[\xi + \sum_{k=i}^{N-1} f(t_k, Y_{k+1}) \Delta_k\right], \\ Z_i &= \mathbb{E}_{\mathbb{P},i}\left[\xi \Theta_N^{(i)} + \sum_{k=i+1}^{N-1} f(t_k, Y_{k+1}) \Theta_k^{(i)} \Delta_k\right] \end{cases} \tag{6}
$$

where (Y_i, Z_i) approximates $(\mathcal{Y}_{t_i}, \mathcal{Z}_{t_i})$ along the time-grid $\pi = \{0 := t_0 < \cdots < t_i < \cdots < t_N := T\}$, $\Delta_i := t_{i+1} - t_i$, $\mathbb{E}_{\mathbb{P},i}[.] := \mathbb{E}_{\mathbb{P}}[. \mid \mathcal{F}_{t_i}]$, and where $\{\Theta_k^{(i)} \; : \; k = i+1 \ldots, N\}$ is a suitable sequence of square integrable random variables. For reasons detailed in Section 2.3, we consider functions f that are independent of z from here on; this is not the case of [15]. It is important to assume that, for all i and k, $\Theta_k^{(i)}$ satisfies

$$\mathbb{E}_{\mathbb{P},i}\left[\Theta_k^{(i)}\right] = 0, \qquad \mathbb{E}_{\mathbb{P},i}\left[|\Theta_k^{(i)}|^2\right]^{1/2} \leq \frac{C_M}{\sqrt{t_k - t_i}},$$

for some finite constant C_M.

Remark. In fact, the MWDP is known to be a 'good' discrete time approximation of (3) in the Markovian framework (1) [28]. We choose a DPE based on Malliavin weights since we know from [15] that it allows better control on convergence. Nevertheless, the subsequent importance sampling scheme could be designed with other DPEs, and this would not greatly affect the arguments of this section which follow.

We now introduce the discrete-time importance sampling scheme on which we base the LSMC scheme later. As will be explained in Subsection 2.3, we assume that f does not depend on z. We first define simplified notation to deal with the discrete-time counterpart of the importance sampling inverse Radon-Nikodym derivative (4). For a given piecewise constant (in time), bounded, adapted process h with $h|_{(t_k, t_{k+1}]} := h_k \in (\mathbb{R}^q)^\top$ $\mathcal{F}_{t_k}$-measurable, let

$$L_j^{(h)} := \exp\left(-\sum_{k=0}^{j-1} h_k \Delta W_k + \frac{1}{2}\sum_{k=0}^{j-1} |h_k|^2 \Delta_k\right), \quad 0 \leq j \leq N,$$

where we write $\Delta W_k = W_{t_{k+1}} - W_{t_k}$. We also define

$$L_{i,j}^{(h)} := \frac{L_j^{(h)}}{L_i^{(h)}} = \exp\left(-\sum_{k=i}^{j-1} h_k \Delta W_k + \frac{1}{2}\sum_{k=i}^{j-1} |h_k|^2 \Delta_k\right).$$

As indicated in Proposition 2.1, h will be computed backward in time using the processes (Y, Z). In principle, we would now set $h_k = Z_k / Y_k$ and define $\mathbb{Q}^h|_{\mathcal{F}_{t_i}} = \left[L_i^{(h)}\right]^{-1} \mathbb{P}|_{\mathcal{F}_{t_i}}$ and

$$Y_i = \mathbb{E}_{\mathbb{Q}^h,i}\left[\xi L_{i,N}^{(h^N)} + \sum_{k=i}^{N-1} f(t_k, Y_{k+1}) L_{i,k+1}^{(h)} \Delta_k\right],$$

as we did in Section 2.1. However, at time point i, the solution (Y_i, Z_i) is not known explicitly, therefore it isn't possible to compute $L_{i,k+1}^{(h)}$ explicitly. In order to have an explicit formulation, we use a modified probability measure defined as follows: for $0 \leq i, j \leq N$, set

$$\mathbb{Q}_i^h|_{\mathcal{F}_{t_j}} := \left(\left[L_{i+1,j}^{(h)}\right]^{-1} \mathbf{1}_{j > i+1} + \mathbf{1}_{j \leq i+1}\right) \mathbb{P}|_{\mathcal{F}_{t_j}}.$$

Since h is bounded, the Girsanov theorem implies that $\mathbb{Q}_i^h$ is a probability measure. Observe that the change of measure is effective after $i+2$ instead of $i+1$. We denote by $W^{\mathbb{Q}_i^h}$ the new Brownian motion under $\mathbb{Q}_i^h$, so that the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_i^h}$ restricted to $\mathcal{F}_{t_j}$ writes

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_i^h}|_{\mathcal{F}_{t_j}} &:= L_{i+1,j}^{(h)}\mathbf{1}_{j>i+1} + \mathbf{1}_{j\le i+1}\\ &= \mathbf{1}_{j>i+1}\exp\left(-\sum_{k=i+1}^{j-1}\left\{h_k\Delta W_k^{\mathbb{Q}_i^h} + \frac{1}{2}|h_k|^2\Delta_k\right\}\right) + \mathbf{1}_{j\le i+1}\end{aligned}$$

for $j \ge i$. Now, we are in a position to represent the MWDP (6) with the above change of probability measure: for $i = N-1, \dots, 0$, set

$$\left.\begin{aligned}Y_i &= \mathbb{E}_{\mathbb{Q}_i^{h^N},i}\left[\xi L_{i+1,N}^{(h^N)} + \sum_{k=i}^{N-1} f(t_k, Y_{k+1})L_{i+1,k+1}^{(h^N)}\Delta_k\right],\\ Z_i &= \mathbb{E}_{\mathbb{P},i}\left[\xi\Theta_N^{(i)} + \textstyle\sum_{k=i+1}^{N-1} f(t_k, Y_{k+1})\Theta_k^{(i)}\Delta_k\right],\\ h_i^N &= Z_i/Y_i,\\ \mathbb{E}_{\mathbb{Q}_i^{h^N},i}[.] &= \mathbb{E}_{\mathbb{Q}_i^{h^N}}[.\mid\mathcal{F}_{t_i}]\end{aligned}\right\} \tag{7}$$

The importance sampling DPEs (7) are the natural discrete-time approximation of (5). They are solved recursively backwards in time using the pseudo-algorithm

$$\Big[Y_N(=\xi)\Big] \to \Big[Z_{N-1} \to Y_{N-1} \to h_{N-1}^N \to \mathbb{Q}_{N-2}^{h^N}\Big] \to \Big[Z_{N-2} \to Y_{N-2} \to h_{N-2}^N \to \mathbb{Q}_{N-3}^{h^N}\Big] \to \dots$$

Observe that there is no importance sampling for the Z component of the solution; we will digress on this in Section 2.3.

In the sequel, we specialize the subsequent algorithm (fully detailed in Section 5) to a Markovian setting. The standing assumptions are the following.

$(\mathbf{H_X})$ The drift coefficient $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ of (2) is bounded and the diffusion coefficient $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^q$ is bounded, uniformly η-Hölder continuous in space, $\eta > 0$, uniformly in time. Furthermore, σ is uniformly elliptic. There exists a constant $C_X \ge 1$ such that for any $t \in [0,T]$, any $x \ne y \in \mathbb{R}^d$, any $\zeta \in \mathbb{R}^q$, we have

$$|b(t,x)| + \frac{|\sigma(t,x)-\sigma(t,y)|}{|x-y|^\eta} \le C_X, \qquad C_X^{-1}|\zeta|^2 \le \zeta.\sigma(t,x)\sigma(t,x)^\top\zeta \le C_X|\zeta|^2.$$

The time grid $\pi = \{0 := t_0 < \cdots < t_i < \cdots < t_N := T\}$ is uniform, i.e. $t_i = Ti/N$ and $\Delta_i = T/N$ for all i. The Euler scheme associated to X with time-step Δ_i is denoted by $X_{t_i}^N := X_i$ and defined iteratively by $X_0 = x_0$ and $X_{i+1} = X_i + b_i(X_i)\Delta_i + \sigma_i(X_i)\Delta W_i$ for $i \ge 0$, where we set $b_i(.) = b(t_i,.)$ and $\sigma_i(.) = \sigma(t_i,.)$.

($\mathbf{H_f}$) In (6), $f(t_i, y, \omega) = f_i(X_i, y)$ a.s., where $f_i(x, y)$ is a deterministic function, globally Lipschitz continuous in y uniformly in i and x (with a Lipschitz constant L_f), and $|f_i(x, 0)| \leq C_f$ uniformly in i and x;

($\mathbf{H}_\xi$) $\xi(\omega) = g(X_N)$, where g is a bounded globally Lipschitz continuous function (with Lipschitz constant L_g);

($\mathbf{H_C}$) $1 \leq Y_i \leq C_y$ and $|Z_i| \leq C_z$ for any i.

($\mathbf{H}_\Theta$) For any i, the Malliavin weights $\mathbf{\Theta^{(i)}} := (\Theta_{i+1}^{(i)}, \ldots, \Theta_N^{(i)})$ in (6) are squared integrable $(\mathbb{R}^q)^\top$-valued random variables and there exist measurable functions $\theta_j^{(i)}$ such that

$$\Theta_j^{(i)} = \theta_j^{(i)}(X_i, \ldots, X_N, \Delta W_i, \ldots, \Delta W_{N-1}).$$

There is a finite constant $C_M \in (0, +\infty)$ such that, for any $0 \leq i < k \leq N$,

$$\mathbb{E}\left[\Theta_k^{(i)} \mid X_i\right] = 0, \qquad \mathbb{E}\left[|\Theta_k^{(i)}|^2 \mid X_i\right]^{1/2} \leq \frac{C_M}{\sqrt{t_k - t_i}}.$$

($\mathbf{H_C}$) is a reinforcement of ($\mathbf{H_a}$)-($\mathbf{H_b}$), which ensures that h^N is bounded. The latter property is frequently used in the subsequent analysis. Observe also that the above assumptions are stronger than those for the MWDP scheme with no importance sampling [15] (for the time grid, $\Delta_i = T/N$ and $R_\pi = 1$); this is in the spirit of simplifying the paper.

In the case where X is a Brownian motion, we can take $\Theta_k^{(i)} = \frac{W_{t_k} - W_{t_i}}{t_k - t_i}$ (see [15, Section 1.4]), which obviously satisfies ($\mathbf{H}_\Theta$). From [22], we know that the continuous time Malliavin weight $\Theta_s^{(t)}$ is a Brownian stochastic integral between t and s with the integrand depending on the SDE. Thus, the assumption ($\mathbf{H}_\Theta$) is an adaptation of this property to a discrete time setting. We allow $\Theta_j^{(i)}$ to depend on the processes between j and N, although this dependence does not appear in explicit examples we are aware of [28], and this dependence is treated in the subsequent analysis.

Under ($\mathbf{H_f}$) and ($\mathbf{H}_\xi$), one can derive the upper bounds in ($\mathbf{H_C}$) from [15, Corollary 2.6]; in particular, one needs the Lipschitz continuity of the function g to ensure that the bound on $|Z_i|$ is uniform in i. ($\mathbf{H_C}$) assumes additionally a lower bound on Y that can be obtained as described in Remark 2.2. Moreover, one can easily obtain the following Markovian property, which follows by applying a reverse change of measure in the expression of Y_i and using the results of [15, Section 3.1].

Lemma 2.3. *Assume* ($\mathbf{H_X}$), ($\mathbf{H_f}$), ($\mathbf{H}_\xi$), ($\mathbf{H}_\Theta$). *For each i, there exist measurable functions $y_i : \mathbb{R}^d \to \mathbb{R}$ and $z_i : \mathbb{R}^d \to (\mathbb{R}^q)^\top$ such that $Y_i = y_i(X_i)$ and $Z_i = z_i(X_i)$.*

2.3 Discussion on the driver not depending on z

In Section 2.1, the optimal importance sampling for computing the conditional expectation of $\mathcal{Y}$ is obtained using the drift $\mathcal{Z}/\mathcal{Y}$. It is not clear how to efficiently transfer these arguments to $\mathcal{Z}$. Below, we suggest two possible approaches. Both approaches have some potential, but suffer from technical difficulties which we do not know how to solve at present. We explore these difficulties to encourage future research on this topic.

1. First, from the representation of Z in (6) using the Malliavin integration-by-parts formula, we obtain the l-th component of $\mathcal{Z}_{t_i}$ (or its approximation Z_i) as the conditional expectation of $\xi_{l,i} := \xi\Theta^{(t_i)}_{l,T} + \int_{t_i}^T \Theta^{(t_i)}_{l,s} f(s, \mathcal{Y}_s, \mathcal{Z}_s)\mathrm{d}s$. Therefore, it can be interpreted as a new BSDE problem on the interval $[t_i, T]$ with zero driver and terminal condition $\xi_{l,i}$. Denote its solution by $(\mathcal{Y}^{\xi_{l,i}}, \mathcal{Z}^{\xi_{l,i}})$. Using the techniques of Proposition 2.1 we obtain that the optimal drift for changing the measure for the evaluation of $\mathcal{Z}_{l,t_i}$ is (formally) $h^{\xi_{l,i}} := \mathcal{Z}^{\xi_{l,i}}/\mathcal{Y}^{\xi_{l,i}}$. This leads to significant difficulties: first from the theoretical point of view, there is no clear set of checkable assumptions ensuring we can reduce to $(\mathbf{H_a})$-$(\mathbf{H_b})$ for the new BSDE. Second, from the numerical point of view, one must solve a BSDE $(\mathcal{Y}^{\xi_{l,i}}, \mathcal{Z}^{\xi_{l,i}})$ for every time-point t_i in order to obtain the optimal probability measure. Computationally, this is extremely expensive and it seems a priori that there is no way such an algorithm may be efficient.

2. Second, instead of the representation of $\mathcal{Z}_{t_i}$ or Z_i using integration by parts formula, we could take advantage of the BSDE-type equation satisfied by $(\mathcal{Z}_t)_t$ (see [18] for a recent account on the subject). However, these equations involve "the $\mathcal{Z}$ of the $\mathcal{Z}$", i.e. Gamma processes. We must add DPEs to (7) to approximate the Gamma (like in 2BSDE [13]). However, a complete error analysis of these DPEs in the context of LSMC algorithms seems especially difficult. Therefore, this approach is beyond the scope of this work.

Finally, if $f(t, y, z)$ depends on z and if the Monte-Carlo estimation of Z in (7) is made without appropriate variance reduction (suited to Z specifically), we would obtain a propagation of "lack of variance reduction" on the Y component due to the Z component through f. Therefore, we would lose the variance reduction on Y. Thus, to keep track of the benefit of importance sampling for Y, it seems necessary to consider a driver independent of z.

3 Stability of L_2-norm under USES

Let ρ be a probability density w.r.t. the Lebesgue measure on $\mathbb{R}^d$, and R be a random variable with such a density. Set

$$\|\varphi\|_\rho := \left(\int_{\mathbb{R}^d} \varphi^2(x)\rho(x)\mathrm{d}x\right)^{1/2} = \left(\mathbb{E}\left[\varphi^2(R)\right]\right)^{1/2} \tag{8}$$

for any measurable function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ in $\mathbf{L}_2(\mathbb{R}^d, \rho)$. We now introduce the USES assumption.

($\mathbf{H}_\rho$) ρ is a continuous density and there is a positive continuous function $C_\rho : \mathbb{R} \to [1, +\infty)$ such that, for any $\Lambda \geq 0$, any $\lambda \in [0, \Lambda]$ and any $y \in \mathbb{R}^d$, we have

$$\frac{\rho(y)}{C_\rho(\Lambda)} \leq \int_{\mathbb{R}^d} \rho(y + z\sqrt{\lambda}) \frac{e^{-|z|^2/2}}{(2\pi)^{d/2}} \mathrm{d}z \leq C_\rho(\Lambda)\rho(y). \tag{9}$$

Observe that (9) implies that ρ must be strictly positive. Examples of such distributions are given later in Proposition 3.3, all having sub-exponential tails. The acronym USES stands for Uniform Sub-Exponential Sandwiching and it can be summerized shortly as follows: by initializing a Euler scheme with a density ρ, the marginal density of the process remains equivalent to ρ (up to constant which is uniform locally w.r.t. time). This stability property is stated as follows.

Proposition 3.1. *Assume* ($\mathbf{H_X}$). *Let* $\mathfrak{h} := (\mathfrak{h}_0, \ldots, \mathfrak{h}_{N-1})$ *be a vector of functions, where* $\mathfrak{h}_k : \mathbb{R}^d \to (\mathbb{R}^q)^\top$ *are bounded and measurable. Let* $i \in \{0, \ldots, N-1\}$, *and* $R^{(i)}$ *be a random variable satisfying* ($\mathbf{H}_\rho$) *which is independent of the Brownian motion* W. *Define by* $X^{(i)}_{\cdot}(\mathfrak{h})$ *be the Markov chain*

$$\begin{aligned} &X^{(i)}_i(\mathfrak{h}) := R^{(i)}, \\ &X^{(i)}_{j+1}(\mathfrak{h}) := X^{(i)}_j(\mathfrak{h}) + \Big[b_j(X^{(i)}_j(\mathfrak{h})) + \sigma_j(X^{(i)}_j(\mathfrak{h}))\mathfrak{h}_j^\top(X^{(i)}_j(\mathfrak{h}))\Big]\Delta_j + \sigma_j(X^{(i)}_j(\mathfrak{h}))\Delta W_j, \end{aligned} \tag{10}$$

for $i \leq j \leq N-1$. *Then, there exist finite positive constants* $\underline{c}_{(11)}$ *and* $\overline{c}_{(11)}$ *(depending only on* d, q, T, $|\mathfrak{h}|_\infty$, C_X, ρ, *but not on* i*) such that, for any* $j \in \{i, \ldots, N\}$ *and any square integrable function* $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$, *we have*

$$\underline{c}_{(11)} \left\|\varphi\right\|_\rho \leq \left(\mathbb{E}\left[\varphi^2(X^{(i)}_j(\mathfrak{h}))\right]\right)^{1/2} \leq \overline{c}_{(11)} \left\|\varphi\right\|_\rho. \tag{11}$$

Our main interest in this result is for having equivalent $\mathbf{L}_2$-norms despite the importance sampling drift, see Remark 5.1. Moreover, randomizing the initial condition is easy to implement in practice. We present tractable examples of ρ in Proposition 3.3, and state how to simulate the corresponding random variable R.

Similar equivalence-of-norm results are established in [2, Proposition 5.1] but they have been derived for time-homogenous diffusion processes (and not Euler schemes) with smooth coefficients. For an extension to time-inhomogeneous diffusion process with smooth in space coefficients, see [14, Proposition 3.8]. Moreover, [14, Proposition 3.9] partially extends this to the Euler scheme, still with smooth coefficients. Our contribution is to consider drift coefficients b and $\mathfrak{h}$ without smoothness condition in space and time, and only Hölder continuity in space for the diffusion coefficient σ. The main application of this is to piecewise-continuous $\mathfrak{h}$, as we encounter in the LSMC scheme (see (48)-(50) later).

The reader will easily check from the proof below that the above equivalence of $\mathbf{L}_2$-norms immediately extends to $\mathbf{L}_p$-norms, $p \geq 1$; however, in this work, only the case $p = 2$ is used.

Proof of Proposition 3.1. Let $0 \leq i < j \leq N$ and $x \in \mathbb{R}^d$. Denote by $p^x_{i,j}(y)$ the density at $y \in \mathbb{R}^d$ of the random variable X^x_j defined iteratively by

$$X^x_i := x, \quad X^x_{j+1} := X^x_j + \left[b_j(X^x_j) + \sigma_j(X^x_j)\mathfrak{h}_j^\top(X^x_j)\right]\Delta_j + \sigma_j(X^x_j)\Delta W_j \text{ for } j \geq i.$$

It is easy to check that, under the ellipticity assumption $(\mathbf{H_X})$, this density exists; actually it can be written as a convolution of Gaussian densities. In view of (10), observe that $X_j^{R^{(i)}} := X_j^{(i)}(\mathfrak{h})$. The following lemma, proved in [23], provides upper and lower bounds on $p^x_{i,j}(y)$ using a Gaussian density (i.e. Aronson-like estimates).

Lemma 3.2 ([23, Theorem 2.1])**.** *Under the assumptions and notation of Proposition 3.1, there exists a finite constant $C_{(12)} \geq 1$ (depending only on d, q, T and C_X) such that*

$$\frac{1}{C_{(12)}} \frac{e^{-C_{(12)}\frac{|y-x|^2}{2(t_j-t_i)}}}{(2\pi(t_j-t_i))^{d/2}} \leq p^x_{i,j}(y) \leq C_{(12)} \frac{e^{-\frac{|y-x|^2}{2C_{(12)}(t_j-t_i)}}}{(2\pi(t_j-t_i))^{d/2}}, \tag{12}$$

for any $0 \leq i < j \leq N$ and any $x, y \in \mathbb{R}^d$.

The upper bound of (11) can now be proved as follows:

$$\begin{aligned}
\mathbb{E}\left[\varphi^2(X_j^{(i)}(\mathfrak{h}))\right] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x) p^x_{i,j}(y)\varphi^2(y)\mathrm{d}x\mathrm{d}y \\
&\leq C_{(12)}^{1+d/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho\left(y + z\sqrt{C_{(12)}(t_j-t_i)}\right) \frac{e^{-|z|^2/2}}{(2\pi)^{d/2}}\varphi^2(y)\mathrm{d}z\mathrm{d}y \\
&\leq C_{(12)}^{1+d/2} C_\rho(C_{(12)}T) \int_{\mathbb{R}^d} \rho(y)\varphi^2(y)\mathrm{d}y.
\end{aligned}$$

The lower inequality is proved similarly. □

To conclude this section, we provide a list of distributions satisfying the USES property $(\mathbf{H}_\rho)$. The proof is postponed to Appendix A.

Proposition 3.3. *The following densities/distributions satisfy Assumption $(\mathbf{H}_\rho)$; in addition their coordinates $(R_1, \ldots, R_d)$ are i.i.d. and each coordinate can be simply sampled using a Rademacher r.v. ε (taking ± 1 with equal probability) and a $[0,1]$-uniformly distributed r.v. U, both being independent.*

(a) Laplace distribution: *For $\mu > 0$ set*

$$\rho(x) := \prod_{i=1}^{d} \left[(\mu/2) e^{-\mu|x_i|} \right]. \tag{13}$$

Each coordinate can be sampled as $\varepsilon \ln(U)/\mu$.

(b) Pareto-type distribution: *For $\mu > 0$ and $k > 0$, set*

$$\rho(x) := \prod_{i=1}^{d} \left[(\mu k/2)(1 + \mu|x_i|)^{-k-1} \right]. \tag{14}$$

Each coordinate can be sampled as $\varepsilon(U^{-1/k} - 1)/\mu$.

(c) Twisted Exponential-type distribution: *For $\mu > 0$ and $\alpha > 2$ set*

$$\rho(x) := \prod_{i=1}^{d} \left[(\mu e/\alpha) e^{-(1+\mu|x_i|)^{2/\alpha}} (1 + \mu|x_i|)^{2/\alpha - 1} \right]. \tag{15}$$

Each coordinate can be sampled as $\varepsilon[(1 - \ln(U))^{\alpha/2} - 1]/\mu$.

4 Regression on piecewise constant basis functions and nonparametric estimates

In this section, we develop a fully explicit, nonparametric error analysis for LSMC scheme on piecewise constant functions for a single-period problem. These results are not specific for BSDEs. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(Y, X) : \Omega \to \mathbb{R} \times \mathbb{R}^d$ be random variables satisfying $\mathbb{E}\left[|Y|^2\right] < +\infty$. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. copies of these random variables, with $n > 1$. The pair (Y, X) is termed the response/observation, and ensemble of the i.i.d copies termed the *sample* from now on. We aim at estimating the regression function $m(x) = \mathbb{E}[Y \mid X = x]$ using the sample. The function m is generally unknown. Nonetheless, we assume that $|m(x)| \leq L$ for a known constant $L > 0$.

We denote by ν the joint law of (Y, X) and let ν_n be the empirical measure associated to the sample. Whenever there is no conflict, ν (resp. ν_n) will stand for (by slight abuse of notation) the law of X (resp. the empirical measure associated to the data $(X_1, \ldots, X_n)$).

For $K \in \mathbb{N}^*$ disjoint sets $H_1, \ldots, H_K$ in $\mathbb{R}^d$, let $\mathcal{K} := \text{span}\{\mathbf{1}_{H_1}, \ldots, \mathbf{1}_{H_K}\}$ be the linear space of functions that are piecewise constants on each H_k. We estimate the function m using sample and the class of functions $\mathcal{K}$ on the domain $D := \bigcup_{i=1}^{K} H_i \subset \mathbb{R}^d$ with least-squares regression and truncation:

$$m(\cdot) \approx m_n(\cdot) := \mathcal{T}_L \left[\arg\min_{\varphi \in \mathcal{K}} \int_{\mathbb{R} \times \mathbb{R}^d} |y - \varphi(x)|^2 \nu_n(\mathrm{d}y, \mathrm{d}x) \right]$$

where $y \in \mathbb{R} \mapsto \mathcal{T}_L[y] = -L \vee y \wedge L$ is the soft-thresolding operator. Observe that $m_n(\cdot)$ is *not* the optimal approximation of $m(\cdot)$ in $\mathbf{L}_2(\nu_n)$. This role is played by the function $m_n^\star(\cdot)$ defined below:

$$m(\cdot) \approx m_n^\star(\cdot) := \arg\min_{\varphi \in \mathcal{K}} \int_{\mathbb{R}^d} |m(x) - \varphi(x)|^2 \nu_n(\mathrm{d}x).$$

We emphasize that functions m_n and $m_n^\star$ are *piecewise constant* on the sets H_k. Thanks to the orthogonal structure of the class $\mathcal{K}$, the functions m_n and $m_n^\star$ are available in closed form [17, Ch. 4]: on each set H_k, $k \in \{1, \dots, K\}$, the approximating functions are defined by

$$m_n(\cdot)|_{H_k} = \mathcal{T}_L\left[\frac{\int_{\mathbb{R}\times\mathbb{R}^d} y \mathbf{1}_{H_k}(z)\nu_n(\mathrm{d}y, \mathrm{d}z)}{\nu_n(H_k)}\right], \quad m_n^\star(\cdot)|_{H_k} = \frac{\int_{\mathbb{R}^d} m(z)\mathbf{1}_{H_k}(z)\nu_n(\mathrm{d}z)}{\nu_n(H_k)} \tag{16}$$

with the convention $0/0 = 0$. In particular, if $\nu_n(H_k) = 0$ (no data in H_k), we set $m_n(\cdot)|_{H_k} = m_n^\star(\cdot)|_{H_k} = 0$.

The error of the LSMC scheme is given by the expected *risk*

$$R(m_n) := \mathbb{E}\left[\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu(\mathrm{d}x)\right]. \tag{17}$$

The inner integral is taken with respect to the true law ν rather than the empirical law ν_n. One can define the empirical version of (17), i.e.

$$R_n(m_n) := \mathbb{E}\left[\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu_n(\mathrm{d}x)\right]. \tag{18}$$

For various reasons, it is important to have switching estimates between $R(m_n)$ and $R_n(m_n)$, i.e. to estimate $R(m_n)$ in terms of $R_n(m_n)$ up to errors, and vice-versa. This is usually achieved using concentration-of-measure inequalities. Let us recall the classical technique to estimate the error. First, estimates on the empirical risk (18) are known [17, Theorem 11.1], and usually take the form

$$R_n(m_n) \le const \times \left\{\sum_{k=1}^{K} [\mathrm{osc}_k^{(m)}]^2 \nu(H_k) + L^2\nu(D^c) + K\frac{\sup_{x\in\mathbb{R}^d} \mathbb{V}\mathrm{ar}(Y \mid X = x)}{n}\right\} \tag{19}$$

where $\mathrm{osc}_k^{(m)}$ is the oscillation of m on H_k, i.e.

$$\mathrm{osc}_k^{(m)} := \sup_{x,y\in H_k} |m(x) - m(y)|. \tag{20}$$

Then, to complete the estimate, one addresses the difference

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu(\mathrm{d}x) - 2\int_{\mathbb{R}^d} |m(x) - m_n(x)|^2 \nu_n(\mathrm{d}x)\right)_+\right]$$

$$\leq \mathbb{E}\left[\sup_{\varphi\in\mathcal{K},|\varphi|_\infty\leq L}\left(\int_{\mathbb{R}^d}|m(x)-\varphi(x)|^2\nu(\mathrm{d}x)-2\int_{\mathbb{R}^d}|m(x)-\varphi(x)|^2\nu_n(\mathrm{d}x)\right)_+\right]$$
$$\leq \frac{2028(K+1)\log(3n)L^2}{n}, \tag{21}$$

where L is the uniform bound on m. These results are obtained using concentration-of-measure inequalities [16, Propositions 4.9 and 4.10], which force the L^2 to appear by the use of Hoeffding-type inequalities. In the above, we can interchange the roles of ν and ν_n and the upper bound (21) keeps the same form, see the arguments in [16]. The main point to observe is that the estimate from concentration-of-measure (21) is less tight with respect to n than the estimate (19) on $R_n(m_n)$, because the conditional variance $\sup_{x\in\mathbb{R}^d}\mathbb{V}\mathrm{ar}(Y \mid X=x)$ may be substantially smaller than L^2. This is particularly true in the context of variance reduction algorithms, such as the importance sampling scheme presented in Section 5, where the aim is to minimize the conditional variance term. Therefore, if one were to use usual concentration-of-measure results like (21), the impact of variance reduction would be lost.

The main result of this section is to demonstrate that, thanks to the structure of the approximation space $\mathcal{K}$, one can obtain switching estimates on the risks for which estimate on the left-hand side of (21) is much smaller than $L^2\frac{K\log(n)}{n}$, and therefore for which the impact of variance reduction is not lost.

Theorem 4.1. *Assume that m is bounded by $L>0$. For each $k\in\{1,\dots,K\}$, define $\mathrm{osc}_k^{(m)}$ as in* (20)*. Define also the upper bound $\sigma^2 := \sup_{x\in\mathbb{R}^d}\mathbb{V}\mathrm{ar}(Y \mid X=x)$. Then*

$$R(m_n) \leq 8R_n(m_n) + 10\sum_{k=1}^K[\mathrm{osc}_k^{(m)}]^2\nu(H_k) + \frac{8\sigma^2}{n}\sum_{k=1}^K\exp\left(-\frac{3n\nu(H_k)}{104}\right)$$
$$+ L^2\sum_{k=1}^K\nu(H_k)\exp\left(-n\nu(H_k)\right), \tag{22}$$
$$R_n(m_n) \leq 8R(m_n) + 10\sum_{k=1}^K[\mathrm{osc}_k^{(m)}]^2\nu(H_k) + \frac{4\sigma^2}{n}\sum_{k=1}^K\exp\left(-\frac{3n\nu(H_k)}{8}\right). \tag{23}$$

Note that the switching-estimates from $R(m_n)$ to $R_n(m_n)$ and from $R_n(m_n)$ to $R(m_n)$ are not symmetric: this reflects that $\nu_n(H_k)>0$ implies $\nu(H_k)>0$ but the converse is false. This will become evident in the proof. Putting (22) together with (19) gives a bound (Corollary 4.2) that improves known estimates (like [17, Theorem 11.3] for instance). The improvement comes from the statistical error which is now essentially $K\frac{\sigma^2}{n}$ as soon as the mean number $n\nu(H_k)$ of simulations in each H_k is large enough:

$$L^2\nu(H_k)\exp\left(-n\nu(H_k)\right) \leq L^2\frac{\sup_{u\geq 0} ue^{-u/2}}{n}\exp\left(-n\nu(H_k)/2\right) \ll \frac{\sigma^2}{n}.$$

Corollary 4.2. *Assume that m is bounded by $L > 0$, that $\sigma^2 := \sup_{x\in\mathbb{R}^d} \mathbb{V}\mathrm{ar}(Y \mid X = x) < +\infty$ and that $(H_1, \dots, H_K)$ are disjoint subsets of $\mathbb{R}^d$, with $\bigcup_k H_k =: D$. For universal constant $C_{(24)} > 0$, we have*

$$R(m_n) \le C_{(24)} \left[\sum_{k=1}^{K} [\mathrm{osc}_k^{(m)}]^2 \nu(H_k) + K\frac{\sigma^2}{n} + L^2 \sum_{k=1}^{K} \nu(H_k) \exp\left(-n\nu(H_k)\right) + L^2\nu(D^c) \right]. \tag{24}$$

Proof of Theorem 4.1. Recall that $m_n = 0$ on D^c; thus both risks admit the decomposition over disjoint sets

$$\begin{aligned} R(m_n) &= \sum_{k=1}^{K} \mathbb{E}\left[\int_{H_k} |m(x) - m_n(x)|^2 \nu(\mathrm{d}x)\right] + \int_{D^c} |m(x)|^2 \nu(\mathrm{d}x) \\ &:= \sum_{k=1}^{K} R(m_n, H_k) + R(0, D^c), \end{aligned} \tag{25}$$

$$\begin{aligned} R_n(m_n) &= \sum_{k=1}^{K} \mathbb{E}\left[\int_{H_k} |m(x) - m_n(x)|^2 \nu_n(\mathrm{d}x)\right] + \mathbb{E}\left[\int_{D^c} |m(x)|^2 \nu_n(\mathrm{d}x)\right] \\ &:= \sum_{k=1}^{K} R_n(m_n, H_k) + R(0, D^c) \end{aligned} \tag{26}$$

where we have used at the last equality that $(X_1, \dots, X_n)$ are i.i.d. with distribution ν. Thus, it is enough to compare $R(m_n, H_k)$ and $R_n(m_n, H_k)$ for any k.

We start by proving (22). Assume first $\nu_n(H_k) > 0$. Using Cauchy's inequality,

$$\int_{H_k} (m(x) - m_n(x))^2 \nu(\mathrm{d}x) \le 2\int_{H_k} (m(x) - m_n^\star(x))^2 \nu(\mathrm{d}x) + 2\int_{H_k} (m_n^\star(x) - m_n(x))^2 \nu(\mathrm{d}x). \tag{27}$$

In view of (16), for any $x \in H_k$,

$$|m(x) - m_n^\star(x)| \le \frac{1}{\nu_n(H_k)} \int_{H_k} |m(x) - m(z)|\, \nu_n(\mathrm{d}z) \le \mathrm{osc}_k^{(m)}. \tag{28}$$

Thus, the first integral on the right hand side of (27) is bounded by $[\mathrm{osc}_k^{(m)}]^2 \nu(H_k)$. For the second integral, observe that, for an arbitrary point $x_k \in H_k$,

$$\int_{H_k} (m_n^\star(x) - m_n(x))^2 \nu(\mathrm{d}x) = (m_n^\star(x_k) - m_n(x_k))^2 \nu(H_k).$$

Combining the above results, it follows that

$$\int_{H_k} (m(x) - m_n(x))^2 \nu(\mathrm{d}x) \le 2[\mathrm{osc}_k^{(m)}]^2 \nu(H_k) + 2(m_n^\star(x_k) - m_n(x_k))^2 \nu(H_k)$$

$$
\begin{aligned}
&\le 2[\mathrm{osc}_k^{(m)}]^2\nu(H_k) + 4(m_n^\star(x_k) - m_n(x_k))^2\nu_n(H_k) \\
&\quad + 2(m_n^\star(x_k) - m_n(x_k))^2(\nu(H_k) - 2\nu_n(H_k))_+. \qquad (29)
\end{aligned}
$$

For the second term on the right hand side above, we again make use of (28):

$$
\begin{aligned}
(m_n^\star(x_k) - m_n(x_k))^2\nu_n(H_k) &= \int_{H_k} (m_n^\star(x) - m_n(x) \pm m(x))^2\nu_n(\mathrm{d}x) \\
&\le 2\int_{H_k} (m_n^\star(x) - m(x))^2\nu_n(\mathrm{d}x) + 2\int_{H_k} (m(x) - m_n(x))^2\nu_n(\mathrm{d}x) \\
&\le 2[\mathrm{osc}_k^{(m)}]^2\nu_n(H_k) + 2\int_{H_k} (m(x) - m_n(x))^2\nu_n(\mathrm{d}x).
\end{aligned}
$$

Collecting the different inequalities and taking the expectation in (29) gives (using also $\mathbb{E}\left[\nu_n(H_k)\right] = \nu(H_k)$) that

$$
\begin{aligned}
&\mathbb{E}\left[\int_{H_k} (m(x) - m_n(x))^2\nu(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)>0}\right] \\
&\le 10[\mathrm{osc}_k^{(m)}]^2\nu(H_k) + 8\mathbb{E}\left[\int_{H_k} (m(x) - m_n(x))^2\nu_n(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)>0}\right] \\
&\quad + 2\mathbb{E}\left[(m_n^\star(x_k) - m_n(x_k))^2(\nu(H_k) - 2\nu_n(H_k))_+\mathbf{1}_{\nu_n(H_k)>0}\right]. \qquad (30)
\end{aligned}
$$

It remains to estimate the last expectation term on the right hand side of (30). Write

$$
\widetilde{m}_n(x) := \arg\min_{\varphi\in\mathcal{K}} \int_{\mathbb{R}\times\mathbb{R}^d} |y - \varphi(x)|^2\nu_n(\mathrm{d}y, \mathrm{d}x), \qquad (31)
$$

so that $m_n(x) = \mathcal{T}_L\left[\widetilde{m}_n(x)\right]$. Clearly, from (16), $m_n^\star$ is bounded by L, therefore $|m_n^\star(x) - m_n(x)| \le |m_n^\star(x) - \widetilde{m}_n(x)|$. Additionally, on $\{\nu_n(H_k) > 0\}$, from definitions (16) and (31) we have $\widetilde{m}_n|_{H_k} = \frac{1}{n\nu_n(H_k)}\sum_{i=1}^n \mathbf{1}_{X_i\in H_k}Y_i$ and $m_n^\star|_{H_k} = \frac{1}{n\nu_n(H_k)}\sum_{i=1}^n \mathbf{1}_{X_i\in H_k}m(X_i)$. Denoting by $\mathbb{E}^{(n)}[\cdot] := \mathbb{E}\left[\cdot \mid X_1, \ldots, X_n\right]$, we obtain

$$
\begin{aligned}
\mathbb{E}^{(n)}[(m_n^\star(x_k) - m_n(x_k))^2\mathbf{1}_{\nu_n(H_k)>0}] &\le \mathbb{E}^{(n)}[(m_n^\star(x_k) - \widetilde{m}_n(x_k))^2\mathbf{1}_{\nu_n(H_k)>0}] \\
&= \frac{\sum_{i=1}^n \mathbf{1}_{X_i\in H_k}\mathbb{E}^{(m)}[(Y_i - m(X_i))^2]}{n^2\nu_n^2(H_k)}\mathbf{1}_{\nu_n(H_k)>0} \\
&\le \frac{\sigma^2}{n\nu_n(H_k)}\mathbf{1}_{\nu_n(H_k)>0} \qquad (32)
\end{aligned}
$$

using that sample is independent and that $m(X_i) = \mathbb{E}\left[Y_i \mid X_i\right]$ for all i. Therefore, conditioning inside the expectation, we are left with

$$
\begin{aligned}
\mathbb{E}\left[(m_n^\star(x_k) - m_n(x_k))^2(\nu(H_k) - 2\nu_n(H_k))_+\mathbf{1}_{\nu_n(H_k)>0}\right] &\le \frac{\sigma^2}{n}\mathbb{E}\left[\left(\frac{\nu(H_k)}{\nu_n(H_k)} - 2\right)_+\mathbf{1}_{\nu_n(H_k)>0}\right] \\
&\le \frac{4\sigma^2}{n}\exp\left(-\frac{3n\nu(H_k)}{104}\right)
\end{aligned}
$$

where the last inequality follows from Lemma B.1 in Appendix. To summarize, from the above and (30) we have proved

$$\begin{aligned}\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)>0}\right] \le\ & 10[\mathrm{osc}_k^{(m)}]^2\nu(H_k)\\ &+8\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu_n(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)>0}\right]\\ &+\frac{8\sigma^2}{n}\exp\left(-\frac{3n\nu(H_k)}{104}\right). \end{aligned} \tag{33}$$

On the other hand, on $\{\nu_n(H_k)=0\}$, $m_n|_{H_k}=0$ and we simply have

$$\begin{aligned}\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)=0}\right] &= \int_{H_k} m(x)^2\nu(\mathrm{d}x)(1-\nu(H_k))^n\\ &\le L^2\nu(H_k)\exp(-n\nu(H_k)). \end{aligned} \tag{34}$$

Moreover, $\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu_n(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)=0}\right]=0$. By summing up (33)-(34) and combining them with (25)-(26), we obtain the announced inequality (22).

We now establish (23)**.** We invert the roles of ν and ν_n in the computations and proceed with the same arguments as before. The inequality (30) becomes

$$\begin{aligned}&\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu_n(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)>0}\right]\\ &\le 10[\mathrm{osc}_k^{(m)}]^2\nu(H_k)+8\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)>0}\right]\\ &\qquad +2\mathbb{E}\left[(m_n^\star(x_k)-m_n(x_k))^2(\nu_n(H_k)-2\nu(H_k))_+\mathbf{1}_{\nu_n(H_k)>0}\right].\end{aligned}$$

On the other hand, $\mathbb{E}\left[\int_{H_k}(m(x)-m_n(x))^2\nu_n(\mathrm{d}x)\mathbf{1}_{\nu_n(H_k)=0}\right]=0$; plugging this and (32) in the above yields

$$R_n(m_n,H_k)\le 10[\mathrm{osc}_k^{(m)}]^2\nu(H_k)+8R(m_n,H_k)+2\mathbb{E}\left[\frac{\sigma^2}{n\nu_n(H_k)}(\nu_n(H_k)-2\nu(H_k))_+\mathbf{1}_{\nu_n(H_k)>0}\right].$$

Observe here the difference with switching from $R(m_n,H_k)$ to $R_n(m_n,H_k)$ for which we needed to handle an additional term associated to the event $\{\nu_n(H_k)=0\}$, see (34). To complete the proof, we apply Lemma B.1 and plug this into (26). □

5 Importance sampling least-squares Monte-Carlo scheme

5.1 Algorithm

In this section, we approximate the value functions $y_i(\cdot)$ and $z_i(\cdot)$ from Lemma 2.3 with numerical counterparts $y_i^{(M)}(\cdot)$ and $z_i^{(M)}(\cdot)$, respectively, using a fully implementable LSMC algorithm; the pseudo-algorithm is stated in Algorithm 5.1. Regression is implemented on piecewise constant functions, like in Section 4. We shall initialize the Euler scheme with IS drift randomly according to a distribution ρ satisfying the so called USES property $(\mathbf{H}_\rho)$. The stability result on the $\mathbf{L}_2$-norm from Proposition 3.1 then enable us to obtain explicit error estimates in Theorem 5.3 below. We call this scheme the Importance Sampling Malliavin Weights LSMC scheme, ISMW-LSMC for short.

As inputs, our algorithm will take independent simulations of random variables (specified explicitly later): these random variables are defined on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. On the other hand, the usual stochastic processes and random variables describing the (discrete) BSDE are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Strictly speaking, we work with the product space $(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}})$ to describe and analyse the algorithm. However, this rigorous treatment complicates the notation without bringing unexpected features. For simplicity, we avoid $(\omega, \widetilde{\omega})$-notation whenever clear.

Let $(R^{(0)}, \dots, R^{(N-1)})$ be i.i.d. copies of a random variable R satisfying the USES property $(\mathbf{H}_\rho)$ and W be a Brownian motion. We assume that they are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and mutually independent; furthermore let $\mathfrak{h} : (\widetilde{\omega}, i, x) \in \widetilde{\Omega} \times \{0, \dots, N-1\} \times \mathbb{R}^d \to h_i(\widetilde{\omega}, x) \in (\mathbb{R}^q)^\top$ be a bounded stochastic drift function. We will write $\mathbf{X}^{(i)}(\mathfrak{h})$ to denote the path

$$\left(X_i^{(i)}(\mathfrak{h}) = R^{(i)}, X_{i+1}^{(i)}(\mathfrak{h}), \dots, X_N^{(i)}(\mathfrak{h}) \right),$$

where $(X_j^{(i)}(\mathfrak{h}))_{i \le j \le N}$ is (conditionally on $\widetilde{\mathcal{F}}$) a Markov chain given by

$$\begin{cases} X_i^{(i)}(\mathfrak{h}) & := R^{(i)}, \qquad X_{i+1}^{(i)}(\mathfrak{h}) := X_i^{(i)}(\mathfrak{h}) + b_i(X_i^{(i)}(\mathfrak{h}))\Delta_i + \sigma_i(X_i^{(i)}(\mathfrak{h}))\Delta W_i, \\ X_{j+1}^{(i)}(\mathfrak{h}) & := X_j^{(i)}(\mathfrak{h}) + \left[b_j(X_j^{(i)}(\mathfrak{h})) + \sigma_j(X_j^{(i)}(\mathfrak{h}))\mathfrak{h}_j^\top(X_j^{(i)}(\mathfrak{h})) \right] \Delta_j + \sigma_j(X_j^{(i)}(\mathfrak{h}))\Delta W_j \\ & \qquad i+1 \le j \le N-1; \end{cases} \tag{35}$$

for simplicity, we do not explicit the $\widetilde{\omega}$-dependency.

Remark 5.1. We explain now why the USES property $(\mathbf{H}_\rho)$ is essential. Let $i_1, i_2 \in \{0, \dots, N-1\}$ be unequal $i_1 \neq i_2$. It is clear from the Euler scheme above that the random variables $X_j^{(i_1)}(\mathfrak{h})$ and $X_j^{(i_2)}(\mathfrak{h})$ will have **unequal** distributions whenever $j > \min(i_1, i_2)$, even with equal drift functions. This implies that the $\mathbf{L}_2$-norms $\|\phi\|_{i,j} := (\mathbb{E}[|\phi|^2(X_j^{(i)}(\mathfrak{h})])^{1/2}$ do not correspond at j: $\|\phi\|_{i_1,j} \neq \|\phi\|_{i_2,j}$, and therefore one cannot make use of Gronwall's inequality to treat the terms due to dynamical programming. However, we express the error estimates of Theorem 5.3 in terms of the $\mathbf{L}_2$-norm $\|\phi\|_\rho$ of (8). Thanks to the $\mathbf{L}_2$-norm equivalence given by the USES property of Proposition 3.1, we recover the ability to use Gronwall's inequality. In usual LSMC algorithms [16, 15], the Markov chains are initialized starting from a fixed point at time zero uniformly for every i, so one does not face this problem.

In the ISMW-LSMC scheme below, we will be interested in the following two drift functions:

$$h_j(x) := \frac{z_j(x)}{y_j(x)}, \qquad h_j^{(M)}(\widetilde{\omega}, x) := \frac{z_j^{(M)}(\widetilde{\omega}, x)}{y_j^{(M)}(x)}; \tag{36}$$

in what follows, we omit to write the dependence on $\widetilde{\omega}$ in the function $h^{(M)}$ for conciseness. The drift $h_j(\cdot)$ satisfies $h_j(X_j) = h_j^N$, where h_j^N is the drift of the Radon-Nikodym derivative for the Importance Sampling DPEs (7) (see Lemma 2.3). $h_j^{(M)}(\cdot)$ is the Monte-Carlo equivalent of $h_j(\cdot)$.

Let $\mathbf{x} = (x_i, \dots, x_N) \in (\mathbb{R}^d)^{N-i+1}$ and $\mathbf{w} = (w_i, \dots, w_{N-1}) \in (\mathbb{R}^q)^{N-i}$ and $\mathfrak{h} : \widetilde{\Omega} \times \{0, \dots, N-1\} \times \mathbb{R}^d \to (\mathbb{R}^q)^\top$. We introduce the function

$$L^{\mathfrak{h}}_{i+1,j}(\mathbf{x}, \mathbf{w}) := \exp\left(- \sum_{k=i+1}^{j-1} \left\{ \mathfrak{h}_k(x_k) w_k + \frac{1}{2} |\mathfrak{h}_k(x_k)|^2 \Delta_k \right\} \right),$$

and set

$$S_{Y,i}(\mathbf{x}, \mathbf{w}) := g(x_N) L^h_{i+1,N}(\mathbf{x}, \mathbf{w}) + \sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}(x_{k+1})\big) L^h_{i+1,k+1}(\mathbf{x}, \mathbf{w}) \Delta_k, \tag{37}$$

$$S_{Z,i}(\mathbf{x}, \mathbf{w}) := g(x_N) \theta_N^{(i)}(\mathbf{x}, \mathbf{w}) + \sum_{k=i+1}^{N-1} f_k\big(x_k, y_{k+1}(x_{k+1})\big) \theta_k^{(i)}(\mathbf{x}, \mathbf{w}) \Delta_k, \tag{38}$$

where $h_j(\cdot) = z_j(\cdot)/y_j(\cdot)$. The following lemma is a preparatory representation of the Markovian functions $(y_i(\cdot), z_i(\cdot))$ as conditional expectations involving the Euler scheme (35) – with or without IS drift h – starting from $R^{(i)}$.

Lemma 5.2. *Assume* $(\mathbf{H_X})$, $(\mathbf{H_f})$, $(\mathbf{H}_\xi)$, $(\mathbf{H_C})$, $(\mathbf{H_\Theta})$. *Let $R^{(i)}$ be a random variable satisfying the USES property $(\mathbf{H}_\rho)$. Recalling the definition* (35) *for $\mathbf{X}^{(i)}(h)$ and $\mathbf{X}^{(i)}(0)$ with a Brownian motion W (independent of $R^{(i)}$), and setting $\boldsymbol{\Delta}\mathbf{W}^{(i)} = (\Delta W_i^{(i)}, \dots, \Delta W_{N-1}^{(i)})$, we have*

$$y_i(R^{(i)}) = \mathbb{E}_{\mathbb{P}}\left[S_{Y,i}(\mathbf{X}^{(i)}(h), \boldsymbol{\Delta}\mathbf{W}^{(i)}) \mid R^{(i)} \right], \tag{39}$$

$$z_i(R^{(i)}) = \mathbb{E}_{\mathbb{P}}\left[S_{Z,i}(\mathbf{X}^{(i)}(0), \boldsymbol{\Delta}\mathbf{W}^{(i)}) \mid R^{(i)} \right]. \tag{40}$$

Proof. The proof is similar to Lemma 2.3. We must take into account two additional facts. First, that the Euler schemes $\mathbf{X}^{(i)}(h)$ and $\mathbf{X}^{(i)}(0)$ both take value $R^{(i)}$ at time t_i and that $R^{(i)}$ is independent of the Brownian motion; using the proof method of Lemma 2.3 given in [15, Section 3.1], this is sufficient to prove (40). Second, the distribution of $(W_j + \sum_{k=i}^{j-1} h^\top(X_k^{(i)}(h)) \mathbf{1}_{k \geq i+1} \Delta_k : i \leq j \leq N)$ under $\mathbb{P}$ is the same as distribution of $(W_j : i \leq j \leq N)$ under $\mathbb{Q}_i^{h^N}$; (39) then follows from (37) and these two additional facts. □

Remark. In (38), we could replace $g(x_N)$ by $g(x_N) - g(x_i)$. This would not change the equality in the conditional expectations because the Malliavin weights are conditionally centered. However this would additionally reduce the variance (see [1]). In the subsequent error analysis (see the proof of Theorem 5.3), a careful inspection shows that this would remove the factor $1/(T-t_i)$ in the definition of $\mathcal{E}^{(Z,i)}$ under the condition, in addition to $(\mathbf{H_\Theta})$, that Malliavin weights have finite fourth moments satisfying $\mathbb{E}[|\Theta_j^{(i)}|^4]^{1/4} \le C_M(t_j-t_i)^{-1/2}$.

In the ISMW-LSMC scheme, in (37)-(38) we will use $h_j^{(M)}(\cdot)$ instead of $h_j(\cdot)$ (both defined in (36)), and $y_{k+1}^{(M)}(\cdot)$ instead of $y_{k+1}(\cdot)$. This leads to the definitions

$$S_{Y,i}^{(M)}(\mathbf{x},\mathbf{w}) := g(x_N)L_{i+1,N}^{h^{(M)}}(\mathbf{x},\mathbf{w}) + \sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}^{(M)}(x_{k+1})\big)\big)L_{i+1,k+1}^{h^{(M)}}(\mathbf{x},\mathbf{w})\Delta_k,$$

$$S_{Z,i}^{(M)}(\mathbf{x},\mathbf{w}) := g(x_N)\theta_N^{(i)}(\mathbf{x},\mathbf{w}) + \sum_{k=i+1}^{N-1} f_k\big(x_k, y_{k+1}^{(M)}(x_{k+1})\big)\theta_k^{(i)}(\mathbf{x},\mathbf{w})\Delta_k.$$

In what follows, we will need the threshold functions

$$\mathcal{T}_{C_y}(y) := 1 \vee y \wedge C_y, \qquad \mathcal{T}_{C_z}(z) := \Big(-C_z \vee z_1 \wedge C_z, \dots, -C_z \vee z_q \wedge C_z \Big)$$

for $y \in \mathbb{R}$ and $z \in (\mathbb{R}^q)^\top$, where the constants C_y and C_z are defined in $(\mathbf{H_C})$.

Algorithm 5.1. *Set $y_N^{(M)}(\cdot) = g(\cdot)$. Starting from $i = N-1$ and working backwards to $i = 0$, let*

- *$K^{(Y,i)} \in \mathbb{N}^*$ and $\{H_1^{(Y,i)}, \dots, H_{K^{(Y,i)}}^{(Y,i)}\}$ be disjoint subsets of $\mathbb{R}^d$, and $D^{(Y,i)} := \bigcup_k H_k^{(Y,i)}$.*
- *$K^{(Z,i)} \in \mathbb{N}^*$ and $\{H_1^{(Z,i)}, \dots, H_{K^{(Z,i)}}^{(Z,i)}\}$ be disjoint subsets of $\mathbb{R}^d$, and $D^{(Z,i)} := \bigcup_k H_k^{(Z,i)}$.*
- *$M_i \in \mathbb{N}^*$ a number of simulations at time t_i, with $M_i \ge \max(K^{(Y,i)}, K^{(Z,i)})$ (to avoid having an under-determined system).*
- *$\mathcal{C}_i := \{(R^{(i,1)}, \mathbf{\Delta W}^{(i,1)}), \dots, (R^{(i,M_i)}, \mathbf{\Delta W}^{(i,M_i)})\}$ be a collection of i.i.d. copies of the starting points $R^{(i)}$ and Brownian increments $\mathbf{\Delta W}^{(i)}$; we term $\mathcal{C}_i$ to be the cloud of simulations at time t_i, used to construct i.i.d. copies of the Markov chain $\mathbf{X}^{(i)}(h^{(M)})$, $\mathbf{X}^{(i)}(0)$ and Malliavin weights $\mathbf{\Theta^{(i)}}$. The clouds of simulations $\{\mathcal{C}_0, \dots \mathcal{C}_{N-1}\}$ are mutually independent, i.e. independently simulated.*

Set the sample dependent functions $y_i^{(M)}(\cdot), z_i^{(M)}(\cdot)$ and $h_i^{(M)}(\cdot)$ recursively as follows.

Approximation of z_i. *For every $k \in \{1, \dots, K^{(Z,i)}\}$, define the set of indices*

$$A_k^{(Z,i)} := \Big\{ m \in \{1, \dots, M_i\} : R^{(i,m)} \in H_k^{(Z,i)} \Big\}.$$

Let $\psi_{Z,i}^{(M)}(\cdot) : \mathbb{R}^d \to (\mathbb{R}^q)^\top$ be a piecewise constant function defined as follows:

1. *if* $A_k^{(Z,i)} = \emptyset$, *set* $\psi_{Z,i}^{(M)}(\cdot)|_{H_k^{(Z,i)}} := 0$;
2. *otherwise, set*

$$\psi_{Z,i}^{(M)}(\cdot)|_{H_k^{(Z,i)}} := \frac{1}{\#(A_k^{(Z,i)})} \sum_{m \in A_k^{(Z,i)}} S_{Z,i}^{(M)}(\mathbf{X}^{(i,m)}(0), \mathbf{\Delta W}^{(i,m)}).$$

To complete, define $\psi_{Z,i}^{(M)}(\cdot)|_{[D^{(Z,i)}]^c} := 0$ *and* $z_i^{(M)}(\cdot) := \mathcal{T}_{C_z}\big(\psi_{Z,i}^{(M)}(\cdot)\big)$.

Approximation of y_i. *For every* $k \in \{1, \ldots, K^{(Y,i)}\}$, *define the set of indices*

$$A_k^{(Y,i)} := \left\{ m \in \{1, \ldots, M_i\} : R^{(i,m)} \in H_k^{(Y,i)} \right\}.$$

Let $\psi_{Y,i}^{(M)}(\cdot) : \mathbb{R}^d \to \mathbb{R}$ *be a piecewise constant function defined as follows:*

1. *if* $A_k^{(Y,i)} = \emptyset$, *set* $\psi_{Y,i}^{(M)}(\cdot)|_{H_k^{(Y,i)}} := 1$;
2. *otherwise, set*

$$\psi_{Y,i}^{(M)}(\cdot)|_{H_k^{(Y,i)}} := \frac{1}{\#(A_k^{(Y,i)})} \sum_{m \in A_k^{(Y,i)}} S_{Y,i}^{(M)}(\mathbf{X}^{(i,m)}(h^{(M)}), \mathbf{\Delta W}^{(i,m)}). \tag{41}$$

Last, set $\psi_{Y,i}^{(M)}(\cdot)|_{[D^{(Y,i)}]^c} := 1$ *and* $y_i^{(M)}(\cdot) := \mathcal{T}_{C_y}\big(\psi_{Y,i}^{(M)}(\cdot)\big)$.

Approximation of h_i. *Having calculated* $z_i^{(M)}(\cdot)$ *and* $y_i^{(M)}(\cdot)$, *define* $h_i^{(M)}(\cdot) = z_i^{(M)}(\cdot)/y_i^{(M)}(\cdot)$.

Remark. The Euler schemes $\mathbf{X}^{(i)}(h^{(M)})$ and $\mathbf{X}^{(i)}(0)$ are re-simulated at every time-point t_i, which is necessary to account for the updated IS drift. At a first sight, this re-simulation step seems computationally expensive. In fact, re-simulation serves the additional purpose of reducing memory consumption, since the memory allocation of each simulation can be immediately removed once its contribution to the regression has been made. This is explained in the introduction of [16]. This is important in practice, because we know from [16, 15] that the memory is the critical point for LSMC-based schemes in large dimension.

5.2 Main error result

Our main error estimates on the ISMW-LSMC algorithm are the following.

Theorem 5.3. *Assume* $(\mathbf{H_X})$, $(\mathbf{H_f})$, $(\mathbf{H}_\xi)$, $(\mathbf{H_C})$, $(\mathbf{H_\Theta})$, $(\mathbf{H}_\rho)$. *For each* i, *define the function*

$$\hat{S}_{Y,i}^{(M)}(\mathbf{x}, \mathbf{w}) := g(x_N) L_{i+1,N}^{h^{(M)}}(\mathbf{x}, \mathbf{w}) + \sum_{k=i}^{N-1} f_k\big(x_k, y_{k+1}(x_{k+1})\big) L_{i+1,k+1}^{h^{(M)}}(\mathbf{x}, \mathbf{w}) \Delta_k$$

and set

$$\operatorname{osc}_k^{(y_i)} := \sup_{x,x'\in H_k^{(Y,i)}} |y_i(x) - y_i(x')|, \qquad \operatorname{osc}_k^{(z_i)} := \sup_{x,x'\in H_k^{(Z,i)}} |z_i(x) - z_i(x')|,$$

$$p_k^{(Y,i)} := \mathbb{P}\left(R^{(i)} \in H_k^{(Y,i)}\right), \qquad p_k^{(Z,i)} := \mathbb{P}\left(R^{(i)} \in H_k^{(Z,i)}\right),$$

$$\sigma^2_{Y,i,M} := \mathbb{E}\left[\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \mathbb{V}\mathrm{ar}\left[\hat{S}^{(M)}_{Y,i}(\mathbf{X}^{(i)}(h^{(M)}), \boldsymbol{\Delta}\mathbf{W}^{(i)}) | R^{(i)} = x, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{N-1}\right]\right],$$

$$\mathcal{E}^{(Y,i)} := \sum_{k=1}^{K^{(Y,i)}} [\operatorname{osc}_k^{(y_i)}]^2 p_k^{(Y,i)} + \sigma^2_{Y,i,M}\frac{K^{(Y,i)}}{M_i} + C_y^2 \sum_{k=1}^{K^{(Y,i)}} p_k^{(Y,i)} \exp\left(-M_i p_k^{(Y,i)}\right) + C_y^2\mathbb{P}(R^{(i)} \notin D^{(Y,i)}),$$

$$\mathcal{E}^{(Z,i)} := \sum_{k=1}^{K^{(Z,i)}} [\operatorname{osc}_k^{(z_i)}]^2 p_k^{(Z,i)} + C_z^2 \frac{K^{(Z,i)} q \log(3M_i)}{M_i} + \frac{C^2_{(49)}}{T-t_i}\frac{K^{(Z,i)}}{M_i} + C_z^2\mathbb{P}(R^{(i)} \notin D^{(Z,i)}).$$

for constant $C_{(49)}$ to be defined in (49) *below. There exists a constant $C_{(42)}$ (resp. $C_{(43)}$) depending only on T, L_f, C_z, $\bar{c}_{(11)}$, $C_{(24)}$ (resp. on T, L_f, C_z, C_M, $\bar{c}_{(11)}$, $C_{(24)}$) such that, for each $i \in \{0, \dots, N-1\}$*

$$\mathbb{E}\left[\left\| y_i(\cdot) - y_i^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \le C_{(42)}\left((\mathcal{E}^{(Y,i)})^{1/2} + \sum_{k=i+1}^{N-1} (\mathcal{E}^{(Y,k)})^{1/2}\Delta_k\right), \tag{42}$$

$$\mathbb{E}\left[\left\| z_i(\cdot) - z_i^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \le C_{(43)}\left((\mathcal{E}^{(Z,i)})^{1/2} + \sum_{k=i+1}^{N-1} \frac{(\mathcal{E}^{(Y,k+1)})^{1/2}}{\sqrt{t_k - t_i}}\Delta_k\right). \tag{43}$$

As in [15, Section 3.5], the above error estimates are sufficient to establish the convergence of the ISMW-LSMC scheme by appropriately choosing the basis functions and the number of simulations. We expect that the importance sampling algorithm will converge faster w.r.t. the simulation effort compared to basic LSMC scheme. The improvement can be captured through the term $\sigma^2_{Y,i,M}$, which is expected to be small; the exponential sums in $\mathcal{E}^{(Y,i)}$ are negligible for sufficiently large M_i. This is confirmed by our following numerical experiments in Section 6. Actually, the finiteness of $\sigma^2_{Y,i,M}$ (with uniform bound w.r.t. i) easily follows from the boundedness of the IS-drift, the boundedness of $y(\cdot)$, and of the boundedness/Lipschitz continuity of f_k. Observe that the function $\hat{S}^{(M)}_{Y,i}(\mathbf{x}, \mathbf{w})$ is constructed with the true function $y_{k+1}(\cdot)$ in the driver f_k, and with the empirical drift $h_k^{(M)}(\cdot) \approx h_k(\cdot)$ in the Radon-Nikodym derivative function $L^{h^{(M)}}_{i+1,k+1}(\mathbf{x}, \mathbf{w})$. Following from the analysis in Section 2.1, we expect $\sigma^2_{Y,i,M}$ to be small. Although it is delicate to precisely quantify the variance reduction, the statistical error of the ISMW-LSMC is presumably much smaller compared to a scheme without IS.

5.3 Error analysis: proof of Theorem 5.3

5.3.1 Proof of (42)

Define the function $\hat{\psi}^{(M)}_{Y,i}(\cdot)$ (constant on each set $H^{(Y,i)}_k$, $k \in \{1, \dots, K^{(Y,i)}\}$) by

$$\hat{\psi}^{(M)}_{Y,i}(\cdot)|_{H^{(Y,i)}_k} := \frac{1}{\#(A^{(Y,i)}_k)} \sum_{m \in A^{(Y,i)}_k} \hat{S}^{(M)}_{Y,i}(\mathbf{X}^{(i,m)}(h^{(M)}), \mathbf{\Delta W}^{(i,m)}), \tag{44}$$

if $\#(A^{(Y,i)}_k) > 0$ and 1 otherwise, and $\hat{\psi}^{(M)}_{Y,i}(\cdot)|_{(D^{(Y,i)})^c} = 1$ on the complement of $D^{(Y,i)}$. First, using the 1-Lipschitz property of $\mathcal{T}_{C_y}(.)$ and the triangle inequality, observe that

$$\mathbb{E}\left[\left\|y_i(\cdot) - y^{(M)}_i(\cdot)\right\|^2_\rho\right]^{1/2} \le \mathbb{E}\left[\left\|y_i(\cdot) - \mathcal{T}_{C_y}(\hat{\psi}^{(M)}_{Y,i}(\cdot))\right\|^2_\rho\right]^{1/2} + \mathbb{E}\left[\left\|\hat{\psi}^{(M)}_{Y,i}(\cdot) - \psi^{(M)}_{Y,i}(\cdot)\right\|^2_\rho\right]^{1/2} \tag{45}$$

In what follows, we first estimate the first term of the r.h.s. above using the results from Section 4, then the second term by direct computations.

Let $\mathcal{F}^{(M)}_i$ be the σ-algebra generated by the simulation clouds $\{\mathcal{C}_{i+1}, \dots, \mathcal{C}_{N-1}\}$ together with $\{R^{(i,m)} \ : \ m = 1, \dots, M_i\}$, and let $\mathbb{E}^{(M)}_i[\cdot]$ be the associated conditional expectation $\mathbb{E}_\mathbb{P}\left[\cdot \mid \mathcal{F}^{(M)}_i\right]$. Then, proceeding as in the proof of (39) in Lemma 5.2, we obtain for any $m \in \{1, \dots, M_i\}$ that

$$y_i(R^{(i,m)}) = \mathbb{E}^{(M)}_i\left[\hat{S}^{(M)}_{Y,i}(\mathbf{X}^{(i,m)}(h^{(M)}), \mathbf{\Delta W}^{(i,m)})\right].$$

Thus, $y_i(\cdot) - \mathcal{T}_{C_y}(\hat{\psi}^{(M)}_{Y,i}(\cdot))$ is the difference between the regression function and its empirical approximation, equivalent to the functions m and m_n in Section 4. Therefore, from Corollary 4.2 (working under the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}^{(M)}_i]$ in the place of the expectation $\mathbb{E}[\cdot]$, and applying the tower law), we obtain the estimate

$$\mathbb{E}\left[\left\|y_i(\cdot) - \mathcal{T}_{C_y}(\hat{\psi}^{(M)}_{Y,i}(\cdot))\right\|^2_\rho\right] \le C_{(24)}\mathcal{E}^{(Y,i)}. \tag{46}$$

We now treat the second term $\mathbb{E}\left[\left\|\hat{\psi}^{(M)}_{Y,i}(\cdot) - \psi^{(M)}_{Y,i}(\cdot)\right\|^2_\rho\right]^{1/2}$ on the r.h.s. of (45). For this we make use of the definitions (41)-(44) and recall that $R^{(i)}$ has the density ρ:

$$\mathbb{E}\left[\left\|\hat{\psi}^{(M)}_{Y,i}(\cdot) - \psi^{(M)}_{Y,i}(\cdot)\right\|^2_\rho\right] = \sum_{k=1}^{K^{(Y,i)}} \mathcal{A}_{k,i}\; p^{(Y,i)}_k \tag{47}$$

where

$$\mathcal{A}_{k,i} := \mathbb{E}\left[\left[\#(A^{(Y,i)}_k)\right]^{-2}\Big(\sum_{m \in A^{(Y,i)}_k} \mathcal{D}_m\Big)^2 \mathbf{1}_{\#(A^{(Y,i)}_k)>0}\right],$$

$$\mathcal{D}_m := \hat{S}^{(M)}_{Y,i}(\mathbf{X}^{(i,m)}(h^{(M)}), \mathbf{\Delta W}^{(m,i)}) - S^{(M)}_{Y,i}(\mathbf{X}^{(i,m)}(h^{(M)}), \mathbf{\Delta W}^{(m,i)}).$$

Now, using the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\mathcal{A}_{k,i} &\le \mathbb{E}\left[\frac{\sum_{m=1}^{M_i} \mathbb{E}^{(M)}_i\left[\mathcal{D}_m^2\right] \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,m)})}{\#(A_k^{(Y,i)})} \mathbf{1}_{\#(A_k^{(Y,i)})>0}\right] \\
&= M_i \mathbb{E}\left[\frac{\mathbb{E}^{(M)}_i\left[\mathcal{D}_1^2\right] \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,1)})}{\#(A_k^{(Y,i)})} \mathbf{1}_{\#(A_k^{(Y,i)})>0}\right] \\
&= M_i \mathbb{E}\left[\mathbb{E}\left[\mathcal{D}_1^2 \mid R^{(i,1)}, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{N-1}\right] \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,1)})\right] \mathbb{E}\left[\frac{1}{1 + \sum_{m=2}^{M_i} \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,m)})}\right] \\
&\le \frac{\mathbb{E}\left[\mathcal{D}_1^2 \mathbf{1}_{H_k^{(Y,i)}}(R^{(i,1)})\right]}{p_k^{(Y,i)}}
\end{aligned}$$

where the equalities follow because the simulations are i.i.d. and the final inequality follows from direct computation using the binomial distribution (see [17, Lemma 4.1]):

$$\mathbb{E}\left[\frac{1}{1 + \mathtt{Bin}(n,p)}\right] \le \frac{1}{(n+1)p}.$$

Substituting this back into (47), we obtain

$$\mathbb{E}\left[\left\|\hat{\psi}^{(M)}_{Y,i}(\cdot) - \psi^{(M)}_{Y,i}(\cdot)\right\|_\rho^2\right]^{1/2} \le \mathbb{E}\left[\mathcal{D}_1^2\right]^{1/2}.$$

Applying the triangle inequality to the $\mathbf{L}_2$-norm and making use of the Lipschitz continuity of f, we obtain an estimate on $\mathbb{E}\left[\mathcal{D}_1^2\right]^{1/2}$ as follows:

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{D}_1^2\right]^{1/2} = \mathbb{E}\left[|\hat{S}^{(M)}_{Y,i}(\mathbf{X}^{(i)}(h^{(M)}), \mathbf{\Delta W}^{(i)}) - S^{(M)}_{Y,i}(\mathbf{X}^{(i)}(h^{(M)}), \mathbf{\Delta W}^{(i)})|^2\right]^{1/2} \\
&\le \sum_{j=i}^{N-1} L_f \mathbb{E}\left[|L^{h^{(M)}}_{i+1,j+1}(\mathbf{X}^{(i)}(h^{(M)}), \mathbf{\Delta W}^{(i)})|^2 |y_{j+1}(X^{(i)}_{j+1}(h^{(M)})) - y^{(M)}_{j+1}(X^{(i)}_{j+1}(h^{(M)}))|^2\right]^{1/2} \Delta_j \\
&\le L_f e^{|h^{(M)}(\cdot)|^2_\infty T/2} \sum_{j=i}^{N-1} \mathbb{E}\left[|y_{j+1}(X^{(i)}_{j+1}(-h^{(M)})) - y^{(M)}_{j+1}(X^{(i)}_{j+1}(-h^{(M)}))|^2\right]^{1/2} \Delta_j
\end{aligned}$$

where the last equality follows from applying the reverse change of probability measure. The USES property from Proposition 3.1 then yields

$$\mathbb{E}\left[\left\|\hat{\psi}^{(M)}_{Y,i}(\cdot) - \psi^{(M)}_{Y,i}(\cdot)\right\|_\rho^2\right]^{1/2} \le \bar{c}_{(11)} L_f e^{C_z^2 T/2} \sum_{j=i}^{N-1} \mathbb{E}\left[\left\|y_{j+1}(\cdot) - y^{(M)}_{j+1}(\cdot)\right\|_\rho^2\right]^{1/2} \Delta_j, \quad (48)$$

where we have used that $|h^{(M)}(\cdot)| \le C_z$. Substituting (46) and (48) into (45), we conclude that

$$\mathbb{E}\left[\left\|y_i(\cdot) - y_i^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \le (C_{(24)}\mathcal{E}^{(Y,i)})^{1/2} + \bar{c}_{(11)}L_f e^{C_z^2 T/2} \sum_{j=i}^{N-1} \mathbb{E}\left[\left\|y_{j+1}(\cdot) - y_{j+1}^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \Delta_j.$$

The proof of (42) is now completed by an application of Gronwall's inequality. □

5.3.2 Proof of (43)

For the computations on Z, we could proceed analogously to Section 5.3.1 in order to obtain estimates in term of conditional variances. On the other hand, since no specific variance reduction is made, there is no interest for such sophistication. Instead, we follow the error analysis of [15] for the z-component. It suffices to first use Proposition 3.9 from this reference to estimate the error between the exact $\mathbf{L}_2-$norm $\mathbb{E}\left[\left\|z_i(\cdot) - z_i^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2}$ and the empirical one – like in (21) above – and then to apply the Steps 2 and 3 of the proof of Theorem 3.10 with the conditional variance bounds of Lemma 3.7. This writes

$$\begin{aligned}\mathbb{E}\left[\left\|z_i(\cdot) - z_i^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \le\ & C_z\sqrt{\frac{2028(K^{(Z,i)}+1)q\log(3M_i)}{M_i}} \\ & + \sqrt{2}\Bigg(\inf_{\varphi\in\mathrm{Span}\{\mathbf{1}_{H_k^{(Z,i)}}:1\le k\le K^{(Z,i)}\}} \mathbb{E}\left[|\varphi(R^{(i)}) - z_i(R^{(i)})|^2\right]^{1/2} + \overline{C}_{Z,i}\sqrt{\frac{K^{(Z,i)}}{M_i}} \\ & + C_M L_f \sum_{k=i+1}^{N-1} \mathbb{E}\left[|y_{k+1}(X_{k+1}^{(i)}(0)) - y_{k+1}^{(M)}(X_{k+1}^{(i)}(0))|^2\right]^{1/2} \frac{\Delta_k}{\sqrt{t_k - t_i}}\Bigg),\end{aligned}$$

with
$$\overline{C}_{Z,i} = \frac{C_{(49)}}{\sqrt{T-t_i}} \tag{49}$$

where the constant $C_{(49)}$ depends only on C_g, C_f, L_f, C_M, T, q (see [15, Lemma 3.7] for the explicit expression). Observe that

$$\inf_{\varphi\in\mathrm{Span}\{\mathbf{1}_{H_k^{(Z,i)}}:1\le k\le K^{(Z,i)}\}} \mathbb{E}\left[|\varphi(R^{(i)}) - z_i(R^{(i)})|^2\right]^{1/2} \le \left(\sum_{k=1}^{K^{(Z,i)}} (\mathrm{osc}_k^{(z_i)})^2 p_k^{(Z,i)} + C_z^2\mathbb{P}\left(R^{(i)} \notin D^{(Z,i)}\right)\right)^{1/2}$$

and that, for all k,

$$\mathbb{E}\left[|y_{k+1}(X_{k+1}^{(i)}(0)) - y_{k+1}^{(M)}(X_{k+1}^{(i)}(0))|^2\right]^{1/2} \le \bar{c}_{(11)}\mathbb{E}\left[\left\|y_{k+1}(\cdot) - y_{k+1}^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \tag{50}$$

using the USES property (Proposition 3.1). We have proved the existence of a universal constant $C_{(51)}$ such that

$$\mathbb{E}\left[\left\|z_i(\cdot) - z_i^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \leq C_{(51)}(\mathcal{E}^{(Z,i)})^{1/2} \qquad (51)$$
$$+ \bar{c}_{(11)} C_M L_f \sum_{k=i+1}^{N-1} \mathbb{E}\left[\left\|y_{k+1}(\cdot) - y_{k+1}^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2} \frac{\Delta_k}{\sqrt{t_k - t_i}}.$$

By plugging into (51) the estimates (42) on $\mathbb{E}\left[\left\|y_{k+1}(\cdot) - y_{k+1}^{(M)}(\cdot)\right\|_\rho^2\right]^{1/2}$, we obtain the announced inequality (43). $\square$

6 Numerical experiments

Consider the Brownian motion model $X = W$ $(d = q)$. Define the function $\omega(t, x) = \exp(t + \sum_{k=1}^q x_k)$. We perform numerical experiments on the BSDE with data $g(x) = 1 + \omega(T, x)(1 + \omega(T, x))^{-1}$ and

$$f(t, x, y) = q\omega(t, x)\left(y - 1 - \frac{2 + q}{2q}\right)(1 + \omega(t, x))^{-2},$$

We shall work with $T = 1$, and $q \in \{2, 4\}$. The BSDE has explicit solutions in this framework, given by

$$y_i(x) = 1 + \omega(t_i, x)(1 + \omega(t_i, x))^{-1}, \qquad z_{k,i}(x) = \omega(t_i, x)(1 + \omega(t_i, x))^{-2},$$

where $z_{k,i}(x)$ is the k-th component of the q-dimensional cylindrical function $z_i(x) \in (\mathbb{R}^q)^\top$.

For USES, we simulate using the random variable R with Laplace distribution, whose density is given in (13) with $\mu = 1$. For the least-squares Monte Carlo, we use a hypercubes basis defined on the domain $[-6.5, 6.5]^q$. The number of basis functions K and the number of simulations M are equal on every time point and parameterized according to the number of time-steps N:

$$K = c_q \times N^{q/2}, \qquad M = c_q \times N^{(2+q)/2},$$

for $c_2 = (3.5)^2$ and $c_4 = (4)^4$. To assess the performance of the algorithm, we compute the average mean squared error (MSE) over 10 independent runs of the algorithm for three error indicators:

$$MSE_{Y,\max} := 10^{-3} \max_{0\le i\le N-1} \sum_{m=1}^{10^3} |y_i(R_m) - y_i^{(M)}(R_m)|^2,$$

$$MSE_{Y,\mathrm{av}} := 10^{-3} N^{-1} \sum_{m=1}^{10^3} \sum_{i=1}^{N-1} |y_i(R_m) - y_i^{(M)}(R_m)|^2,$$

$$MSE_{Z,\mathrm{av}} := 10^{-3} N^{-1} \sum_{m=1}^{10^3} \sum_{i=1}^{N-1} |z_i(R_m) - z_i^{(M)}(R_m)|^2,$$

where the simulations $\{R_m; m = 1, \dots, 10^3\}$ are i.i.d. and independently drawn from the simulations used for the LSMC scheme.

Results for the importance sampling scheme are presented in Table 1 for dimension 2, respectively in Table 5 for dimension 4, and rates of convergence w.r.t. to the number of time points and computational time are presented in Table 2 (resp. 6). The contrasting results for the scheme without importance sampling are to be found in Tables 3 and 4 for $q = 2$, and 7 and 8 for $q = 4$. The tests have been performed on a processor 2.3 GHz Intel Core i7 in dimension 2, and 2.9GHz in dimension 4, with code written in `C`.

We first observe that for the same values of M, N and K, the algorithm with IS provides smaller errors on the Y-component than using the algorithm without IS. This is coherent with the estimates of Theorem 5.3. Moreover, the error on Z is unaltered, which is expected since no importance sampling is applied in this part of the algorithm. The convergence rate of the error on Y w.r.t. the number of time steps is higher for the ISMW-LSMC scheme compared to the scheme without IS. Although the computational time of the IS-based scheme is slightly larger, we observe that the convergence rate of the error on Y w.r.t. the computational time is still in favour of the scheme of IS. These results demonstrate the gain of efficiency using the ISMW-LSMC scheme.

Table 1: **Importance sampling,** $q = 2$

Max Y	Av Y	Av Z	Comp. time	Time Pts
–3.107593	-3.339273	-1.625611	-5.874215	1.609438
-3.809944	-4.216414	-1.957937	-2.863354	2.302585
-4.036548	-4.458091	-2.033387	-1.372861	2.708050
-4.434476	-4.860591	-2.150866	-0.175684	2.995732
-4.771406	-5.185585	-2.368438	1.523607	3.401197

Table 2: **Rates for importance sampling,** $q = 2$

	Max Y	Av Y	Av Z
Time points	-1.01	-1.11	-0.46
Comp. time	-0.21	-0.24	-0.09

Table 3: **No importance sampling,** $q = 2$

Max Y	Av Y	Av Z	Comp. time	Time Pts
-3.102519	-3.343045	-1.546175	-6.155726	1.609438
-3.514528	-3.912367	-1.856532	-3.319166	2.302585
-3.634813	-4.104047	-2.000568	-1.806159	2.708050
-3.648408	-4.235227	-2.122356	-0.661470	2.995732
-3.785295	-4.418884	-2.342785	1.027265	3.401197

A Distributions satisfying $(\mathbf{H}_\rho)$: proof of Proposition 3.3

Observe that thanks to the product form of the densities (13)-(14)-(15) (due to the independence of the coordinates), the d-dimensional result follows from the case $d = 1$. Thus we shall only prove the one-dimensional result.

Case (a). For $r \in \mathbb{R}$, set $\mathcal{I}(r) := \int_{\mathbb{R}} e^{r|z|} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} \mathrm{d}z$. Then a direct triangle inequality gives

$$\rho(y)\mathcal{I}(-\mu\sqrt{\lambda}) \leq \int_{\mathbb{R}} \rho(y + z\sqrt{\lambda}) \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} \mathrm{d}z \leq \rho(y)\mathcal{I}(\mu\sqrt{\lambda}).$$

Clearly, $\mathcal{I}(.)$ is a positive and continuous function, thus bounded from below and from above on the compact $[-\mu\sqrt{\Lambda}, \mu\sqrt{\Lambda}]$. □

Case (b). We prove only the case $\mu = 1$; the general case $\mu > 0$ is similar and is left to the reader. Set

$$\mathcal{J}(y, \lambda) := \int_{\mathbb{R}} \rho(y + z\sqrt{\lambda}) \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} \mathrm{d}z. \tag{52}$$

First, observe that $\mathcal{J}(.)$ and $\rho(.)$ are both positive and continuous: hence, for any given $y_0 > 0$, $\mathcal{J}(y, \lambda)/\rho(y)$ is bounded from above and from below uniformly on $[-y_0, y_0] \times [0, \Lambda]$. Fixing $y_0 = 1$, it remains to check (9) only for $(y, \lambda) \in [-y_0, y_0]^c \times [0, \Lambda]$.

Upper bound. Write $\mathcal{J} := \mathcal{J}_1 + \mathcal{J}_2$ where $\mathcal{J}_1$ and $\mathcal{J}_2$ correspond respectively to the integral on $A_\lambda := \{z : |z|\sqrt{\lambda} \leq |y|/2\}$ and on $[A_\lambda]^c$. On the one hand on A_λ, use $(1+|y+z\sqrt{\lambda}|) \geq (1 + |y|/2) \geq \frac{1}{2}(1 + |y|)$ to get $\mathcal{J}_1(y, \lambda) \leq 2^{k+1}\rho(y)$. On the other hand, obviously we

Table 4: **Rates for no Importance Sampling,** $q = 2$**.**

	Max Y	Av Y	Av Z
Time points	-0.51	-0.73	-0.49
Comp. time	-0.07	-0.13	-0.10

Table 5: **Importance sampling,** $q = 4$

Max Y	Av Y	Av Z	Comp. time	Time Pts
-3.439640	-3.530475	-1.516734	-0.865253	1.609438
-3.788365	-4.144437	-1.861130	2.839795	2.302585
-4.097266	-4.506245	-2.046588	4.915364	2.708050
-4.354352	-4.778275	-2.192688	6.316752	2.995732

have

$$\mathcal{J}_2(y,\lambda) \leq \frac{k}{2}\int_{[A_\lambda]^c} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}}\mathrm{d}z \leq k\mathcal{N}\left(-\frac{|y|}{2\sqrt{\Lambda}}\right) \leq c(k,\Lambda)\rho(y)$$

where $\mathcal{N}(\cdot)$ is the cumulative probability function of the standard normal distribution, and $c(k,\Lambda) > 0$ depends only on k and Λ and ensures the last inequality is valid for any $|y| > y_0 := 1$.

Lower bound. By integrating only on $B_\lambda := \{z : |z|\sqrt{\lambda} \leq |y|\}$ and using $(1 + |y + z\sqrt{\lambda}|) \leq (1 + 2|y|) \leq 2(1 + |y|)$ on that set, we obtain

$$\begin{aligned}\mathcal{J}(y,\lambda) \geq (\frac{k}{2})(1+|y|)^{-k-1}2^{-k-1}\int_{B_\lambda} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}}\mathrm{d}z &\geq 2^{-k}\rho(y)\left(\frac{1}{2} - \mathcal{N}\left(-\frac{|y|}{\sqrt{\Lambda}}\right)\right)\\ &\geq 2^{-k}\rho(y)\left(\frac{1}{2} - \mathcal{N}\left(-\frac{1}{\sqrt{\Lambda}}\right)\right)\end{aligned}$$

for any $|y| \geq |y_0| := 1$ and $\lambda \in [0,\Lambda]$. □

Case (c). As for the case (b), we give the proof only for $\mu = 1$, the general case being analogous. Using the same arguments as before, (9) easily holds for $(y,\lambda) \in [-1,1] \times [0,\Lambda]$ and it remains to prove it for $|y| > 1$ and $\lambda \in [0,\Lambda]$. We define $\mathcal{J}$ as in (52), but with the density ρ defined in (15).

Upper bound. Write $\mathcal{J} := \mathcal{J}_1 + \mathcal{J}_2$ where $\mathcal{J}_1$ and $\mathcal{J}_2$ correspond respectively to the integral on $A_\lambda := \{z : |z|\sqrt{\lambda} \leq |y|/2\}$ and on its complement. On A_λ, use $(1 + |y + z\sqrt{\lambda}|) \geq \frac{1}{2}(1+|y|)$ and $(1+|y+z\sqrt{\lambda}|)^{2/\alpha} \geq (1+|y|-|z|\sqrt{\lambda})^{2/\alpha} \geq (1+|y|)^{2/\alpha} - (|z|\sqrt{\lambda})^{2/\alpha}$ (owing to $2/\alpha < 1$), to get

$$\mathcal{J}_1(y,\lambda) \leq \int_{A_\lambda} \frac{e}{\alpha}e^{-(1+|y|)^{2/\alpha}}e^{(|z|\sqrt{\lambda})^{2/\alpha}}(1+|y|)^{2/\alpha-1}2^{1-2/\alpha}\frac{e^{-|z|^2/2}}{(2\pi)^{1/2}}\mathrm{d}z$$

Table 6: **Rates for Importance Sampling,** $q = 4$**.**

	Max Y	Av Y	Av Z
Time points	-0.81	-1.04	-0.56
Comp. time	-0.16	-0.21	-0.11

Table 7: **No importance sampling,** $q = 4$

Max Y	Av Y	Av Z	Comp. time	Time Pts
-2.992670	-3.172890	-1.515503	-1.110783	1.609438
-3.259278	-3.614908	-1.855790	2.533719	2.302585
-3.253435	-3.708148	-2.036607	4.649199	2.708050
-3.298297	-3.895621	-2.206599	6.056020	2.995732

$$\leq \rho(y) 2^{1-2/\alpha} \int_{\mathbb{R}} e^{(|z|\sqrt{\Lambda})^{2/\alpha}} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} \mathrm{d}z.$$

Using the same arguments as for the case (b), we show that, for all $|y| \geq |y_0| := 1$, $\mathcal{J}_2(y, \lambda) \leq c(\alpha, \Lambda)\rho(y)$ for some constant $c(\alpha, \Lambda) > 0$ depending only on α and Λ.

Lower bound. It is obtained by integrating only on $B'_\lambda := \{z : |z|\sqrt{\lambda} \leq 1\} \supset B'_\Lambda$. On B'_λ, since $|y| > 1$ we have $|z|\sqrt{\lambda} \leq |y|$, and therefore $(1 + |y + z\sqrt{\lambda}|) \leq 2(1 + |y|)$ and $(1 + |y + z\sqrt{\lambda}|)^{2/\alpha} \leq (1 + |y|)^{2/\alpha} + (|z|\sqrt{\Lambda})^{2/\alpha}$. We then deduce

$$\begin{aligned}
\mathcal{J}(y, \lambda) &\geq \int_{B'_\lambda} \frac{e}{\alpha} e^{-(1+|y|)^{2/\alpha}} e^{-(|z|\sqrt{\Lambda})^{2/\alpha}} (1 + |y|)^{2/\alpha - 1} 2^{2/\alpha - 1} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} \mathrm{d}z \\
&\geq \rho(y) 2^{2/\alpha - 1} \int_{B'_\Lambda} e^{-(|z|\sqrt{\Lambda})^{2/\alpha}} \frac{e^{-|z|^2/2}}{(2\pi)^{1/2}} \mathrm{d}z.
\end{aligned}$$

The proof is complete. □

B Large deviation estimates for binomial distribution

Lemma B.1. *Let X be a random variable with distribution* $\mathtt{Bin}(n, p)$ *with $p \in [0, 1]$ and $n \geq 1$. Then,*

$$\mathbb{E}\left[\left(\frac{np}{X} - 2\right)_+ \mathbf{1}_{X>0}\right] \leq 4 \exp\left(-\frac{3np}{104}\right), \tag{53}$$

$$\mathbb{E}\left[\left(1 - \frac{2np}{X}\right)_+ \mathbf{1}_{X>0}\right] \leq 2 \exp\left(-\frac{3np}{8}\right). \tag{54}$$

Proof. If $p = 0$ (respectively $p = 1$), then $X = 0$ (resp. $X = n$) a.s. and the above inequalities are obvious. Assume from now on that $p(1 - p) > 0$.

Table 8: **Rates for no Importance Sampling,** $q = 4$**.**

	Max Y	Av Y	Av Z
Time points	-0.41	-0.67	-0.56
Comp. time	-0.07	-0.13	-0.11

We start by proving (53)**.** Firstly, observe that X and $(npX^{-1} - 2)$ are both positive if and only if $0 < X < np/2$. Therefore, denoting by X' the random variable with distribution $\mathtt{Bin}(n+1, p)$, we have

$$\begin{aligned}
\mathcal{I} := \mathbb{E}\left[\left(\frac{np}{X} - 2\right)_+ \mathbf{1}_{X>0}\right] &= \sum_{i=1}^{\lfloor np/2 \rfloor} (\frac{np}{i} - 2)\binom{n}{i} p^i (1-p)^{n-i} \\
&\leq \sum_{i=1}^{\lfloor np/2 \rfloor} \frac{n(i+1)}{(n+1)i} \times \frac{(n+1)!}{(i+1)!(n+1-i-1)!} p^{i+1}(1-p)^{(n+1)-(i+1)} \\
&\leq 2\mathbb{P}\left(2 \leq X' \leq np/2 + 1\right) \\
&\leq 2\mathbb{P}\left(X' - (n+1)p \leq np/2 + 1 - (n+1)p\right) \\
&= 2\mathbb{P}\left(X' - \mathbb{E}\left[X'\right] \leq 1 - p - np/2\right).
\end{aligned}$$

Now, assuming that $1 - p - np/2$ is smaller than $-(n+1)p/4$, i.e. $np/4 \geq 1 - 3p/4$, one can apply Bernstein's inequality [17, Lemma A.2] above to determine

$$\begin{aligned}
\mathcal{I} \leq 2\mathbb{P}\left(\frac{X' - \mathbb{E}[X']}{n+1} \leq -\frac{p}{4}\right) &\leq 4\exp\left(-\frac{(n+1)p^2/16}{2p(1-p) + 2p/12}\right) \\
&\leq 4\exp\left(-\frac{3np}{8(13-12p)}\right) \leq 4\exp\left(-\frac{3np}{104}\right).
\end{aligned}$$

In particular, the inequality is valid for $n \geq 4/p$. On the other hand, for $n \leq 4/p$, observe that

$$\mathcal{I} \leq \mathbb{E}\left[\left(\frac{4}{X} - 2\right)_+ \mathbf{1}_{X>0}\right] \leq 2 \leq 2\exp\left(\frac{3}{26}\right)\exp\left(-\frac{3np}{104}\right) \leq 4\exp\left(-\frac{3np}{104}\right).$$

Thus, (53) is proved.

Now we justify (54)**.** We simply observe that

$$\mathbb{E}\left[\left(1 - \frac{2np}{X}\right)_+ \mathbf{1}_{X>0}\right] \leq \mathbb{P}\left(\frac{X - \mathbb{E}[X]}{n} > p\right) \leq 2\exp\left(-\frac{np^2}{2p(1-p) + 2p/3}\right)$$

using the Bernstein inequality, which gives (54) after simplification. $\square$

References

[1] S. Alanko and M. Avellaneda. Reducing variance in the numerical solution of BSDEs. *Comptes Rendus Mathematique*, 351(3):135–138, 2013.

[2] V. Bally and A. Matoussi. Weak solutions for SPDEs and Backward Doubly Stochastic Differential Equations. *Journal of Theoretical Probability*, 14:125–164, 2001.

[3] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.*, 41(3B):1831–1863, 2013.

[4] T. Ben Zineb and E. Gobet. Preliminary control variates to improve empirical regression methods. *Monte-Carlo methods and Applications*, 19(4):331–354, 2013.

[5] C. Bender and T. Moseler. Importance sampling for backward SDEs. *Stochastic Analysis and Applications*, 28:226–253, 2010.

[6] C. Bender and J. Steiner. Least-squares Monte Carlo for BSDEs. In R. Carmona, P. Del Moral, P. Hu, and N. Oudjane, editors, *Numerical Methods in Finance*, pages 257–289. Series: Springer Proceedings in Mathematics, Vol. 12, 2012.

[7] J. Blanchet, P. Glynn, and Y. Shi. On Lyapunov inequalities and subsolutions for efficient importance sampling. *ACM TOMACS Transactions in Modeling and Computer Simulation*, 22(3):13, 2012.

[8] P. Briand and C. Labart. Simulation of BSDEs by Wiener chaos expansion. *Annals of Applied Probability*, 24(4):1120–1171, 2014.

[9] J.A. Bucklew. *Introduction to Rare Event Simulation.* Springer, 2004.

[10] J.-F. Chassagneux and D. Crisan. Runge-Kutta schemes for backward stochastic differential equations. *The Annals of Applied Probability*, 24(2):679–720, 2014.

[11] S. Crepey. *Financial modeling: a Backward Stochastic Differential Equations perspective.* Springer. Springer-Verlag, Berlin, 2013.

[12] N. El Karoui, S. Hamadène, and A. Matoussi. Backward stochastic differential equations and applications. In R. Carmona, editor, *Indifference pricing: theory and applications*, chapter 8, pages 267–320. Springer-Verlag, 2008.

[13] A. Fahim, N. Touzi, and X. Warin. A probabilistic numerical method for fully nonlinear parabolic PDEs. *Ann. Appl. Probab.*, 21(4):1322–1364, 2011.

[14] E. Gobet and C. Labart. Solving BSDE with adaptive control variate. *SIAM Numerical Analysis*, 48(1):257–277, 2010.

[15] E. Gobet and P. Turkedjiev. Approximation of BSDEs using Malliavin weights and least-squares regression. *To appear in Bernoulli*, 2014.

[16] E. Gobet and P. Turkedjiev. Linear regression MDP scheme for discrete backward stochastic differential equations under general conditions. *To appear in Mathematics of Computation*, 2014.

[17] L. Gyorfi, M. Kohler, A. Krzyzak, and H. Walk. *A distribution-free theory of nonparametric regression.* Springer Series in Statistics, 2002.

[18] Y. Hu, D. Nualart, and X. Song. Malliavin calculus for backward stochastic differential equations and application to numerical solutions. *Ann. Appl. Probab.*, 21(6):2379–2423, 2011.

[19] B. Jourdain and J. Lelong. Robust adaptive importance sampling for normal random vectors. *Ann. Appl. Probab.*, 19(5):1687–1718, 2009.

[20] V. Lemaire and G. Pagès. Unconstrained recursive importance sampling. *Ann. Appl. Probab.*, 20(3):1029–1067, 2010.

[21] J. Ma and J. Yong. *Forward-Backward Stochastic Differential Equations.* Lecture Notes in Mathematics, 1702, Springer-Verlag, 1999. A course on stochastic processes.

[22] J. Ma and J. Zhang. Representation theorems for backward stochastic differential equations. *The Annals of Applied Probability*, 12(4):1390–1418, 2002.

[23] S. Menozzi and V Lemaire. On some non asymptotic bounds for the euler scheme. *Electron. J. Probab.*, 15:no. 53, 1645–1681, 2010.

[24] N.J. Newton. Variance reduction for simulated diffusions. *SIAM Journal on Applied Mathematics*, 54(6):1780–1805, 1994.

[25] G. Rubino and B. Tuffin. *Rare Event Simulation using Monte Carlo Methods.* Wiley, 2009.

[26] R.Y. Rubinstein and D.P. Kroese. *Simulation and the Monte Carlo method.* Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2008.

[27] J. Smoller. *Shock waves and reaction-diffusion equations*, volume 258 of *Fundamental Principles of Mathematical Sciences.* Springer-Verlag, New York, second edition, 1994.

[28] P. Turkedjiev. Two algorithms for the discrete time approximation of Markovian backward stochastic differential equations under local conditions. *To appear in Electronic Journal of Probability, available on* *http://hal.archives-ouvertes.fr/hal-00862848*, 2015.