Pricing Higher-Dimensional American Options Using The Stochastic Grid Method

Shashi Jain * Cornelis W. Oosterlee †

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Abstract

This paper considers the problem of pricing options with early-exercise features whose payoff depends on several sources of uncertainty. We propose a *stochastic grid method* for estimating the upper and lower bound values of high-dimensional American options. The method is a hybrid of the *least squares method* of Longstaff and Schwartz (2001) [22], the *stochastic mesh method* of Broadie and Glasserman (2004) [11], and *stratified state aggregation along the pay-off* method of Barraquand and Martineau (1995) [3]. Numerical results are given for single asset Bermudan options, Bermudan max options, Bermudan options on the arithmetic mean of a collection of stocks.

1 Introduction

Pricing of Bermudan [1] options especially for multi-dimensional processes is a challenging problem owing to its path-dependent settings. The traditional valuation methods, such as lattice and tree-based techniques are often impractical in such cases due to the curse of dimensionality and hence are used only in the low-dimensional cases. In recent years many simulation-based algorithms have been proposed for pricing Bermudan options, most of which use a combination of Monte Carlo simulations and dynamic programming to estimate the option price.

The method which we call *stochastic grid method* (SGM) here, for pricing Bermudan options is in line with the *stochastic mesh method* (SMM) of Broadie and Glasserman (1997) [10], where moving backwards in time (starting from the expiration time), the option prices are approximated at each mesh point (representing an underlying state) and this information is used to compute the conditional continuation value at mesh points in the previous time step.

*TU Delft, Delft Institute of Applied Mathematics, Delft, the Netherlands, email: s.jain@cwi.nl, Nuclear Research Group, Petten, and thanks to CWI–Centrum Wiskunde & Informatica, Amsterdam.

†CWI–Centrum Wiskunde & Informatica, Amsterdam, the Netherlands, email: c.w.oosterlee@cwi.nl, and TU Delft, Delft Institute of Applied Mathematics

[1] A Bermudan option is an option where the buyer has the right to exercise at a set (discretely spaced) of times. This is intermediate between a European option which allows exercise at a single time, namely expiry and an American option, which allows exercise at any time. With an increasing number of exercise opportunities Bermudan option values approach the value of an American option

The SGM has several advantages over the existing methods. The LSM, although computationally fast and very simple to implement, uses a large number of paths to obtain a good exercise policy. Also the number of basis functions required for regression grows almost exponentially with the dimensions of the problem. SGM on the other hand can be used to obtain a good exercise policy using far fewer paths. The number of basis functions used in the SGM is independent of the dimensions of the problem. The method is fast and accurate compared to the SMM of Broadie and Glasserman. SGM doesn't suffer from the limitations, pointed out by Boyle *et al.*[7], of the SSAP method of Barraquand and Martineau, making it an efficient algorithm for handling options with large number of underlying assets.

The paper is organized as follows, Section 2 is devoted to the description of the *stochastic grid method*, its similarity with some existing methods and unique elements of the method. In Section 3 we deal with some specific examples in order of increasing problem complexity, discuss and compare the results with the other available models. In Section 4 we conclude, make observations about some existing open problems and directions in which the future research efforts can be made.

2 The Method of Stochastic Grid

The stochastic grid method (SGM) solves a general optimal stopping problem using a hybrid of dynamic programming and Monte Carlo methods. The method provides both upper and lower bound values on the option price. Upper bound values are attributed to the Jensen's inequality, while the lower bound values are computed by discounting the pay-off obtained by following a sub-optimal exercise policy. We describe in detail how these bounds are obtained in the sections to follow.

2.1 Problem Formulation

We assume complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and finite time horizon $[0, T]$. Ω is the set of all possible realizations of the stochastic economy between 0 and T. $\mathcal{F}_T$ is the sigma field of distinguishable events at time T, and $\mathcal{P}$ is the probability measure on elements of $\mathcal{F}$. The information structure in this economy is represented by an augmented filtration $\mathcal{F}_t : t \in [0, T]$. We assume $\mathcal{F}_t$ is generated by Z_t, a d-dimensional standard Brownian motion, and the state of economy is represented by an $\mathcal{F}_t$-adapted Markovian process $S_t = (S_t^1, \ldots, S_t^d) \in \mathcal{R}^d$, where $t \in [t_0 = 0, \ldots, t_i, \ldots, t_k = T]$. Let $h_t = h(t, S_t)$ be a non-negative adapted process representing the pay-off of the option, i.e. the holder of the option receives h_t if the option is exercised at time t. Let the risk-less savings account process be $B_t = \exp(\int_0^t r_s \, ds)$, where r_t denotes the instantaneous risk-free rate of return. The problem is then to compute

$$V_0 = \max_{\tau} \mathbb{E} \left[\frac{h(\tau, S_\tau)}{B_\tau} \right], \tag{1}$$

where τ is a stopping time taking values in the finite set $\{0, t_1, \ldots, t_k = T\}$. The value of the option at the terminal time T is equal to the products pay-off.

$$V(T, x) = h(T, x). \tag{2}$$

The conditional continuation value $Q(t_{i-1}, S_{t_{i-1}} = x)$, i.e. the expected future pay-off at time t_{i-1} and state $S_{t_{i-1}} = x$ is given by

$$Q(t_{i-1}, S_{t_{i-1}} = x) = \frac{B_{t_{i-1}}}{B_{t_i}} \mathbb{E}\left[V(t_i, S_{t_i}) | S_{t_{i-1}} = x\right], \tag{3}$$

The Bermudan option value at time t_{i-1} and state $S_{t_{i-1}} = x$ is given by

$$V(t_{i-1}, S_{t_{i-1}}) = \max(h(t_{i-1}, S_{t_{i-1}}), Q(t_{i-1}, S_{t_{i-1}})). \tag{4}$$

We are interested in finding the value of the option at the initial state S_0, i.e. $V(0, S_0)$.

2.2 Literature Overview

Regression-based approaches for pricing Bermudan options are proposed by Carriere (1996) [13], Tsitsiklis and Van Roy (1999) [27] and Longstaff and Schwartz (2001) [22]. The Longstaff and Schwartz least squares method (LSM) computes the option price by first determining the optimal exercise policy for a set of simulated paths and then finding the expected value of the discounted pay-off obtained by following this exercise policy. The option price obtained is the lower bound on the true option price as the exercise policy obtained would either be inferior or equal to the optimal exercise policy. Clément *et al.*-(2002) [15] analyze the convergence of the LSM. Glasserman and Yu (2002) [18] analyze a *regression later* approach, compared to the *regression now* strategy implied in the LSM algorithm. Glasserman and Yu (2004) [19] study the trade off between the number of basis functions and the number of paths. A drawback of the LSM algorithm is that the number of basis functions grows with the dimensions of the problem. In absence of an upper bound value, there is furthermore no valid confidence interval for the LSM algorithm. Belomestny *et al.*(2010) [4], compare the local regression estimators which are popular for computing Greeks with global regression estimators, which is a generalization of the methods of Tsitsiklis and Van Roy (1999) and Longstaff and Schwartz (2001). They also present an algorithm where instead of regressing continuation functions, the control and stopping times are backwardly constructed on a set of simulated trajectories.

As algorithms based on determining the optimal exercise strategy estimate the lower bound it is difficult to determine how well they estimate the true option value. Broadie and Glasserman (1997, 2004) [11, 10] propose simulated trees and the stochastic mesh method to generate both upper and lower bounds on the true option values.

Duality-based approaches for Bermudan option pricing are proposed independently by Haugh and Kogan (2004) [20] and Rogers (2002) [25] which can be used to construct an upper bound on the option value. Andersen and Broadie (2004) [1] improved the practical implementation of duality-based methods by proposing a simulation algorithm for obtaining the upper bounds from any given exercise policy. The duality-based algorithms work by first computing the lower bounds using some exercise policy (a sub-optimal policy) and then adding a non-negative quantity that penalizes potentially incorrect exercise decisions made by the sub-optimal policy. In theory the upper bound values are tight if the initial approximation is close to the true price of the option. Broadie and Cao

(2008) [8] introduce new variance reduction techniques and computational improvements to Monte Carlo methods for both lower bound (primal algorithms) and duality-based upper bound methods.

The *stochastic mesh method* of Broadie and Glasserman (2004) [11] approximates the option values using a dynamic programming-style backward recursion for approximating the price and optimal exercise policy. A stochastic mesh which denotes the state of underlying assets is generated for each exercise opportunity. The option values for the mesh points at the terminal time would be equal to their pay-off upon exercise. For the previous time interval (exercise opportunity) the Bermudan option value is approximated as the maximum between its intrinsic value (value on immediate exercise) and discounted continuation value. Each mesh point at a given time interval is connected to all the mesh points in the next time interval with some associated weight. The continuation value at each mesh point is then computed as the weighted sum of option values attained due to all possible transitions to mesh points in the next time step. In the original mesh method, the weights were computed from the transition density of the underlying process. For complex models the transition density may be unknown, and in singular problems like the pricing of American-Asian options the transition density fails to exist altogether [12].

In an improvement to the original stochastic mesh method, Broadie, Glasserman and Ha (2000) [12] avoid the use of the transition density of the underlying process of asset prices and other state variables by choosing mesh weights through optimization of a convex objective function subject to known conditional expectations.

Bally *et al.*-(2005) [2] come up with the *quantization tree method*, a spatial discretization scheme, where rather than settling the grids *a priori*, at each time step they produce a grid Γ_k^* of size N_k, which optimally fits to a large simulated sample of S_{t_k} among all grids with size N_k such that the closest neighbor rule projection of S_{t_k} onto the grid Γ_k^* is the best least squares approximation of S_{t_k}.

In an important attempt to circumvent the curse of dimensionality problem associated with pricing of Bermudan options, Barraquand and Martineau (1995) [3] introduce the *state aggregation* technique, in which they partition the space of underlying assets (state space) into a tractable number of cells, and compute an approximate early-exercise strategy that is constant over those cells. They limit their search to strategies that depend upon a stratification map (a real-valued function mapping the state), rather than upon the entire state itself. Particularly in the case of Bermudan options they use the payoff as stratification map, and call this technique as *stratified state aggregation along the payoff* (SSAP). Boyle *et al.*-(1997) [7] draw attention to some drawbacks of using SSAP.

2.3 Method Details of the SGM

The stochastic grid method defined here begins by generating N sample paths originating from the initial state S_0 by an Euler discretization. The stochastic grid points (S_{t_k}, t_k) can be interpreted as the intersections of the sample paths with a plane representing different intermediate time steps t_k. Figures 1 and 2 show the grid points for an option with two underlying assets $S_{t_i} = (S_1, S_2)$ starting from the initial state $S_{t_0} = (100, 100)$. Figure 1 shows a projection of the spatial distribution of the grid points at exercise period t which is close

to the initial time t_0 and Figure 2 shows the spatial distribution of the grid points at s which is close to the final exercise time $t_k = T$. The number of grid points in the vicinity of the initial state $S_{t_0} = (100, 100)$, the point where we are interested to find the option value, increases as we approach t_0, providing a natural refinement around the point of interest. This method of grid generation is closely related to the binomial tree approach, where only grid points associated with the initial state are generated. The value of the option at the expiration

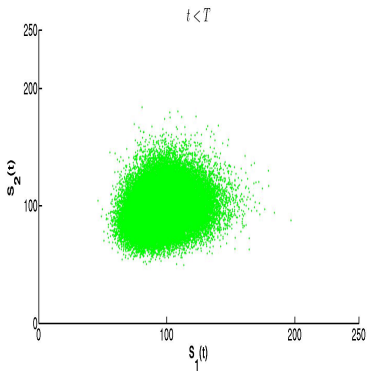

Figure 1: Grid Points (30000×30000) at t, where $t < T$.

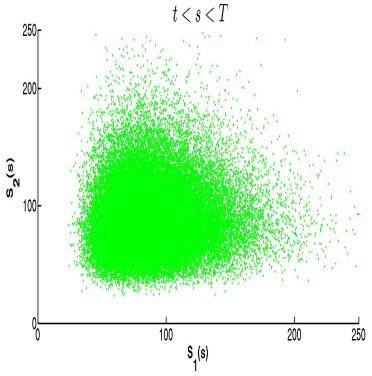

Figure 2: Grid Points (30000×30000) at s, where $t < s < T$.

time $t_k = T$ will be equal to its pay-off given by $h(T, S_T)$. We restrict our attention to financial derivatives with pay-offs that are element of the space of square integrable or finite variance functions. Examples of pay-off functions on multiple assets include, for a basket call option, $h(t, S_t) = (a_1 S_t^1 + \cdots + a_n S_t^n - K)^+$, for an out-performance option $h(t, S_t) = (\max(a_1 S_t^1, \ldots, a_n S_t^n) - K)^+$, where the notation x^+ is short for $\max(x, 0)$.

2.4 Upper Bound On the Option Price

The SGM, like the stochastic mesh method of Broadie and Glasserman, approximates the option value at each grid point (mesh point in the latter case), starting from the grid points at the expiration time $t_k = T$ and moving backwards in time. The SGM however differs from the stochastic mesh method in the following:

- In the stochastic mesh method, to approximate the option values at mesh points at time t_{k-1}, only transition to mesh points at t_k is allowed. In SGM each grid point at time t_{k-1} acts as a source node for a new stochastic process (real or imaginary set of paths) that can attain any value in the next time step.

- In the stochastic mesh method, continuation value $\widehat{Q}(t_{k-1}, S_{t_{k-1}})$ for the mesh points at t_{k-1} is the weighted sum of option values realized due to the transition to the mesh points in the next time step, where the weights denote the probability of such conditional transitions. In SGM, instead of using discrete weights, we use the probability density function of the transition for computing the discounted expected option value. Using a

continuous distribution function to compute the expectation is possible as we will have the functional approximation of the option values (discussed in Section 2.5) by means of regression at t_k and are not restricted only to the grid points at t_k.

Figures 3 and 4 gives an illustration comparing the stochastic mesh method with the method of stochastic grid.

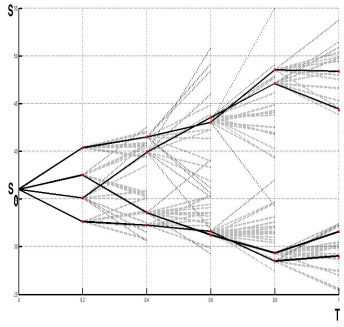

Figure 3: Stochastic grid: Each node acts as a source for a set of imaginary paths (represented by the dotted lines) until the next time step .

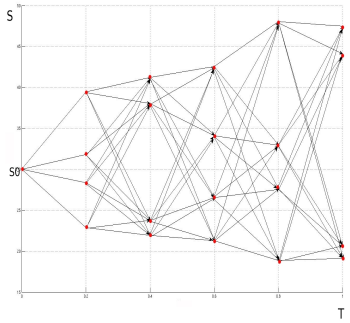

Figure 4: Stochastic Mesh: Each node is connected to the nodes at the next time step with arcs having certain weights

The method of stochastic grid is equivalent to the stochastic mesh, when an arbitrarily large number of mesh points is available in the next time step, in which case the associated weights approximate the continuous transition probability distribution function. This enables us to use some of the results (theorems 1 and 2) of Broadie and Glasserman [11].

Result 1 *The method of stochastic grid is biased high,i.e.,*
$\mathbb{E}[\widehat{V}(0, S_0)] \geq V(0, S_0)$.

Result 2 *The method of stochastic grid converges to the true value ,i.e.,*
$\|\widehat{V}(t, S_t) - V(t, S_t)\| \to 0$.

2.5 Parametrization of the option values

In order to compute the continuation value at time t_i and state $S_{t_i} = x$, i.e. $Q(t_i, S_{t_i} = x)$, from Equation (3) we need to know the functional form of the option value at time t_{i+1}. At the expiration time, the option value is given by Equation (2).

We write the pay-off function in the following form

$$h(t, S_t) = \max(g(S_t) + X, 0),$$

where $g : [0, T] \times \mathcal{R}^d \to \mathcal{R}$ is the mapping function. Thus in the case of a simple call on a single asset with strike K, $g(S_t) = S_t$ and $X = -K$, for a put on the maximum of d assets and strike K, $g(S_t^1, \ldots, S_t^d) = -\max(S_t^1, \ldots, S_t^d)$ and $X = K$. We assume that the unknown functional form of the option value, $V(t, S_t)$ can be represented by a linear combination of a countable set of $\mathcal{F}_t$-measurable basis functions, where $\mathcal{F}_t$ is the information set at time t.

Similar to the regression-based algorithms (Tsitsiklis and Van Roy (1999), Longstaff and Schwartz (2001)) the *method of stochastic grid* approximates the unknown functional form of the option values $V(t, S_t)$ by projecting them on the first $M(< \infty)$ polynomial basis functions. We denote this approximation as $V_M(t, S_t)$. The method uses separate sets of basis functions for regressing the option value at grid points which are in the exercise region and in the continuation region, allowing an accurate approximation using far fewer basis functions, as compared to using one set of basis functions for all the grid points. As shown by Longstaff and Schwartz [22] the fitted value of this regression, $\widehat{V}_M(t, S_t)$, converges in mean square sense and in probability to $V_M(t, S_t)$ as the number N of grid points in the simulation goes to infinity.

The regression in the *method of stochastic grid* differs from the LSM algorithm in the following aspects.

- The method does not approximate the functional form of the *continuation value*, rather it directly approximates the functional form of the *option values* at the exercise dates t_i.
- In the LSM algorithm the regression is done only for the in-the-money paths, whereas in the method of stochastic grid there are two separate regressions, for the grid points in the exercise region and for the grid points in the continuation region.
- In the LSM algorithm for high-dimensional options the set of basis functions should include terms in the underlying assets (state variables), as well as *cross-products* of these terms. The number of basis functions needed grows with the dimensionality of the problem (at-least 19 terms for a five-dimensional option). In the SGM the high-dimensional state space is mapped to a single-dimensional adapted process, which is used for regression. We found that irrespective of the dimensions of the problem, we need to use upto four basis functions (including the constant) for regression, as using more than four basis functions brings no significant improvement in the solution.

We approximate Equation (4) over a set of M polynomial basis functions, as

$$\widehat{V}(t, S_t) = \sum_{m=0}^{M-1} a_m \Psi_m(g(S_t)), \tag{5}$$

such that at each time step

$$r = \min_{a_m} |\widehat{V}(t, g(S_t)) - V(t, S_t)|^2, \tag{6}$$

where $\{\Psi(\cdot)\}_{m=0}^{\infty}$ form a set of basis functions, and r is the residual error.

2.5.1 Mapping high-dimensional state-space to single-dimensional $g(\cdot)$ space

In an approach similar to Barraquand and Martineau [3], we reduce the dimensions of the problem by using $g(S_t)$ rather than the cross-products of the underlying states for regression. At t_{i+1} we obtain option values as a function of $g(S_{t_{i+1}})$, i.e. $\widehat{V}(t_{i+1}, g(S_{t_{i+1}}))$. In order to approximate the option value at any

arbitrary underlying state $S_{t_{i+1}}$ we first find its corresponding mapping using the function $g(S_{t_{i+1}})$ and then determine the option value using $\widehat{V}(t_{i+1}, g(S_{t_{i+1}}))$. Thus $g(\cdot)$ is the stratification map and $\widehat{V}$ corresponds to the function that stores the constant option value for each bin (partition). Unlike Barraquand *et al.*we do *not* have an approximation of exercise strategy that is constant over each cell, rather we have an approximation of the option value which is constant for each partition.

Boyle *et al.*(1997) [7] and Broadie and Detemple (1996) [9] show that the pay-off value is not a sufficient statistic for determining the optimal exercise decision for options on the maximum of several assets. This argument however is specific to the Barraquand and Martineau approach of SSAP and would not apply to the present model. In the SSAP method the state space is first mapped to the partitions (cells) along the pay-off space $h(t, S_t)$ and then the same exercise decision is applied for all underlying states that fall into a particular cell or partition. This results in seemingly far off state points (like (100,90),(100,100) and (100,50)) to have the same exercise decision. In SGM *first* the exercise decision is made for each underlying state S_t (or grid point) at time step t and *then* the state space is reduced to $g(S_t)$. The functional approximation of the option value at t_{i+1}, i.e. $\widehat{V}(t_{i+1}, g(S_{t_{i+1}}))$ is used to compute the expected continuation value (3) at t_i and *not* make the exercise decision at t_{i+1}.

In order to give a better intuition about our method and allay the concerns raised by Boyle *et al.* [7], we use the same example given by them. Figures 5 to 8 shows the evolution of two asset prices (S_1, S_2) in a two time step computation. The option pay-off $h(S_1, S_2) = g(S_1, S_2) = \max(S_1, S_2)$ and for convenience the risk-less interest rate is taken to be zero. The steps followed at each time step starting from the final expiration time t_2 are

- Step 1: Compute the continuation value at each state point.
- Step 2: Make the exercise decision, based on the greater of immediate exercise $h(t, S_t = x)$ or *continuation value* $Q(t, S_t = x)$.
- Step 3: Regress the option value obtained over $g(S_1, S_2) = \max(S_1, S_2)$, to be used in the previous exercise time step (as we move backwards in time) to compute the continuation value.
- Step 4: In the previous exercise time step, compute the transition probability from each state point to the $g(\cdot)$- space in the next time step, i.e. $\mathbb{P}(g(S_{t_{i+1}})|S_{t_i} = x)$.

Focusing on the example figure 5 shows that at time t_2 the option values $V(t_2, S_{t_2} = (14, 2))$ and $V(t_2, S_{t_2} = (2, 14))$ are 14 and $V(t_2, S_{t_2} = (4, 2))$ is 4. On regressing these values over $\max(S_1, S_2)$ we get $\widehat{V}(t_2, g(S_{t_2}) = 14) = 14$ and $\widehat{V}(t_2, g(S_{t_2}) = 4) = 4$, as shown in figure 6. Moving to exercise time step t_1 we first compute the transition probability for each state point (grid point) at t_1 to the $g(\cdot)$ space in t_2. In the present example the state $S_{t_1} = (8, 8)$ transitions to $g(S_{t_2}) = 14$ with probability 1. Similarly, the conditional transition probability for $S_{t_1} = (8, 4)$ is $\mathbb{P}(g(S_{t_2}) = 4|S_{t_1} = (8, 4)) = 1$. Together with these conditional transition probabilities and the approximation of the option values at t_2, we compute the continuation value for the state points at

t_1. The continuation value at $S_{t_1} = (8,8)$ equals 14, computed by:

$$Q(t_1, S_{t_1}) = \sum_i \widehat{V}(t_2, g(S_{t_2}) = i) \cdot \mathbb{P}(g(S_{t_2}) = i | S_{t_1} = (8,8)).$$

The continuation value at $S_{t_1} = (8,4)$ is 4, determined as:

$$Q(t_1, S_{t_1}) = \sum_i \widehat{V}(t_2, g(S_{t_2}) = i) \cdot \mathbb{P}(g(S_{t_2}) = i | S_{t_1} = (8,4)).$$

Figure 7 shows that the option value at S_{t_1} is the *maximum* of immediate exercise and continuation, i.e. $\max(8,14)$ for $S_{t_1} = (8,8)$ and $\max(8,4)$ for $S_{t_1} = (8,4)$. Thus it is optimal to exercise in state $S_{t_1} = (8,4)$ and to continue in the state $S_{t_1} = (8,8)$. On regressing these values over $\max(S_1, S_2)$ we get $\widehat{V}(t_1, g(S_{t_1}) = 8)$ is 11, as shown in figure 8. Finally for time step t_0 state $(8,6)$ evolves to $g(S_{t_1}) = 8$ with probability 1. Therefore, the conditional continuation value is 11.

$$\sum_i \widehat{V}(t_1, g(S_{t_1}) = i) \cdot \mathbb{P}(g(S_{t_1}) = i | S_{t_0} = (8,6)),$$

and the option value $V(t_0, (8,6)) = \max(8,11)$, which is the correct value.

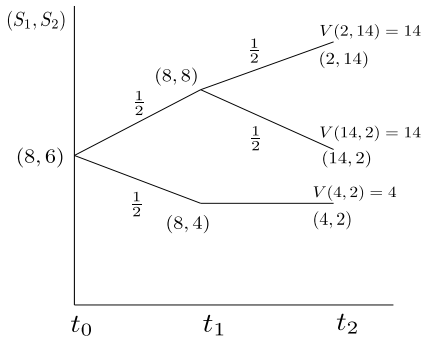

Figure 5: Step I: Compute the option values at t_2 as function of (S_1, S_2)

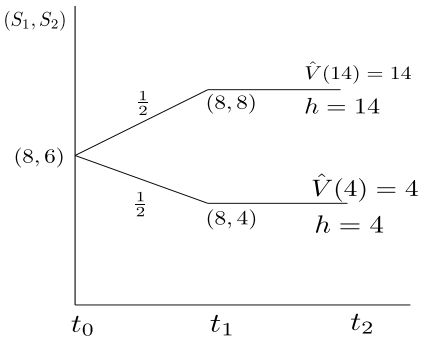

Figure 6: Step II: Map the option prices to $\max(S_1, S_2)$

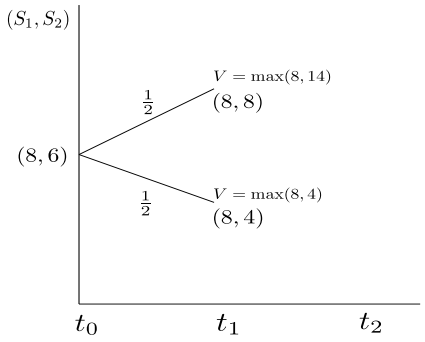

Figure 7: Step III: Compute the option values at t_1 as function of (S_1, S_2)

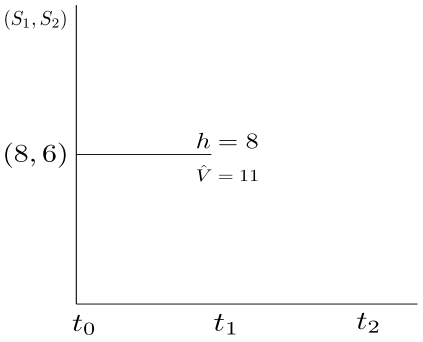

Figure 8: Step IV: Map the option price to $\max(S_1, S_2)$

Although this example is over simplified, it gives a basic understanding of our approach. In figures 9 and 10 we plot the shape of typical exercise regions ε^X

for an Bermudan call option on the max of two underlying assets obtained using SGM. The figures are in agreement with those deduced by Broadie and Detemple (1996) [9]. Interestingly we can see, as was deduced by Boyle *et al.*that, prior to maturity exercise is not optimal when the prices of the underlying assets are equal.

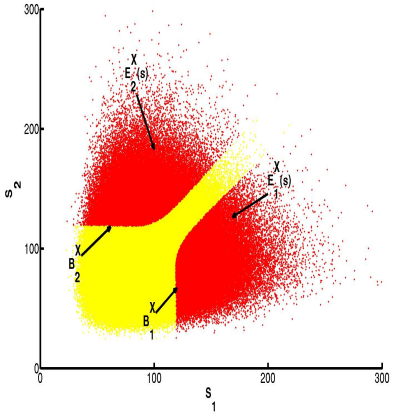

Figure 9: Illustration of $\varepsilon^X(t)$ for a max option at time t with $t < T$

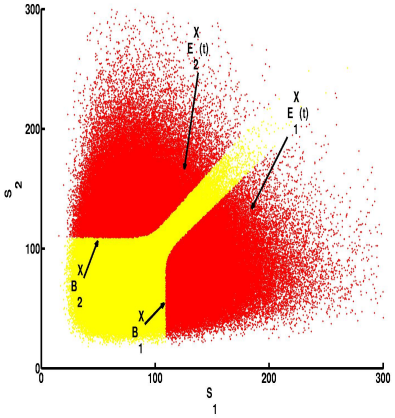

Figure 10: Illustration of $\varepsilon^X(s)$ for a max option at time s with $t < s < T$

A drawback of the SSAP method given by Boyle *et al.*is that the error in SSAP cannot be bounded, neither from above nor from below. However, in SGM, as the option value at each state point is the maximum of continuation and immediate exercise, the measurement error in estimation of the continuation value results in the maximum operator being upward biased (the max operator being convex), leading to the option value being biased high.

2.6 Computing the Continuation Value

The continuation values at the grid points (underlying states) for time step t_{i-1} are the expected option values in the next time step given the present state of the underlying assets. This can be written as,

$$Q(t_{i-1}, S_{t_{i-1}} = x) = \frac{B_{t_{i-1}}}{B_{t_i}} \mathbb{E}[V(t_i, S_{t_i}) | S_{t_{i-1}} = x].$$

As mentioned in Section 2.5 we approximate the option value at t_i as a polynomial function of a single-dimensional $g(\cdot)$ space, (Equation (5)), rather than the complete state space S_{t_i}. The continuation value can then be approximated as

$$\widehat{Q}(t_{i-1}, S_{t_{i-1}} = x) = \frac{B_{t_{i-1}}}{B_{t_i}} \mathbb{E}[\widehat{V}(t_i, g(S_{t_i})) | S_{t_{i-1}} = x]. \tag{7}$$

Here $\widehat{V}_{t_i}$ is a polynomial function of the adapted process $g(S_{t_i})$ and hence we need to determine the conditional probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}} = x)$ in order to compute its expectation. Using Equation (5), Equation (7) can

be written as

$$\widehat{Q}(S_{t_{i-1}} = x) = \frac{B_{t_{i-1}}}{B_{t_i}} \int\limits_{S_{t_i} \in \mathcal{R}^d} \left(\sum_{m=0}^{M-1} a_m \Psi_m(g(S_{t_i})) \right) d\mathbb{P}(g(S_{t_i}) | S_{t_{i-1}} = x). \quad (8)$$

There are three distinct possibilities for computing the distribution of $g(S_{t_i})$ given state $S_{t_{i-1}}$:

1. The exact transition probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}} = x)$ is known, for example a call or put on a single asset in the Black-Scholes framework, a call or put on the geometric mean of d assets.

2. The transition probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}} = x)$ is unknown, however, the moments of the distribution are known, for example for a call or put on the *Max* or *Min* of d assets in the Black Scholes framework.

3. The transition probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}} = x)$ and its moments are both unknown.

Case 1 is the trivial case where the density function is already known. This case can also be handled efficiently by Fourier techniques, particularly when the conditional density function is not known but when the characteristic function (the Fourier transform of the conditional density) is (Fang and Oosterlee (2008) [16]). Case 3 can be reduced to Case 2, by computing the moments with the help of Monte Carlo sub-simulations. For each grid point at time step t_{i-1}, we generate sub-paths until time t_i and compute the first four non-central moments ($\mu_1' = \mu$, $\mu_2' = \sigma^2$, μ_3', μ_4') of $g(S_{t_i})$. Once we have these moments for $g(S_{t_i})$ corresponding to the grid points at t_{i-1}, we approximate the conditional density function $f(g(S_{t_i})|S_{t_{i-1}} = x)$ using the *Gram Charlier Series* (See Kendall and Stuart (1969) [21]). Given the moments of a distribution, the Gram Charlier series approximates the density function $f(x)$ as,

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right] \left[1 + \frac{\kappa_3}{3!\sigma^3} H_3\left(\frac{x-\mu}{\sigma} \right) + \frac{\kappa_4}{4!\sigma^4} H_4\left(\frac{x-\mu}{\sigma} \right) \right], \quad (9)$$

where $H_3(x) = x^3 - 3x$ and $H_4(x) = x^4 - 6x^2 + 3$ are Hermite polynomials. $\kappa_1 = \mu$, $\kappa_2 = \mu_2' = \sigma^2$, $\kappa_3 = \mu_3'$, $\kappa_4 = \mu_4' - 3\mu_2'^2$ are the first four cumulants. More details about computing the probability density function are given in the specific examples in the sections to follow.

2.7 Need for Peripheral Paths

The coefficients a_i for the functional approximation of the option value at t_k in Equation (5) are chosen such that the overall residual error is minimized over the $g(\cdot)$ - space mapping of the grid points at t_k. Although the probability of transition from grid points at t_{k-1} to S_{t_k}, such that $g(S_{t_k})$ lies beyond the region where $\widehat{V}(t_k, \cdot)$ has been optimized is low (i.e. $\mathbb{P}(g(S_{t_k})|S_{t_{k-1}})$ is low), the error in the approximation of option value $\widehat{V}(t_k, g(S_{t_k}))$ can be high enough such that the error ε in the computation of the continuation value,

$$\varepsilon = |\widehat{V}(t_k, S_{t_k}) - V(t_k, S_{t_k})| \mathbb{P}(g(S_{t_k})|S_{t_{k-1}}),$$

is significant enough to influence the overall solution. We observed that this is especially significant in the case of high-dimensional options. In order to avoid such errors, in the high-dimensional cases we generate additional paths (from points on the periphery of source point) such that the error $|\widehat{V}(t_k, S_{t_k}) - V(t_k, S_{t_k})|$ is significant only when $\mathbb{P}(g(S_{t_k})|S_{t_{k-1}})$ is extremely low.

2.8 Lower Bound Values

The solution from the SGM can validated by computing the lower bound on the option price, using the exercise policy obtained from it. To compute the lower bound on the option price, we simulate a number of sample paths (either the same set of paths used for the upper bound algorithm or a fresh set of paths[2] can be used) originating from S_0 using the same discretization scheme. The continuation value at the new grid points is obtained using

$$Q(t_{i-1}, S_{t_{i-1}} = x) = \frac{B_{t_{i-1}}}{B_{t_i}} \mathbb{E}\left[\widehat{V}(t_i, g(S_{t_i}))|S_{t_{i-1}} = x\right],$$

where the functional approximation of the option values $\widehat{V}(t_i, g(S_{t_i}))$ is obtained from the upper bound algorithm. For each sample path, we find the first exercise period t_i, if it exists, for which $h(S_{t_i}) \geq \widehat{Q}(t_i, S_{t_i})$. The option is then exercised and its discounted pay-off is given by $h(S_{t_i})/B_{t_i}$. The lower bound on the option price is now obtained as

$$\underline{V_0} = \mathbb{E}_0\left[\frac{h_{\tilde{\tau}}}{B_{\tilde{\tau}}}\right], \tag{10}$$

where $\tilde{\tau} = \min\{t \in [0, T] : \widehat{Q_t} \leq h_t\}$. The option value achieved by following any exercise strategy is dominated by the optimal strategy. In other words, as the option value is obtained by following a stopping rule $\tilde{\tau}$ it gives the lower bound on the true price (see Andersen and Broadie (2004) [1]).

2.9 Algorithm

We briefly summarize the *stochastic grid method* here.

- Step I: Generate N sample paths $\{S_{t_0}, \ldots, S_{t_k}\}$, where $[t_0 = 0, \ldots, t_k = T]$ and $S_{t_i} \in \mathcal{R}^d$, starting from $S_{t_0} = S_0$. The paths are discretized in time using *Euler's discretization scheme*. Each of the N asset prices S_{t_i} represent the grid points in t_i;
- Step II: Compute the option value for grid points in $t_k = T$ as $V(T, S_T) = \max(g(S_T) + X, 0)$;
- Step III: Compute the approximate functional form $\widehat{V}(T, S_T)$ by regressing the option value at the grid points over polynomial basis functions of $g(S_T)$;

[2]Longstaff and Schwartz (2001), Broadie and Glasserman (1997), Raymar and Zwecher (1997) , Garcia (1999) , and others suggest that convergence of a simulation algorithm can be tested by applying the simulation results obtained to an out-of-sample set of paths. In our case the simulation results can be applied by using the functional approximation of the options values from the simulation to compute the continuation value and eventually an exercise policy to out-of-sample paths.

- Step IV: Perform the following steps for each time step t_i moving backwards in time, starting from t_{k-1} until we reach t_0 to obtain the upper bound value $V(t_0, S_{t_0} = S_0)$;
  - Compute the continuation value for grid points in t_i using the functional approximation of option values $\widehat{V}_{t_{i+1}}$ in t_{i+1}

$$\widehat{Q}(t_i, S_{t_i}) = \frac{B_{t_i}}{B_{t_{i+1}}} \mathbb{E}[\widehat{V}(t_{i+1}, g(S_{t_{i+1}}))|S_{t_i})];$$

  - Compute the option value for grid points in t_i as $V(t_i, S_{t_i}) = \max(g(S_{t_i}) + X, \widehat{Q}(t_i, S_{t_i}))$;
  - Compute the functional approximation of the option value in t_i by regressing the option value obtained at each grid point in t_i over a set of polynomial basis function of $g(S_{t_i})$;
  - Go to the previous time step ($i = i - 1$).
- Step V: Using the exercise strategy obtained while computing the upper bound, for each path determine the earliest time to exercise $\tilde{\tau} = \min\{t \in [0, T] : \widehat{Q_t} \leq h_t\}$. Obtain the lower bound option value as $\mathbb{E}_0 \left[\frac{h_{\tilde{\tau}}}{B_{\tilde{\tau}}}\right]$.

3 Numerical Results

In this section we illustrate our methodology by pricing single asset Bermudan put options, Bermudan options on the max of two, three and five assets, and a basket option on an arithmetic mean of four and five assets. The underlying assets are assumed to follow the standard single and multi-asset Black-Scholes model (geometric Brownian motion, GBM). Here we assume that the option expires at time T and that there are k equally-spaced exercise dates in the interval $[0, T]$.

3.1 American Put on a Single Asset

We consider the problem of pricing an American put option on a single asset, where the risk-neutral asset price follows the stochastic differential equation

$$dS = rSdt + \sigma SdW, \tag{11}$$

r being the continuously compounded risk-free interest rate, σ the annualized volatility. Here we assume r and σ to be constant. W is the standard Brownian motion. We assume that the option is exercisable a finite number of times (k) per year at a strike price of K up-to and including the final expiration time T. Using the Euler discretization scheme we generate N sample paths $\{S_{t_0}, \ldots, S_{t_k}\}$. The asset values S_{t_i} represent the grid points in t_i.

We describe each of the steps followed for this specific example in SGM.

3.2 Parametrization of the Option Value for a Single Asset

The option price at any time t prior to the expiration time T is given by

$$V(t, S_t) = \max(g(S_t) + X, Q(t, S_t)).$$

For a *put option*, $g(S_t) = -S_t$ and $X = K$ and $Q(t, S_t)$ is the continuation value.

To compute the functional approximation of the option value at time t, we regress the option values obtained at the grid points on polynomial basis function of $|g(S_t)|$. The option value at the early exercise boundary, where $g(S_t) + X = Q(t, S_t)$, has a discontinuity in its curvature (*gamma*) [13] and hence cannot be approximated by a single polynomial function. We perform a segmented least squares regression with the break point at $\mathcal{X}_t^* = g(S_t)$, when

$$g(S_t) = Q(t, S_t) - X \tag{12}$$

Thus we have separate regression for the grid points in the *early exercise region* and grid points in the *continuation region*. For each of these segments we regress the option value as

$$\widehat{V}(t, S_t) = \mathbf{1}_{\{g(S_t) < \mathcal{X}_t^*\}} \sum_{m=0}^{M-1} a_m (|g(S_t)|)^m + \mathbf{1}_{\{g(S_t) \geq \mathcal{X}_t^*\}} \sum_{m=0}^{M-1} b_m (|g(S_t)|)^m \tag{13}$$

with the coefficients a_m and b_m chosen so that residuals r_1 and r_2 are minimized,

$$r_1 = \min_{a_m} \left(\mathbf{1}_{\{g(S_t) < \mathcal{X}_t^*\}} |V(t, S_t) - \widehat{V}(t, |g(S_t)|)|^2 \right)$$

$$r_2 = \min_{b_m} \left(\mathbf{1}_{\{g(S_t) \geq \mathcal{X}_t^*\}} |V(t, S_t) - \widehat{V}(t, |g(S_t)|)|^2 \right)$$

We choose the first four polynomials (including the constant) as basis functions as more than four does not bring any significant improvement in the quality of regression.

3.3 Continuation Value for the Single Asset Case

In order to compute the continuation value for the grid points at t_{i-1} using Equation (7), we need the transition probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}})$. For a single asset following a stochastic process given by Equation (11), the conditional transition density function is given by

$$\mathbb{P}(|g(S_{t_i})| = x | S_{t_{i-1}}) = S_{t_{i-1}} e^{\left((r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}\, X\right)} \mathbb{P}(X = x^*) \tag{14}$$

where $\Delta t = t_i - t_{i-1}$, $X \sim \mathcal{N}(0, 1)$ and

$$x^* = \frac{1}{\sigma\sqrt{\Delta t}} \left[\log\left(\frac{x}{S_{t_{i-1}}}\right) - (r - \frac{\sigma^2}{2})\Delta t \right].$$

Equation (8) can then be written as

$$\begin{aligned} \widehat{Q}(t_{i-1}, S_{t_{i-1}}) \quad = \quad & \frac{B_{t_{i-1}}}{B_{t_i}} \left(\int_{-\infty}^{K^*} \sum_{m=0}^{M-1} a_m (f(X))^m d\mathbb{P}(X) \right. \\ & \left. + \int_{K^*}^{\infty} \sum_{m=0}^{M-1} b_m (f(X))^m d\mathbb{P}(X) \right) \end{aligned} \tag{15}$$

where

$$K^* = \frac{1}{\sigma\sqrt{\Delta t}}\left[\log\left(\frac{|\mathcal{X}^*_{t_i}|}{S_{t_{i-1}}}\right) - (r - \frac{\sigma^2}{2})\Delta t\right],$$

$$f(X) = S_{t_{i-1}} e^{\left((r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}\, X\right)}$$

$$d\mathbb{P}(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}} dX.$$

Solving Equation (15) we get the continuation value at each grid point as

$$\widehat{Q}(t_{i-1}, S_{t_{i-1}}) = \frac{B_{t_{i-1}}}{B_{t_i}}\left[\sum_{m=0}^{M-1} Z^m_{t_i}\left((a_m - b_m)\Phi\left(K^* - m\sigma\sqrt{\Delta t}\right) + b_m\right)\right], \quad (16)$$

where

$$Z^m_{t_i} = \left(S^m_{t_{i-1}} e^{m\left(\left(r-\frac{\sigma^2}{2}\right)+\frac{m}{2}\sigma^2\right)\Delta t}\right),$$

and

$$\Phi(x) = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right].$$

In order to compute the value of $\mathcal{X}^*_{t_{i-1}} = |g((S_{t_{i-1}})|$, we need to solve the non-linear equation

$$g(S_{t_{i-1}}) = K - Q(t_{i-1}, S_{t_{i-1}}), \quad (17)$$

where the value of $Q(t_{i-1}, S_{t_{i-1}})$ is obtained from Equation (16). When the volatility σ of the asset is low, the value of $\mathcal{X}^*_{t_{i-1}}$ can be approximated as,

$$\mathcal{X}^*_{t_{i-1}} = \max\left(|g(S_{t_{i-1}})|\mathbf{1}_{g(S_{t_{i-1}})\geq Q(t_{i-1},S_{t_{i-1}})-X}\right),$$

i.e., we find the maximum value of the asset price for the grid points lying in the *early-exercise* region, or alternatively the minimum value of the asset price for grid points in the *continuation* region. When the volatility of the underlying asset is high, or if the asset is deeply OTM, good results can be obtained by simply dividing the domain into *in-the-money* and *out-of-money* regions, rather than in *exercise* and *continuation* regions. When we use *in-the-money* and *out-of-money* regions for regression, the value of $\mathcal{X}^*_t$ is a constant and equal to the strike price K.

3.4 Results for Single Asset Put option

To illustrate the results, Table 1 reports the value of the early-exercise option implied by both the finite difference method and SGM. The finite difference method is a commonly used second-order accurate scheme, without any advanced features, based on implicit finite differences with 40,000 time steps per year and 1000 steps for the stock price.

We give both the upper bound and lower bound values. The SGM estimates are based on 10000 (5000 plus and 5000 antithetic) paths using 50 exercise points per year. Figure 11 compares the standard errors for the SGM and the LSM algorithm with varying number of paths. The dotted horizontal and vertical lines show that the number of paths required to obtain the same standard error for the LSM is 10^6, while that for the SGM upper bounds is less than 10^4. Figure

12 compares the convergence for the SGM with the LSM. Tighter bounds are obtained from the SGM algorithm, when separate regression is done for grid points in *exercise-continuation* regions, rather than *ITM-OTM* regions.

The differences between the finite difference and SGM algorithms are typically very small in this case. The standard errors for the simulated values for the upper bound range from 0.05 to 0.23 cents which is significantly less than that for LSM, and the lower bound range from 0.31 to 0.69 cents which is comparable to the LSM. The time taken for each simulation is few seconds on a system with Intel(R) Duo-Core 2.13 GHz processors and 2 GB RAM.

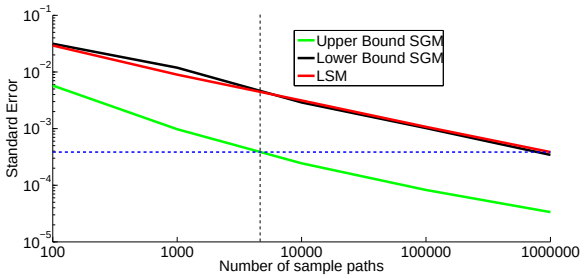

Figure 11: Standard errors values for the American put option on a single asset, comparison of upper and lower bound error values from the SGM and LSM algorithms.

S_0	σ	T	Finite Difference American	SGM Lower Bound(s.e)	SGM Upper Bound(s.e)	LSM (s.e)	Closed form European
36	0.2	2	4.840	4.837 (0.35)	4.8530 (0.07)	4.821 (0.32)	3.763
36	0.4	2	8.508	8.491 (0.57)	8.583 (0.14)	8.488 (0.51)	7.700
38	0.2	2	3.745	3.740 (0.31)	3.776 (0.06)	3.735 (0.32)	2.991
38	0.4	2	7.670	7.653 (0.50)	7.743 (0.20)	7.669 (0.50)	6.979
40	0.2	2	2.885	2.873 (0.40)	2.918 (0.05)	2.879 (0.37)	2.356
40	0.4	2	6.920	6.904 (0.69)	6.989 (0.13)	6.921 (0.55)	6.326
42	0.2	2	2.212	2.202 (0.41)	2.243 (0.07)	2.206 (0.32)	1.841
42	0.4	2	6.248	6.233 (0.61)	6.312 (0.23)	6.243 (0.51)	5.736
44	0.2	2	1.690	1.685 (0.40)	1.718 (0.05)	1.675 (0.32)	1.429
44	0.4	2	5.647	5.640 (0.57)	5.705 (0.14)	5.622 (0.51)	5.202

Table 1: Comparison of the SGM upper and lower bound values with the LSM and finite difference results for an American put option on a single asset, where the option is exercisable 50 times per year. The strike price of the put is 40, the short term interest rate is 0.06. The simulation for SGM is based on 10000 (5000 plus 5000 antithetic) paths for the asset price process, and for LSM is based on 100000 (50000 plus and 50000 antithetic) paths. The standard error for the simulation (s.e) is in cents while the option values are in dollars

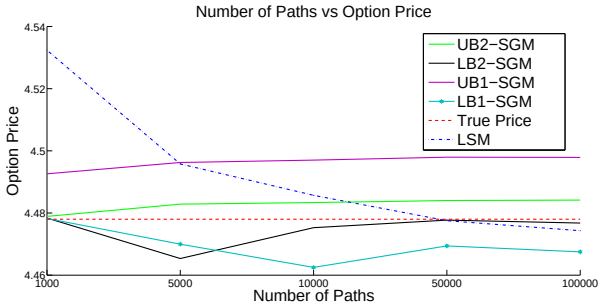

Figure 12: Number of paths vs option price for SGM and LSM. UB2 and LB2 are upper and lower bounds from SGM when there is separate regression for *exercise* and *continuation* region. UB1 and LB1 are upper and lower bounds when there is separate regression for ITM and OTM regions. The option price is on a stock with initial price 36, strike = 40, $\sigma = 0.2$, short-term interest rate $r = 0.06$ and expiration time T = 1 year. There are 50 equally spaced exercise opportunities

3.5 American Call on Maximum of d Assets

A Bermudan max-option is a discretely-exercisable option on multiple underlying assets whose pay-off depends on the maximum among all asset prices. We assume that the asset prices follow correlated geometric Brownian motion processes, i.e.,

$$\frac{dS_t^i}{S_t^i} = (r - q_i)dt + \sigma_i dW_t^i, \tag{18}$$

where each asset pays a dividend at a continuous rate of q_i. $W_t^i, i = 1, \ldots, d$, are standard Brownian motions and the instantaneous correlation between W_t^i and W_t^j is ρ_{ij}. We assume that the option expires at time T and there are k equally spaced exercise dates in the interval $[0, T]$. If we use K to denote the strike price of the option, then the pay-off for d underlying asset is $\max(g(S_t^i) + X, 0)$, where $X = -K$ and,

$$g(S_t^i) = \max(S_t^1, \ldots, S_t^d). \tag{19}$$

We start by generating N sample grid points $(S_{t_i}^1, \ldots, S_{t_i}^d)$ at each time step t_i, using the discretization scheme

$$S_{t_i}^j = S_{t_{i-1}}^j \exp\left((r - q_i - \frac{1}{2}|\sigma_i|^2)\Delta t + \sum_{1 \le k \le d} \sigma_{jk} W_{\Delta t}^k \right), 1 \le j \le d, \tag{20}$$

where $\Delta t = t_i - t_{i-1}$. As explained in Section 2.7, for high-dimensional options additional peripheral paths are required to have a better functional approximation of the option values in regions where transition from grid points in the previous time step has significant probability. To obtain these peripheral paths, we add additional source around the initial point and generate paths from these. In the present example we generate additional sample paths from two points around initial source point S_0, the points selected as shown in Table 2. The peripheral paths are used only to obtain the exercise-policy from the upper bound method and are not used for obtaining the lower bound values. Figure 13 gives the empirical relation for α values (see Table 2) for which the best lower bound values are obtained. We rely only on the lower bound values

and not on the upper bound values to deduce this empirical relation, as the lower bound value is an independent validation of the exercise policy obtained.

Peripheral point	Deeply OTM (α)	Just OTM (α)	ATM (α)	Just ITM (α)	Deeply ITM (α)
$S_0 e^{\alpha\sigma\sqrt{\Delta t}}$	(0.7,0.8)	(0.8,0.9)	(0.9,1)	(1,1.2)	(1.2,1.6)
$S_0 e^{-\alpha\sigma\sqrt{\Delta t}}$	1	1	1	1	1

Table 2: The location of peripheral points and empirical range of α values for different regions for a call option.

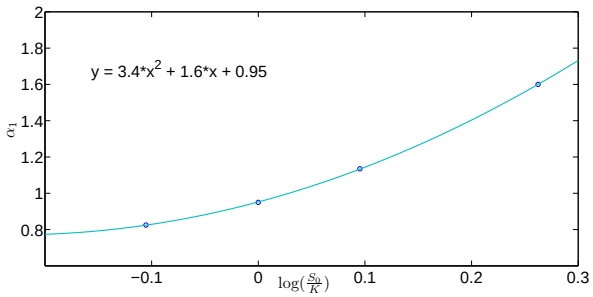

Figure 13: Empirical relation for a call option on five underlying assets between alpha values and $x = \log\left(\frac{S_0}{K}\right)$, α values selected to maximize the lower bound values.

Additional peripheral paths are required because in their absence the regressed function values around peripheral grid points becomes a source of error. In the subsequent section we discuss the scheme of parametrization and computing the continuation values specific to the Bermudan max call option.

3.6 Parametrization of the Option Value for Max Options

In order to compute the functional form of the option value at t, we regress the option values obtained at the grid points over the polynomial basis function of $g(S_t)$. The option values have a discontinuous curvature (gamma) at the early exercise boundary. We use segmented regression, with the break point at $\mathcal{X}_t^* = g(S_t)$, where $g(S_t) + X = Q(t, S_t)$. The regression scheme can be written as,

$$\widehat{V}(T, S_T) = \mathbf{1}_{\{g(S_t)<\mathcal{X}_t^*\}} \sum_{m=0}^{M-1} a_m(\Psi_m(g(S_T))) + \mathbf{1}_{\{g(S_t)\geq\mathcal{X}_t^*\}} \sum_{m=0}^{M-1} b_m(\Psi_m(g(S_T))), \tag{21}$$

where Ψ are the basis functions. The coefficients a_m and b_m are chosen such that residuals r_1 and r_2 are minimized,

$$r_1 = \min_{a_m} \left(\mathbf{1}_{\{g(S_t)<\mathcal{X}_t^*\}} |V(T, S_T) - \widehat{V}(T, g(S_T))|^2 \right),$$

$$r_2 = \min_{b_m} \left(\mathbf{1}_{\{g(S_t)\geq\mathcal{X}_t^*\}} |V(T, S_T) - \widehat{V}(T, g(S_T))|^2 \right).$$

We use a set of four (including the constant) *Hermite polynomial* basis functions of $g(S_t)$ for regression in our example. The choice of other types and greater number of basis functions did not bring about significant improvement in the regression.

3.7 Computing the Continuation Value for Max Options

In order to compute the continuation value for grid points at t_{i-1} using Equation (7), we need to know the transition probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}})$. For a call on the maximum of d underlying assets, $(g(S_{t_i}) = \max(S^1_{t_i}, \ldots, S^d_{t_i}))$, it is difficult to compute the exact transition density function. Like Boyle and Tse (1990) [6], we use Clark's algorithm to compute the first four moments of this distribution. The approximation of the transition probability density function can be obtained from these moments using the *Gram Charlier Expansion*. Clark's algorithm ([14]) gives the exact expression for the first four moments of the maximum of a pair of jointly normal variates as well as the correlation coefficient between the maximum of the pair and the third normal variate. The details of Clark's algorithm are given in Appendix A. S_{t_i} being a log-normal process given by Equation (20), we can write

$$\begin{aligned}\mathbb{P}\left(g(S_{t_i}) = X|S_{t_{i-1}}\right) &= \mathbb{P}\left(\max_{1\le j\le d}(S^j_{t_i}) = X|S_{t_{i-1}}\right)\\ &= \mathbb{P}\left(\max_{1\le j\le d}(Z^j_{t_i}) = \log(X)|S_{t_{i-1}}\right), \qquad (22)\end{aligned}$$

where $Z^j_{t_i}, 1 \le j \le d$ has multivariate normal distribution. Using Clark's algorithm we can obtain the first four moments of the random variable $Y = \max(Z^1_{t_i}, \ldots, Z^d_{t_i})$. If ${\kappa_i}^3 (1 \le i \le 4)$ are the first four cumulants of Y then using the *Gram Charlier Expansion* we can write the approximate probability density function of Y as Equation (9). The continuation value given by Equation (7) can then be given by,

$$\begin{aligned}\widehat{Q}(t_{i-1}, S_{t_{i-1}}) &= \frac{B_{t_{i-1}}}{B_{t_i}}\left(\int_{-\infty}^{K^*}\sum_{m=0}^{M-1} a_m\Psi_m(e^x)d\mathbb{P}(Y = x|S_{t_{i-1}})\right.\\ &\left.+\int_{K^*}^{\infty}\sum_{m=0}^{M-1} b_m\Psi_m(e^x)d\mathbb{P}(Y = x|S_{t_{i-1}})\right), \qquad (23)\end{aligned}$$

where $K^* = \log(\mathcal{X}^*_{t_i})$ The solution of Equation (23) is given in Appendix C.

[3]

$$\kappa_1 = \mu = \mu'_1$$

$$\kappa_2 = \sigma^2 = \mu'_2 - {\mu'_1}^2$$

$$\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3$$

$$\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3{\mu'_2}^2 + 12\mu'_2{\mu'_1}^2 - 6{\mu'_1}^4$$

where μ'_i is the i *th* non-central moment

3.8 Results for Bermudan Call on Max of several Assets

To illustrate the results Table 3 compares the result of a Bermudan max option on 2, 3 and 5 underlying assets. The results reported in Table 3 are fairly remarkable given the simplicity of the method. The bounds obtained from the SGM are not as tight as compared to the duality-based methods, however, the number of paths required to obtain a fairly accurate exercise policy (as reflected by the lower-bound values) is far less than required to obtain the exercise policy for the duality-based methods. Also the time for each simulation is less than a minute on a system with Intel(R) Duo-Core 2.13 GHz processors and 2 GB RAM. The number of basis functions required for regression, irrespective of the dimensions of the problem, is upto 4 (including the constant).

S_0	SGM LB(s.e)	SGM UB (s.e)	SGM 95% CI	Point Est.	Binomial Value	95% CI AB	95 % CI BC
n = 2 assets							
90	8.069 (0.026)	8.123 (0.003)	[8.021,8.130]	8.075	8.075	[8.053, 8.082]	-
100	13.904 (0.024)	13.964 (0.004)	[13.845,13.972]	13.909	13.902	[13.892, 13.934]	-
110	21.284 (0.028)	21.351 (0.003)	[21.229, 21.357]	21.293	21.345	[21.316, 21.359]	-
n = 3 assets							
90	11.224 (0.023)	11.324 (0.003)	[11.178, 11.318]	11.25	11.29	[11.265, 11.308]	-
100	18.626 (0.031)	18.715 (0.005)	[18.565, 18.725]	18.65	18.69	[18.661, 18.728]	-
110	27.483 (0.036)	27.623 (0.006)	[27.412, 27.635]	27.52	27.58	[27.512, 27.663]	-
n = 5 assets							
90	16.6053 (0.028)	16.703 (0.005)	[16.551, 16.713]	16.632	-	[16.602, 16.655]	[16.620, 16.653]
100	26.045 (0.033)	26.152 (0.006)	[25.975, 26.163]	26.069	-	[26.109, 26.292]	[26.115, 26.164]
110	36.638 (0.047)	36.803 (0.005)	[36.545, 36.813]	36.679	-	[36.704, 36.832]	[36.710, 36.798]

Table 3: Bermudan Max-Call option on 2, 3 and 5 underlying assets: The results are compared with Andersen and Broadie (2004) [1] and Broadie and Cao (2008) [8]. The parameters are: $K = 100, r = 5\%, q = 10\%, \rho = 0, T = 3, \sigma = 20\%$. There are ten exercise opportunities equally spaced in time. Values in parentheses are standard errors. The 95% confidence interval is given by $[L - 1.96\sigma_l, U + 1.96\sigma_u]$, where L and U are lower and upper bound values and σ_l, σ_u are their respective standard errors. The total number of grid points at each time step was 30,000 with an equal number of paths generated from the 3 source grid points (two peripheral and one initial point)

3.9 American Put on Arithmetic Mean of d Assets

A Bermudan basket option is a discretely-exercisable option on multiple underlying assets whose pay-off depends on the weighted average of the underlying asset prices. We assume that the asset prices follow correlated geometric Brownian motion processes given by Equation (18). The pay-off for d underlying asset is $\max(g(S_t^i) + X, 0)$, where $X = K$ and,

$$g(S_t^i) = -(w_1 S_t^1 + \cdots + w_d S_t^d), \tag{24}$$

such that

$$\sum_{i=1}^{d} w_i = 1.$$

The discretization and parametrization scheme for a Bermudan put on a basket is the same as that for Bermudan call on max of several assets. However, we use the case of *basket option* to show how the conditional continuation value can be computed for a general case.

3.10 Computing the Continuation Value for Bermudan Basket Options

In order to compute the continuation value for grid points at t_{i-1} using Equation (7), the transition probability density function $\mathbb{P}(g(S_{t_i})|S_{t_{i-1}})$ is required. For a put on the weighted mean of d underlying assets, the exact transition density function is unknown. The moments for the distribution of $g(S_{t_i})$ can be obtained using *sub-simulations*, which can be used to approximate the density function using the *Gram Charlier series* (Equation (9)). For each grid point at t_{i-1}, sub-paths are generated until the next time step t_i, and the first four non-central moments of the distribution of $g(S_{t_i})$ so obtained are computed. In order to reuse the results obtained for the Bermudan max option, we find the distribution $\mathbb{P}(\log(|g(S_{t_i})|) = x|S_{t_{i-1}})$, rather than determining $\mathbb{P}(|g(S_{t_i})| = x|S_{t_{i-1}})$. The continuation value is then given by Equation (23).

3.11 Results for Bermudan Basket Option

To illustrate the results Table 4 compares the result of a Bermudan put option on 4 underlying assets. In order to compute the continuation value, we generate $N_S = 1000$, sub-paths for each of the underlying assets. The computational effort increases linearly with the number of exercise opportunities k, the number of paths N_S in the sub-simulations, and the dimension of the problem d. Although computationally more expensive than the case where the moments of the distribution can be computed analytically, this example shows a generic case when its not easy to compute the transition probability density or its moments directly. The time taken for each simulation was a few (< 5) minutes. Table 5 compares the results of a Bermudan put option on 5 underlying assets with those reported by Bender *et al.*(2006). The LSM values obtained lie in the confidence interval proposed by us, and except for the *in-the-money* case our confidence intervals agree with those reported by Bender *et al.*

4 Conclusion

In this paper we have presented the algorithm of the stochastic grid method for pricing and exercising Bermudan options. Our approach is based on dynamic programming and linear least squares regression. One of the main achievements of the algorithm is its ability to truly reduce a multi-dimensional problem to a single-dimensional case, and yet avoid some of the associated short comings as were discussed by Boyle *et al.* [7]. Unlike the duality-based algorithm, we do-not need an exercise policy from other methods to compute the upper bounds.

S_0	SGM LB(s.e)	SGM UB (s.e)	SGM 95% CI	Point Est.	FFT Value	LSM (s.e.)
40	1.739 (0.37)	1.740 (0.16)	[1.732, 1.744]	1.738	1.739	1.739 (0.08)

Table 4: Bermudan put option on arithmetic mean of 4 underlying assets: The results are compared with CONV method of Lord *et al.*(2008) [23] and the LSM values. The parameters are: $K = 40, r = 6\%, q = 2\%, \rho = 0.25, T = 1, \sigma = 20\%$. There are ten exercise opportunities equally spaced in time. Values in parentheses are standard errors. The 95% confidence interval is given by $[L - 1.96\sigma_l, U + 1.96\sigma_u]$, where L and U are lower and upper bound values and σ_l, σ_u are their respective standard errors (in cents). The total number of grid points at each time step were 30,000 with equal number of paths generated from the 3 source grid points (two peripheral and one initial point). For the LSM algorithm there were 300,000 paths for each asset, and 18 set of basis functions were used

S_0	SGM LB(s.e)	SGM UB (s.e)	SGM 95% CI	Point Est.	LSM (s.e)	BKS 95 % CI
90	6.874 (0.011)	6.881 (0.009)	[6.853, 6.898]	6.876	6.867 (0.001)	[10.000, 10.004]
100	2.134 (0.012)	2.141 (0.008)	[2.114, 2.167]	2.140	2.163 (0.001)	[2.154, 2.164]
110	0.540 (0.010)	0.550 (0.006)	[0.520, 0.562]	0.541	0.540 (0.001)	[0.535, 0.540]

Table 5: Bermudan put option on arithmetic mean of 5 underlying assets: The results are compared with the intervals reported by Bender *et al.*(2006) [5] and the LSM values. The parameters are: $K = 100, r = 5\%, \rho = 0, T = 3, \sigma = 20\%$. There are three exercise opportunities equally spaced in time. Values in parentheses are standard errors. The total number of grid points at each time step were 3000 with equal number of paths generated from the 3 source grid points (two peripheral and one initial point). For the LSM algorithm there were 120,000 paths for each asset, and 24 set of basis functions were used

The stochastic grid method gives both the upper bound values and the exercise policies to obtain the lower bounds. We showed through examples that method is accurate for the sample problems.

An important outcome of the SGM is that unlike the LSM, a good exercise policy can be obtained using very few sample paths. This makes it an attractive option when dealing with options with many underlying assets. Once the exercise policy is obtained, good lower bound values can be obtained using large number of paths. The method is computationally fast, although when the transition probability density function or its moments are unavailable, subsimulations required to obtain it can make the method somewhat slower. The method is very flexible and can be modified for specific problems (e.g. Bermudan call on max of several assets) to improve the computational performance.

In order to compute the functional approximation of option values at any time step, the only information required is the regression coefficients from the next time step, making this simulation technique highly suitable for parallel implementation.

Although we did not look into the computation of price sensitivities for hedging purposes here, we believe that the method is well suited for computing the Greeks. Once the functional approximation of the option values at the next time step is available, it can be used to compute the continuation values and option values in the close neighbourhood of the initial point, thus allowing for fast approximation of some of the Greeks.

An aspect not yet explored is the choice of more refined methods to generate the initial grid points. The stochastic method for grid generation is a convenient method of grid generation, such that the density of the generated grid points at different time steps is closely related to the transition probability density function, when the transitions happen from the initial point. Since once the grid points are generated, the method employed for generating the grid points has no effect on the solution, it allows using more sophisticated methods of grid generation, like the *quantization tree method* of Bally *et al.* [2] in future work.

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A The Clark Algorithm

The Clark Algorithm (1961) [14], computes the first four moments of the maximum of a pair of jointly normal variates as well as the correlation coefficient between the maximum of the pair and a third normal variate. Let X_1 and X_2 have a bivariate normal distribution, with means μ_1 and μ_2, and standard deviations σ_1 and σ_2, respectively. The correlation coefficient between the two is ρ. Y denotes the maximum of (X_1,X_2). Let ν_i denote the ith non-central moment for the distribution of Y, then

$$\nu_1 = \mu_1\Phi(\alpha) + \mu_2\Phi(-\alpha) + a\phi(\alpha) \tag{25}$$

$$\nu_2 = \left(\mu_1^2+\sigma_1^2\right)\Phi(\alpha) + \left(\mu_2^2+\sigma_2^2\right)\Phi(-\alpha) + (\mu_1+\mu_2)\,a\phi(\alpha) \tag{26}$$

$$\begin{aligned}\nu_3 \;=\;& \left(\mu_1^3+3\mu_1\sigma_1^2\right)\Phi(\alpha) + \left(\mu_2^3+3\mu_2\sigma_2^2\right)\Phi(-\alpha)\\ &+\left[\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)a + \left(2\sigma_1^4+\sigma_1^2\sigma_2^2+2\sigma_2^4\right.\right.\\ &\left.\left.-2\sigma_1^3\sigma_2\rho-2\sigma_1\sigma_2^3\rho-\sigma_1^2\sigma_2^2\rho^2\right)a^{-1}\right]\phi(\alpha),\end{aligned} \tag{27}$$

$$\begin{aligned}\nu_4 \;=\;& \left(\mu_1^4+6\mu_1^2\sigma_1^2+3\sigma_1^4\right)\Phi(\alpha) + \left(\mu_2^4+6\mu_2^2\sigma_2^2+3\sigma_2^4\right)\Phi(-\alpha)\\ &+\left\{\left(\mu_1^3+\mu_1^2\mu_2+\mu_1\mu_2^2+\mu_2^3\right)a - 3\alpha\left(\sigma_1^4-\sigma_2^4\right)\right.\\ &+4\mu_1\sigma_1^3\left[3\left(\frac{\sigma_1-\sigma_2\rho}{a}\right)-\left(\frac{\sigma_2-\sigma_1\rho}{a}\right)^3\right]\\ &\left.+4\mu_2\sigma_2^3\left[3\left(\frac{\sigma_2-\sigma_1\rho}{a}\right)-\left(\frac{\sigma_1-\sigma_2\rho}{a}\right)^3\right]\right\}\phi(\alpha),\end{aligned} \tag{28}$$

If X_3 is a random variable with normal distribution, and the correlation coefficients between X_3 and X_1, X_2 are ρ_1, ρ_2 respectively, then the correlation coefficient ρ_{X_3Y} between X_3 and $Y = \max(X_1, X_2)$ is given by

$$\rho_{X_3Y} = \left[\sigma_1\rho_1\Phi(\alpha) + \sigma_2\rho_2\Phi(-\alpha)\right]/(\nu_2-\nu_1^2)^{\frac{1}{2}}, \tag{29}$$

where

$$a^2 = \sigma_1^2+\sigma_2^2-2\sigma_1\sigma_2\rho$$

$$\alpha = \frac{\mu_1-\mu_2}{a}$$

$$\phi(x) = (2\pi)^{-\frac{1}{2}}\exp\left(-\frac{x^2}{2}\right)$$

$$\Phi(x) = \int_{-\infty}^{x}\phi(t)\,dt.$$

B Approximation for Estimating the Moments of the Greatest of a Finite Set of Normally Distributed Variables

Clark's method can be used to obtain the exact moments of Y and its correlation with X_3, however, as the distribution of Y is not exactly normal, the method can be only used to obtain the approximation of the first four moments of the maximum of a set of d normal variates. If $X_1, \ldots, X_d$ are the d jointly normal variates, and Y is the maximum of these d variates then by using the recursive scheme

$$Y_i = \max(X_1, X_2, \ldots, X_{i+1}) = \max(Y_{i-1}, X_{i+1})$$

and applying Clark's approximation at each step we can compute the approximation of the first four moments for the distribution of Y. It is easy to deduce how Clark's method can be used to obtain the moments for the *minimum* of d assets as well (see Boyle *et al.*(1990) [6]).

C Solution for Continuation Value

The solution to equation,

$$\int_{-\infty}^{K^*} \sum_{m=0}^{M-1} a_m \Psi_m(e^x) d\mathbb{P}(Y = x|S_{t_{i-1}}),$$

where $\Psi(\cdot)$ is polynomial basis function, can be written as the linear combination of,

$$\int_{-\infty}^{K^*} (e^{mx}) d\mathbb{P}(Y = x|S_{t_{i-1}}),$$

$m = (0, \ldots, (M-1))$, and $\mathbb{P}(Y = x|S_{t_{i-1}})$ is

$$\phi\left(\frac{x-\mu}{\sigma}\right)\left[1 + \frac{\kappa_3}{3!\sigma^3}H_3\left(\frac{x-\mu}{\sigma}\right) + \frac{\kappa_4}{4!\sigma^4}H_4\left(\frac{x-\mu}{\sigma}\right)\right].$$

Here $\phi(x)$ is,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

We need to solve,

$$\int_{-\infty}^{K^*} (e^{mx})\phi\left(\frac{x-\mu}{\sigma}\right)\left[1 + \frac{\kappa_3}{3!\sigma^3}H_3\left(\frac{x-\mu}{\sigma}\right) + \frac{\kappa_4}{4!\sigma^4}H_4\left(\frac{x-\mu}{\sigma}\right)\right] dx. \quad (30)$$

Equation (30) can be written as,

$$A\int_{-\infty}^{K^*} \phi\left(\frac{x-\theta}{\sigma}\right)\left[1 + \frac{\kappa_3}{3!\sigma^3}H_3\left(\frac{x-\mu}{\sigma}\right) + \frac{\kappa_4}{4!\sigma^4}H_4\left(\frac{x-\mu}{\sigma}\right)\right] dx, \quad (31)$$

where,

$$A = e^{\left(\mu m + \frac{m^2\sigma^2}{2}\right)},$$

and

$$\theta = \left(\mu + m\sigma^2\right).$$

This can be written in a form easy to integrate,

$$A \qquad \int_{-\infty}^{K^*} \phi\left(\frac{x-\theta}{\sigma}\right)\left[1 + \frac{\kappa_3}{3!\sigma^3}\sum_{j=0}^{3}\binom{3}{j}\left(\frac{\theta-\mu}{\sigma}\right)^j H_{3-j}\left(\frac{x-\theta}{\sigma}\right)\right.$$
$$\left. + \frac{\kappa_4}{4!\sigma^4}\sum_{k=0}^{4}\binom{4}{k}\left(\frac{\theta-\mu}{\sigma}\right)^k H_{4-k}\left(\frac{x-\theta}{\sigma}\right)\right] dx. \tag{32}$$