High Dimensional American Options

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity Term 2005

I dedicate this thesis to my lovely wife Ruth.

Acknowledgements

I acknowledge the help of both my supervisors, Dr. Jeff Dewynne and Dr. William Shaw and thank them for their advice during my time in Oxford. Also, Dr. Sam Howison, Prof. Jon Chapman, Dr. Peter Carr, Dr. Ben Hambly, Dr. Peter Howell and Prof. Terry Lyons have spared time for helpful discussions.

I especially thank Dr. Gregory Lieb for allowing me leave from Dresdner Kleinwort Wasserstein to complete this research.

I would like to thank Dr. Bob Johnson and Prof. Peter Ross for their help, advice, and references over the years. Andy Dickinson provided useful ideas and many cups of tea.

The EPSRC and Nomura International plc made this research possible by providing funding.

Finally, I would like to thank my parents for their love and encouragement over the years, and my wife, Ruth, for her unfailing and invaluable love and support throughout the course of my research.

Abstract

Pricing single asset American options is a hard problem in mathematical finance. There are no closed form solutions available (apart from in the case of the perpetual option), so many approximations and numerical techniques have been developed. Pricing multi–asset (high dimensional) American options is still more difficult.

We extend the method proposed theoretically by Glasserman and Yu (2004) by employing regression basis functions that are martingales under geometric Brownian motion. This results in more accurate Monte Carlo simulations, and computationally cheap lower and upper bounds to the American option price. We have implemented these models in QuantLib, the open–source derivatives pricing library. The code for many of the models discussed in this thesis can be downloaded from `quantlib.org` as part of a practical pricing and risk management library.

We propose a new type of multi–asset option, the "Radial Barrier Option" for which we find analytic solutions. This is a barrier style option that pays out when a barrier, which is a function of the assets and their correlations, is hit. This is a useful benchmark test case for Monte Carlo simulations and may be of use in approximating multi–asset American options. We use Laplace transforms in this analysis which can be applied to give analytic results for the hitting times of Bessel processes.

We investigate the asymptotic solution of the single asset Black–Scholes–Merton equation in the case of low volatility. This analysis explains the success of some American option approximations, and has the potential to be extended to basket options.

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Chapter 1

Introduction

Financial options were first traded on an exchange in 1973 at the Chicago Board Options Exchange (CBOE). That same year the breakthrough papers of Black and Scholes, and Merton, were published. They provided pricing formulae for European options on a single stock where the stock price follows a geometric Brownian motion. European options can only be exercised on the maturity date of the contract. American style options, however, can be exercised at any time between the start of the contract and the maturity date, which makes finding the price much harder. The American option pricing problem has been a fertile area of research for the past thirty years.

Most research has focused on the pricing of single asset American options. There are many ways to formulate the problem (Barone-Adesi, 2005; Myneni, 1992). As we shall see in the literature survey of this thesis (Chapter 2) there are a number of different approximation techniques and numerical methods for solving the single asset American option problem in the Black–Scholes–Merton (BSM) framework. As one might expect there is a trade off between the speed and accuracy required.

Since the initial markets in single asset options, multi–asset options have been introduced and have proved popular with investors, as they are generally cheaper than buying a portfolio of options on single assets. Formulae for European options on the maximum or minimum of a basket of stocks were first published by Stulz (1982) and Johnson (1987). However, there has been little analytic investigation of basket options with American style options. In the case of an option on the geometric average of a basket of stocks the problem can be reduced to a single asset option on a modified process. This is because a product of lognormal random variables is itself lognormal. Apart from this result there has been only limited success in research into analytic solutions for multi–asset American options (Broadie and Detemple, 1997).

Many numerical methods developed for single asset American options can be applied to multi–asset American options. The binomial tree method is applied in two and three underlying assets by Boyle et al. (1989). Finite difference solutions can also be used in the case of two underlying assets. However, these methods suffer from exponentially increasing computational cost as the number of dimensions (assets) increases. Grid based methods are difficult to extend practically to more than three underlying assets. Some promising research into increasing the number of dimensions handled includes the irregular grid approach in Berridge and Schumacher (2002), wavelets (Dempster and Eswaran, 2001), and sparse grid methods (Reisinger, 2004).

The $N^{-1/2}$ convergence rate of Monte Carlo methods (where N is the number of paths) is independent of the number of dimensions. Therefore they are the natural choice for high dimensional European style, or path dependent, options (Boyle et al., 1997). However, to price American options it is necessary to have knowledge about the optimal exercise boundary. Until Tilley (1993), this blocked research into Monte Carlo methods for early exercise problems. Since then there has been progress and now a number of simulation methods are available (these are reviewed in Glasserman (2004)). The method of Longstaff and Schwartz (2001) has proved to be versatile and easy to implement. However, these methods are still computationally expensive. Monte Carlo methods can be used to give both lower and upper bounds to the option price. This is more expensive still. Andersen and Broadie (2004) find that about 60% of computional time is spent on calculating the upper bound. Glasserman and Yu (2004) have provided theoretical justification for expecting that regression methods using basis functions that are martingales may be more computationally efficient. Unfortunately, this method has not be implemented in the existing literature.

One issue that has not received as much attention is pricing American options under models other than the standard Black–Scholes–Merton framework. Under the BSM assumptions the volatility of the asset used in the pricing formula should be the same for all strikes. This is not what is observed in financial markets. A number of different volatility models have been developed to capture this behaviour. Recently a number of papers have investigated pricing American options in models where the asset price follows a more general stochastic process (see Detemple and Tian (2002) for more general diffusion processes and Eberlein and Papapantoleon (2004) for Lévy processes, and references therein). While pricing high dimensional American options is a hard problem even within the simplified BSM framework we need to bear in mind which methods have the potential to be extended to more realistic models.

1.1 Research objectives

This research had two objectives. The first was to evaluate and augment the existing Monte Carlo methods for American style basket options. The second was to investigate the possibility of finding analytic methods for pricing multi–asset American options.

We wanted to improve the understanding of existing methods for finding lower and upper bounds for the price of multi–asset American options using Monte Carlo simulation. A contribution in this area was to test the ideas of Glasserman and Yu (2004) and see whether, as suggested theoretically, using martingale basis functions in the regression produced any benefits.

We wanted to see what kind of option structures on multiple assets could be valued analytically. Barrier options and European capped calls have provided the basis for approximation schemes in the single asset case. Therefore we wanted to find analytic solutions to multi–asset problems that could be used either as test cases for numerical methods, or the base for new approximations to multi–asset option pricing problems.

1.2 Contributions

This thesis presents a number of original contributions in the field of Mathematical Finance. These include:

- An extension and implementation of the method of Glasserman and Yu (2004), by suggesting basis functions that are martingales under geometric Brownian motion (Chapter 3)[1].
- A new type of multi–asset option, that we call the "Radial Barrier Option", for which we give closed form solutions in one, three and five dimensions. In other

[1] see Firth (2004b), presented at Stochastic Finance 2004.

cases numerical solution of an integral may be required (Chapter 4)[2].

- We invert the Laplace transform for the hitting time of a Bessel process to a level. The results are valid for all $\nu \in \mathbb{R}$. A number of stochastic processes used in finance, such as CIR and CEV processes, can be written as Bessel processes (Chapter 5).

- A new asymptotic analysis for the single asset American call option in the presence of a constant dividend yield (Chapter 6)[3]. This approach could be applied to the multi–asset problem. The details of the asymptotic analysis are given in Appendix C.

Full details of presentations and publications based on these contributions are given in Appendix A. We give more details on these contributions in the following subsections.

1.2.1 Monte Carlo methods for high dimensional American options

High dimensional American options have no analytic solution and are difficult to price numerically. Progress has been made in using Monte Carlo simulation to give both lower and upper bounds on the price (Andersen and Broadie, 2004; Kogan and Haugh, 2004; Rogers, 2002). Building on an idea of Glasserman and Yu (2004) we investigate the utility of martingale basis functions in regression based approximation methods. Regression methods are known to give lower bounds easily, however upper bounds are usually computationally expensive. Martingale basis functions enable fast calculation of upper bounds on the price, and also speed up the calculation of the lower bound. These methods can easily be extended to more general Lévy processes.

1.2.2 Radial barrier options

We derive an analytic expression for a new type of multi-asset barrier option using Laplace transform methods. The solution is assumed to be radially symmetric in the normalized dimensionless variables, hence the name "Radial Barrier Options". In the single-asset case our results reduce to published results for American binary, single

[2]published as Firth and Dewynne (2004), presented at the Bachelier Finance Society Third World Congress 2004

[3]published as Firth et al. (2004), presented at ECMI 2002.

or double, barrier options (Kunimoto and Ikeda, 1992; Pelsser, 2000; Rubinstein and Reiner, 1991a).

These options payoff if a barrier, defined as a function of the parameters describing the process for the underlying assets, is hit. As we have found closed form expressions for the value they can be valued rapidly. Radial options are a useful, and hard, benchmark test case for verifying Monte Carlo implementations. Radial options may be useful to approximate other, actively traded, financial products, or as a control variate.

1.2.3 Bessel process hitting times

The Laplace inversion used in pricing radial barrier options can also be applied to calculating the hitting time to a level of a Bessel process (Borodin and Salminen, 1996; Göing-Jaeschke and Yor, 2003). We provide the analytic formulas for the hitting time up and hitting time down (which have been derived independently via an eigenfunction expansion approach in Linetsky (2004b)). We also derive expressions for the small time asymptotic behaviour as previous results do not converge well for small values of the time parameter. We verify the results through Monte Carlo simulation of the underlying Brownian motions, an Euler discretization of the SDE, and exact simulation by sampling from the appropriate non-central chi-squared distribution (Glasserman, 2004).

1.2.4 An asymptotic analysis for the American call option

We present a new asymptotic analysis of an American call option where the diffusion term (volatility) is small compared to the drift terms (interest rate and continuous dividend yield). We show that in the limit where diffusion is negligible, relative to drift, then, at leading order, the American call's behaviour is the same as a perpetual American call option (except in a boundary layer about the option's expiry date). This result gives some insight into the utility of perpetual American option in approximating vanilla American options. This idea has been investigated independently in Widdicks et al. (2005). This approach could be applied to the multi–asset American option problem.

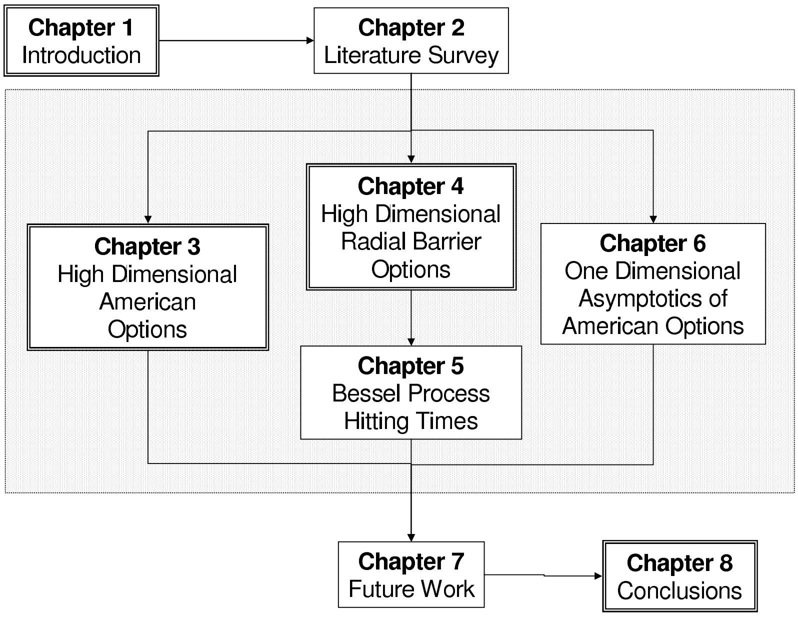


Figure 1.1: Organisation of this thesis

1.3 Organisation of this thesis

This document is organised as in Figure 1.1. The body of this thesis consists of the four main contributions (in the shaded box in the figure) introduced in the previous section (Section 1.2). The contents of each chapter is as follows:

Chapter 2: Literature Survey. We describe the various ways to formulate the American option problem, and cover the different methods that have been used to solve, or approximate, the value of American options. First we describe methods for the single asset American option, then we describe methods for multi–asset American options.

Chapter 3: High Dimensional American Options. We give an extension, using basis functions that are martingales under geometric Brownian motion, to the existing Monte Carlo methods for high dimensional American options. This allows faster and more accurate valuation of multi–asset American options. We present numerical results from an implementation of the theory. We show that convergence is faster in general, and specifically that upper bounds can be found more quickly.

Chapter 4: High Dimensional Radial Barrier Options. We introduce a new multi–asset option, called the "Radial Barrier Option", and develop an analytic solution for the option value. We present closed form pricing formulae for options defined inside, and outside, the barrier. We then investigate numerically the analytic expressions. We present graphs giving a financial interpretation for "Radial Barrier Options" and evaluate them.

Chapter 5: Bessel Process Hitting Times. The Laplace transform inversions performed in the previous chapter can be applied to the hitting times of Bessel processes. We follow through this argument and present results for large and small values of time. We compare the analytic results with Monte Carlo simulation of the underlying Brownian motions, and simulation of the stochastic differential equation.

Chapter 6: One Dimensional Asymptotics of American Options. We present a new asymptotic approximation for a single asset American call option, with a constant dividend yield, in the case of low volatility, relative to the risk neutral drift. We show that, to leading order, the location of the optimal exercise boundary is the same as in the perpetual option case. We detail the implementation, and evaluate the success of the approach.

Chapter 7: Future Work. We present some future directions for this research.

Chapter 8: Conclusions. We have made a number of contributions to research in Mathematical Finance; martingale basis functions to provide more accurate and more efficient bounds for high dimensional American option prices, analytic solutions for multi–asset "Radial Barrier Options", expressions for the hitting times of Bessel processes, and an asymptotic analysis for single asset American options in the presence of low volatility .

Chapter 2

Literature Survey

2.1 Introduction

A derivative security is a contract derived from the price S of an underlying asset. The underlying asset may be a stock, the price of oil, a foreign exchange rate, a forward contract, or some other measurable value. For a general introduction to mathematical finance theory see, among others, Björk (1998), Baxter and Rennie (1996), Hull (2000) or Wilmott et al. (1995).

An *option* gives the holder the right, but not the obligation, to buy or sell the underlying asset. An option is valid up to the maturity date T. A *call* option is the right to buy at a *strike* price E and has a payoff of $\max(S-E,0)$, also written $(S-E)^+$. The 0 corresponds to the scenario where the option holder does not exercise their right to buy. We write $h(S)$ for a general payoff function. A *put* option with the same strike has payoff $h(S) = \min(E-S,0)$, also written $h(S) = (E-S)^+$.

A *European* option can be exercised only at the maturity date T. *American* options can be exercised at any time $t \leq T$. A *Bermudan* option can be exercised at a finite set of dates $t_i \leq T$.

The derivatives pricing theory of Black and Scholes (1973) and Merton (1973) assumes the now standard model for stock price processes – geometric Brownian motion (GBM) (Samuelson, 1965). The returns on the assets are governed by the stochastic differential equation (SDE)

$$\frac{dS}{S} = (\mu - q)\,dt + \sigma\,dW\,, \tag{2.1}$$

where S is the price of the underlying asset at time t. μ is the constant expected return of the asset, q is the constant continuous dividend yield proportional to the asset price, and σ is the constant volatility of asset returns. W is a Wiener process, or Brownian motion. If $q < 0$ we can think of a constant continuous cost of carry proportional to the asset price S, or a foreign interest rate. This has been such a successful model partly because the SDE can be solved analytically.

We now derive the Black–Scholes–Merton equation. $V(S, t)$ is a general option value, applying Itô's lemma (Øksendal, 2000) gives

$$dV = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S}\, dS\,.$$

Consider a portfolio Π consisting of an option V and an amount Δ of the underlying asset S.

$$\Pi = V - \Delta S\,.$$

Notice that we are long the option, V, and therefore, if allowed, it is possible for us to exercise the option early (in the case of American or Bermudan options). If this portfolio was set up as $\Delta S - V$ we would not hold the option and so could not choose when to exercise it. The amount of stock Δ that we hold is fixed at the start of each time step. We cannot anticipate stock movements. The value of the portfolio changes according to

$$d\Pi = dV - \Delta\, dS - \Delta\, q\, S\, dt\,.$$

Substituting dV into this expression gives

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S}\, dS - \Delta\, dS - \Delta\, q\, S\, dt\,.$$

If we now choose

$$\Delta = \frac{\partial V}{\partial S}$$

we eliminate all randomness from the evolution of the portfolio. All dS terms cancel and only dt terms remain. This is called *delta hedging*. We argue, by arbitrage, that this must be equal to the change in value of the equivalent amount of money in a risk free bank account

$$d\Pi = r\Pi\, dt\,.$$

Substituting the value for Δ and equating these two expressions for $d\Pi$ gives the celebrated Black–Scholes–Merton equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV = 0\,. \tag{2.2}$$

The terminal condition is $V(S,T) = h(S_T)$. We also have boundary conditions for $S = 0$ and $S \to \infty$ (Lipton, 2001; Wilmott et al., 1993). The standard Black–Scholes–Merton model makes a number of assumptions, including no transaction costs and trading in continuous time. We do not consider the break down of these assumptions in this thesis. Black and Scholes (1973) show that the option price satisfies the partial differential equation (PDE). It will be convenient to write this in operator form as

$$\mathcal{L}_{bs}[V] = 0,$$

where

$$\mathcal{L}_{bs} = \frac{\partial}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r\,. \tag{2.3}$$

The price, $V(S,t)$, of a European option, at time t, is given by the expected payoff of the option discounted under the risk free interest rate (assumed constant),

$$V(S,t) = \mathbb{E}\left[e^{-rT}\, h(S_T)\right]\,,$$

S_T is the value of the underlying asset at the option expiry date. This, *risk neutral valuation* approach to option pricing was suggested by Cox and Ross (1976). The option price is the discounted expected value of the option payoff at maturity, where the asset price follows the risk-neutral price process,

$$\frac{dS}{S} = (r-q)\,dt + \sigma\,dW \tag{2.4}$$

where the statistical, real world, drift μ, is replaced by the risk free interest rate r. The theoretical framework was put on a rigorous footing in two papers by Harrison and Kreps (1979) and by Harrison and Pliska (1981). The existence of a unique probability measure is equivalent to the absence of arbitrage in the market.

The value, at time t, of the American option can be written as,

$$V(S,t) = \sup_{\tau} \mathbb{E}\left[e^{-r\tau}\, h(S_\tau)\right]\,, \tag{2.5}$$

where the maximum is taken over all stopping times $0 \le \tau \le T$. There are many equivalent formulations for the American option pricing problem. We will consider them later in this chapter. Recent reviews of American option research include Barone-Adesi (2005) and Broadie and Detemple (2004).

2.1.1 Multi–asset framework

The single asset model for asset prices, given in equation (2.1), can easily be generalized to deal with an option with multiple underlying assets. Each asset price, S_i, is driven by a geometric Brownian motion

$$\frac{dS_i}{S_i} = (\mu_i - q_i)\, dt + \sigma_i\, dW_i\,. \tag{2.6}$$

The W_i are standard Brownian motions that are correlated, with the correlation between W_i and W_j being written ρ_{ij}. The PDE for the value, V, of an option that depends on the evolution of n different underlying assets, all in the same country, with price $0 < S_i < \infty$, where $i = 1 \ldots n$, is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sum_{ij}^{n} \sigma_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i}^{n} (r - q_i) S_i \frac{\partial V}{\partial S_i} - rV = 0\,, \tag{2.7}$$

where r is the risk free rate, q_i is the dividend yield of the i^{th} asset, t is time, and σ_{ij} is the covariance of the i^{th} asset with the j^{th} asset. Bold face capital letters, such as $\mathbf{X}$, represent matrices, and bold face lower case letters, such as $\mathbf{x}$, represent column vectors. The covariance matrix with elements σ_{ij}, denoted by $\mathbf{COV}$, is symmetric, $\sigma_{ij} = \sigma_{ji}$. The element σ_{ii} is the square of the volatility of the i^{th} asset, σ_i^2. We write the volatility as a diagonal matrix, $\mathbf{\Sigma}$, with the σ_i on the diagonal and zeros off the diagonal. The correlation between assets i and j is written ρ_{ij}, we write this as a symmetric matrix, $\mathbf{P}$, with unity on the diagonal and ρ_{ij} as the off diagonal entries. The covariance, volatility and correlation are related by $\sigma_{ij} = \sigma_i\, \rho_{ij}\, \sigma_j$ We write this in matrix notation as $\mathbf{COV} = \mathbf{\Sigma}\,\mathbf{P}\,\mathbf{\Sigma}$.

2.2 Analytic solutions

Many option valuation problems can be solved to obtain closed form solutions. In this section we survey some of the existing analytic solutions to option pricing problems, both for single asset options and multi–asset options.

2.2.1 European options

The Black–Scholes–Merton solution (Black and Scholes, 1973; Merton, 1973) for the value, $C(S, t)$, of a European call option at time t on underlying asset S, expiring at time T is given by

$$C(S,t) = Se^{-q(T-t)}\,\mathrm{N}\,(d_+) - Ee^{-r(T-t)}\,\mathrm{N}\,(d_-)\,, \tag{2.8}$$

where $\mathrm{N}\,(.)$ is the cumulative normal distribution function and the parameters $d_\pm$ are defined

$$d_\pm = \frac{\log(\frac{S}{E}) + (r - q \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,. \tag{2.9}$$

The value for a put option is

$$P(S,t) = Ee^{-r(T-t)}\,\mathrm{N}\,(-d_-) - Se^{-q(T-t)}\,\mathrm{N}\,(-d_+)\;.$$

2.2.2 Barrier options

Barrier options can also be valued in closed form. A typical barrier option is a down-and-out call option. This is otherwise the same as a standard vanilla call option (Hull, 2000) but it ceases to exist if the spot price falls below a barrier. The first analytic formula for a barrier option was the case of a down-and-out call option in Merton (1973). Barrier options are popular because they are cheaper than the corresponding European vanilla options. Variants include the down-and-in call option, where a vanilla call option is activated if the barrier is hit. These barrier options can have "up" versions where the barrier is above the strike, and put versions where a vanilla put is knocked-out, or activated. Closed-form formulae for all types of barrier options can be found in Rubinstein and Reiner (1991a).

A rebate may be paid if the barrier is hit. The payment may be made when the barrier is hit, or may be delayed until the expiry date of the option. This case can also be solved analytically for European options, but the formulae are more complicated (Shaw, 1998).

There has been extensive work on the pricing of double barrier options, or corridor options, where the option has a barrier above and below the spot price (Davydov and Linetsky, 2001b; Geman and Yor, 1996; Kunimoto and Ikeda, 1992; Linetsky, 2002; Lipton, 2001; Pelsser, 2000). These formulae involve the summation of series of terms, and vary in their convergence properties.

2.2.3 Perpetual American options

It is possible to value perpetual American call options in closed form (Merton, 1973; Wilmott, 1998). Removing the time dependence from the Black–Scholes–Merton equation leaves an ordinary differential equation for the perpetual option value, $C_\infty(S)$, with solution

$$C_\infty(S) = (S^* - E)\left(\frac{S}{S^*}\right)^{m_+}, \tag{2.10}$$

where

$$S^* = \frac{E\, m_+}{m_+ - 1}, \tag{2.11}$$

and

$$m_\pm = \frac{1}{\sigma^2}\left(-(r - q - \tfrac{1}{2}\sigma^2) \pm \sqrt{(r - q - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2}\right). \tag{2.12}$$

2.2.4 Geometric average options

Asian options have a payout dependent on an average of the asset price. European Asian options on a continuously sampled geometric average can be valued exactly (Kemna and Vorst, 1990) in the Black–Scholes–Merton framework. This is because the geometric average of a lognormal random variable is itself lognormal. European options on the maximum or minimum of two average rates are considered by Wu and Zhang (1999).

2.2.5 Multi–asset options

Many multi–asset options can be priced analytically. An option to exchange one option for another (payoff $h(S_1, S_2) = (S_1(T) - S_2(T))^+$) was priced in Margrabe (1978). The perpetual version of this contract was considered by Gerber and Shiu (1996).

An outperformance option may pay out on the maximum or minimum of two assets, for example $h(S_1, S_2) = (\max(S_1(T), S_2(T)) - E)^+$. This option was priced in Stulz (1982), and generalised to an arbitrary number of assets in Johnson (1987). To evaluate this formula it is necessary to have good approximations to the multi-variate cumulative normal (see Genz (1993), Genz (2004) and references therein).

There has been only limited research into analytic solutions for multi-asset American or compound options (Broadie and Detemple, 1997; Buchen and Skipper, 2003; Villeneuve, 1999).

2.3 Problem formulations for American options

The crucial difference between European and American options is that an American option can be exercised, by its holder, at any time up to and including expiry. This makes their pricing and hedging mathematically challenging and few closed form solutions have been found. American options are important because they are very widely traded. At least as important as the pricing and hedging issues, for American options, is the problem of determining the optimal exercise strategy.

Of course, the theory does not match perfectly with financial practice. We refer the interested reader to Broadie et al. (2000a) and references therein for an investigation into exercise policies and prices observed in the market.

2.3.1 Inequalities

An American option conveys more rights to the holder than an otherwise equivalent European option, so the option can not be worth less than its European equivalent. As an American option can be exercised at any time until expiry the payoff function must be defined for all times.

In the case of an American put option we have the opportunity to exercise early and realise a profit at a time before expiry. This money is worth more to us now than it would be in the future, because of the time value of money (assuming a positive interest rate).

If the underlying asset pays dividends, we give up the income stream from the dividend payments when we exercise a put option. The greater the income from the dividends the less likely we will be to exercise early.

In the case of the American call option with no dividends we gain nothing by exercising early, so we would rather pay the money to buy the asset as late as possible, that is at expiry. Therefore we never exercise the option early, and the price of the option is the same as in the European case.

When dividends are present we gain the dividend stream when we exercise the option early. There is a balance between an increasing risk free rate, which makes us exercise later, an increasing dividend stream, which makes us exercise earlier, and the volatility. Increasing volatility makes us less likely to exercise early, as we might make more profit if we hold on to the option.

An American option must be worth at least as much as its payoff, otherwise there would be an arbitrage opportunity. Assume that a put option is worth less than its payoff, $V < h$, then we buy the put for V, and buy the asset for S, then immediately exercise the option and get E for selling our asset. We therefore have $(E-S)-V > 0$, or equivalently, $h-V > 0$, and have made a risk-free profit. In reality this gap would be closed by arbitrage traders who would push the price of the option up by buying it in quantity. In general, this principle for the American option can be expressed as the following inequality,

$$V(S,t) \geq h(S,t)\,. \tag{2.13}$$

The critical price at which we should exercise the American option is called the *optimal exercise boundary*, $S^*(t)$. This boundary depends on time. It is also called the *free boundary* in the free boundary formulation for the American option. This boundary splits state space into two regions:

- the *continuation region*, $\mathcal{C}$, where we should hold the option and not exercise early, and
- the *stopping region*, $\mathcal{S}$, where we should exercise immediately.

In one dimension the stopping region, if it exists, is usually smooth and simply connected (Myneni, 1992). However, in higher dimensions, with more than one asset, the stopping region can split into two or more separate regions and take on more complex shapes (Broadie and Detemple, 1997).

There are many ways to formulate the American option pricing problem (Barone-Adesi, 2005). The most intuitive is the Optimal Stopping Formulation.

2.3.2 Optimal stopping formulation

We generalise equation (2.5), and write the Optimal Stopping formulation for the value of an American option as

$$V(S,t) = \max_{\tau} \mathbb{E}\left[e^{-r\tau}h(S,\tau)\right] \tag{2.14}$$

where the maximum is taken over all stopping times $0 \leq \tau \leq T$. As we have seen the payoff of an American call is $h_C(S) = \max(S-E,0)$ at all times over the life of the contract. Bensoussan (1984) and Karatzas (1988) proved that no arbitrage ensures the above formulation for the true price.

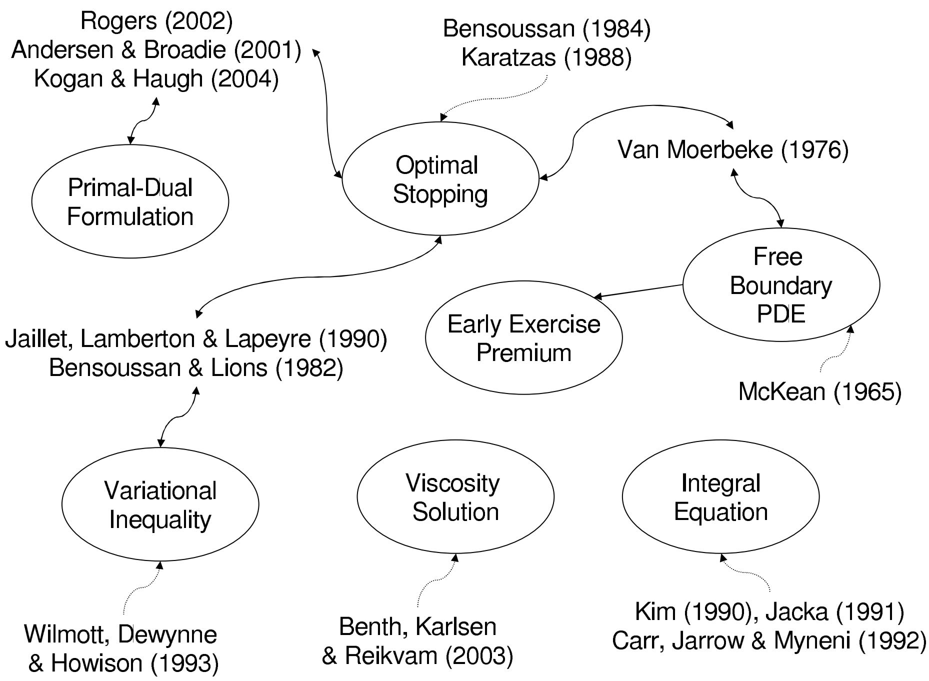


Figure 2.1: American option problem formulations

Informally, we see that the pricing equation is the maximum of an expected value. The option pay-out under the risk neural measure gives the expectation, as in the European pricing formula. In the American case we can exercise at any time. The option holder should use the optimal exercise policy so the option will have the maximum value from all possible stopping times.

There are two other major ways of formulating the American option pricing problem. The earliest, the free boundary formulation, is due to McKean (1965). The other is the variational inequality formulation, or linear complementarity problem, due to Jaillet et al. (1990). The relationships between different formulations for the American option problem are shown in Figure 2.1.

2.3.3 Free boundary formulation

McKean (1965) formulated the American call option as a free boundary problem (see also Samuelson (1967)). The free boundary formulation and the optimal stopping formulation are equivalent (Carr et al., 1992; Peskir, 2005; Van Moerbeke, 1976). See Wilmott et al. (1993) for a description of the free boundary formulation. For complex payoffs there may be more than one free boundary.

In the single asset case we have a *smooth pasting* condition. The value of the option meets the payoff function smoothly, so $\partial V/\partial S = 1$ for a call, and $\partial V/\partial S = -1$ for a put. This is not the most general formulation, as options with discontinuous payoffs do not satisfy the smooth pasting condition. They have to be formulated in another way, such as an optimal stopping problem, or linear complementarity problem.

The smooth pasting formulation for the value, $V(S,t)$, of an American call option on an underlying stock that pays a continuous dividend yield is,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,, \tag{2.15}$$

with boundary conditions

$$\begin{aligned} V(S,T) &= \max(S-E,0) && (2.16)\\ V(S^*(t),t) &= S^*(t) - E && (2.17)\\ \frac{\partial V}{\partial S}(S^*(t),t) &= 1, && (2.18) \end{aligned}$$

and $t < T$. $S^*(t)$ represents the location of the free boundary at time t.

2.3.4 Linear complementarity formulation

Jaillet et al. (1990) used work by Bensoussan and Lions (1982) to rigorously link the optimal stopping problem with variational inequalities. This justified the use of linear complementarity methods in the solution of American option pricing problems.

We present the more concise formulation of Dewynne (2000), based on the "obstacle problem" from continuum mechanics. The presentation is similar to that for the European Black–Scholes–Merton equation above. Let S represent the price of an asset. As above we assume that the price process follows the geometric Brownian motion, (2.1). Let $V(S,t)$ represent the value of the option at time t when the underlying asset has value S. Then, applying Itô's lemma (Björk, 1998; Øksendal, 2000) we have

$$dV = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S} dS\,.$$

Assuming that the option has not been exercised, we construct a delta-hedged portfolio worth,

$$\Pi = V - \Delta S\,.$$

Notice that we are long the option, V, and therefore it is possible for us to exercise the option early. If this portfolio was set up as $\Delta S - V$ we would not hold the option and so could not choose when to exercise it. Choosing

$$\Delta = \frac{\partial V}{\partial S}$$

in the value of the portfolio during a time step, dt, of

$$d\Pi = \left(\frac{\partial V}{\partial t} - qS\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

In the case of the European option it is possible to argue, by arbitrage, that this must be equal to the change in value of the equivalent amount of money in a risk free bank account. This may only be possible in one direction for the American option, because of the early exercise feature.

If we hold the portfolio with the option in it, then we are free to exercise the option when it is under-performing in comparison to the equivalent bank account and deposit the money so it earns the risk free rate, therefore

$$d\Pi \leq r\Pi dt,$$

which can be written, using the Black–Scholes–Merton operator, $\mathcal{L}_{bs}$, defined in (2.3), as

$$\mathcal{L}_{bs}[V]dt = (d\Pi - r\Pi dt) \leq 0\,. \tag{2.19}$$

It is not always possible to construct the opposite argument, because when we short the portfolio containing the option the other party can exercise the option if it starts to under-perform. In the European case it is not possible to exercise early. As we saw in (2.13), the value of an American option has to be greater than or equal to its payoff.

$$V(S,t) \geq h(S,t)\,. \tag{2.20}$$

We now consider making the two inequalities, (2.19) and (2.20), strict inequalities. First we consider the implications of assuming $V > h$, then we consider the implications of assuming $\mathcal{L}_{bs}[V] < 0$.

1. If the option value is strictly greater than the current payoff $V > h$ we can sell the option for more than we would get by exercising it. Therefore it is not optimal to exercise the option, and it is possible to short the option without it

being exercised. Thus we can enforce both sides of the no-arbitrage step in the derivation of the Black–Scholes–Merton equation so that

$$V(S,t) > h(S,t) \quad \Rightarrow \quad \mathcal{L}_{bs}[V] = 0\,.$$

2. When $\mathcal{L}_{bs}[V] < 0$ the portfolio is under-performing when compared to the bank account. We unwind the portfolio and invest the money at the risk free rate. We know that $V(S,t) \geq h(S,t)$ at all times. We have just seen that $V(S,t) > h(S,t)$ implies that $\mathcal{L}_{bs}[V] = 0$. Therefore, in this case, we must have $V(S,t) = h(S,t)$, so

$$\mathcal{L}_{bs}[V] < 0 \quad \Rightarrow \quad V(S,t) - h(S,t) = 0\,.$$

In either case one of the conditions has to be equal to zero, therefore these two cases can be expressed concisely as

$$\begin{gathered} \mathcal{L}_{bs}[V] \leq 0, \qquad V - h \geq 0, \\ (V-h)\mathcal{L}_{bs}[V] = 0. \end{gathered} \tag{2.21}$$

This formulation does not uniquely determine the functional form for V unless some regularity conditions are also satisfied. V must be continuous, to prevent the possibility of arbitrage at discontinuities. Also, the holder of the option must choose the exercise strategy that maximises the value of the option, without admitting the chance of arbitrage.

2.3.5 Primal–dual formulation

The optimal stopping formulation is usually employed when American options are discussed in the martingale framework. Rogers (2002), and also Kogan and Haugh (2004) and Andersen and Broadie (2004), have given some new ideas on how to find upper bounds for American option prices. Glasserman and Yu (2004) also propose a method for finding upper bounds for the American option price, using basis functions that are martingales. However, they do not implement or evaluate their method.

2.3.6 Early exercise premium formulation

Carr et al. (1992) first presented the American option price split into the value of an equivalent European option and the additional value due to the possibility of early exercise. Using $\tau = T - t$ as the time remaining until the option expires, we can write this for call options, following Kim (1990), as,

$$V(S,\tau) = C(S,\tau) + u(S,\tau)\,,$$

where V is the full American call option value, $C(S,\tau)$ is the value of the equivalent European call option, and $u(S,\tau)$ is the value of the early exercise. As we have seen, the value of the call option is given by,

$$C(S,\tau) = \mathbb{E}\left[e^{-r\tau}\left(S-E\right)^{+}\right]\,.$$

The value of being able to exercise early can be written

$$\begin{aligned} u(S,\tau) = \; & \textstyle\int_0^\tau \Big[qSe^{-q(\tau-s)}\mathrm{N}\left(d_+(S,\tau-s;S^*(s))\right) \\ & \qquad -rEe^{-r(\tau-s)}\mathrm{N}\left(d_-(S,\tau-s;S^*(s))\right)\Big]\,ds\,, \end{aligned} \tag{2.22}$$

where S^* is again the location of the optimal exercise boundary and $\mathrm{N}\left(.\right)$ is the cumulative normal distribution function and where d_+ and d_- are similar to those parameters seen before,

$$\begin{aligned} d_+(S,\tau;S^*) &= \frac{\log(S/S^*) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\,, \\ d_-(S,\tau;S^*) &= d_+(S,\tau;S^*) - \sigma\sqrt{\tau}\,. \end{aligned}$$

The early exercise premium can be seen to be the value of a contingent claim that comes into effect whenever the asset price exceeds the optimal exercise boundary. This contingent claim pays the greater of the interest earned on the exercise price and the dividends paid by the asset.

2.3.7 Integral equation formulation

In the previous section we saw the American option problem expressed as an integral. This can be useful as the dimensionality of the problem can be reduced. Also boundary conditions can be incorporated into the integral equation.

One such integral equation is given by Kim (1990),

$$\begin{aligned} S^*(s) - E \;=\; & C(S^*(s),s) \\ & + \textstyle\int_0^s \Big[qS^*(s)e^{-q(s-\xi)}\mathrm{N}\left(d_+(S^*(s),s-\xi;S^*(\xi))\right) \\ & \qquad -rEe^{-r(s-\xi)}\mathrm{N}\left(d_-(S^*(s),s-\xi;S^*(\xi))\right)\Big]\,d\xi\,. \end{aligned}$$

The notation is as in the previous section. This equation represents the fact that the option value on the optimal exercise boundary is equal to the value of immediate exercise. Solving this equation will give an expression for the free boundary, S^*. This can then be used in (2.22) to find the option value.

Kim (1990) reproduced the results of McKean (1965) for warrants and extended them to include the American put. Jacka (1991) used the integral equation formulation to enable a numerical solution to be found, also see Carr et al. (1992).

Gao et al. (2000) and Karatzas and Wang (2000) investigate pricing American barrier options analytically using integral equations.

2.3.8 Viscosity solution formulation

Recently the American option problem has been formulated as a viscosity solution (Benth et al., 2003; Martini, 2000). Details of the numerical solution procedure are given in Benth et al. (2001). This formulation allows solution on a fixed domain, with no explicit free boundary, the "price" for this is the fact that there is a non-linearity in the solution. As noted in Benth et al. (2003), the partial differential equation resulting from the viscosity solution formulation can be interpreted as the infinitesimal version of the early exercise premium representation. A related solution formulation is also proposed in Badea and Wang (2004a) and Badea and Wang (2004b).

2.3.9 Other formulations

Kuske and Keller (1998) transformed the free boundary formulation for the American put value from Wilmott et al. (1993), by using Green's theorem, into an integral equation for the optimal exercise boundary. This equation was then solved asymptotically for times close to expiry. The results improved on those in Barles et al. (1995). Ševčovič (2001) has also done an analysis using transforms to give a non-linear integral equation. This equation is then solved iteratively, to give an approximation to the free boundary, which is then used to give a value for the option.

Mallier and Alobaidi (2000) also produce an integral equation representation by manipulating the free boundary formulation into a Fredholm type integral.

2.3.10 Alternate stochastic processes

Detemple and Tian (2002) consider American option valuation for a class of diffusion processes. They consider American options in stochastic volatility models, with stochastic interest rates, American bond options and American options on the CEV model.

2.3.10.1 Jump diffusion

Pricing American options in a jump diffusion model is considered in Pham (1997). Chesney and Jeanblanc (2003) extends the BAW approach to jump diffusions, published as Chesney and Jeanblanc (2004)

2.3.10.2 Lévy processes

There has recently been much research into American options under various Lévy processes. Standard references for Lévy processes are Bertoin (1996) and Sato (1999). Schoutens (2003) considered Lévy processes in finance specifically. A good survey of option pricing under Lévy processes is given in Eberlein and Papapantoleon (2004).

A number of different Lévy processes have been proposed for modelling financial markets, such as the symmetric Variance Gamma process (Madan and Seneta, 1990), asymmetric Variance Gamma process (Madan et al., 1998), the Normal Inverse Gaussian process (Barndorff-Nielsen, 1998).

American options are considered theoretically in Boyarchenko and Levendorskii (2004). Mordecki (2002) investigates perpetual American options, as do Boyarchenko and Levendorskii (2002). American options under variance gamma are considered in (Hirsa and Madan, 2003).

A number of numerical methods have been applied to American options; Almendral and Oosterlee (2005a) use a second order finite difference scheme for the PIDE for the CMGY process. In Almendral and Oosterlee (2005b) a linear complementarity formulation is used perform numerical pricing using the PIDE. A lattice has been used by Këllezi and Webber (2004) for valuing Bermudan options on Lévy driven assets.

2.4 Analytic relationships

Although there are no analytic expressions for the American option value it is still possible to say some things about American options.

Villeneuve (1999) built on the work of Broadie and Detemple (1997) investigating the properties of the optimal exercise boundary for multiple assets.

2.4.1 Put call parity

The well known parity results between call options and put options can be extremely useful in practice. For example, they sometimes allow lower variance Monte Carlo estimates for option prices to be found.

The usual parity statement between European call and put options is

$$C - P = Se^{-q(T-t)} - Ee^{-r(T-t)}\,. \tag{2.23}$$

Put call parity for American options is discussed in McDonald and Schroder (1998), Chesney and Gibson (1993) and Carr and Chesney (1996). They show that under geometric Brownian motion the American put and call option values are related by

$$C(S, E, r, q, T) = P(E, S, q, r, T)\,.$$

This can be seen intuitively by considering option FX options, where we can think of r as the local interest rate, and q as the foreign interest rate. If we identify the parameters: $S \to E$, $E \to S$, $r \to q$ and $q \to r$ then the American put price equals the American call price.

There are a number of put call parity results for American barrier options. Gao et al. (2000) prove a result relating American barrier options. American barrier options are also considered in Karatzas and Wang (2000), Dai and Kwok (2004) and Haug (2001).

2.4.2 Multi–asset options

Two dimensional exercise regions for American options are considered in Broadie and Detemple (1997). The case of American lookback options is investigated by Lai and Lim (2004). Two dimensional exercise regions are also considered for Quanto lookback American options in Dai et al. (2004). Perpetual American options on two underlying assets have been analysed by Gerber and Shiu (1996). Wong and Kwok (2003) consider barrier option and multi–asset derivatives.

2.5 Numerical methods

As there are few closed form solutions available to value American options it is necessary to solve the equations numerically. We survey methods for single asset options, barrier options, and multi–asset options.

2.5.1 Single asset American options

We discuss the existing solution methods and approximations for single asset options in order of speed. We place the fastest approximations first. Closed form approximations are the fastest solution method. Next fastest, generally, are convergent analytic approximations, or methods based on the evaluation of an integral. Numerical methods such as the binomial tree, or finite differences are convergent, but are computationally expensive. Finally, Monte Carlo simulation is not competitive for single asset options.

Broadie and Detemple (1996) and AitSahlia and Carr (1997) compare a number of approximation techniques. There is a trade off between speed and accuracy. Both papers recommend, in order of increasing accuracy and computation time, the use of:

- The Lower Bound Approximation (LBA) and the Lower and Upper Bound Approximation (LUBA) of Broadie and Detemple (1996).
- The Analytic Method of Lines (MoL), especially the modified 3-point extrapolation (Carr and Faguet, 1996).
- The Black–Scholes–Merton modified Binomial method with Richardson extrapolation (BBSR) (Broadie and Detemple, 1996).

More recently there has been progress on the understanding of the integral representation for the early exercise premium (AitSahlia and Lai, 2001; Chen and Chadam, 2001; Chen et al., 2001; Goodman and Ostrov, 2002; Jacka, 1991; Ju, 1998; Little et al., 2000; Shaw, 1998). These integral representations may be numerically simpler to solve than the method of Subrahmanyam and Yu (1993) considered by AitSahlia and Carr (1997) and the method of Kim (1990) evaluated by Broadie and Detemple (1996). The integral equation approach is also attractive because analytic expressions for the hedge parameters can be found. Integral equation representations have the potential to give very fast and accurate results to the single asset American option valuation and hedging problem.

2.5.1.1 Analytic approximations

One of the earliest approximation methods is Geske (1979), who approximated the American option price by considering a compound option with two exercise dates. Geske and Johnson (1984) improved the accuracy by considering more exercise dates and introduced the use of Richardson extrapolation to option pricing. The two point version of this method is denoted by GJ2 in Figure 2.2, which compares many approximation methods, as surveyed by AitSahlia and Carr (1997). GJ4 denotes the 4 point method in Figure 2.3, which reproduces results reported in Ju (1999). The general method involves an infinite series of multidimensional cumulative normals which requires care to evaluate (Genz, 1993). Bunch and Johnson (1992) improve the accuracy and efficiency of this method, using only two points. This is known as the modified Geske–Johnson method (MGJ2 in Figures 2.2 and 2.3). Preferred methods, that are both fast and accurate, are in the lower left corner of the figure.

Another early approach is the quadratic approximation of Barone-Adesi and Whaley (1987) (BAW) and MacMillan (1986). This approach is accurate for short and long dated options, but can give poor estimates between one and five years, an issue addressed in the improved method of Ju (1999) (Ju). The (Ju) point on Figure 2.2 is scaled against the BAW method using the results reported in Ju (1999).

It is also possible to approximate the American option price by approximating the optimal exercise boundary. Omberg (1987) approximates the free boundary by an exponential function, for which it is possible to find a closed form solution. Bjerksund and Stensland (1993) use a flat exercise boundary. This gives a closed form solution of reasonable accuracy. Bjerksund and Stensland (2002) extend this approach by assuming an exercise boundary consisting of two flat pieces.

The Lower Bound Approximation (LBA) and Lower and Upper Bound Approximation (LUBA) are described in Broadie and Detemple (1996). The lower bound uses the value for the American capped call option, derived by Broadie and Detemple (1995). As shown in Figure 2.2, this is a fast and accurate method. However, it is

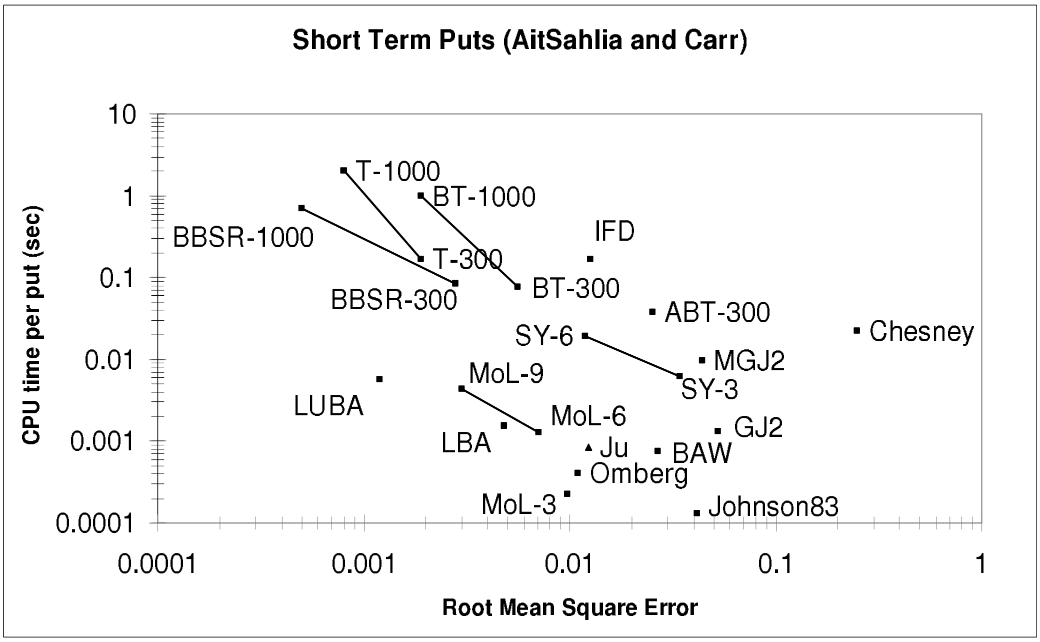


Figure 2.2: Comparison of approximation methods for short term American put options, using the results reported in AitSahlia and Carr (1997). Preferred methods are in the lower left corner. BBSR is the modified binomial method, using Richardson extrapolation and the European Black–Scholes–Merton formula for the first step (Broadie and Detemple, 1996), with 300 or 1000 steps. T is the trinomial tree method (Kamrad and Ritchken, 1991; Parkinson, 1977), with 300 or 1000 steps. BT is the standard binomial tree Cox et al. (1979) with 300 or 1000 steps. ABT is the accelerated binomial tree method of Breen (1991). MoL is the Analytical Method of Lines (Carr and Faguet, 1996), with 3, 6 or 9 steps (the 3 step method uses Richardson extrapolation). BAW is the method of Barone-Adesi and Whaley (1987). IFD is the implicit finite difference method, with a 200 by 300 grid (Brennan and Schwartz, 1977). LBA is the Lower Bound Approximation (Broadie and Detemple, 1996). LUBA is the Lower and Upper Bound Approximation of Broadie and Detemple (1996). Omberg and Chesney are the exponential optimal exercise boundary methods of Omberg (1987) and Chesney (1989), respectively. SY is the integral method of Subrahmanyam and Yu (1993), using 3 or 6 points. MG2 is the two point method of Geske and Johnson (1984) and MGJ2 is the modified two point approximation of Bunch and Johnson (1992). Johnnson83 is the method of Johnson (1983). Ju is the result of Ju (1999), scaled against the BAW method.

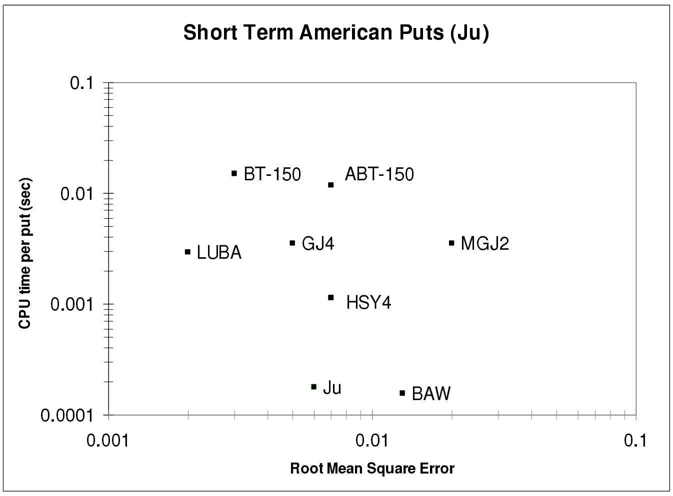


Figure 2.3: Comparison of approximation methods for short term American put options using results from Ju (1999). We use the same legend as in Figure 2.2, the only difference being that HSY4 is the 4 point method of Huang et al. (1996). The results are computed over different portfolios of short term American put options.

not convergent, in that there is no parameter that can be increased to ensure a more accurate result.

Carr and Faguet (1996) use the analytic method of lines (MoL) to approximate the American option value. This method approximates the time derivative, but solves the resulting ordinary differential equation at each time step analytically. This is equivalent to the randomization method descried in Carr (1998). The randomization method randomizes the time to maturity of the option, and creates an approximation using the mean maturity time. The analytic method of lines is an accurate, fast and convergent method. They use Richardson extrapolation to improve convergence. An alternative randomization method to that of Carr (1998) is explored in Kimura (2004).

Johnson (1983) is another early approximation, extended by Blomeyer (1986) to deal with the case of dividends. Roll (1997) deals with the case of known dividends (Geske, 1979; Whaley, 1981).

Method of Bjerksund and Stensland (1993) We will consider approximations using a flat exercise boundary later in this thesis, so we include some details on the method of Bjerksund and Stensland (1993) here. Bjerksund and Stensland (1993) consider two boundaries. The first is given by

$$X_T = S_0^* + (S_\infty^* - S_0^*)(1 - e^{f(T)}) ,$$

where

$$f(T) \equiv -((r-q)T + 2\sigma\sqrt{T}) \left(\frac{S_0^*}{S_\infty^* - S_0^*} \right) ,$$

the level of the perpetual exercise boundary, S_∞^*, is given by

$$S_\infty^* = \frac{m_+}{m_+ - 1} E ,$$

and the initial level of the exercise boundary, S_0^*, is given by,

$$S_0^* = \max(E, \frac{r}{q} E) .$$

The second boundary that they consider is X_T^* where

$$X_T^* = \max_{X \in \{S_0^*, \ldots, S_\infty^*\}} V .$$

Putting even a few intermediate points in the set of possible flat boundaries greatly improves the accuracy of the flat boundary approximation.

2.5.1.2 Asymptotic approximations and integral methods

There are many papers performing asymptotic analyses on the American option problem. These papers generally investigate the structure of the free boundary close to expiry. Few papers investigate the accuracy of approximations for the American option price. Once an approximation for the free boundary has been found the American option can be priced using the methods in Carr et al. (1992); Jacka (1991); Kim (1990).

Matched Asymptotic expansions are useful in finance Howison (2005b). See Howison (2005a) for an application of matched asymptotic expansions to the difference between true American options and Bermudan options.

Barles et al. (1995) investigate the critical stock price near expiry and find upper and lower bounds. They then use this to price American options.

Chen et al. (2001) evaluate the approximations for the free boundary of Barone-Adesi and Whaley (1987); Bunch and Johnson (2000); Kuske and Keller (1998) and their own approach Chen and Chadam (2001). They recommend the use of their ordinary differential equation approximation to find accurate values for the exercise boundary, reporting an accuracy of 10^{-7} for all parameter ranges of σ and r, in minutes.

Evans et al. (2002) investigate the behaviour near expiry of options on an asset paying a constant continuous dividend yield. Their results are consistent with those reported independently in Goodman and Ostrov (2002); Little et al. (2000). Chen et al. (2001) do not consider the problem of using their approximation for the free boundary to actually price American options.

Mallier and Alobaidi (2000) use Laplace transforms to find an integral equation. Knessl (2001a,b) also use Laplace transforms to reduce the partial differential equation to the heat equation and investigate the problem for various small parameters.

Huang et al. (1996) approximate the integrand, rather than the free boundary. Ju (1998) approximated the free boundary itself, using a piecewise exponential function. However, this method still does not use a continuous function to approximate the boundary, as shown in AitSahlia and Lai (1999). This is improved upon by AitSahlia and Lai (2001) who use linear splines with few knots to find a continuous piecewise linear approximation to the free boundary. They find that this gives good approximations to the American option price. Lai and Lim (2003) gives a survey of the work of Chernoff, as it has been applied to these problems.

Underwood and Wang (2000) also use an integral representation, and they go on to compute American option values, but they do not evaluate their method against competing methods. Ševčovič (2001) also transforms the free boundary formulation into an integral equation. He obtains a non-linear integral equation using Fourier sine and cosine integral transforms.

Gao et al. (2000) introduces an integral representation of the early exercise premium for American style barrier options an extends the method of Ju (1998) to this case.

Recent developments include Lamberton and Villeneuve (2003), and Zhang and Li (2003) who extend the results of Chen and Chadam (2001) and test the accuracy.

Chevalier (2005) extends previous results on the behaviour of the exercise boundary near expiry to the case of local volatility models.

Of particular relevance to this thesis is the paper by Widdicks et al. (2005) who have performed a singular perturbation analysis on the American option problem. They reduce the number of parameters in the canonical representation of the problem sufficiently so that lookup tables can be used.

2.5.1.3 Trees and lattices

Using a binomial tree to value options was suggested by Cox et al. (1979) and Rendleman and Bartter (1979). This method approximates the underlying stochastic process as a lattice. American options can then be valued by dynamic programming. Convergence of this method is proved in Amin and Khanna (1994). Lattice methods work well for single asset options, and have been extended to higher dimensions (Boyle, 1988; Boyle et al., 1989). In Figure 2.2 this method is denoted by BT.

The accelerated binomial tree method of Breen (1991), using Richardson extrapolation on a standard binomial tree, has been shown to give worse performance than the regular method (Broadie and Detemple, 1996). This is due to the oscillations in the convergence. Richardson extrapolation assumes that the functional form of the error term is known. In Figure 2.2 this method is denoted by ABT.

Broadie and Detemple (1996) suggest improving the binomial method by using the analytic Black–Scholes–Merton formula to evaluate the first dynamic programming step. They find that this smooths the convergence of the price, and suggest using Richardson extrapolation in Figure 2.2 (BBSR). In particular they use m steps and set the price to $V = 2V_m - V_{m/2}$. This gives a faster rate of convergence, and for vanilla American options this is one of the best methods.

Binomial trees have a branching factor of 2. At each time step the tree splits in two. Using a branching factor of 3 gives a trinomial tree (Kamrad and Ritchken, 1991; Parkinson, 1977). Figure 2.2 shows the effectiveness of the trinomial method (T).

Recently Figlewski and Gao (1999) have improved the efficiency and accuracy of the binomial method. Broadie and Yamamoto (2003) applies the Fast Gauss Transform to speed up calculations involving the summation of Gaussians. This method is applied to American options in Broadie and Yamamoto (2004).

2.5.1.4 Finite differences

Finite difference methods were suggested for use in finance by Brennan and Schwartz (1977, 1978). This was extended in Courtadon (1982). Jaillet et al. (1990) prove the convergence of the Brennan and Schwartz (1977) method. This discretisation produces a tridiagonal matrix system which can be solved efficiently. The implicit finite difference scheme is denoted by IFD in Figure 2.2 (using a 200 by 300 grid). Finite difference methods can be used to solve the variational inequality formulation. Standard references for finite difference solutions to partial differential equations (PDEs) are Smith (1985) and Morton and Mayers (1994). Tavella and Randall (2000) apply finite difference methods to derivative pricing problems.

Oscillations in the solution due to the discontinuities in the derivative of the payoff can be lessened by using some fully implicit time steps before changing to a higher order scheme. This is idea behind the BBSR method in binomial tree framework (Broadie and Detemple, 1996).

Second order upwind schemes produce pentadiagonal matrices (Huang and Pang, 1998). Jump diffusion models for the underlying asset price process produce denser matrices, which are more computationally expensive to solve (Zhang, 1997). The same problem occurs with general Levy models Matache et al. (2003).

Wu and Kwok (1997) proposed a front–fixing method for valuing American options. Penalty methods include Nielsen et al. (2002); Zvan et al. (1998).

Scott (1987) first applied Projected Successive Over Relaxation (PSOR) (Crank, 1984) to option pricing problems. Also see Wilmott et al. (1993) for details on the method applied to American options.

Dempster and Hutton (1999) notes that PSOR fails to converge for some typical volatility values. They use a simplex to solve a linear programming problem to value the American option. Problems fitting the observed volatility smile are also considered by Dempster and Richards (2000).

Crank–Nicolson (Crank and Nicolson, 1947) is generally not the best method, as severe oscillations may occur in the calculation of hedge parameters (Coleman et al., 2002; Shaw, 1998). Shaw (1998) recommends use of the Douglas scheme. Coleman et al. (2002) proposes a Newton method to solve the linear complementarity problem. Duffy (2004) proposes an exponentially fitted scheme to prevent oscillations.

Another difficulty for standard discretisation schemes is the case of low diffusion. This occurs for the diffusion operator for the PDE representation of the Asian option price (Zvan et al., 1997).

Multigrid methods for free boundary problems (Brandt and Cryer, 1983) were applied to the case of American options on assets with stochastic volatility by Clarke and Parrott (1999).

2.5.1.5 Monte Carlo

Financial options can also be priced by Monte Carlo simulation (Boyle (1977) and Boyle et al. (1997)). However, Monte Carlo simulation converges slowly, so is generally only competitive when other methods cannot be easily applied. The standard approach in numerical methods for American options, such as the binomial tree or finite differences, is to evaluate the option payoff at the option expiry time and use dynamic programming to work backwards through time until the initial time is reached. This is problematic for Monte Carlo methods, which rely on simulating paths forward through time. For many years this stymied research into Monte Carlo methods for American options.

Recently this problem has been overcome (Longstaff and Schwartz, 2001). However, Monte Carlo methods are still only competitive for high dimensional problems, so we postpone discussion of them until later.

2.5.2 Barrier options

There are similarities between barrier options and American options. If the location of the free boundary were known the American option could be valued in the same way as barrier options. Therefore we survey the methods of solving barrier options.

As seen above, many analytic formulae for barrier options are available (Merton, 1973; Rubinstein and Reiner, 1991a). Many methods of analytic solution are available for double barriers, with varying convergence properties. Solutions for more complex options, or underlying asset price processes, generally require numerical solution.

2.5.2.1 Trees and lattices

Binomial methods (Cox et al., 1979; Rendleman and Bartter, 1979) converge slowly to the correct price for the case of barrier options. To overcome this Boyle and Lau (1994) ensured that the barrier lay on lattice nodes.

The extra flexibility of the trinomial tree approach is used by Ritchken (1995) to ensure that lattice nodes line up with the barrier for double barriers and curved barriers, however, a large number of timesteps were still required when the initial price was close to the barrier. Cheuk and Vorst (1996b) improved on Ritchken's method by including a time dependent shift in the trinomial lattice. Figlewski and Gao (1999) introduced an adaptive mesh method that increased the number of lattice nodes near to the barrier. This greatly reduced the number of timesteps required for accurate pricing. Ahn et al. (1999) extended this approach to deal with discretely monitored barrier options. Other papers using lattice methods include Derman et al. (1995); Rogers and Stapleton (1998); Rogers and Zane (1997).

Many papers have studied different stochastic processes for the underlying asset. Jump diffusion models were considered in Leisen (1999). Barrier options priced under the CEV process using a trinomial lattice by Boyle and Tian (1999) A general diffusion process on a trinomial lattice was considered by Tian (1999). Barrier swaptions were considered in Cheuk and Vorst (1996a).

2.5.2.2 Finite differences

Explicit finite difference methods were used to price barrier options by Boyle and Tian (1998), implicit methods were used in Zvan et al. (2000).

Finite element methods have been used for discretely monitored two asset barrier options in Pooley et al. (2000). The continuously monitored case is considered in Topper (1998).

2.5.2.3 Monte Carlo

When using Monte Carlo methods to price barrier options it is important to distinguish between the case of continuously and discretely monitored barriers.

When the financial option being simulated involves a barrier feature, such as an option that knocks out if a barrier level is reached during the life of the option a bias is introduced. This bias is due to the fact that the stochastic process has to be discretised into a finite number of time steps. At each time there is a chance that the continuous process crossed the barrier when the discretization says the barrier was not touched. A correction for this bias was given in Andersen and Brotherton-Ratcliffe (1996) (see also El Babsiri and Noel (1998) and Beaglehole et al. (1997)).

These results are extended to double barriers by Baldi et al. (1999). This method has been extended to the jump diffusion model by Metwally and Atiya (2002). Barone-Adesi and Sorwar (2002) extended the approach to interest rate derivatives. Ribeiro and Webber (2003) use a gamma bridge to correct for the bias under the variance-gamma process (Madan et al., 1998). Bias corrections for multi–asset barriers are considered by Shevchenko (2003).

Related to this issue is the fact that in the financial marketplace most options with barrier features are indeed discretely monitored. For example, a knock-out call option may only knock out if the asset closing price exceeds the barrier at the close of business. Monitoring the barrier daily implies a significant price difference to the continuous monitoring case, as noted by Cheuk and Vorst (1996b), corrections to the continuous analytic solution to account for this effect are given in Broadie et al. (1997b) (see also Hörfelt (2003)). This effect is investigated via matched asymptotic expansions in Howison and Steinberg (2005).

As mentioned above, the discontinuities in hedge parameters cause difficulties for numerical methods for barrier options. The standard way to calculate the delta of an option when using Monte Carlo simulation is to run two Monte Carlo simulations with the same random numbers and use a finite difference approximation to the derivative. Recently there has been much work on calculating the Greeks from Monte Carlo simulation by Malliavin calculus, initiated by Fournié et al. (1999) (see also Fournié et al. (2001), Benhamou (2003) and Bernis et al. (2003)). Benhamou (2003) showed that the optimal Malliavin weight coincides with the maximum likelihood ratio method of Broadie and Glasserman (1996).

2.5.3 High dimensional American options

Pricing American options on multiple underlying assets, or with multiple risk factors is a difficult problem. Many methods have been proposed, but none is perfect.

An important simulation method for American options was Tilley (1993). This and other early methods such as Barraquand and Martineau (1995); Bossaerts (1989); Broadie and Glasserman (1997, 2004); Fu and Hu (1995) are reviewed by Boyle et al. (1997). Since then the stochastic mesh method (Broadie and Glasserman, 2004) has been modified to use low-discrepancy sequences (Boyle et al., 2000). Other methods include; the parametrization of the optimal exercise boundary (Bossaerts, 1989; Fu and Hu, 1995; Ibáñez and Zapatero, 2001), a quantization tree algorithm (Bally et al., 2005), wavelets (Dempster et al., 2000; Dempster and Eswaran, 2001), sparse grids (Reisinger, 2004), and an irregular grid approximation (Berridge and Schumacher, 2002). Regression methods include Carriere (1996); Longstaff and Schwartz (2001); Tsitsiklis and Van Roy (2001). Primal-Dual representations of the American option problem allow both an upper and lower bound to be calculated (Andersen and Broadie, 2004; Kogan and Haugh, 2004; Rogers, 2002). Papers using a Malliavin calculus approach include (Bally et al., 2003; Fournié et al., 2001, 1999; Lions and Regnier, 2001). A comparison of some approaches can be found in (Fu et al., 2001).

2.5.3.1 Correlation

When modelling multi–asset options it is necessary to consider how the underlying assets move in relation to each other. We specify the correlation and covariance of the assets. However, due to difficulties with the data the matrix that results may not be a valid correlation matrix. Some eigenvalues may be negative (Jäckel, 2002). A reliable solution to this problem is given by Higham (2002). This method involves alternating a projection onto the space of matrices with unit diagonal, and a projection onto the space of symmetric matrices. For the remainder of this thesis we assume that we have a valid correlation matrix.

2.5.3.2 Trees and lattices

The binomial tree (Cox et al., 1979; Rendleman and Bartter, 1979) and other lattice methods have been extended to price multi–asset options. Boyle (1988) used a trinomial tree for two underlying variables. This method was modified to deal with k underlyings in Boyle et al. (1989). Kamrad and Ritchken (1991) use a more general trinomial tree method for k underlying state variables. Ekvall (1996) extends the method of Rendleman and Bartter (1979). Gamba and Trigeorgis (2004) extends the method of Trigeorgis (1991), and shows that the method converges monotonically, allowing the efficient use of Richardson extrapolation. However, none of these methods are effective for options on more than three underlying assets.

2.5.3.3 Finite differences

Two dimensional finite difference schemes have been used to price American options under stochastic volatility (Clarke and Parrott, 1999). Zvan et al. (1998) solve this problem using finite elements and a penalty method. Nielsen et al. (2000) use a penalty method to solve American options depending on two underlying assets.

Recently some techniques have been proposed to extend grid based methods beyond two dimensions. Berridge and Schumacher (2002) proposes an irregular grid approximation. Wavelets are used in Dempster et al. (2000); Dempster and Eswaran (2001). Sparse grid methods are proposed by Reisinger (2004). This method builds up a hierarchy of grids that can be solved in parallel, and then the solutions are combined to give a solution to the original problem.

2.5.3.4 Monte Carlo

An important simulation method for American options was Tilley (1993). This encouraged people to investigate the possibility of pricing American options using Monte Carlo methods. This and other early methods such as (Barraquand and Martineau, 1995; Bossaerts, 1989; Broadie and Glasserman, 1997, 2004; Fu and Hu, 1995) are reviewed in Boyle et al. (1997). A comparison of methods is made in Fu et al. (2001). Many of these methods are also explained in Glasserman (2004) and Tavella (2002).

It is important to understand the biases in simulation methods for pricing American options. Many methods work by approximating the optimal exercise boundary, and terminating paths dependent on this exercise policy. This general method can introduce both low and high biases.

High bias is introduced by using information from the future in simulated paths to make the exercise decision. In real life future information is not available, so this introduces a high bias.

Low bias is introduced by choosing to exercise by using a sub-optimal exercise policy. Resimulating using independent paths and using the sub-optimal exercise strategy gives an estimate for the option value that has definite low bias.

2.5.3.5 Parametric approximations

The unknown location of the optimal exercise boundary is a major problem when using Monte Carlo simulation to price American options. One approach to this problem is to make a parametrization of an approximate exercise boundary, and then to optimize the boundary over the parameters.

For a single asset we saw that approximating the exercise boundary by a piecewise exponential curve (Ju, 1998), or a four piece linear spline (AitSahlia and Lai, 2001), gives a good approximation for the American option price. However, the structure of the exercise boundary in higher dimensions is not simple, as investigated by Broadie and Detemple (1997); Villeneuve (1999).

If a parametrization can be found we have to consider bias in the estimator. As the optimization process involves using future information a high bias is introduced. Estimating a sub-optimal exercise policy introduces a low bias. Overall we have a mixed bias, therefore it may be worthwhile to use the estimated exercise policy with an independent set of paths to find a low biased estimator.

Andersen (2000) uses a parametric approach to value Bermudan swaptions in the Libor Market Model. Garcia (2003) uses a parametrization method. This approach is also used in Fu and Hu (1995) and Ibáñez and Zapatero (2001). A survey is given in Glasserman (2004).

2.5.3.6 Markov chain methods

Carr and Yang (2000) extend the Markov Chain method of Barraquand and Martineau (1995) to the Heath–Jarrow–Morton (HJM) setting for pricing callable bond options. Bally and Pagès (2000); Bally et al. (2005) also use state-space partitioning, they call their method a quantization tree algorithm. Kay et al. (2002) have a related method, and also use an iterated integral for two asset American options, extending the method of Geske and Johnson (1984).

2.5.3.7 Stochastic tree methods

Broadie and Glasserman (1997) suggests using a bushy tree to estimate American option prices. This method generates both high biased and low biased estimates, both of which converge to the true value. The amount of work grows exponentially with the number of exercise opportunities, so is still computationally expensive. Broadie, Glasserman, and Jain (1997a) improved the approach. Recently Tomek (2004) modified this approach to limit the computational burden. This method is discussed in Glasserman (2004).

2.5.3.8 Stochastic mesh methods

The stochastic mesh method (Broadie and Glasserman, 2004) can be thought of as a recombining stochastic tree. This limits the computational burden. The method has been modified to use low-discrepancy sequences (Boyle et al., 2000). Broadie and Yamamoto (2003) improves the performance of the stochastic mesh method by using a fast Gauss transform. Avramidis and Hyden (1999) and Avramidis et al. (2000) also propose improvements to the method.

2.5.3.9 Regression based methods

Regression methods include Longstaff and Schwartz (2001), Carriere (1996) and Tsitsiklis and Van Roy (2001). The method of Longstaff and Schwartz (2001) has been particularly popular and has been investigated by a number of authors (Moreno and Navas, 2003; Stentoft, 2004). Proof of convergence is given in Clément et al. (2001). Chaudhary (2005) and Lemieux (2004) have applied quasi-random sequences to the valuation of American options.

Picazo (2002) notes that regression methods try to approximate too much detail. It is only necessary to decide between two options, exercise, or not, and that does not require full knowledge of the functional form of the continuation value.

2.5.3.10 Duality methods

Primal-dual representations of the American option problem allow both an upper and lower bound to be calculated (Andersen and Broadie, 2004; Kogan and Haugh, 2004; Rogers, 2002). These methods are also described in Glasserman (2004). Strategies for choosing a good martingale for the method in Rogers (2002) are investigated in Lamper and Howison (2003).

Glasserman and Yu (2004) present theoretical support to suggest that using martingale basis functions in the regression may give more accurate price estimates. However, they do not suggest any such basis functions, and they do not evaluate their suggestions numerically.

2.5.3.11 Variance reduction techniques

All Monte Carlo methods can benefit from variance reduction techniques. They are essential for implementing practical Monte Carlo methods. For detail on antithetic variates, control variates, importance sampling, and other methods, we refer the reader to Boyle et al. (1997), Glasserman (2004) and Jäckel (2002).

2.5.3.12 Quasi Monte Carlo

Quasi Monte Carlo methods have been shown to be very useful in finance. Paskov and Traub (1994) used the method. Details are given in Glasserman (2004), Jäckel (2002) and Tavella and Randall (2000). As noted above Chaudhary (2005) and Lemieux (2004) have applied quasi-random sequences to the valuation of American options.

2.6 Software issues

If a technique is to be of practical use we need to consider the complexity of the algorithm, and theoretical convergence properties. However, we also need to consider the practical implementation of the algorithm. This can have a major effect on the overall speed.

As well as speed to run, we also need to consider speed of development, and code maintenance. There are many differences between development for academic purposes and development for real world applications. However, as the field of Mathematical Finance come to use more computational techniques academics would do well to learn from practitioners in this area.

2.6.1 C++

In investment banks and quantitative teams most numerical work is done in the programming language C++ (Stroustrup, 1997). C++ is a very powerful, flexible and general language. Algorithms may be prototyped in Matlab (2004) or Mathematica (2004), but production systems are not usually run on these platforms.

Other general purpose languages, such as Java, C# and Fortran have their uses. However, as C++ is dominant and extremely versatile, we implemented models using it. C++ has been standardised, and many companies, (and open-source projects) implement standards compliant compilers. This, coupled with the massive installed base of C++ systems, make C++ a safe bet for implementing analytics libraries.

2.6.2 Design patterns

Design patterns allow discussions on design to take place at a level above the actual code. Often used solutions to particular problems are written down, as a toolbox to be used, and reused, by future developers facing similar design problems. Gamma et al. (1995) is the best known book on design patterns. However, there are now many books available (Buschmann et al., 1996; Schmidt et al., 1996). Design patterns books are now being written specifically for the problems that arise in financial derivatives systems (Duffy, 2004; Joshi, 2004).

2.6.3 QuantLib

While it is possible to design and write new code independently it is easier to use and build upon existing projects, if these projects are well implemented. QuantLib is such a derivatives pricing library, implemented in C++ . As the project website (QuantLib, 2004) says

> The QuantLib project is aimed at providing a comprehensive software framework for quantitative finance. QuantLib is a free/open–source library for modelling, trading, and risk management in real-life.

QuantLib is released under the modified BSD license (QuantLib, 2004), this enables anyone to modify and redistribute the code. As discussed in Appendix D and Firth (2004a,c) there are many advantages to open-source projects. Working with an established base of code enables researchers to focus on interesting research problems, rather than getting each new student to recode existing models.

Working code will be verified, tested, and adopted more quickly than academic papers alone. Academics researching in computational areas such as mathematical finance can greatly increase the impact of their work if both their papers and software implementations are available on the internet (Lawrence, 2001).

2.6.4 Parallel computational methods

There are only a few papers specifically on the parallel implementation of derivative pricing models (Avramidis et al., 2000; Pauletto, 2000). We did not make extensive use of parallel computation, so we discuss parallel methods no further. Parallel quasi–Monte Carlo methods are considered in Okten and Srinivasan (2002).

2.7 Conclusions

There is a vast body of literature representing research into single asset American option pricing. This problem has been formulated in many ways and many numerical methods have been devised. However, it remains an open problem as does pricing multi–asset American options.

Though much research has been done there are three areas in the literature that are missing:

1. faster and more efficient Monte Carlo methods for estimating upper bounds for American basket options

2. a good understanding of the asymptotics of multi–asset American options

3. analytic solutions for multi–asset options

Chapter 3

High Dimensional American Options

3.1 Introduction

As we saw in Chapter 2, Glasserman and Yu (2004) investigate the relative merits of 'regression now' versus 'regression later'. 'Regression now' involves using basis functions defined at the current time step and regressing discounted option values from the next period. 'Regression later' uses option values and basis functions defined one time step ahead. Glasserman and Yu (2004) prove a theorem indicating that, under certain conditions, 'regression later' should produce more accurate estimates than 'regression now'. This involves using basis functions that are martingales in the regression. However, they do not suggest specific basis functions and do not implement their method. In this chapter we propose basis functions that are martingales under geometric Brownian motion, and implement and evaluate the 'regression later' scheme.

Primal-dual representations of the American option problem allow both an upper and lower bound to be calculated (Andersen and Broadie, 2004; Kogan and Haugh, 2004; Meinshausen and Hambly, 2004; Rogers, 2002). However, the upper bound involves calculating an expectation, which has been done using another Monte Carlo simulation. This "simulation on simulation" is computationally expensive. Using basis functions that are martingales in the regression allows the immediate calculation of the required conditional expectation. Again, there is no numerical evaluation of this idea in Glasserman and Yu (2004). Therefore, in this chapter we evaluate the speed and accuracy of using martingale basis functions to find upper bounds for American option prices.

This chapter is structured as follows: we formulate the American option pricing problem, following Glasserman and Yu (2004), as an optimal stopping problem whose solution can be found by dynamic programming, in Section 3.2. In Section 3.3 we present algorithms for the implementation of the dynamic programming problem, as well as theoretical statements on the accuracy of the approximation. In Section 3.4 we state Theorem 1 from Glasserman and Yu (2004), which suggests that 'regression later' may produce more accurate estimates than 'regression now'. We construct regression basis functions that are martingales under geometric Brownian motion in section 3.5 to enable us to test Theorem 1. In Sections 3.6 and 3.7 we present the theory and algorithms for calculating low and high biased estimates for American option prices, respectively. The software implementation is discussed in Section 3.8. Numerical results are presented in Section 3.9.

3.2 Problem formulation

We use the notation of Glasserman and Yu (2004) and formulate the American option pricing problem as an optimal stopping problem. X_0 is the fixed initial financial information, $X_0, X_1, \ldots, X_m$ is a $\mathbb{R}^d$ valued Markov chain describing all relevant financial information. If exercised at time i, $i = 0, 1, \ldots, m$, the option pays out $h_i(X_i)$, each h_i being a function from $\mathbb{R}^d$ into $[0, \infty)$. $\mathcal{T}_i$ denotes the set of randomized stopping times taking values in $\{i, i+1, \ldots, m\}$. We define

$$V_i^*(x) = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}\left[h_\tau(X_\tau) \,|\, X_i = x\right], \quad x \in \mathbb{R}^d, \quad i = 0, 1, \ldots, m, \tag{3.1}$$

where $V_i^*(x)$ is the value of the option at time i and in state x. All expectations are under the risk neutral measure. Note that V^* indicates the true option value. As noted in Glasserman and Yu (2004) restricting τ to be an ordinary stopping time, such that each event $\{\tau = i\}$ be determined by $X_1, \ldots, X_i$, does not allow for stopping rules estimated through simulation. Therefore we allow randomized stopping times to depend on other random variables, independent of $X_{i+1}, \ldots, X_m$. We want to find $V_0^*(X_0)$. The option value can be written

$$V_m^*(x) = h_m(x) \tag{3.2}$$

$$V_i^*(x) = \max\left(h_i(x), \mathbb{E}\left[V_{i+1}^*(X_{i+1}) | X_i = x\right]\right), \tag{3.3}$$

$i = 0, 1, \ldots, m-1$. The dynamic programming equations can also be written in terms of the continuation value

$$C_i^*(x) = \mathbb{E}\left[V_{i+1}^*(X_{i+1})|X_i = x\right] , \quad i = 0, 1, \ldots, m-1 ,$$

as

$$C_m^*(x) = 0 \tag{3.4}$$

$$C_i^*(x) = \mathbb{E}\left[\max\left(h_{i+1}(X_{i+1}), C_{i+1}^*(X_{i+1})\right)|X_i = x\right] , \tag{3.5}$$

$i = 0, 1, \ldots, m-1$. The option values satisfy

$$V_i^*(x) = \max\left(h_i(x), C_i^*(x)\right) .$$

As in Glasserman and Yu (2004), we absorb discount factors into the definitions of X_i and h_i. In algorithms we include discounting explicitly.

3.3 Approximate dynamic programming

As in Glasserman and Yu (2004) consider approximations using time dependent basis functions $\psi_i(X_i)$ that are functions from $\mathbb{R}^d$ to $\mathbb{R}$. We approximate (3.2)–(3.3) and (3.4)–(3.5) as follows. Consider approximations of the form

$$V_i^*(x) \approx \sum_{k=0}^{K} \beta_{ik}\psi_{ik}(x)$$

and

$$C_i^*(x) \approx \sum_{k=0}^{K} \gamma_{ik}\psi_{ik}(x)$$

for some constants β_{ik} and γ_{ik}. We approximate these coefficients by projection onto the span of $\psi_{ik}(X_i)$, $k = 0, 1, \ldots, K$. We follow Glasserman and Yu (2004) and define, for any square–integrable random variable Y, the projection

$$\Pi_i Y = \mathbb{E}\left[Y\psi_i\left(X_i\right)^T\right] \left(\mathbb{E}\left[\psi_i(X_i)\,\psi_i(X_i)^T\right]\right)^{-1}$$

as

$$\Pi_i Y = \sum_{k=0}^{K} a_k \psi_{ik}(X_i) , \tag{3.6}$$

where

$$(a_0, \ldots, a_K) = \mathbb{E}\left[Y \psi_i\left(X_i\right)^T\right] \left(\mathbb{E}\left[\psi_i(X_i)\, \psi_i(X_i)^T\right]\right)^{-1}, \tag{3.7}$$

and the residual $Y - \Pi_i Y$ is uncorrelated with the basis functions. We also write the function defined by the coefficients (3.7), as in Glasserman and Yu (2004),

$$(\Pi_i Y)(x) = \sum_{k=0}^{K} a_k \psi_{ik}(x)\,.$$

Following Glasserman and Yu (2004) we impose the condition
(**C1**). For each $i = 1, \ldots, m$, $\psi_{i0} \equiv 1$, $\mathbb{E}\left[\psi_{ik}(X_i)\right] = 0$, $k = 1, \ldots, K$, and

$$\mathbb{E}\left[\psi_i(X_i)\, \psi_i(X_i)^T\right] = \begin{pmatrix} 1 & & & & \\ & \sigma_{i1}^2 & & & \\ & & \sigma_{i2}^2 & & \\ & & & \ddots & \\ & & & & \sigma_{iK}^2 \end{pmatrix},$$

with $0 < \sigma_{il}^2 < \infty$ for all i, l.

This condition ensures that the matrix is finite and non-singular. The basis functions must be linearly independent and have finite variance. The requirement that they also be uncorrelated can be achieved through a linear transformation.

3.3.1 Regression now

Glasserman and Yu (2004) interpret approximate methods as regressions in this framework (see also Glasserman (2004)). In particular, they distinguish between 'regression now' and 'regression later'. In 'regression now' we perform the regression at the current time step. In 'regression later' we perform the regression at the next time step.

We define an approximation to (3.4)–(3.5) by

$$\begin{aligned} C_m(x) &= 0 && (3.8)\\ C_i(x) &= \left(\Pi_i \max\left(h_{i+1}(X_{i+1}), C_{i+1}(X_{i+1})\right)\right)(x)\,. && (3.9) \end{aligned}$$

We use (3.6) and write the continuation value as a linear combination of basis functions

$$C_i(x) = \sum_{k=0}^{K} \beta_{ik} \psi_{ik}(x)\,,$$

where the coefficients β_{ik} are defined as in (3.7) with Y replaced by

$$V_{i+1}(X_{i+1}) \equiv \max\left(h_{i+1}(X_{i+1}), C_{i+1}(X_{i+1})\right) .$$

Glasserman and Yu (2004) define the residual ε_{i+1} by writing

$$V_{i+1}(X_{i+1}) = \sum_{k=0}^{K} \beta_{ik}\psi_{ik}(X_i) + \varepsilon_{i+1} ,$$

and prove that under the condition
(**C2**). For all $i = 0, \dots, m-1$,

$$\mathbb{E}\left[\varepsilon_{i+1} \,|\, X_i\right] = 0 ,$$

the approximate dynamic programming problem (3.8)–(3.9) is exact.

Approximation If (C2) does not hold we can use a sample estimation of the projection through simulation. We denote the approximation to the option value V_i^* by $\hat{V}_i$, and the approximation to the continuation value C_i^* by $\hat{C}_i$. For $j = 1, \dots, N$ let $(X_{1j}, \dots, X_{mj})$ be independent replications of the underlying Markov chain. Set $\hat{C}_m = 0$ and

$$\hat{C}_i(x) = \sum_{k=0}^{K} \hat{\beta}_{ik}\psi_{ik}(x) ,$$

where $\hat{\beta}_i^T = (\hat{\beta}_{i0}, \dots, \hat{\beta}_{iK})$ is the vector of regression coefficients

$$(\hat{\beta}_{i0}, \dots, \hat{\beta}_{iK}) = \left(\sum_{j=1}^{N} \hat{V}_{i+1}(X_{i+1,j})\, \psi_i(X_{ij})^T\right) \left(\sum_{j=1}^{N} \psi_i(X_{ij})\, \psi_i(X_{ij})^T\right)^{-1} .$$

Also

$$\hat{V}_{i+1} = \max(h_{i+1}, \hat{C}_{i+1}) ,$$

$i = 0, \dots, m-1$. The initial state X_0 is fixed, we set

$$\hat{C}_0(X_0) = \frac{1}{N} \sum_{j=1}^{N} \hat{V}_1(X_{1j})$$

and $\hat{V}_0(X_0) = \max(h_0(X_0), \hat{C}_0(X_0))$. As this method uses future information, but also estimates a sub–optimal exercise strategy the estimate has mixed bias. Estimators with definite low bias and definite high bias will be given in Sections 3.6 and 3.7, respectively. Convergence results are given in Clément et al. (2001) and Tsitsiklis and Van Roy (2001).

Algorithm 1 We make the method concrete by describing the computation in Algorithm 1. This describes the Least Squares Monte Carlo (LSMC) method of Longstaff and Schwartz (2001) in this setting, as well as the method developed in Tsitsiklis and Van Roy (2001). In Glasserman and Yu (2004) discounting is not explicitly considered. We explicitly denote the discount factor between times i and $i+1$ by $D_{i,i+1}$. In the case of a continuously compounded constant short interest rate r the discount factor is given by $D_{i,i+1} = e^{-r(t_{i+1}-t_i)}$. The option exercise value, h_i, is denominated in time i dollars. The asset price tensor $x_{ij\lambda}$ is the price of asset λ at time i for path j.

Algorithm 1 Least Squares Monte Carlo — Regression Now

Choose the number of asset paths, N, to simulate, $j = 1, \ldots, N$
$x_{ij\lambda}$ generate the asset price tensor (e.g. under geometric Brownian motion) from initial data X_0
$\hat{V}_j \leftarrow h_m(x_{mj\lambda})$ evaluate the final time payoff for each path
while Loop backwards through time steps from $i = m-1$ to 1 **do**
  $\hat{V}_j \leftarrow D_{i,i+1}\hat{V}_j$ discount the estimated option value to the current time
  $A_{jk} \leftarrow \psi_{ik}(x_{ij\lambda})$ the design matrix
  $\hat{\beta}_k \leftarrow (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\hat{V}_j$ the regression
  $\hat{C}_j \leftarrow A_{jk}\hat{\beta}_k$ the estimated continuation value
  $h_j \leftarrow h_i(x_{ij\lambda})$ the value of immediate exercise (payoff) for each path
  if $h_j > \hat{C}_j$ **then**
    $\hat{V}_j \leftarrow h_j$
  end if
end while
$\hat{C}_0 \leftarrow \sum D_{0,1}\hat{V}_j/N$ final discount step and initial continuation value
$\hat{V}_0 \leftarrow \max(h_0(X_0), \hat{C}_0)$ the estimated option value

3.3.2 Regression later

Glasserman and Yu (2004) interpret the stochastic mesh method of Broadie et al. (2000b) as a regression method, but with the regression taking place one time step ahead. They refer to this as ‘regression later’. To do this Glasserman and Yu (2004) introduce the idea of using basis functions in the regression which are martingales. This condition is expressed as
(**C3**). Martingale property :

$$\mathbb{E}\left[\psi_{i+1}(X_{i+1}) \,|\, X_i\right] = \psi_i(X_i)\,, \qquad i = 0, 1, \ldots, m-1\,.$$

We write the continuation value as a linear combination of basis functions

$$C_i^+(x) = \sum_{k=0}^{K} \gamma_{ik}\psi_{ik}(x) \quad i = 1, \dots, m-1\,.$$

Also $V_{i+1}^+ = \max(h_{i+1}, C_{i+1}^+)$, $i = 0, \dots, m-1$. We define the residual ε_{i+1}^+ through (Glasserman and Yu, 2004)

$$V_{i+1}^+(X_{i+1}) = \sum_{k=0}^{K} \gamma_{ik}\psi_{i+1,k}(X_{i+1}) + \varepsilon_{i+1}^+$$

with $\varepsilon_{i+1}^+ = V_{i+1}^+(X_{i+1}) - \Pi_{i+1}V_{i+1}^+(X_{i+1})$ uncorrelated with the components of $\psi_{i+1}(X_{i+1})$. Consider the conditions:

(**C4**). For all $i = 0, \dots, m-1$ and $k = 0, \dots, K$,

$$\mathbb{E}\left[\varepsilon_{i+1}^+\left(\psi_{i+1,k}(X_{i+1}) - \psi_{ik}(X_i)\right)\right] = 0\,.$$

(**C4′**). For all $i = 0, \dots, m-1$,

$$\mathbb{E}\left[\varepsilon_{i+1}^+ \,|\, X_i\right] = 0\,.$$

Glasserman and Yu (2004) prove that if (C3) and (C4) hold then $C_i^+ = C_i$ for all i. This means that the solution satisfies the approximate dynamic programming problem (3.8)–(3.9) exactly. They also prove that if (C3) and (C4') hold then in addition $C_i^+ = C_i^*$ for all i.

Approximation We follow Glasserman and Yu (2004) and write $\hat{V}^+(x)$ and $\hat{C}^+(x)$ to indicate that the estimates have been found using 'regression later'. As in the 'regression now' case above, for $j = 1, \dots, N$ let $(X_{1j}, \dots, X_{mj})$ be independent replications of the underlying Markov chain. Define $\hat{C}_m^+ = 0$. Under this assumption Glasserman and Yu (2004) prove

$$\hat{C}_i^+(x) = \sum_{k=0}^{K} \hat{\gamma}_{ik}\psi_{ik}(x) \quad i = 1, \dots, m-1\,,$$

converges to the true value, where $\hat{\gamma}_i^T = (\hat{\gamma}_{i0}, \dots, \hat{\gamma}_{iK})$ is the vector of regression coefficients,

$$(\hat{\gamma}_{i0}, \dots, \hat{\gamma}_{iK}) = \left(\sum_{j=1}^{b} \hat{V}_{i+1}^+(X_{i+1,j})\,\psi_{i+1}(X_{ij})^T\right)\left(\sum_{j=1}^{b} \psi_{i+1}(X_{ij})\,\psi_{i+1}(X_{ij})^T\right)^{-1}.$$

Note that the regression coefficients are estimated using later basis functions ψ_{i+1}, rather than ψ_i. Also $\hat{V}_{i+1}^+ = \max(h_{i+1}, \hat{C}_{i+1}^+)$, $i = 0, \dots, m-1$.

Algorithm 2 Algorithm 2 details the computation. Note that the discounting of the estimated continuation value is done implicitly using the martingale condition (C3) on the basis functions. However, we have not yet defined any martingale basis functions. The implementation can be made more efficient by storing the A_{jk} from the previous time step for use in the next time step.

Algorithm 2 Least squares Monte Carlo — regression later

Require: Basis functions $\psi_{ik}(x)$ are martingales satisfying condition (C3)
  Choose the number of asset paths, N, to simulate, $j = 1, \dots, N$
  $x_{ij\lambda}$ generate the asset price tensor (e.g. under geometric Brownian motion) from initial data X_0
  $\hat{V}_j^+ \leftarrow h_m(x_{mj\lambda})$ evaluate the final time payoff for each path
  while Loop backwards through time steps from $i = m - 1$ to 1 **do**
    $A_{jk} \leftarrow \psi_{i+1,k}(x_{i+1,j\lambda})$ the design matrix - later
    $\hat{\gamma}_k \leftarrow (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\hat{V}_{i+1,j}^+$ the regression
    $A_{jk} \leftarrow \psi_{ik}(x_{ij\lambda})$ the design matrix - now
    $\hat{C}_j^+ \leftarrow A_{jk}\hat{\gamma}_k$ the estimated continuation value
    $h_j \leftarrow h_i(x_{ij\lambda})$ the value of immediate exercise (payoff) for each path
    if $h_j > \hat{C}_j^+$ **then**
      $\hat{V}_j^+ \leftarrow h_j$
    else
      $\hat{V}_j^+ \leftarrow D_{i,i+1}\hat{V}_j^+$ discount the estimated option value to the current time
    end if
  end while
  $\hat{C}_0^+ \leftarrow \sum D_{0,1}\hat{V}_j^+/N$ final discount step and initial continuation value
  $\hat{V}_0^+ \leftarrow \max(h_0(X_0), \hat{C}_0^+)$ the estimated option value

---

3.4 Comparison

To state Theorem 1 from Glasserman and Yu (2004) we need to define stronger conditions on the residuals:
(**C5a**).

$$\mathbb{E}\left[\varepsilon_{i+1}^+ \,|\, \psi_{i+1}(X_{i+1})\right] = 0 \quad \text{and} \quad \mathbb{E}\left[(\varepsilon_{i+1}^+)^2 \,|\, \psi_{i+1}(X_{i+1})\right] = \text{var}\left[\varepsilon_{i+1}^+\right]$$

(**C5b**).

$$\mathbb{E}\left[\varepsilon_{i+1} \,|\psi_i(X_i)\right] = 0 \quad \text{and} \quad \mathbb{E}\left[(\varepsilon_{i+1})^2 \,|\, \psi_i(X_i)\right] = \text{var}\left[\varepsilon_{i+1}\right]$$

We also define the coefficients of determination

$$\begin{aligned} R^2_\beta &= \operatorname{var}\left[\beta_i^T \psi_i(X_i)\right] / \operatorname{var}\left[V_{i+1}(X_{i+1})\right] \\ R^2_\gamma &= \operatorname{var}\left[\gamma_i^T \psi_{i+1}(X_{i+1})\right] / \operatorname{var}\left[V_{i+1}(X_{i+1})\right] . \end{aligned}$$

Informally we can think of the coefficient of determination as giving an indication of the percentage of the variation explained by the regression. Therefore a higher coefficient of determination indicates a better regression fit.

We write the covariance matrix of $\hat{\beta}$ as $\operatorname{cov}[\hat{\beta}]$ and let $\Sigma_\beta = \lim_{N\to\infty} N\operatorname{cov}[\hat{\beta}]$ whenever the limit exists. Similarly denote the limiting covariance matrix of $\hat{\gamma}$ by Σ_γ. The existence of these limits is implied by the following uniform integrability conditions:

(**C6**). As $N \to \infty$,

$$N\,\mathbb{E}\left[\left(\sum_{j=1}^{N} \psi_i(X_{ij})\,\psi_i(X_{ij})^T\right)^{-1}\right] \to \left(\mathbb{E}\left[\psi_i(X_i)\,\psi_i(X_i)^T\right]\right)^{-1}$$

and

$$N\,\mathbb{E}\left[\left(\sum_{j=1}^{N} \psi_{i+1}(X_{i+1,j})\,\psi_{i+1}(X_{i+1,j})^T\right)^{-1}\right] \to \left(\mathbb{E}\left[\psi_{i+1}(X_{i+1})\,\psi_{i+1}(X_{i+1})^T\right]\right)^{-1} .$$

We can now state Theorem 1 of Glasserman and Yu (2004)

Theorem 1 *If (C1) and (C3)–(C4) hold, then* $R^2_\beta \leq R^2_\gamma$. *If also (C5)–(C6) hold then* $\Sigma_\gamma \leq \Sigma_\beta$.

Note that this only holds over a single time step.

3.5 Martingale basis functions

Particular martingale basis functions satisfying (C3) are not given by Glasserman and Yu (2004). We suggest basis functions that are martingales under geometric Brownian motion.

3.5.1 One dimensional martingale basis functions

We want to find basis functions that satisfy the martingale property (C3). Let X_i be a scalar describing the evolution of a single asset. The simplest basis function is a constant, we see immediately this satisfies the condition (C3)

$$\mathbb{E}\left[1|X_i\right] = 1\,.$$

To consider what happens for other basis functions we need to specify a model for the asset price process. The risk–neutral asset price process follows the stochastic differential equation (SDE)

$$\frac{dX}{X} = (r-q)\,dt + \sigma\,dW$$

where r is the constant risk free interest rate, q is the continuous dividend yield on the asset and σ is the constant volatility of asset returns. W is a standard Brownian motion. The initial asset price is X_0. We can solve the SDE exactly Glasserman (2004) to find

$$X_t = X_0 e^{(r-q-\sigma^2/2)t+\sigma\sqrt{t}N(0,1)}\,,$$

where $N(0,1)$ is the normal distribution with mean 0 and standard deviation 1. We expect the monomial

$$\psi_i(X_i) = e^{-(r-q)(t_i-t_0)}X_i\,,$$

to satisfy the martingale condition (C3) under GBM. To verify this we calculate

$$\begin{aligned}
\mathbb{E}\left[\psi_{i+1}(X_{i+1})|X_i\right] &= \mathbb{E}\left[e^{-(r-q)(t_{i+1}-t_0)}X_{i+1}|X_i\right] && (3.10)\\
&= e^{-(r-q)(t_{i+1}-t_0)}\mathbb{E}\left[X_{i+1}|X_i\right] && (3.11)\\
&= X_i\,, && (3.12)
\end{aligned}$$

as the discount factor and dividend yield are constant.

Now we find general monomial basis functions that are martingales under geometric Brownian motion. Consider monomial basis functions of the form

$$\psi_{ik}(X_i) = a_{ik}\,(X_i)^k\,,$$

where we consider each a_{ik} fixed. We investigate the condition (C3),

$$\mathbb{E}\left[\psi_{i+1,k}(X_{i+1})|X_i\right] = \mathbb{E}\left[a_{i+1,k}(X_{i+1})^k\,|\,X_i\right]\,. \qquad (3.13)$$

In general we can write

$$X_{i+\delta} = X_i\epsilon_\delta\,,$$

where

$$\epsilon_\delta = e^{(r-q)\delta}\, e^{\sigma W_\delta - \sigma^2\delta/2}\,.$$

Hence, from (3.13),

$$\begin{aligned}\mathbb{E}\left[\psi_{i+1,k}(X_{i+1})|X_i\right] &= a_{i+1,k}\,\mathbb{E}\left[X_i^k \epsilon_\delta^k \,|\, X_i\right] \\ &= a_{i+1,k}\,(X_i)^k\,\mathbb{E}\left[\epsilon_\delta^k \,|\, X_i\right]\,, \end{aligned} \tag{3.14}$$

and we will have a martingale basis function satisfying the condition (C3) if

$$a_{ik} = a_{i+1,k}\,\mathbb{E}\left[\epsilon_\delta^k \,|\, X_i\right]\,. \tag{3.15}$$

Now in general we have

$$\begin{aligned}\mathbb{E}\left[\epsilon_\delta^k\right] &= e^{k(r-q)\delta}\,\mathbb{E}\left[\left(e^{\sigma W_\delta}\right)^k\right] e^{-k\sigma^2\delta/2} \\ &= e^{k(r-q)\delta}\,\mathbb{E}\left[e^{k\sigma\sqrt{\delta}Z}\right] e^{-k\sigma^2\delta/2} \\ &= e^{k(r-q)\delta}\, e^{k^2\sigma^2\delta/2 - k\sigma^2\delta/2} \\ &= e^{k(r-q)\delta}\, e^{k(k-1)\sigma^2\delta/2}\,.\end{aligned}$$

Substituting into (3.15) we have a martingale if

$$a_{ik} = a_{i+1,k}\, e^{k(r-q)\delta}\, e^{k(k-1)\sigma^2\delta/2}\,.$$

Hence, a general formula for martingale basis functions under GBM is

$$\psi_{ik}(X_i) = (X_i)^k\, e^{-\left(k(r-q)+k(k-1)\sigma^2/2\right)(t_i-t_0)}\,. \tag{3.16}$$

The first three basis functions generated using $k = 0, 1, 2$ in this formula are,

$$\psi_{i0}(X_i) = 1 \tag{3.17}$$
$$\psi_{i1}(X_i) = X_i\, e^{-(r-q)(t_i-t_0)} \tag{3.18}$$
$$\psi_{i2}(X_i) = (X_i)^2\, e^{-(2(r-q)+\sigma^2)(t_i-t_0)}\,. \tag{3.19}$$

European options Many studies have found that including the payoff function as a basis function improves the accuracy of regression methods. We look for a general way of constructing martingales from payoff functions that does not require solving the full option pricing problem. Therefore, we fix a payoff function $h(X_i)$ and consider the basis function

$$\psi_{ik}(X_i) = e^{-r(t_m-t_i)}\,\mathbb{E}\left[h(X_m)|X_i\right]\,,$$

which is the value at time i of a European option paying $h(X_i)$ at time t_m. This can be used as a basis function satisfying (C3)

$$\begin{aligned}
\mathbb{E}\left[\psi_{i+1}(X_{i+1})|X_i\right] &= e^{-r(t_m-t_{i+1})}\,\mathbb{E}\left[\mathbb{E}\left[h(X_m)|X_{i+1}\right]\,|\,X_i\right] \\
&= e^{-r(t_m-t_{i+1})}\,e^{-r(t_{i+1}-t_i)}\,\mathbb{E}\left[h(X_m)|X_i\right] \\
&= e^{-r(t_m-t_i)}\,\mathbb{E}\left[h(X_m)|X_i\right] \\
&= \psi_{ik}(X_i)\,.
\end{aligned}$$

3.5.2 Two dimensional martingale basis functions

We consider correlated geometric Brownian motions,

$$\frac{dX_\lambda}{X_\lambda} = (r-q_\lambda)\,dt + \sigma_\lambda\,dW_\lambda \qquad \lambda = 1,2\,,$$

where r is the constant risk free interest rate, q_λ is the constant continuous dividend yield on asset X_λ, and σ_λ is the constant volatility of returns for asset X_λ. Each W_λ is a standard Brownian motion, and W_λ and W_μ have correlation ρ. The initial asset price is X_0. We can solve each SDE exactly to find the representation

$$X_\lambda(t) = X_\lambda(0)e^{(r-q_\lambda-\sigma_\lambda^2/2)t+\sigma_\lambda W(t)}\,.$$

We now consider X_i to be two dimensional, with components x and y, evolving under correlated geometric Brownian motion. Expectations are found by integrating against the joint probability density

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2}-\frac{2\rho xy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)\right]\,,$$

where σ_1 is the volatility of the variable x, σ_2 is the volatility of the variable y, and ρ is the correlation between the two variables. We want to find basis functions that satisfy the martingale property (C3)

$$\mathbb{E}\left[\psi_{i+1}(X_{i+1})|X_i\right] = \psi_i(X_i)\,,$$

in two dimensions. As in the one dimensional case the simplest basis function is a constant. The one dimensional martingale monomials suggested in the previous section remain martingales in this two dimensional setting. We require martingale basis functions that involve products of the two underlying assets.

Gerber and Shiu (1996) show, by solving a quadratic equation, that

$$\psi_i(X_i) = e^{-r(t_i - t_0)}\, X_2 \left(\frac{X_1}{X_2}\right)^{\alpha_\pm}$$

are martingales, where

$$\alpha_\pm = \omega \pm \Delta\,,$$

and

$$\omega = \tfrac{1}{2} + \frac{q_1 - q_2}{\nu^2} \quad \text{and} \quad \Delta = \sqrt{\omega^2 + \frac{2q_2}{\nu^2}}\,,$$

where

$$\nu^2 = \sigma_1^2 - \rho\sigma_1\sigma_2 + \sigma_2^2\,.$$

Other options Various multi-dimensional options can be priced easily analytically and potentially used as martingale basis functions. European options on the maximum or minimum of two asset are priced in Stulz (1982), and the results extended in Johnson (1987). A well known example is an option on the geometric average of a basket of stocks Glasserman (2004).

3.6 Low biased estimators

The LSMC estimator has undetermined bias because it both generates a sub-optimal exercise strategy and uses future information (as described in Glasserman (2004)). If we fix a sub-optimal exercise policy generated by the LSMC algorithm and perform a Monte Carlo simulation with a new set of independent simulated paths we have a low biased estimator of the American option value. The low biased estimator for the option value is denoted by $\hat{V}_i^{\downarrow}(x)$, and the high biased estimator by $\hat{V}_i^{\uparrow}(x)$.

Algorithm 3 To produce an estimate for the American option value with definite low bias we use Algorithm 1 but on each time step we store the vector of basis function coefficients $\hat{\beta}_{ik}$. We now have a matrix of coefficients β_{ik} for the coefficient of the k^{th} basis function at time step i. This matrix defines a function at each time step i for the estimated continuation value of the option. We define an exercise strategy by exercising the option if the value of immediate exercise is greater than the estimated continuation value.

Algorithm 3 Low Biased Monte Carlo for American Options — Regression Now

Require: Basis function coefficients $\hat{\beta}_{ik}$ stored from Algorithm 1
Choose the number of asset paths, N, to simulate, $j = 1, \ldots, N$
$x_{ij\lambda}$ generate the asset price tensor (e.g. under geometric Brownian motion) from initial data X_0
for j = 0 to N **do**
  for i = 0 to m **do**
    $\hat{C}_j \leftarrow \hat{\beta}_{ik}\psi_{ik}(x_{ij\lambda})$ the estimated continuation value using stored basis function coefficients
    $h_j \leftarrow h_i(x_{ij\lambda})$ the value of immediate exercise (payoff)
    if $h_j > \hat{C}_j$ **then**
      $\hat{V}_j^{\downarrow} = D_{0,i}\, h_j$ discount the payoff
      break from time loop
    end if
    if $i = m$ **then**
      $\hat{V}_j^{\downarrow} = D_{0,m}\, h_j$ exercise at the final time
    end if
  end for
end for
$\hat{V}_0^{\downarrow} \leftarrow \sum \hat{V}_j^{\downarrow}/N$ the low biased estimated option value

3.7 High biased estimators

Many methods have now been suggested to give high biased estimates of the American option value. The simplest is simulating each path and choosing the optimal exercise time. This simple estimate uses future information and is biased high. The stochastic mesh method Broadie and Glasserman (2004) gives lower and upper bounds which both converge asymptotically to the true value. Duality approaches have been suggested in a number of papers (Andersen and Broadie, 2004; Kogan and Haugh, 2004; Rogers, 2002), however these methods tend to converge even more slowly than the low biased estimators.

We view

$$\hat{C}_i^{+}(.) = \sum_{k=0}^{K} \hat{\gamma}_{ik}\, \psi_{ik}(.)\,, \qquad i = 1, 2, \ldots, m-1\,,$$

and

$$\tilde{V}_{i+1}^{+}(.) \triangleq \sum_{k=0}^{K} \hat{\gamma}_{ik}\, \psi_{i+1,k}(.)\,, \qquad i = 0, 1, \ldots, m-1\,,$$

as deterministic functions.

Define both $M_0 = 0$ and

$$M_n = \sum_{i=0}^{n-1} \left[\tilde{V}_{i+1}^{+}(X_{i+1}) - \hat{C}_i(X_i) \right] , \qquad n = 1, \dots, m \, . \tag{3.20}$$

Each summand is

$$\tilde{V}_{i+1}^{+}(X_{i+1}) - \hat{C}_i(X_i) = \sum_{k=0}^{K} \hat{\gamma}_{ik} \left[\psi_{i+1,k}(X_{i+1}) - \psi_{ik}(X_i) \right] .$$

We now quote Theorem 2 from Glasserman and Yu (2004).

Theorem 2 *If (C3) holds then*

$$V_0^{\downarrow}(X_0) = \mathbb{E}\left[h_{\hat{\tau}}(X_{\hat{\tau}})\right] \leq V_0^{*}(X_0) \leq \mathbb{E}\left[\max_{n=0,1,\dots,m} (h_n(X_n) - M_n) \right] = V_0^{\uparrow}(X_0) \, .$$

As we assume (C3) and use "regression later" we have all the information we require to calculate (3.20). However, basis functions satisfying (C3) are not suggested by Glasserman and Yu (2004).

Modified martingale The martingale defined in equation (3.20) is an estimate for the upper bound that does not take advantage of all available information. Andersen and Broadie (2004) model the duality gap, rather than the upper bound itself. Once an estimate for the lower bound has been found we can write the upper bound as

$$V_0^{\uparrow}(X_0) = V_0^{\downarrow}(X_0) + \Delta_0$$

where Δ_0 is the duality gap. If we modify the initial value of the martingale M_n so that $M_0 = V_0^{\downarrow}(X_0)$ then we can produce an estimate for the duality gap with less variance.

Algorithm 4 To produce an estimate for the American option value with definite high bias we first use Algorithm 2 and store the vector of martingale basis function coefficients $\hat{\gamma}_k$ at each time step i. This gives a matrix of coefficients $\hat{\gamma}_{ik}$. We define an exercise strategy by comparing the value of immediate exercise to the estimated continuation value. This gives a low biased estimate.

To obtain the high biased estimate we use martingale basis functions as proposed in Glasserman and Yu (2004). As the high biased estimator builds on the the previous algorithms we obtain a LSMC estimate, a low biased estimate, and a high biased estimate from the algorithm.

Algorithm 4 Low and High Biased Monte Carlo for American Options — Regression Later

Require: Basis functions $\psi_{ik}(x)$ are martingales satisfying condition (C3)
Require: Martingale basis function coefficients $\hat{\gamma}_{ik}$ stored from Algorithm 2.
Choose the number of asset paths, N, to simulate, $j = 1, \ldots, N$
$x_{ij\lambda}$ generate the asset price tensor (e.g. under geometric Brownian motion) from initial data X_0
for j = 0 to N **do**
  $M \leftarrow 0$, $U \leftarrow 0$
  for i = 0 to m **do**
    $h_j \leftarrow h_i(x_{ij\lambda})$ the value of immediate exercise
    $\hat{C}_j^+ \leftarrow \hat{\gamma}_{ik}\,\psi_{ik}(x_{ij\lambda})$ the estimated continuation value
    $\tilde{V}_j^+ \leftarrow \hat{\gamma}_{ik}\,\psi_{i+1,k}(x_{i+1,j\lambda})$ the estimated future option value
    if $h_j > \hat{C}_j^+$ **then**
      $\hat{V}_j^{\downarrow+} \leftarrow h_j$
      do not repeat
    end if
    $M \leftarrow M + \tilde{V}_j^+ - \hat{C}_j^+$
    $U \leftarrow \max(U, h_j - M)$
  end for
  $\hat{V}_j^{\uparrow+} \leftarrow U$
end for
$V_0^{\downarrow+} \leftarrow \sum \hat{V}_j^{\downarrow+}/N$ low biased estimator
$V_0^{\uparrow+} \leftarrow \sum \hat{V}_j^{\uparrow+}/N$ high biased estimator

3.8 Implementation

The methods derived in Chapter 3 were implemented in QuantLib and used to obtain the numerical results presented in this chapter. There was no existing framework from basket options in QuantLib, therefore we implemented a new `Instrument` class `basketoption.cpp`. We implemented the following `PricingEngines` to value basket options:

- `stulzengine.cpp` implements the closed form valuation formulae for European call and put options on the maximum and minimum of two underlying assets (Stulz, 1982). Results were verified against Visual Basic code implementing the algorithm[1], and values in Haug (1997, page 58). The tests verified results to an absolute accuracy of 10^{-3} using `basketoption.cpp` in the QuantLib test suite.

[1] available from `http://www.maths.ox.ac.uk/~firth/computing/excel.shtml`

- `mcamericanbasketengine.cpp` implements Algorithm 1, a regression method similar to Longstaff and Schwartz (2001) for American basket options. This was tested against in the single asset case against Table 1 appearing in Longstaff and Schwartz (2001), and for a three asset American max call against the values reported in Tavella (2002, page 198) (to a relative error accuracy of 1%), using `basketoption.cpp` in the QuantLib test suite.

- `mcamericanbasketenginelater.cpp` implements the 'regression later' approach in Algorithm 4, using basis functions that are martingales under geometric Brownian motion. Results were verified against those in Table 1 of Longstaff and Schwartz (2001) and Example 8.6.1 in Glasserman (2004), using the test file `basketoption.cpp` in the QuantLib test suite.

The basis functions were implemented as C++ classes. A standard 550 MHz Athlon PC with 256 Mb of RAM and a 1.8 GHz Intel Centrino laptop with 512 Mb of RAM were used to run the simulations.

The Monte Carlo simulations were run using the QuantLib framework. The default random number generator is the Mersenne twister random number generator (Matsumoto and Nishimura, 1998), and normally distributed random numbers were generated by inversion (Moro, 1995).

3.9 Numerical results

We ran simulations using Algorithms 1 and 2 to test the practical impact of Theorem 1. This says that 'regression later' will produce a better fit and less variable estimates of coefficients than 'regression now', which should translate into a more accurate estimate for the option value.

To test the accuracy of using martingale basis functions to produce upper bounds to the American option price, as discussed in Chapter 3, we used Algorithm 4.

3.9.1 Regression precision

Under certain conditions on the residuals, 'regression later' gives an estimator with less variance than 'regression now' (Theorem 1). In this section we test this theoretical result numerically.

First we check that 'regression later' provides good estimates for the option value. The results in Table 3.1 compare 'regression now' with 'regression later', using single asset put option test cases from Table 1 in Longstaff and Schwartz (2001). Only in-the-money paths are used in the regression. Both regressions use the first three martingale basis functions described above. We find that 'regression later' does indeed provide a good estimate for the option value. 50,000 paths and 50,000 antithetic paths are used.

Next we compare the accuracy of 'regression now' to 'regression later'. We ran the simulations 50 times with 1000 paths for each simulation. Table 3.2 shows that the estimates of the option values using 'regression later' are on average three times more accurate than estimates using 'regression now'.

3.9.2 Lower and upper bounds

We use Algorithm 4 to obtain lower and upper bounds for the American option price using Theorem 2, for single asset, two asset and five asset max call basket options.

3.9.2.1 Single asset options

We test the upper bounds for single asset options, again using single asset put options from Longstaff and Schwartz (2001). We use 1000 paths to estimate the regression coefficients, then another 1000 independent paths to estimate the lower bounds. 100 paths are used to estimate the upper bound. For the dual method based on a single step estimation of the continuation value we use either 10, or 100 paths. 100 paths giving a better upper bound. The final column shows the results of the martingale basis function method. The entire process is repeated 50 times and standard errors are shown in parentheses.

Table 3.3 shows that the results for the $n = 100$ case are comparable to the results for the martingale basis function case. However, to run the four test cases took over 3 hours for the Dual-V method with $n = 100$ sub-paths, whereas the martingale method took a mere 10 minutes. The Dual-V method with 10 sub-paths took 30 minutes to complete. Avoiding "simulation on simulation" gives a substantial speed improvement.

X_0	σ	T	Finite difference American	Closed form Euro	LSMC est.	(s.e)	Regression now	(s.e)	Regression later	(s.e)
36	0.2	1	4.478	3.844	4.472	(.010)	4.467	(.009)	4.466	(.009)
36	0.2	2	4.840	3.763	4.821	(.012)	4.835	(.011)	4.839	(.011)
36	0.4	1	7.101	6.711	7.091	(.020)	7.104	(.019)	7.100	(.019)
36	0.4	2	8.508	7.700	8.488	(.024)	8.495	(.023)	8.495	(.023)

Table 3.1: Comparison of 'regression now' and 'regression later' with the results for American style put option in Table 1 in Longstaff and Schwartz (2001). X_0 is the initial asset price, σ is the volatility of returns, and T is the number of years until the option expiry date. The continuously compounded short–term interest rate is $r = 0.06$, and the strike price of all the put options is $E = 40$. 'Regression now' uses a constant and the first three Laguerre polynomials. 'Regression later' uses a constant and the first three martingale basis functions under geometric Brownian motion. The estimates are made using 100,000 simulations (50,000 and 50,000 antithetic). (s.e) denotes the standard error of the value to the left. Only in-the-money paths were used in the regression.

				Regression now			Regression later		
X_0	σ	T	American	value	absolute error	relative error (%)	value	absolute error	relative error (%)
36	0.2	1	4.478	4.525	.047	1.050	4.496	.018	.402
36	0.2	2	4.840	4.888	.048	0.992	4.860	.020	.413
36	0.4	1	7.101	7.240	.139	1.957	7.075	.026	.366
36	0.4	2	8.508	8.629	.121	1.422	8.579	.071	.835

Table 3.2: Improved accuracy of 'regression later'. We use the same parameters as in Table 3.1, but run 1000 samples (500 and 500 antithetic) 50 times and average the estimated option value to assess the accuracy. Both regressions use a constant and the first three martingale basis functions under geometric Brownian motion. X_0 is the initial asset price, σ is the volatility of returns, and T is the number of years until the option expiry date. The continuously compounded short–term interest rate is $r = 0.06$, and the strike price of all the put options is $E = 40$. The relative error is calculated by dividing the absolute error by the true option price. Only in-the-money paths were used in the regression.

X_0	σ	T	true	LSM	Low	$n = 10$	$n = 100$	Martingale
36	0.2	1	4.478	4.499	4.451	5.276 (.01)	4.777 (.01)	4.668 (.00)
36	0.2	2	4.840	4.847	4.784	5.752 (.01)	5.228 (.01)	5.054 (.01)
36	0.4	1	7.101	7.106	7.032	8.455 (.02)	7.855 (.01)	7.732 (.02)
36	0.4	2	8.508	8.467	8.412	10.228 (.02)	9.619 (.01)	9.767 (.03)

Table 3.3: Dual estimates for a single asset option. We use the same parameters as in Table 3.1, but run 1000 samples (500 and 500 antithetic) 50 times and average the estimated option value to assess the accuracy. Both regressions use a constant and the first three martingale basis functions under geometric Brownian motion. The four test cases took 30 minutes for $n = 10$, 3 hours 40 minutes for $n = 100$, and 10 minutes for the martingale case.

				$n = 10$	$n = 100$	
X_0	Exact	LSM	Low	Dual-V	Dual-V	Martingale
90	8.08	8.07	7.99	8.91	8.17	8.60
100	13.90	13.90	13.80	15.20	14.01	14.21
110	21.34	21.32	21.16	23.28	21.54	21.40

Table 3.4: Dual estimates for a two asset American max call option from example 8.6.1 in Glasserman (2004). This is a call option on the maximum of two assets, with equal initial stock prices are strike price $E = 100$. The interest rate $r = 0.05$, dividend yields $q_1 = q_2 = 0.1$ and volatilities $\sigma_1 = \sigma_2 = 0.2$. The assets are independent. The option expires in $T = 3$ years, and can be exercised at nine equally spaced dates. Standard errors for estimates are approximately 0.01–0.02.

3.9.2.2 Two asset options

We use Example 8.6.1 from Glasserman (2004), as used in Andersen and Broadie (2004) and Broadie and Glasserman (1997, 2004). This is a call option on the maximum of two assets, X_1 and X_2. The initial stock prices are equal and the strike price is $E = 100$. The interest rate $r = 0.05$, dividend yields $q_1 = q_2 = 0.1$ and volatilities $\sigma_1 = \sigma_2 = 0.2$. The assets are independent. The option expires in $T = 3$ years, and can be exercised at nine equally spaced dates.

Glasserman (2004) use three cases, $X_i(0) = 90, 100$ and 110 for which the option values are $8.08, 13.90$ and 21.34, respectively. They calculate the exact value using the multi–asset binomial method of Boyle et al. (1989). The numerical results in Table 3.4 use the methodology from Glasserman (2004) by using 4000 paths to estimate the regression coefficient, followed by 4000 independent paths for the low biased estimate, followed by 100 paths to estimate the upper bound. The simulation is run 100 times

X_0	95% Interval	LSM	Low	$n = 10$ Dual-V	$n = 100$ Dual-V	Martingale
90	[16.602, 16.655]	16.76	16.45	19.72	19.09	21.73
100	[26.109, 26.292]	26.28	26.00	30.39	29.17	32.51
110	[36.704, 36.832]	36.89	36.54	41.71	40.10	46.29

Table 3.5: Dual estimates for a five asset American max call option, showing dual estimates using $n = 10$ and $n = 100$ subpaths (Glasserman, 2004) and martingale basis function results. The parameters are $r = 0.05$, $q_i = 0.1$, $\sigma_i = 0.2$, $\rho_{ij} = 0$ and $T = 3$. The option can be exercise at any of nine equally spaced dates. The 95% confidence interval is as reported in Andersen and Broadie (2004). Standard errors for estimates are approximately 0.04 for LSM, 0.08 for $n = 10$ runs, 0.02 for $n = 100$ runs, and 0.11 for martingale runs.

to calculate standard errors.

The basis functions used in the method of Kogan and Haugh (2004) are those from the first row in Table 8.1 of Glasserman (2004), namely 1, X_j, X_j^2 and X_j^3. For the martingale basis function method we use the equivalent functions to those above, namely 1, $X_j\, e^{-r(t_i-t_0)}$, $X_j^2\, e^{-(2(r-q)+\sigma^2)(t_i-t_0)}$, and $X_j^3\, e^{-3(r-q+\sigma^2)(t_i-t_0)}$ where $j = 1, 2$.

3.9.2.3 Five asset options

The results for an American five asset max call option are shown in Table 3.5. Again, the initial stock prices are equal and the strike price is $E = 100$. The interest rate $r = 0.05$, dividend yields are all equal $q_i = 0.1$ and volatilities $\sigma_i = 0.2$. The assets are independent. The option expires in $T = 3$ years, and can be exercised at nine equally spaced dates.

The basis functions used in the method of Kogan and Haugh (2004) are all polynomials up to order 5 on the two highest asset, i.e. X_1, X_2, X_1^2, X_2^2, X_1^3, X_2^3, X_1^4, X_1^4, X_1^5, X_2^5, and combinations X_1X_2, $X_1^2X_2$, $X_1X_2^2$, $X_1^2X_2^2$, $X_1^3X_2$, $X_1X_2^3$, $X_1^3X_2^2$, $X_1^2X_2^3$, $X_1^4X_2$ and $X_1X_2^4$. For the martingale basis function method we use the same basis functions as in the two asset case, namely 1, $X_j\, e^{-r(t_i-t_0)}$, $X_j^2\, e^{-(2(r-q)+\sigma^2)(t_i-t_0)}$, and $X_j^3\, e^{-3(r-q+\sigma^2)(t_i-t_0)}$ where $j = 1, 2$. This reduced set of basis functions does not produce a tight upper bound. Including martingale basis functions on a combination of assets, and the two asset European max call option should produce tighter bounds.

We used 4000 paths to estimate the regression coefficients, 4000 paths to estimate the lower bound and 100 paths to estimate the upper bound. These runs are repeated 100 times. To find their confidence intervals Andersen and Broadie (2004) use 200,000 paths to estimate regression coefficients, 2,000,000 paths to generate the lower bound and 10,000 sub-paths in the calculation of the upper bound. Further research, and more extensive computer simulations are needed to compare these methods.

3.10 Evaluation

Theorem 1 in Glasserman and Yu (2004) suggests that 'regression later' should produce more accurate estimates than 'regression now'. The 'regression later' approach is on average three times more accurate, as shown in Table 3.2.

The use of martingale basis functions and duality in regression methods for pricing American options enables the fast calculation of upper bounds. The fact that basis functions are time dependent makes the implementation slightly more complex.

However, the 'regression later' method depends, through condition (C3), on the availability of basis functions that are martingales. We have obtained results for options on underlying assets driven by geometric Brownian motion. It may not be so easy to find martingale basis functions for other stochastic processes.

3.11 Conclusions

We have built on the contribution of Glasserman and Yu (2004) by presenting basis functions satisfying the martingale condition under geometric Brownian motion. The numerical results have indicated that 'regression later' gives more accurate option price estimates than 'regression now', and generates upper bounds more quickly.

Chapter 4

High Dimensional Radial Barrier Options

4.1 Introduction

Numerical methods for multi–asset options can be very slow. To speed up methods it is necessary to find analytic solutions, or analytic approximations. We saw in Chapter 2 that a number of approximations for single asset American options make use of barrier style options that are easier to value. This is the motivation for studying what multi–asset barrier style options can be valued in closed form.

American options are difficult to value with simulation as the paths are generated forwards through time, so it is non-trivial to determine the optimal exercise strategy for the option using dynamic programming.

We look for multi-asset options with American style features that can be solved analytically, and therefore quickly. We call this class of options "Radial Barrier Options" as they depend on the assumption of radial symmetry in the solution method. These options payoff if a barrier, defined as a function of the parameters describing the process for the underlying assets, is hit. These options are a particularly hard test case for the verification of Monte Carlo methods, due to the strange barriers that arise. Radial options may be useful in the financial market place themselves, or it may be possible to use them to approximate other, actively traded, financial products. They may also be useful as a control-variate in Monte Carlo simulation.

In Section 4.2 we formulate the problem, and detail the reduction, via a series of transformations, of the multi-asset Black–Scholes equation to the standard high dimensional heat equation. In Section 4.3 we present the radially symmetric problem which we go on to solve by Laplace transforms in the outer case (Section 4.4) and in the inner case (Section 4.5). We generalize the boundary conditions for these solutions in Section 4.6 using the Laplace convolution theorem. In Section 4.7 we reverse the transformations we have made to find the analytic value of these options in the original financial variables and verify the results in the case of one asset. We present numerical results comparing the analytic formulae to Monte Carlo simulation in Section 4.8. We discuss the qualitative behaviour of the options we have defined in Section 4.9. Finally, we conclude, relate this work to work on Bessel processes (Göing-Jaeschke and Yor, 2003), and suggest possible future directions for this work (Section 4.10).

4.2 The multi–asset Black–Scholes–Merton equation

We consider a Black–Scholes–Merton economy for each asset. The partial differential equation for the value, V, of an option that depends on the evolution of n different underlying assets with price $0 < S_i < \infty$, where $i = 1 \ldots n$, is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sum_{ij}^{n} \sigma_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i}^{n}(r - q_i) S_i \frac{\partial V}{\partial S_i} - rV = 0\,, \tag{4.1}$$

where r is the risk free rate, q_i is the dividend yield of the i^{th} asset, t is time, and σ_{ij} is the covariance of the i^{th} asset with the j^{th} asset. Throughout this chapter bold face capital letters, such as $\mathbf{X}$, represent matrices, and bold face lower case letters, such as $\mathbf{x}$, represent column vectors. The covariance matrix with elements σ_{ij}, denoted by $\mathbf{COV}$, is symmetric, $\sigma_{ij} = \sigma_{ji}$. The element σ_{ii} is the volatility squared of the i^{th} asset, σ_i^2. We write the volatility as a diagonal matrix, $\mathbf{\Sigma}$, with the σ_i on the diagonal and zeros off the diagonal. The correlation between assets i and j is written ρ_{ij}, we write this as a symmetric matrix, $\mathbf{P}$, with unity on the diagonal and ρ_{ij} as the off diagonal entries. The covariance, volatility and correlation are related by $\sigma_{ij} = \sigma_i\, \rho_{ij}\, \sigma_j$ We write this in matrix notation as $\mathbf{COV} = \mathbf{\Sigma}\,\mathbf{P}\,\mathbf{\Sigma}$.

We non-dimensionalise in a similar manner to Wilmott et al. (1993). Let E be some representative price scale of the option, and let $\sigma = \max(\sigma_i)$. We transform the value function and variables using

$$v = \frac{V}{E}\,, \qquad x_i = \log\frac{S_i}{E} \quad \text{and} \quad \tau = \tfrac{1}{2}\sigma^2(T - t)\,, \tag{4.2}$$

where T is the expiry date of the option. Therefore τ is the risk remaining until expiry. We also non-dimensionalise the parameters by

$$\alpha_{ij} = \frac{\sigma_{ij}}{\sigma^2}\,, \quad k_0 = \frac{r}{\frac{1}{2}\sigma^2} \quad \text{and} \quad k_i = \frac{r - q_i}{\frac{1}{2}\sigma^2} \quad i > 0\,, \tag{4.3}$$

where σ is a representative volatility. This gives

$$\frac{\partial v}{\partial \tau} = \sum_{ij}^{n} \alpha_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i}^{n} (k_i - \alpha_{ii}) \frac{\partial v}{\partial x_i} - k_0 v\,, \tag{4.4}$$

where $0 < \tau < \frac{1}{2}\sigma^2 T$ and $-\infty < x < \infty$. Note that the normalized covariance matrix of the α_{ij}, let us call it $\mathbf{A}$, is symmetric positive definite, as dividing by σ^2 does not affect the symmetry or positive definiteness. Also, when all $\sigma_i = \sigma$ then we have $\mathbf{A} = \mathbf{P}$, the correlation matrix.

4.2.1 Fixed boundary

We now reduce this equation to the heat equation. One way to achieve this is to make the transformation

$$v(\mathbf{x}, \tau) = e^{\mathbf{a}.\mathbf{x} - b\tau} u(\mathbf{x}, \tau)\,. \tag{4.5}$$

This allows us to eliminate the u and $\partial u / \partial x_i$ terms by solving equations for $\mathbf{a}$ and b. Making this substitution we get the equation,

$$\begin{aligned} -bu + \frac{\partial u}{\partial \tau} &= \textstyle\sum_{ij} \alpha_{ij} \left(\dfrac{\partial^2 u}{\partial x_i \partial x_j} + a_j \dfrac{\partial u}{\partial x_i} + a_i \dfrac{\partial u}{\partial x_j} + a_i a_j u \right) \\ &+ \textstyle\sum_i (k_i - \alpha_{ii}) \left(a_i u + \dfrac{\partial u}{\partial x_i} \right) - k_0 u\,. \end{aligned}$$

We use the fact that α_{ij} is symmetric, and that we can swap the order of summation to write this as

$$\begin{aligned} -bu + \frac{\partial u}{\partial \tau} &= \textstyle\sum_{ij} \alpha_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} \\ &+ \textstyle\sum_i \left(\left(2 \sum_j \alpha_{ij} a_j \right) + k_i - \alpha_{ii} \right) \dfrac{\partial u}{\partial x_i} \\ &+ \left(\left(\textstyle\sum_{ij} a_i a_j \alpha_{ij} \right) + \left(\sum_i (k_i - \alpha_{ii}) a_i \right) - k_0 \right) u\,. \end{aligned}$$

Setting the coefficients of $\partial u / \partial x_i$ and u to zero gives n equations for the coefficients of the first derivatives and a single equation for the coefficient of u. First consider the coefficients of $\partial u / \partial x_i$. For each $i = 1, \ldots, n$, we have,

$$\sum_j \alpha_{ij} a_j = -\tfrac{1}{2} (k_i - \alpha_{ii})\,.$$

We can write this in terms of matrices as

$$\mathbf{A}\mathbf{a} = -\tfrac{1}{2}\tilde{\mathbf{k}}\,,$$

where $\tilde{\mathbf{k}}$ is the column vector whose i^{th} entry is $k_i - \alpha_{ii}$. We can solve this for $\mathbf{a}$ by inverting $\mathbf{A}$,

$$\mathbf{a} = -\tfrac{1}{2}\mathbf{A}^{-1}\tilde{\mathbf{k}}\,. \tag{4.6}$$

Now let us consider the equation for the coefficient of u,

$$b = -\sum_{ij} a_i \alpha_{ij} a_j - \sum_i (k_i - \alpha_{ii})\, a_i + k_0\,.$$

Writing this as matrices gives

$$b = -\mathbf{a}^T\mathbf{A}\mathbf{a} - \mathbf{a}^T\tilde{\mathbf{k}} + k_0\,,$$

which, as $\mathbf{A}$ is symmetric, can be written

$$b = -\tfrac{1}{4}\tilde{\mathbf{k}}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\tilde{\mathbf{k}} + \tfrac{1}{2}\tilde{\mathbf{k}}^T\mathbf{A}^{-1}\tilde{\mathbf{k}} + k_0\,,$$

hence,

$$b = \tfrac{1}{4}\tilde{\mathbf{k}}^T\mathbf{A}^{-1}\tilde{\mathbf{k}} + k_0 = -\tfrac{1}{2}\tilde{\mathbf{k}}^T\mathbf{a} + k_0\,. \tag{4.7}$$

Therefore we have reduced (4.4) to an n dimensional heat equation,

$$\frac{\partial u}{\partial \tau} = \sum_{ij}^{n} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}\,. \tag{4.8}$$

Note that when $r = q_i = 0$, $\sigma_i = \sigma$ and $\rho_{ij} = 0$ we have $a_i = 1/2$ and $b = n/4$. Now, as $\mathbf{A}$ is symmetric positive definite we can write

$$\mathbf{A} = \mathbf{Q}^T\,\mathbf{D^2}\,\mathbf{Q} \tag{4.9}$$

where $\mathbf{Q}$ is a rotation with $\mathbf{Q}\,\mathbf{Q}^T = 1$ and $\mathbf{Q}^T$ has columns which are the eigenvectors of $\mathbf{A}$. $\mathbf{D}$ is a diagonal matrix of the corresponding n eigenvalues, d_i. Therefore we can rotate to a new basis

$$\bar{\mathbf{z}} = \mathbf{Q}\mathbf{x}\,, \tag{4.10}$$

where $\mathbf{x}$ is the column vector of elements x_i. which, when combined with a scaling, $z_i = d_i\bar{z}_i$ (in matrix notation $\mathbf{z} = \mathbf{D}\,\bar{\mathbf{z}}$), will convert our x_i basis into an orthonormal one, z_i. This gives the result

$$\frac{\partial u}{\partial \tau} = \sum_{i}^{n} \frac{\partial^2 u}{\partial {z_i}^2}\,. \tag{4.11}$$

4.2.1.1 The one dimensional problem

We shall be able to compare our general results with published results for the case of a single underlying asset. To this end it will be useful to note that

$$k_0 = \frac{r}{\frac{1}{2}\sigma^2} \quad \text{and} \quad k_1 = \frac{r-q}{\frac{1}{2}\sigma^2}\,.$$

We find the one dimensional versions of (4.6) and (4.7),

$$a = -\tfrac{1}{2}(k_1 - 1) \quad \text{and} \quad b = \tfrac{1}{4}(k_1 - 1)^2 + k_0\,. \tag{4.12}$$

Writing the constants a and b in financial variables gives

$$a = -\frac{1}{2}\left(\frac{2(r-q)}{\sigma^2} - 1\right) \quad \text{and} \quad b = \frac{1}{4}\left(\frac{2(r-q)}{\sigma^2} - 1\right)^2 + \frac{2r}{\sigma^2}\,. \tag{4.13}$$

4.2.2 Moving boundary

Another way to obtain an n dimensional heat equation is to transform the non-dimensionalised Black-Scholes equation (4.4) using

$$v = e^{-k_0\tau} w\,, \tag{4.14}$$

to obtain

$$\frac{\partial w}{\partial \tau} = \sum_{ij}^{n} \alpha_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i}^{n} (k_i - \alpha_{ii}) \frac{\partial w}{\partial x_i}\,.$$

Next we remove the drifts by

$$y_i = x_i + (k_i - \alpha_{ii})\tau \tag{4.15}$$

to get

$$\frac{\partial w}{\partial \tau} = \sum_{ij}^{N} \alpha_{ij} \frac{\partial^2 w}{\partial y_i \partial y_j}\,.$$

Note that this transformation will cause a fixed boundary in x space to move in S space. Also, an option payoff which is a function of x in x space will be a function of time in S space.

This is in the same form as (4.8), but in the y_i basis, rather than the x_i basis. Again we use the eigenvector - eigenvalue decomposition (4.9) to find a new basis

$$\bar{\hat{\mathbf{z}}} = \mathbf{Q}\mathbf{y}\,,$$

where the elements of the vector $\mathbf{y}$ are y_i, which, when combined with one final transformation $z_i = d_i \bar{\hat{z}}_i$ (in matrix notation $\mathbf{z} = \mathbf{D}\,\bar{\hat{\mathbf{z}}}$), gives the result

$$\frac{\partial w}{\partial \tau} = \sum_i^n \frac{\partial^2 w}{\partial \hat{z}_i^2} \,.$$

For the rest of this chapter we will only consider the fixed boundary case.

4.3 The radial problem for u

We look for radially symmetric solutions, $u(\rho, \tau)$, which depend on the radial distance, $\rho^2 = \sum z_i^2$, and τ. In radial co-ordinates we have

$$\frac{\partial u}{\partial \tau} = \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \left(\rho^{n-1} \frac{\partial u}{\partial \rho} \right) = \frac{\partial^2 u}{\partial \rho^2} + \frac{n-1}{\rho} \frac{\partial u}{\partial \rho} \,. \tag{4.16}$$

We choose boundary conditions that allow this problem to be solved analytically,

$$u(1, \tau) = 1 \tag{4.17}$$

$$u(\infty, \tau) = 0 \tag{4.18}$$

$$u(\rho, 0) = 0 \,. \tag{4.19}$$

We solve the problem where u initially has a value of 1 on the sphere $\rho = 1$ and a value of 0 outside. We then let our time-like variable evolve. This is an option which pays 1 when the boundary is hit before the expiry date, but otherwise expires worthless. There are two cases, the outer problem and the inner problem. The outer problem is when $\rho \geq 1$ meaning that the assets are at a radial distance outside the barrier. The inner problem is when $0 \leq \rho \leq 1$ meaning the assets relate to a radial distance inside the barrier. We consider these two cases in turn.

4.4 Outer problem

For the case of one underlying asset ($n = 1$) the solution

$$u_1(\rho, \tau) = \operatorname{erfc}\left(\frac{\rho - 1}{2\sqrt{\tau}} \right) \tag{4.20}$$

is well known (Carslaw and Jaeger, 1959). We use u_1 to indicate the solution to the one dimensional problem, similarly u_i indicates the solution to the i dimensional problem. u_n or u indicates the general solution.

In three dimensions the problem can be transformed to the the one dimensional case using the transformation $F = u_3\rho$, to reduce the problem to the heat equation for F. Section 9.10 of Carslaw and Jaeger (1959) gives the solution valid for $\rho \geq 1$, the region bounded internally by the sphere $\rho = 1$. We find the result for an initial value of zero, and a constant surface value of 1, by using the 1-dimensional solution (4.20), to find,

$$u_3(\rho, \tau) = \frac{1}{\rho}\,\mathrm{erfc}\left(\frac{\rho - 1}{2\sqrt{\tau}}\right). \tag{4.21}$$

4.4.1 Solution in n dimensions

We can solve the problem (4.16) – (4.19) in n dimensions using a Laplace transform in time. In general the solution involves Bessel functions. We solve the problem in the region $\rho \geq 1$. Initially we have $u(1, 0) = 1$ and $u(\rho, 0) = 0$ for $\rho > 1$. The solution for the two dimensional case can be found in Carslaw and Jaeger (1959) 13.5(I). We take the Laplace transform in time and find the subsidiary equation

$$\frac{d^2\bar{u}}{d\rho^2} + \frac{n-1}{\rho}\frac{d\bar{u}}{d\rho} - s\,\bar{u} = 0\,, \tag{4.22}$$

for the region $\rho \geq 1$, where s is the Laplace parameter. The surface $\rho = 1$ is held at a constant value 1. The transformed boundary condition is

$$\bar{u}(1, s) = 1/s\,.$$

The general solution to the modified Bessel equation (4.22) is

$$\bar{u}(\rho, s) = \rho^{-\nu}\left[AI_\nu(\sqrt{s}\,\rho) + BK_\nu(\sqrt{s}\,\rho)\right]\,,$$

where A and B are constants and the parameter ν is defined to be[1]

$$\nu = \frac{n}{2} - 1\,.$$

We require a finite solution as $\rho \to \infty$, so $A = 0$. We find B from the transformed boundary condition and obtain the solution

$$\bar{u}(\rho, s) = \frac{\rho^{-\nu}K_\nu(\sqrt{s}\,\rho)}{s\,K_\nu(\sqrt{s})}\,. \tag{4.23}$$

[1]This definition has changed by a minus sign from earlier versions of this research, so as to match the existing definition of ν in the literature on Bessel processes.

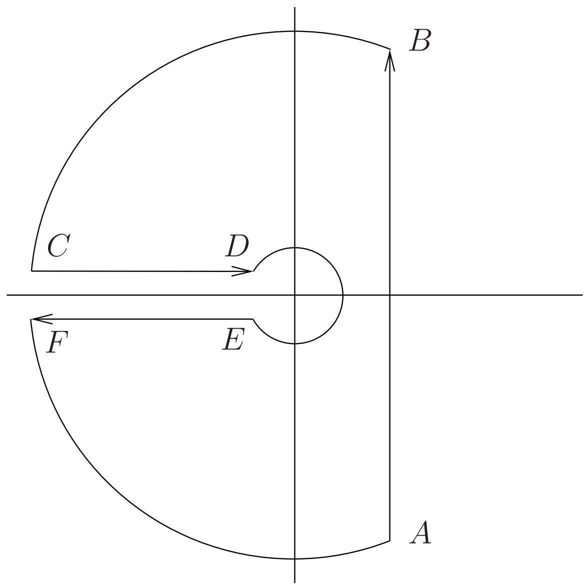


Figure 4.1: The keyhole contour for inverting the Laplace transform

A similar expression appears in Section 2.1 of Göing-Jaeschke and Yor (1999) (see also the references therein). Next we invert this solution using the Laplace Inversion Theorem, so

$$u_n(\rho,\tau) = \frac{\rho^{-\nu}}{2\pi\, i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\tau}\frac{K_\nu(\sqrt{s}\,\rho)}{K_\nu(\sqrt{s})}\frac{ds}{s}\,. \tag{4.24}$$

Note that our series of Bessel functions, for increasing n, is

$$K_{-1/2}, K_0, K_{1/2}, K_1, K_{3/2}, \ldots$$

As $K_{-\nu}(z) = K_\nu(z)$, we can see that the $n = 1$ and $n = 3$ cases will differ only in factors of ρ, which is indeed the case.

The integrand has a branch point at $s = 0$, so we compute the integral using the contour in Figure 4.1. $K_\nu(z)$ has no zeros for $|\arg z| \leq \frac{1}{2}\pi$ (Watson, 1944, section 15.7.). The Bessel function in the denominator of the integrand in (4.24) has argument $\sqrt{s}$, therefore we know that the integrand has no poles within the contour in Figure 4.1. The integral along AB, that we wish to calculate is equal to the integral

$$u = \frac{\rho^{-\nu}}{2\pi\, i}\int_{AF+FE+ED+DC+CB} e^{s\tau}\frac{K_\nu(\sqrt{s}\,\rho)}{K_\nu(\sqrt{s})}\frac{ds}{s}\,.$$

We now calculate these integrals:

- AF: The integral along this contour goes to zero. See Appendix B.1 for details.

- FE: To calculate the integral over FE we put $s = \zeta^2 e^{-i\pi}$ which gives

$$2\int_{\infty}^{0} e^{-\zeta^2\tau} \frac{K_\nu(\rho\zeta e^{-i\pi/2})}{K_\nu(\zeta e^{-i\pi/2})} \frac{d\zeta}{\zeta} = -2\int_{0}^{\infty} e^{-\zeta^2\tau} \frac{J_\nu(\zeta\rho) + iY_\nu(\zeta\rho)}{J_\nu(\zeta) + iY_\nu(\zeta)} \frac{d\zeta}{\zeta},$$

  since

$$K_\nu(ze^{\pi i/2}) = \tfrac{1}{2}\pi i H_\nu^{(1)}(z) = \tfrac{1}{2}\pi i[J_\nu(z) + iY_\nu(z)].$$

- ED: We write $s = \epsilon e^{i\theta}$ so the integral around the circle ED, from $-\pi$ to π, becomes

$$\int_{-\pi}^{\pi} e^{\epsilon\tau e^{i\theta}} \frac{K_\nu(\sqrt{\epsilon}\, e^{i\theta/2}\rho)}{K_\nu(\sqrt{\epsilon}\, e^{i\theta/2})} i\, d\theta.$$

  We take the limit as $\epsilon \to 0$ and used Abramowitz and Stegun (1974, 9.6.9) to find that the integral evaluates to $2\pi i\rho^{-|\nu|}$.

- DC: For the integral on DC we put $s = \zeta^2 e^{i\pi}$, and use

$$K_\nu(ze^{\pi i/2}) = -\tfrac{1}{2}\pi i H_\nu^{(2)}(z) = -\tfrac{1}{2}\pi i[J_\nu(z) - iY_\nu(z)],$$

  since we have a complex argument of $\pi/2$ rather than $-\pi/2$ as in the integral FE, and obtain

$$2\int_{0}^{\infty} e^{-\zeta^2\tau} \frac{K_\nu(\rho\zeta e^{i\pi/2})}{K_\nu(\zeta e^{i\pi/2})} \frac{d\zeta}{\zeta} = 2\int_{0}^{\infty} e^{-\zeta^2\tau} \frac{J_\nu(\zeta\rho) - iY_\nu(\zeta\rho)}{J_\nu(\zeta) - iY_\nu(\zeta)} \frac{d\zeta}{\zeta},$$

  which is minus the complex conjugate of the integral that we found for FE.

- CB: This integral goes to zero. The argument is similar to the case AF. See B.1 for details.

Combining these results we obtain

$$u_n(\rho,\tau) = \rho^{-\nu-|\nu|} + \frac{2\rho^{-\nu}}{\pi} \int_{0}^{\infty} e^{-\zeta^2\tau} \frac{J_\nu(\zeta\rho)Y_\nu(\zeta) - J_\nu(\zeta)Y_\nu(\zeta\rho)}{J_\nu(\zeta)^2 + Y_\nu(\zeta)^2} \frac{d\zeta}{\zeta}, \tag{4.25}$$

which agrees with the solution for the two dimensional case in Carslaw and Jaeger (1959) 13.5(I).

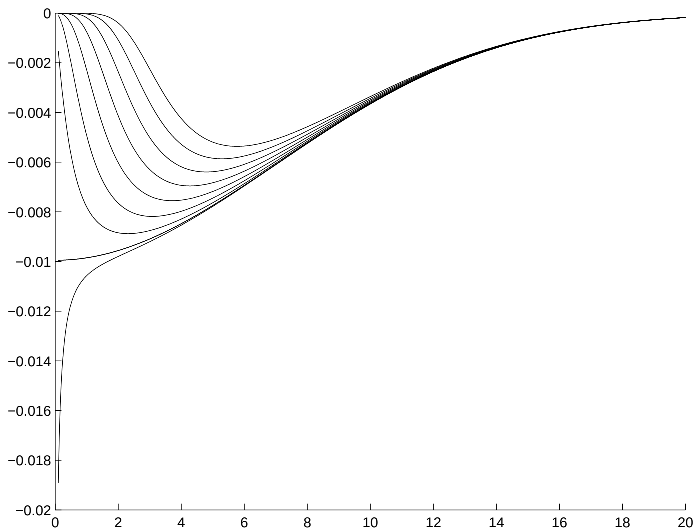


Figure 4.2: A graph of the integrand in (4.25) with $\rho = 1.01$ and $\tau = 0.01$ over a range of values for the dummy variable ζ. The lowest line is for dimension $n = 1$ proceeding to the topmost line where $n = 10$.

4.4.1.1 Spherical Bessel functions

It is informative to note that in one dimension the Bessel functions in the integrand have order of a half and therefore can be rewritten as trigonometric functions. We can verify our solution before and after the Laplace inversion in this way. We can also do this in the three dimensional case.

In five dimensions the situation is slightly more complicated. However, we can write the Bessel functions in (4.25) as

$$\frac{\zeta(\rho - 1)\cos(\zeta - \zeta\rho) + (\rho\zeta^2 + 1)\sin(\zeta - \zeta\rho)}{(\zeta^2 + 1)\rho^{3/2}},$$

we split this into three parts and find one relatively simple integral, and two others that may be found in Gradshteyn and Ryzhik (2000) 3.954. Combining these and simplifying (4.25) we find

$$u_5(\rho, \tau) = \frac{1}{\rho^3}\left[\operatorname{erfc}\left(\frac{\rho - 1}{2\sqrt{\tau}}\right) + (\rho - 1)\, e^{(\rho-1)+\tau} \operatorname{erfc}\left(\sqrt{\tau} + \frac{\rho - 1}{2\sqrt{\tau}}\right)\right], \tag{4.26}$$

which we can verify satisfies the partial differential equation for $n = 5$ dimensions.

4.4.1.2 Even dimensions

When we try to solve the integral in even dimensions we have to expand the Bessel functions of integer order. It has been proved (Watson, 1944) that this is not possible in finite terms. A finite expansion is only possible when the order of the Bessel functions is half an odd integer.

4.4.1.3 Numerical solution

We can calculate the value of the function u from (4.25). We used the adaptive numerical integration function `NIntegrate` in Mathematica (2004) to evaluate the integral. The results are shown in Figure 4.3. Individual data points were calculated using `NIntegrate`, solid lines indicate the evaluation of the closed form solutions available for an odd number of underlying assets. We find good agreement between solutions using `NIntegrate` and the analytic solutions for $n = 1$, $n = 3$ and $n = 5$ dimensions.

Note that as the dimension increases the value of u for a particular value of ρ decreases, which is as we expect. All lines come together at 1 when $\rho = 1$, also as required, and all lines decay toward zero as ρ increases. As we increase the time the influence of the fixed value diffuses away from $\rho = 1$. For small times the diffusion is less. For very small τ we find numerical instability in the solution, due to the oscillatory nature of the integral.

As we increase the time the influence of the fixed value diffuses away from $\rho = 1$. For small times the diffusion is less. For $\tau < 0.1$ we find numerical instability in the solution.

4.4.1.4 Asymptotic solution for small values of τ

Unfortunately we can see that (4.25) does not converge well for small values of τ. To find solutions for small τ we make an asymptotic expansion, as in Carslaw and Jaeger (1959) 13.5. Expanding (4.23), and keeping $\nu = \frac{n}{2} - 1$, we have

$$\bar{u} = \frac{1}{s\rho^{(n-1)/2}} e^{-\sqrt{s}\,(\rho-1)} \left\{1 + \frac{(4\nu^2-1)}{8\rho\sqrt{s}} + \frac{(4\nu^2-1)((4\nu^2-9))}{2(8\rho)^2 s} + \mathcal{O}(\tfrac{1}{s^{3/2}})\right\} \Big/ \left\{1 + \frac{(4\nu^2-1)}{8\sqrt{s}} + \frac{(4\nu^2-1)((4\nu^2-9))}{2(8)^2 s} + \mathcal{O}(\tfrac{1}{s^{3/2}})\right\},$$

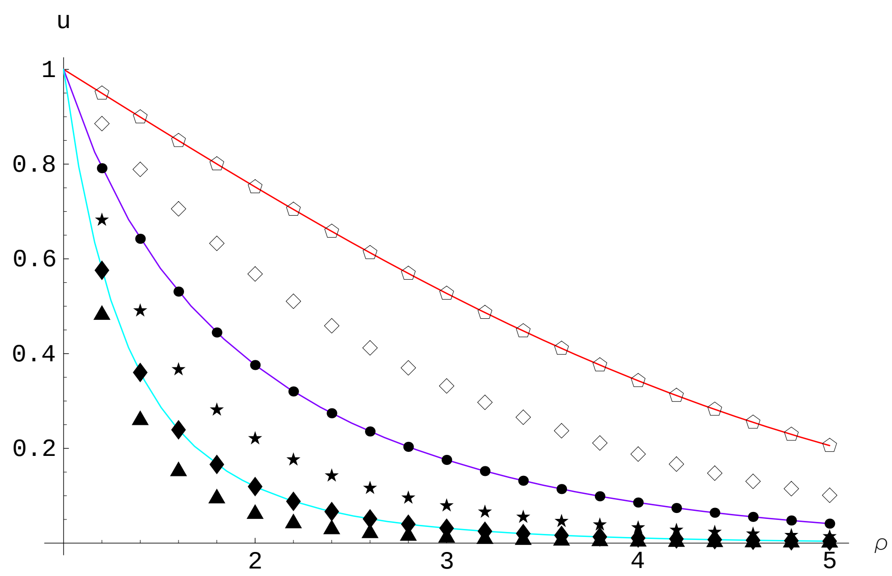


Figure 4.3: A graph of the function u for a range of dimensions. The integral in (4.25) was evaluated over $[0, \infty)$ using Mathematica, $\tau = 5$. The horizontal axis is ρ and the vertical u. The top line is for one dimension, the dimension increases until $n = 6$ for the bottom line. The smooth lines are the exact solutions, the points are the numerical solution.

which we can expand as

$$\bar{u} = \frac{1}{s\rho^{(n-1)/2}} e^{-\sqrt{s}\,(\rho-1)} \left\{ 1 - \frac{(4\nu^2-1)(\rho-1)}{(8\rho)\sqrt{s}} \right. \\ \left. + \frac{(4\nu^2-1)[\rho^2(4\nu^2+7) - 2\rho(4\nu^2-1) + (4\nu^2-9)]}{2(8\rho)^2 s} + \mathcal{O}(\frac{1}{s^{3/2}}) \right\}.$$

Inverting the Laplace transform term by term gives

$$\begin{aligned} u = & \frac{1}{\rho^{(n-1)/2}} \operatorname{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) \\ & - \frac{\sqrt{\tau}\,(4\nu^2-1)(\rho-1)}{2^2\rho^{(n+1)/2}} \,\mathrm{i}^1\mathrm{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) \\ & + \frac{\tau\,(4\nu^2-1)[4\nu^2(\rho-1)^2 + 7\rho^2 + 2\rho - 9]}{2.2^4\rho^{(n+3)/2}} \,\mathrm{i}^2\mathrm{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) + \dots, \end{aligned} \tag{4.27}$$

where $\mathrm{i}^n\mathrm{erfc}\,(.)$ is the iterated error function (Abramowitz and Stegun, 1974). This approximation is valid, according to Carslaw and Jaeger (1959), for $\tau < 0.02$ and ρ not small. We know that $\rho \geq 1$ so the second condition is not restrictive.

4.5 Inner problem

We now consider the inner problem. We look for radially symmetric solutions to (4.16) – (4.19) in the region $0 \le \rho \le 1$. As before we have $\rho^2 = \sum z_i^2$, and τ. The problem is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial u}{\partial \rho}\,. \tag{4.28}$$

We choose boundary conditions that allow this problem for $u(\rho,\tau)$ to be solved analytically,

$$u(1,\tau) = 1 \tag{4.29}$$
$$u_\rho(0,\tau) = 0 \tag{4.30}$$
$$u(\rho,0) = 0\,. \tag{4.31}$$

We solve the problem where initially $u = 1$ on the boundary $\rho = 1$ and $u = 0$ on $0 \le \rho < 1$. If the boundary is not hit before the expiry date then the option expires worthless.

We could also consider options where a radially symmetric payoff, $f(\rho)$, occurs if the option is not knocked out before the expiry date.

The boundary condition

$$\frac{\partial u}{\partial \rho}(0,\tau) = 0\,,$$

is the natural one to consider for a multi–asset radial option. If one asset drops in price then the logarithm of the non-dimensionalised price will fall toward zero. This boundary condition ensures a smooth value function through this point. If that particular asset continues to fall in price the radial measure ρ will start to increase again, and will eventually hit the boundary at $\rho = 1$ and the option will payout.

There are a number of well known results in this case. In one dimension we have the solution

$$u_1(\rho,\tau) = \sum_{n=0}^{\infty} (-1)^n \left(\operatorname{erfc}\left(\frac{2n+1-\rho}{2\sqrt{\tau}}\right) + \operatorname{erfc}\left(\frac{2n+1+\rho}{2\sqrt{\tau}}\right)\right)\,,$$

for small τ and the solution (check Carslaw and Jaeger (1959, 3.5(6)))

$$u_1(\rho,\tau) = 1 - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, e^{-(2n+1)^2\pi^2\tau/4} \cos\tfrac{1}{2}(2n+1)\pi\rho\,,$$

for large τ. In three dimensions we have (Carslaw and Jaeger, 1959, 9.3(5))

$$u_3(\rho,\tau) = \frac{1}{\rho}\sum_{n=0}^{\infty}\left(\operatorname{erfc}\left(\frac{2n+1-\rho}{2\sqrt{\tau}}\right) - \operatorname{erfc}\left(\frac{2n+1+\rho}{2\sqrt{\tau}}\right)\right),$$

for small τ and (Carslaw and Jaeger, 1959, 9.3(4))

$$u_3(\rho,\tau) = 1 + \frac{2}{\pi\rho}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,e^{-n^2\pi^2\tau}\,\sin n\pi\rho\,,$$

for large τ.

4.5.1 Solution in n dimensions

We take the Laplace transform in time of the problem to find the subsidiary equation as before

$$\frac{d^2\bar{u}}{d\rho^2} + \frac{n-1}{\rho}\frac{d\bar{u}}{d\rho} - s\,\bar{u} = 0\,.$$

with boundary conditions

$$\bar{u}(1,s) = 1/s \tag{4.32}$$

$$\bar{u}_\rho(0,s) = 0 \tag{4.33}$$

$$\bar{u}(\rho,0) = 0\,. \tag{4.34}$$

The general solution to is

$$\bar{u}(\rho,s) = \rho^{-\nu}\left[AI_\nu(\sqrt{s}\,\rho) + BK_\nu(\sqrt{s}\,\rho)\right],$$

where I and K are modified Bessel functions and

$$\nu = \frac{n}{2} - 1\,.$$

We are solving for $\bar{u}$ in the region $0 \le \rho \le 1$. The Bessel function K is unbounded as $\rho \to 0$ so that solution must be disallowed and therefore $B = 0$. Bessel functions I are bounded as $\rho \to 0$, so that is the solution we require. We find A from the transformed boundary condition and obtain the solution

$$\bar{u}(\rho,s) = \frac{\rho^{-\nu}\,I_\nu(\sqrt{s}\,\rho)}{s\,I_\nu(\sqrt{s})}\,. \tag{4.35}$$

The Laplace Inversion Theorem says

$$u(\rho,\tau) = \frac{\rho^{-\nu}}{2\pi\, i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{s\tau}\frac{I_\nu(\sqrt{s}\,\rho)}{I_\nu(\sqrt{s})}\frac{ds}{s}\,. \tag{4.36}$$

Note that our series of Bessel functions, for increasing n, is

$$I_{-1/2}, I_0, I_{1/2}, I_1, I_{3/2}, \ldots$$

It is well known that $I_\nu(z)$ is an entire function of z when $\nu \in \mathbb{Z}$. We can then use the contour in Figure 4.4 to invert the Laplace transform. In general $I_\nu(z)$ has a branch cut along the negative real axis. However, the ratio of Bessel functions $I_\mu(\rho z)/I_\mu(z)$ does not actually have a branch cut. We can see this using the series representation for $I_\mu(z)$ (Abramowitz and Stegun, 1974, 9.6.10). We have

$$\frac{I_\mu(z\rho)}{I_\mu(z)} = \rho^\mu \frac{\sum_{k=0}^\infty(\frac{1}{2}\rho^2z^2)^k/(k!\,\Gamma(\mu+k+1))}{\sum_{k=0}^\infty(\frac{1}{2}z^2)^k/(k!\,\Gamma(\mu+k+1))}\,,$$

and the non-integral powers of z outside the sum have cancelled. We have a ratio of two holomorphic functions defined as power series, so the ratio itself is holomorphic. Notice also that for $x \in \mathbb{R}$ with $x > 0$ all terms are positive so the ratio is completely monotonic.

The zeros of $I_\nu(z)$ are discussed in Watson (1944). We are concerned with the right half plane as the Bessel function in the denominator of the integrand has argument $\sqrt{s}$. This will always have $|\arg(\sqrt{s})| \le \frac{1}{2}\pi$.

We have established that the integrand has no branch cut so we invert the Laplace transform using the contour in Figure 4.4. The integral along AB, that we wish to calculate, is equal to the integral

$$u = \frac{\rho^{-\nu}}{2\pi\, i}\left(\int_{ACB}\frac{I_\nu(\sqrt{s}\,\rho)}{I_\nu(\sqrt{s})}\frac{ds}{s} + 2\pi\, i\sum \text{residues}\right)\,.$$

The integral along the contour ACB converges to zero (see Appendix B.2). We calculate the residues of the poles inside the contour.

- For the pole at $s = 0$ we calculate the Laurent series

  $$\frac{\rho^{|\nu|}}{s} + \mathcal{O}(1)\,,$$

  so the residue is

  $$\operatorname{Res}(\bar{u}(\rho,s); s=0) = \rho^{|\nu|}\,.$$

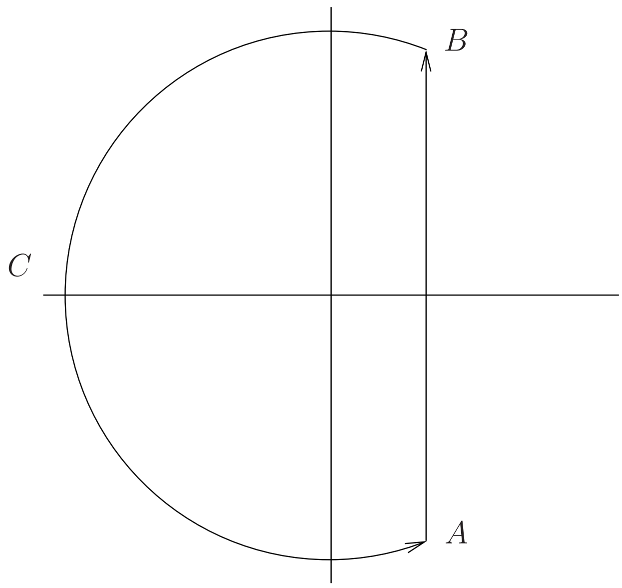


Figure 4.4: The contour for inverting the Laplace transform when there is no branch cut in the integrand.

- We calculate residues at simple poles as follows, let $f(z) = h(z)/k(z)$. If $f(z)$ has a pole at a ($k(a) = 0$) and $h(a) \neq 0$ then the residue is given by $h(a)/k'(a)$ so long as $k'(a) \neq 0$. We evaluate the residues due to zeros of $I_{-\nu}(\sqrt{s})$ by using

$$I_\nu(z) = e^{-\nu\pi i/2} J_\nu(ze^{\pi i/2}) \qquad (-\pi < \arg z \leq \tfrac{1}{2}\pi)$$

so we evaluate the residues at $J_\nu(i\sqrt{s})$ where $s = -\alpha_m^2$ are the zeros of $J_\nu(\alpha_m) = 0$. There are an infinite number of such zeros with positive real part.

$$h(s) = e^{s\tau}\,\frac{I_\nu(\rho\sqrt{s})}{s} \quad \text{and} \quad k(s) = I_\nu(\sqrt{s})\,.$$

The numerator is not zero at the residues. We take the derivative $k'(s)$ evaluated at its zeros

$$\begin{aligned} \left.\frac{\partial k}{\partial s}\right|_{s=-\alpha_m^2} &= \left.\frac{d}{ds} I_\nu(\sqrt{s})\right|_{s=-\alpha_m^2} \\ &= \left.\frac{1}{4\sqrt{s}}\left(I_{\nu-1}(\sqrt{s}) + I_{\nu+1}(\sqrt{s})\right)\right|_{\sqrt{s}=i\alpha_m}. \end{aligned}$$

We use the recurrence relation (Abramowitz and Stegun, 1974, 9.1.27)

$$I_{\nu-1}(z) = \frac{2\nu}{z}\, I_\nu(z) - I_{\nu+1}(z)$$

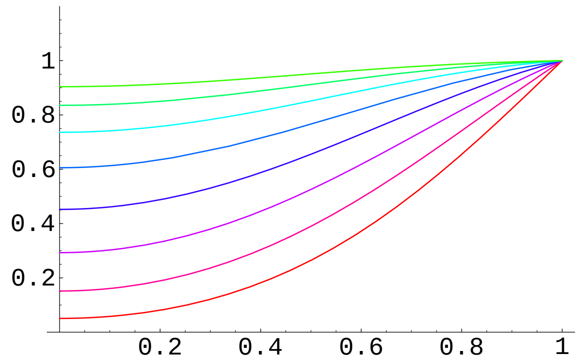


Figure 4.5: u at $\tau = 0.1$ against ρ. The one dimensional solution is the lowest line, going upwards to eight dimensions.

and the fact that $I_\nu(i\alpha_m) = 0$ to obtain

$$\mathrm{Res}\,(\bar{u}(\rho, s); s = -\alpha_m^2) = \frac{h(-\alpha_m^2)}{k'(-\alpha_m^2)} = -\frac{2ie^{-\alpha_m^2\tau}}{\alpha_m}\frac{I_\nu(i\rho\alpha_m)}{I_{\nu+1}(i\alpha_m)}\,.$$

This can be rewritten as

$$\mathrm{Res}\,(\bar{u}(\rho, s); s = -\alpha_m^2) = -\frac{2e^{-\alpha_m^2\tau}}{\alpha_m}\frac{J_\nu(\rho\,\alpha_m)}{J_{\nu+1}(\alpha_m)}\,.$$

The general solution is

$$u(\rho, \tau) \quad = \quad \rho^{-\nu+|\nu|} - 2\,\rho^{-\nu}\sum_{m=1}^{\infty}\frac{e^{-\alpha_m^2\tau}}{\alpha_m}\frac{J_{|\nu|}(\rho\,\alpha_m)}{J_{|\nu|+1}(\alpha_m)} \tag{4.37}$$

This appears in the literature as Carslaw and Jaeger (1959, 13.2(7)).

We can verify this solution satisfies the PDE and boundary conditions by differentiation and substitution, and by using (Watson, 1944, 18.12(1))

$$x^\nu = \sum_{m=1}^{\infty}\frac{2J_\nu(j_m x)}{j_m\,J_{\nu+1}(j_m)}\,,$$

where $0 \leq x < 1$ and $J_\nu(j_m) = 0$.

We show the value of the function $u(\rho, \tau)$ for a number of dimensions in Figure 4.5. We can see that it has value 1 on the barrier $\rho = 1$, and the derivative $u_\rho = 0$ at $\rho = 0$. The lowest line is the $n = 1$ case and the top curve is the $n = 8$ case.

4.5.2 Solution in five dimensions

There are simplified solutions in the case of one and three dimensions, due the the properties of Bessel functions of half-integer order. In five dimensions we can also take advantage of the properties of spherical modified Bessel functions. We have $n = 5$, so $\nu = 3/2$. The transform solution is given by

$$\bar{u}(\rho, s) = \frac{\rho^{-3/2}\, I_{3/2}(\sqrt{s}\,\rho)}{s\, I_{3/2}(\sqrt{s})}\,.$$

We use

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi}}\left(\frac{-\cosh x + x\sinh x}{x^{3/2}}\right),$$

to write this as

$$\bar{u}(\rho, s) = \frac{-\cosh(\sqrt{s}\rho) + \sqrt{s}\rho\sinh(\sqrt{s}\rho)}{\rho^3\left(-s\cosh(\sqrt{s}) + s^{3/2}\sinh(\sqrt{s})\right)}\,.$$

Rewriting the hyperbolic functions as exponentials gives

$$\bar{u}(\rho, s) = \frac{e^{-\sqrt{s}(\rho-1)}\left(-1 - \sqrt{s}\rho + e^{2\sqrt{s}\rho}(-1 + \sqrt{s}\rho)\right)}{s\rho^3\left(-1 - \sqrt{s} + e^{2\sqrt{s}}(-1 + \sqrt{s})\right)}\,.$$

We try to invert this term by term using the binomial theorem. We obtain a problem of the form

$$\bar{u}(\rho, s) = \frac{1}{\rho^3 s(\sqrt{s} - 1)}\left(-(1 + \sqrt{s}\rho)e^{-\sqrt{s}(\rho+1)} + (-1 + \sqrt{s}\rho)e^{-\sqrt{s}(-\rho+1)}\right)$$
$$\sum_{m=0}^{\infty}(-1)^{m+1}\left(\frac{\sqrt{s}+1}{\sqrt{s}-1}\right)^m e^{-2m\sqrt{s}}\,,$$

which, unfortunately, does not appear to be readily inverted.

4.6 Convolutions

In this section we use the Laplace Convolution Theorem

$$\mathcal{L}[g * h] = \mathcal{L}[g]\mathcal{L}[h]$$

to allow us to change the boundary condition when the radial barrier at $\rho = 1$ is hit from a unit payoff to a payoff of the form $\phi(\tau)$. Recall the transform (4.5),

$$v(\mathbf{x}, \tau) = e^{\mathbf{a}.\mathbf{x} - b\tau}u(\mathbf{x}, \tau)\,.$$

We would like to change the boundary condition at $\rho = 1$ to be a function, $\phi(\tau)$, of τ. In particular, using $\phi(\tau) = e^{b\tau}$ will remove the time dependence in the solution once we transform back to financial variables.

4.6.1 Outer problem

We have a solution $u(\rho,\tau)$ to the radial problem

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial u}{\partial \rho} && (4.38)\\ u(1,\tau) &= 1 && (4.39)\\ u(\infty,\tau) &= 0 && (4.40)\\ u(\rho,0) &= 0\,. && (4.41)\end{aligned}$$

We find the solution, $\hat{u}(\rho,\tau)$, to the above problem with the modified boundary condition $u(1,\tau)=\phi(\tau)=e^{b\tau}$. Let us shift the space variable by

$$\rho = \xi + 1\,, \tag{4.42}$$

to obtain the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} + \frac{n-1}{\xi+1}\frac{\partial u}{\partial \xi}\,.$$

The boundary condition at $\rho = 1$ for the original problem is now $u(0,\tau) = 1$ at $\xi = 0$. If we make the same change of variable (4.42) in the $\hat{u}(\rho,\tau)$ problem we have the partial differential equation

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{\partial^2 \hat{u}}{\partial \xi^2} + \frac{n-1}{\xi+1}\frac{\partial \hat{u}}{\partial \xi}\,,$$

with boundary condition $\hat{u}(0,\tau) = \phi(\tau) = e^{b\tau}$. The second and third boundary conditions remain unchanged.

Taking the Laplace transform in τ of the u problem we can find the transformed solution $\bar{u}$. We also take the Laplace transform of the $\hat{u}$ problem. Let s be the transform variable, we write the solution to the $\bar{\hat{u}}$ problem in terms of $\bar{u}$

$$\bar{\hat{u}}(\xi,s) = s\,\overline{\phi}(s)\,\bar{u}(\xi,s)\,.$$

Using the properties of Laplace transforms, and the convolution theorem, this is equal to

$$\bar{\hat{u}}(\xi,s) = \overline{\phi}(s)\,\overline{\frac{\partial u}{\partial \tau}}(\xi,s) = \mathcal{L}\left[\phi(\tau) * \frac{\partial u}{\partial \tau}(\xi,\tau)\right]\,,$$

hence

$$\hat{u}(\xi,\tau) = \phi(\tau) * \frac{\partial u}{\partial \tau}(\xi,\tau) = \int_0^\tau \phi(\tau-\eta)\,\frac{\partial u}{\partial \tau}(\xi,\eta)\,d\eta\,.$$

Alternatively we could choose to use

$$\hat{u}(\xi,\tau) = \frac{\partial \phi}{\partial \tau}(\tau) \, * \, u(\xi,\tau) = \int_0^\tau \frac{\partial \phi}{\partial \tau}(\eta)\, u(\xi,\tau-\eta)\, d\eta\,.$$

4.6.1.1 One dimension

In one dimension we have, from (4.20) with the transform (4.42),

$$u_1(\xi,\tau) = \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right)\,.$$

We set $\phi(\tau) = e^{b\tau}$ to simplify the option after we invert the non-dimensionalising transforms, hence

$$\hat{u}_1(\xi,\tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau e^{\alpha(\tau-\eta)}\; \eta^{-3/2}\, e^{-\xi^2/4\eta}\, d\eta\,,$$

in which we substitute $y = \eta^{-1/2}$, solve the integral, and reverse the transform (4.42), to obtain

$$\hat{u}_1(\rho,\tau) = \frac{e^{b\tau}}{2}\,\left[e^{(\rho-1)\sqrt{b}}\operatorname{erfc}(d_+) + e^{-(\rho-1)\sqrt{b}}\operatorname{erfc}(d_-)\right]\,,$$

where

$$d_\pm = \frac{\rho-1}{2\sqrt{\tau}} \pm \sqrt{b\tau}\,. \tag{4.43}$$

In terms of cumulative normals, $\mathrm{N}\,(.)$, instead of error functions, this is

$$\hat{u}_1(\rho,\tau) = e^{b\tau}\,\left[e^{(\rho-1)\sqrt{b}}\,\mathrm{N}\left(-\sqrt{2}\,d_+\right) + e^{-(\rho-1)\sqrt{b}}\,\mathrm{N}\left(-\sqrt{2}\,d_-\right)\right]\,. \tag{4.44}$$

4.6.1.2 Three dimensions

In three dimensions we transform (4.21) using (4.42),

$$u_3(\xi,\tau) = \frac{1}{\xi+1}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right)\,,$$

which is simply the one dimensional solution divided by a factor of $\xi+1$, so for the boundary condition $\phi(\tau) = e^{b\tau}$ we can solve the convolution as in the one dimensional case to obtain

$$\hat{u}_3(\rho,\tau) = \frac{e^{b\tau}}{\rho}\,\left[e^{(\rho-1)\sqrt{b}}\,\mathrm{N}\left(-\sqrt{2}\,d_+\right) + e^{-(\rho-1)\sqrt{b}}\,\mathrm{N}\left(-\sqrt{2}\,d_-\right)\right]\,, \tag{4.45}$$

where $d_\pm$ is as in (4.43). We can verify that this satisfies (4.16) when $n = 3$.

4.6.1.3 Five dimensions

In five dimensions we convolve the solution (4.26) with the boundary condition $\phi(\tau) = e^{b\tau}$ and find the solution

$$\begin{aligned}\hat{u}_5(\rho,\tau) = \frac{1}{2(b-1)(\rho)^3}\Bigg(&-2\,(\rho-1)\,e^{(\rho-1)+\tau}\,\mathrm{erfc}\left(\frac{(\rho-1)}{2\sqrt{\tau}}+\sqrt{\tau}\right)\\ &+(b-1+\sqrt{b}\,(\rho-1)+b(\rho-1))\,e^{b\tau+(\rho-1)\sqrt{b}}\,\mathrm{erfc}\,(d_+)\\ &+(b-1-\sqrt{b}\,(\rho-1)+b(\rho-1))\,e^{b\tau-(\rho-1)\sqrt{b}}\,\mathrm{erfc}\,(d_-)\Bigg)\,,\end{aligned} \tag{4.46}$$

4.6.1.4 General convolution

We perform the convolution on the general solution to change the boundary condition to $e^{b\tau}$. This follows through as usual and we obtain

$$\hat{u}_n(\rho,\tau) = \frac{2e^{b\tau}\rho^{\nu}}{\pi}\int_0^{\infty}\frac{\zeta}{b+\zeta^2}\left(e^{-\tau(\zeta^2+b)}-1\right)\frac{J_\nu(\zeta\rho)Y_\nu(\zeta)-J_\nu(\zeta)Y_\nu(\zeta\rho)}{J_\nu(\zeta)^2+Y_\nu(\zeta)^2}\,d\zeta\,, \tag{4.47}$$

4.6.2 Inner problem

We can perform a similar analysis for the inner problem. We use the Laplace convolution theorem, or Duhamel's theorem, to modify the boundary condition to be $u(1,\tau) = e^{b\tau}$. There are a number of well known results available.

4.6.2.1 One dimension

In one dimension we have (cf. Carslaw and Jaeger (1959, 3.5 (3)))

$$\begin{aligned}\hat{u}_1(\rho,\tau) = 2\sum_{n=0}^{\infty} e^{-(2n+1)^2\pi\tau/4}&\\ \cos\frac{(2n+1)\pi\rho}{2}&\left\{\frac{(2n+1)\pi(-1)^n}{2((2n+1)^2\pi+b)}\left(\exp(((2n+1)^2\pi+b)\tau)-1\right)\right\}.\end{aligned}$$

4.6.2.2 Two dimensions

In two dimensions we have from Carslaw and Jaeger (1959) 7.6 (12)

$$u(\rho,\tau) = 2\sum_{n=1}^{\infty} e^{-\alpha_n^2\tau}\frac{\alpha_n J_0(\rho\alpha_n)}{J_1(\alpha_n)}\int_0^{\tau} e^{\alpha_n^2\lambda}\phi(\lambda)\,d\lambda\,.$$

With the modified boundary condition $\hat{u}(1,\tau) = \phi(\tau) = e^{b\tau}$ we have

$$\hat{u}_2(\rho,\tau) = 2\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_n^2 + b} \frac{J_0(\rho\,\alpha_n)}{J_1(\alpha_n)} \left(e^{b\tau} - e^{-\alpha_n^2\tau}\right) .$$

4.6.2.3 Three dimensions

Large τ From Carslaw and Jaeger (1959) 9.3 (3) we can find the solution with $e^{b\tau}$ on the boundary for small τ

$$\hat{u}_3(\rho,\tau) = \frac{2\pi}{\rho} \sum_{n=1}^{\infty} (-1)^n \sin(n\pi\rho) \left(\frac{n}{n^2\pi^2 + b}\right) \left(e^{-n^2\pi^2\tau} - e^{b\tau}\right) .$$

Small τ We want to find the convolution solution in the three dimensional case. We start from the solution with $u = 1$ on the boundary.

$$u_3(\rho,\tau) = \frac{1}{\rho} \sum_{n=0}^{\infty} \left(\operatorname{erfc}\left(\frac{2n+1-\rho}{2\sqrt{\tau}}\right) - \operatorname{erfc}\left(\frac{2n+1+\rho}{2\sqrt{\tau}}\right)\right) ,$$

We use the convolution theorem from above. Putting $c_- = 2n+1-\rho$ and $c_+ = 2n+1+\rho$, taking the differential with respect to τ, substituting $y = \eta^{-1/2}$, and using

$$\int_{\tau^{-1/2}}^{\infty} k e^{-k^2y^2/4 - b/y^2}\, dy = \frac{\sqrt{\pi}}{2} \left(e^{-k\sqrt{b}}\operatorname{erfc}\left(\frac{k}{2\sqrt{\tau}} - \sqrt{b\tau}\right) + e^{k\sqrt{b}}\operatorname{erfc}\left(\frac{k}{2\sqrt{\tau}} + \sqrt{b\tau}\right)\right)$$

we obtain

$$\hat{u}_3(\rho,\tau) = \frac{e^{b\tau}}{2\rho} \sum_{n=0}^{\infty} \left(e^{-c_-\sqrt{b}}\operatorname{erfc}(d_1) + e^{c_-\sqrt{b}}\operatorname{erfc}(d_2) - e^{-c_+\sqrt{b}}\operatorname{erfc}(d_3) - e^{c_+\sqrt{b}}\operatorname{erfc}(d_4)\right) ,$$

where

$$d_1 = -\frac{c_-}{2\sqrt{\tau}} + \sqrt{b\tau} \quad \text{and} \quad d_2 = \frac{c_-}{2\sqrt{\tau}} + \sqrt{b\tau}$$
$$d_3 = -\frac{c_+}{2\sqrt{\tau}} + \sqrt{b\tau} \quad \text{and} \quad d_4 = \frac{c_+}{2\sqrt{\tau}} + \sqrt{b\tau} .$$

4.6.2.4 n dimensions

In this section we take the general solution and perform the convolution. If the boundary condition is modified to be $\hat{u}(1,\tau) = \phi(\tau) = e^{b\tau}$ then we have

$$\hat{u}_n(\rho,\tau) = 2\rho^{-\nu} \sum_{m=1}^{\infty} \frac{\alpha_m}{\alpha_m^2 + b} \frac{J_{|\nu|}(\rho\,\alpha_m)}{J_{|\nu|+1}(\alpha_m)} \left(e^{b\tau} - e^{-\alpha_m^2\tau}\right) .$$

Note that in the case of one and three dimensions we write the Bessel functions in terms of sin and cos and the zeros have the simple expression $\alpha_m = m\pi$. We can verify that the solution reduces to those stated above. It is also straightforward to reduce the general case to the two dimensional result. We can verify that in the case $b = 0$ this reduces to the solution $u_n(\rho, \tau)$ by again using (Watson, 1944, 18.12(1)).

4.7 Reverse transformations

Now that we have the solution to some high dimensional problems we transform our results back to the financial variable space. We investigate what classes of financial products we have obtained valuation equations for.

In the case of a single underlying asset we can compare our results with published results. For options on more than one underlying asset the option definition is new.

4.7.1 Single asset case

For the outer problem we compare with single-asset American binary barrier options, by which we mean an option that pays a cash amount at the time a fixed barrier value is hit during the life of the option. Otherwise the option expires worthless. For the inner problem we have an American binary barrier option that pays out when either the upper or lower barrier is hit.

4.7.1.1 Outer problem

In the case of the outer problem we have an American binary barrier option with the barrier below the initial asset price.

If we take the convolution solution in one dimension (4.44) and reverse the transformation (4.5) and the non-dimensionalisations (4.2) we cause the $b\tau$ term in the exponent of (4.5) to cancel. By inspection of (4.13), b is always positive, so the square roots that appear are not problematic. After simplification we find the solution in financial variables to be

$$V(S,t) = E\,e^{a}\left[\left(\frac{E\,e}{S}\right)^{\mu_+}\mathrm{N}\left(z\right) + \left(\frac{E\,e}{S}\right)^{\mu_-}\mathrm{N}\left(z - 2\sigma\sqrt{b(T-t)}\right)\right].$$

where

$$\mu_\pm = -a \pm \sqrt{b} \quad \text{and} \quad z = \frac{\log(Ee/S)}{\sigma\sqrt{T-t}} + \sigma\sqrt{b(T-t)}\,.$$

This is the solution to a cash-at-hit binary barrier option which pays $\$Ee^a$ if the barrier located at $\$Ee$ is hit. This agrees with Rubinstein and Reiner (1991b).

4.7.1.2 Inner problem

When we reverse the transformations to financial variables in the case of the inner problem we obtain double barrier options. For the single asset case we have a double barrier option with barriers at e and $1/e$, both of which relate to a radial distance of $\rho = 1$. The particular style of double barrier option that we require is the double barrier with different rebates on the barriers, and no payoff at maturity. The payoff when the barrier is hit is $\exp(1/2)$.

Equivalent results for the one dimensional case are presented in Pelsser (2000). Results are presented for double barrier options with an initial asset price of $S(t)$ which pay a rebate of K_U when the upper barrier at U is hit, and a rebate of K_U when the upper barrier at U is hit. The value of an option that pays K_U as soon as the barrier at U is hit is given by $V_{RAHU}(t)$,

$$V_{RAHU}(t) = K_U e^{\frac{\mu}{\sigma^2}(l-x)} \left(\frac{\sinh(\frac{\mu'}{\sigma^2} x)}{\sinh(\frac{\mu'}{\sigma^2} l)} - \frac{\sigma^2}{\lambda^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda'_k (T-t)}}{\lambda'_k} k\pi \sin\left(k\pi \frac{l-x}{l} \right) \right),$$

where $l = \log(U/L)$, $x = \log(S(t)/L)$, $\mu = r_d - r_f - \frac{1}{2}\sigma^2$,

$$\mu' = \sqrt{\mu^2 + 2\sigma^2 r_d} \quad \text{and} \quad \lambda'_k = \frac{1}{2}\left(\frac{\mu'^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2} \right).$$

The value of the corresponding lower barrier option is $V_{RAHL}(t)$,

$$V_{RAHL}(t) = K_L e^{\frac{\mu}{\sigma^2} x} \left(\frac{\sinh(\frac{\mu'}{\sigma^2}(l-x))}{\sinh(\frac{\mu'}{\sigma^2} l)} - \frac{\sigma^2}{\lambda^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda'_k (T-t)}}{\lambda'_k} k\pi \sin\left(k\pi \frac{x}{l} \right) \right).$$

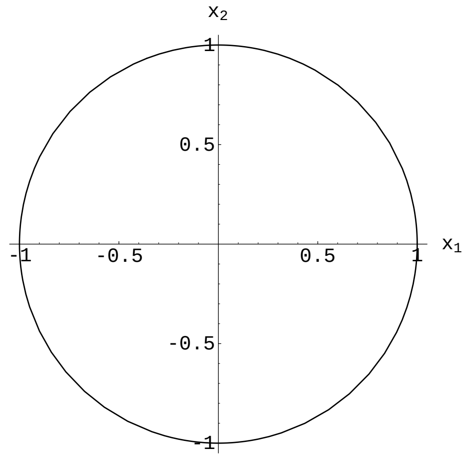
The value of a double barrier option equivalent to a single asset radial option is given by $V_{RAHL}(t) + V_{RAHU}(t)$.

4.7.2 General radial barrier option

Recall the transform (4.5),

$$v(\mathbf{x}, \tau) = e^{\mathbf{a}.\mathbf{x} - b\tau} u(\mathbf{x}, \tau).$$

We have obtained radially symmetric solutions for u. We made additional transformations to change $\mathbf{x}$ into an orthonormal basis $\mathbf{z}$. We have found solutions $u(\rho, \tau)$



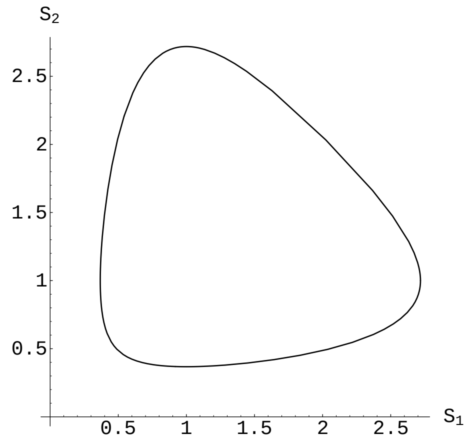


Figure 4.6: The location of the radial barrier for two independent assets. The option can be defined either inside, or outside, the barrier. The assets have equal volatilities $\sigma_1 = 1$ and $\sigma_2 = 1$, and the correlation between the assets is $\rho_{12} = 0$. The interest rate is $r = 0.05$ and the continuous dividend yields are $q_1 = q_2 = 0$, and the strike is $E = 1$.

where $\rho^2 = \sum z_i^2$. In matrix notation this is $\rho^2 = \mathbf{z}^T\mathbf{z}$. In terms of the $\mathbf{x}$ basis we have

$$\rho^2 = (\mathbf{DQx})^T(\mathbf{DQx}) = \mathbf{x}^T\mathbf{Ax}\,,$$

using (4.9) and the facts that $\mathbf{Q}^T\mathbf{Q} = 1$ and $\mathbf{A}$ is symmetric.

Now, we can write

$$\mathbf{a}.\mathbf{x} = |\mathbf{a}||\mathbf{x}|\cos\theta$$

where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{x}$. We have defined our option to have value $u = 1$ when $\rho = 1 = \mathbf{x}^T\mathbf{A}^2\mathbf{x} = |\mathbf{Ax}|$. This is a barrier option which pays out when the barrier is hit. Therefore, reversing the transformation (4.5) and the non-dimensionalisation, and using the convolution theorem, we have payoffs of the form

$$V(|\mathbf{Ax}| = 1, \tau) = E\,e^{|\mathbf{a}||\mathbf{x}|\cos\theta}\,,$$

when the radial barrier is hit. When $\mathbf{a}.\mathbf{x} = 0$ the vectors $\mathbf{a}$ and $\mathbf{x}$ are perpendicular, and the cash payoff when the barrier is hit at this point is simply $\$E$. However, general results for modifying the payoff to something more financially intuitive on the remainder of the barrier is left for future research.

4.7.2.1 Two asset radial option

We can however investigate the location of the barrier. The barrier is distorted due to the transformations involved in reducing the multi–asset Black–Scholes equation to the heat equation. We first consider the case of independent assets with equal volatilities. The left-hand picture in Figure 4.6 shows that the barrier is a circle in x co-ordinates and z co-ordinates. After the transformation back to S co-ordinates, using $x_i = \log(S_i/E)$, we see that the boundary has been stretched by the exponential function.

Figure 4.7 shows the location of the barrier for another set of parameter values. In normalised z coordinates the barrier is still a circle of radius 1. The option can be defined inside or outside the barrier. Once we reverse the transformations of the stretching matrix $\mathbf{D}$ and rotation $\mathbf{Q}$ the barrier location in x coordinates is an ellipse. However, once back in S coordinates the barrier is severely stretched due to the exponential relationship between x and S. We can see in the third diagram in Figure 4.7 shows that it is possible for the asset prices to follow a path around the outside of the barrier, without hitting the barrier, and approach the origin. In practice, the stretching is so severe that the barrier acts like a down-and-out barrier.

The barrier is defined to be at radial distance $\rho = 1$. In the outer case the option is defined for $\rho \geq 1$. In the inner case the option is defined for $0 \leq \rho \leq 1$. However, due to the coordinate transformations the value of ρ is not intuitive in S coordinates. It is instructive to plot the value of ρ for some values of the asset prices. Figure 4.8 shows that the value of ρ grows very slowly away from the barrier, due to the

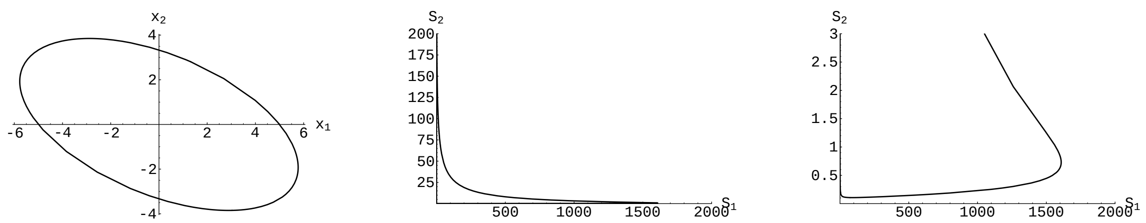


Figure 4.7: The location of the barrier for a radial barrier option on two assets. The option can be defined either inside or outside the barrier. The left graph shows the location of the barrier in the x coordinates, the middle graph shows the barrier in S coordinates, and the right hand graph shows a zoomed in portion of the barrier in S coordinates. The parameters are, interest rate $r = 0.05$, continuous dividend yields $q_1 = q_2 = 0$, volatilities $\sigma_1 = 0.2$ and $\sigma_2 = 0.3$, correlation between the assets $\rho_{12} = 0.5$, and strike price $E = 10$.

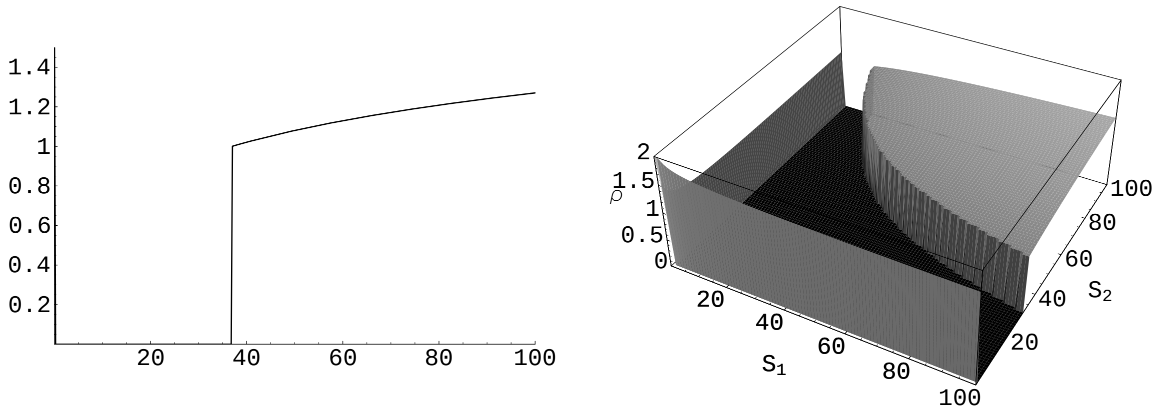


Figure 4.8: The value of ρ for different asset values. The left graph shows values of ρ for a slice through the asset value $S_2 = 80$, the right hand graph shows the value of ρ for both S coordinates. The parameters are interest rate $r = 0.05$, continuous dividend yields $q_1 = q_2 = 0$, volatilities $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, correlation between the assets $\rho_{12} = 0.5$, and strike price $E = 5$.

logarithm in the transform between x and S coordinates.

4.7.2.2 Three asset radial option - outer problem

We take the three asset solution (4.45), and reverse the transformations, as in the previous section,

$$V = \frac{E\, e^{\mathbf{a}.\mathbf{x}}}{\sqrt{\mathbf{x^T A x}}} \left[e^{\sqrt{b\,\mathbf{x^T A x}}} \mathrm{N}\left(z\right) + e^{-\sqrt{b\,\mathbf{x^T A x}}} \mathrm{N}\left(z - 2\sigma\sqrt{b(T-t)}\right)\right] \, ,$$

where

$$z = \frac{\sqrt{\mathbf{x^T A x}} - 1}{\sigma\sqrt{T-t}} + \sigma\sqrt{b(T-t)}\, .$$

The value of the option is again distorted by the $e^{\mathbf{a}.\mathbf{x}}$ factor. For these parameter values we have $\mathbf{a} = \{-2.83642, -2.83642, -2.83642\}^T$, and $b = 5.45556$. We can see the value of the option intersecting the plane $V = 1$ in Figure 4.9.

4.7.2.3 Three dimensional option - Inner problem

The expression in financial variables for the three asset option is rather long. We just comment that the payoff when the boundary $\rho = 1$ is hit is $E\, e^{\mathbf{a}.\mathbf{x}}$.

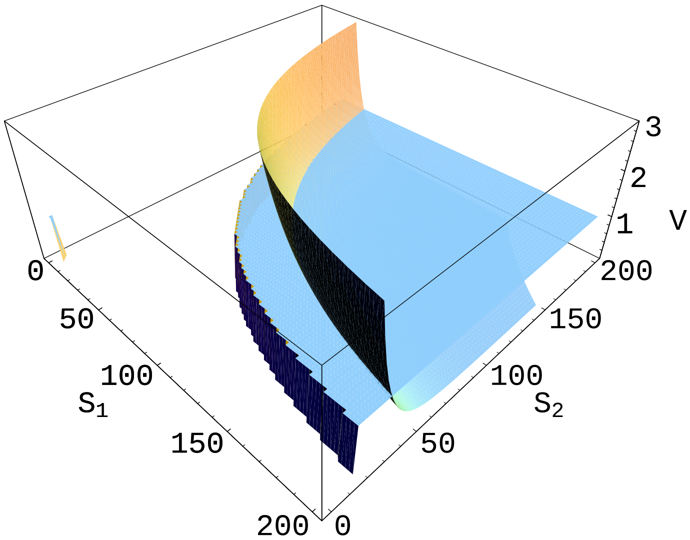


Figure 4.9: The value of the radial option for three assets. The parameters are, interest rate $r = 0.05$, continuous dividend yields $q_1 = q_2 = 0$, volatilities $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$ correlation between the assets $\rho_{ij} = 0.5, i \neq j$, and strike price $E = 20$.

4.8 Numerical results

Though the barriers have a strange shape we can verify the analytic solutions via Monte Carlo simulation. We investigate a number of numerical examples for the inner and outer problems. We provide comparisons between the analytic solution and approximate solutions obtained by Monte Carlo simulation. We do this for the case of a fixed boundary.

4.8.1 Implementation

The analytic results were verified using Mathematica (2004) to do symbolic differentiation and ensure that the solution satisfied the partial differential equation. The numerical integrals, required in the case of an even number of underlying assets, were done using `NIntegrate`, an adaptive integration function in Mathematica. Graphs were exported directly from Mathematica.

The Monte Carlo simulations we performed using Matlab (2004). A number of simulations were performed to calculate standard errors and confidence intervals. The random numbers were generated using the built-in routine `randn`.

	ρ	τ	1	2	3	5
u	0.9	0.02	0.61708	0.65177	0.68564	0.74970
u	1.1	0.02	0.61708	0.58914	0.56098	0.50461

Table 4.1: Values of the function u for inner and outer problems. A payoff of 1 is given when the barrier at 1 is hit. $\rho = 0.9$ corresponds to the inner problem and $\rho = 1.1$ corresponds to the outer problem.

n	r	q_i	ρ_{ij}	a_i	b
1	0	0	-	0.5	0.25
2	0	0	0	0.5	0.5
3	0	0	0	0.5	0.75
5	0	0	0	0.5	1.25
1	0.05	0.02	-	-0.25	0.6875
2	0.05	0.02	0	-0.25	2.6250
3	0.05	0.02	0	-0.25	2.6875
5	0.05	0.02	0	-0.25	2.8125

Table 4.2: Values of the parameters a and b. All volatilities are equal, $\sigma_i = 0.2$.

4.8.2 Numerical results

First we investigate realistic values for the parameters involved in our model. The volatility of asset returns, σ, is usually in the range 0.1 to 0.4. The time to expiry, in years, will be between 0.1 and 10. This means that the non-dimensional time to expiry, τ, will be in the range 0.0005 to 0.8. For an intermediate value we use $\tau = 0.02$. In Table 4.1 we show numeric values for the function u for the inner and outer problem in a number of dimensions. For the inner problem we have $\rho = 0.9$, for the outer problem $\rho = 1.1$. We verified these Laplace inversions numerically using Mathematica.

We show values for the intermediate parameters a and b in Table 4.2. As all the q_i are equal all the a_i are equal. b is a scalar.

Table 4.3 shows the results for a selection of options where the asset is initially outside the barrier. We set the parameter $E = 1$ and obtain a payment of $e^{a.x}$, except in the first two cases, where we compare to a regular digital option. In higher dimensions we start with all S_i equal as such a level as to give a radial distance of $\rho = 1.1$ outside the barrier at $\rho = 1$. We use 10,000 time steps in batches of 100

r	q_i	ρ_{ij}	S_0	Pay	n	Analytic	M.C. (bounds)
0	0	-	1.1	1	1	0.6636	0.6696 (0.6524, 0.6868)
0.05	0.02	-	1.1	1	1	0.6091	0.6098 (0.5931, 0.6266)
0	0	-	$e^{1.1}$	e^a	1	1.0678	1.0635 (1.0360, 1.0909)
0	0	0	2.1767	$e^{a.x}$	2	1.2779	1.2676 (1.2326, 1.3025)
0	0	0	1.8872	$e^{a.x}$	3	1.4472	1.4369 (1.3969, 1.4770)
0	0	0	1.6355	$e^{a.x}$	5	1.7125	1.6838 (1.6382, 1.7294)
0.05	0.02	-	$e^{1.1}$	e^a	1	0.4611	0.4577 (0.4444, 0.4709)
0.05	0.02	0	2.1767	$e^{a.x}$	2	0.3930	0.3872 (0.3759, 0.3985)
0.05	0.02	0	1.8872	$e^{a.x}$	3	0.3424	0.3361 (0.3251, 0.3470)
0.05	0.02	0	1.6355	$e^{a.x}$	5	0.2680	0.2687 (0.2582, 0.2792)

Table 4.3: Values of outer radial options analytically and by Monte Carlo simulation. $\sigma = \sigma_i = 0.2$, $E = 1$, $S_0 = 1.1$, $S_i = E, i \neq 0$, $T = 1$. The Monte Carlo simulations were run with 10,000 time steps in 100 batches of 100 runs. The bounds are to a 99.95% confidence interval. We ensure that $\rho = 1.1$ for all initial data.

r	q_i	ρ_{ij}	S_0	Pay	n	Analytic	M.C. (bounds)
0	0	-	$e^{0.9}$	e^a	1	0.9662	0.9528 (0.9247, 0.9810)
0	0	0	1.8896	$e^{a.x}$	2	1.2098	1.2057 (1.1727, 1.2388)
0	0	0	1.6814	$e^{a.x}$	3	1.4960	1.4739 (1.4380, 1.5098)
0	0	0	1.4956	$e^{a.x}$	5	-	1.8870 (1.8448, 1.9291)
0.05	0.02	-	$e^{0.9}$	e^a	1	0.4848	0.4768 (0.4650, 0.4885)
0.05	0.02	0	1.8896	$e^{a.x}$	2	0.43	0.4662 (0.4548, 0.4777)
0.05	0.02	0	1.6814	$e^{a.x}$	3	0.4563	0.4518 (0.4407, 0.4630)
0.05	0.02	0	1.4956	$e^{a.x}$	5	-	0.4585 (0.4489, 0.4682)

Table 4.4: Values of inner radial options analytically and by Monte Carlo simulation. $\sigma = \sigma_i = 0.2$, $E = 1$, $S_0 = 0.9$, $S_i = E, i \neq 0$, $T = 1$. The Monte Carlo simulations were run with 10,000 time steps in 100 batches of 100 runs. The bounds are to a 99.95% confidence interval. Inner options always have the radial distance set to $\rho = 0.9$.

paths and repeat each batch 100 times. The confidence intervals are to 99.95%, but note that the Monte Carlo estimator is biased due to the discretization at the barrier. However, all Monte Carlo estimates are close to the analytic solution.

Table 4.4 shows results in financial variables for the inner problem. Note that in the single asset case the equivalent regular financial option is a double barrier option with rebates, not an up pay-at-hit digital option. We can use the results in Pelsser (2000), for example, to calculate the option value in the case of a single asset. We need to set the upper barrier at e and the lower barrier at $1/e$. The asset starts at a radial distance on 0.9, which means $S = e^{0.9}$. The payment of e^a is made at the time the barrier is hit. We again use 10,000 time steps in batches of 100 paths, and repeat 100 times to calculate the confidence intervals.

Here the series solution did not converge well for two and five dimensions. This point requires future investigation. However, the one and three dimensional cases are within the confidence interval.

4.9 Qualitative behaviour

The variation in the option value due to the radial distance ρ is dominated by the contribution from the transformation (4.5),

$$v(\mathbf{x}, \tau) = e^{\mathbf{a}.\mathbf{x} - b\tau} u(\mathbf{x}, \tau)\,.$$

For the unequal volatilities and correlations used in Section 4.6 we find parameter values $\mathbf{a} = \{-27.6111, -4.18529\}^T$ and $b = 41.5048$. We have seen that it is possible to remove the $b\tau$ factor by using a convolution. However, the $e^{\mathbf{a}.\mathbf{x}}$ factor is more problematic. The left hand image in Figure 4.10 shows the option value for this set of parameters. The exponential factor causes a extreme change in the option values, resulting in unrealistic values for a financial option.

Notice that the option value slopes downwards, from above 1, to below one. The value function is analytic, so by the intermediate value theorem the value 1 must have been crossed. Therefore it is possible to find a curve on the surface where the option payoff is 1. We can see this in the right hand image in Figure 4.10.

Having done the numerical convolution in the two dimensional case we know that the value on the barrier $\rho = 1$ is fixed, and that the value on the barrier itself is constant in time. When the line $\mathbf{a}.\mathbf{x} = 0$ intersects the barrier, where $u(1, \tau) = 1$), we

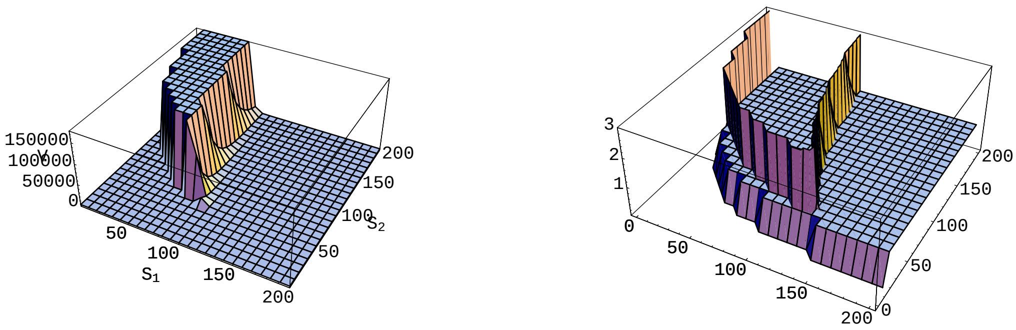


Figure 4.10: The value of a two asset radial option. The left hand graph is the value, and the right hand graph shows the intersection with a plane of value 1. The parameters are; interest rate $r = 0.05$, continuous dividend yields $q_1 = q_2 = 0.0$, volatilities $\sigma_1 = 0.2$ and $\sigma_2 = 0.3$, correlation $\rho_{12} = 0.5$, strike price $E = 5$. The value strike modifier is 10^{44}. The two asset prices are given over the range $S_1 = [1, 200]$ and $S_2 = [1, 200]$. The nondimensional time to expiry is $\tau = 0.02$, which relates to an expiry $T = 1$. These values give parameters $\mathbf{a} = \{-27.6111, -4.18529\}^T$ and $b = 41.5048$.

have a barrier option that pays out \$1 when the barrier is hit. Finding the location of this barrier analytically is left for future research.

4.9.1 Three asset options

When there are three underlying assets we have a closed form solution for the asset price, so do not need to use numerical integration to evaluate the solution.

The location of the barrier behaves similarly to the two asset case. The barrier is an ellipsoid in x coordinates, on the left of Figure 4.11. Again we have extreme stretching in the S coordinates (center and right of Figure 4.11).

The value of the option is again distorted by the $e^{\mathbf{a}.\mathbf{x}}$ factor. For these parameter values we have $\mathbf{a} = \{-2.83642, -2.83642, -2.83642\}^T$, and $b = 5.45556$. We can see the value of the option intersecting the plane $V = 1$ in Figure 4.9. This graph seems relatively well behaved.

4.10 Conclusions

We have derived analytic valuation expressions for multi-asset barrier options. These "Radial Barrier Options" may be of use in the financial markets, as benchmark cases

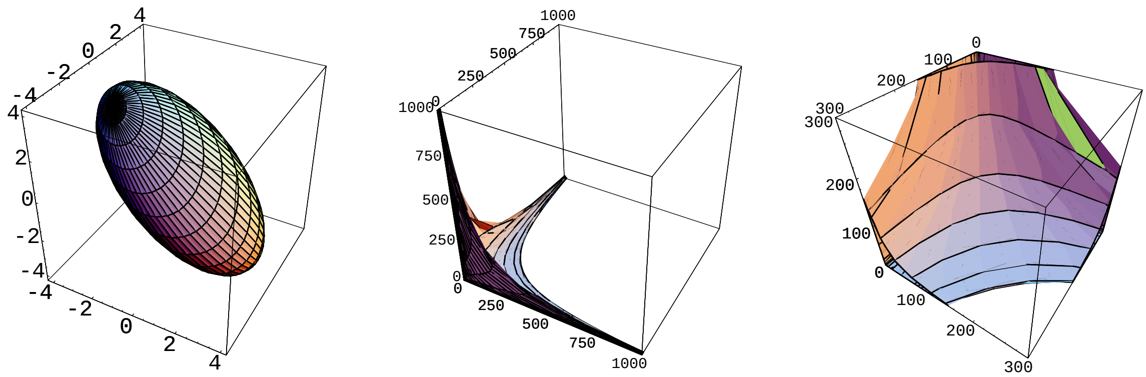


Figure 4.11: The location of the barrier for a radial barrier option on three assets. The option is defined outside the barrier. The left graph shows the location of the barrier in the x coordinates, the middle graph shows the barrier in S coordinates, and the right hand graph shows a zoomed in portion of the barrier in S coordinates. The parameters are, interest rate $r = 0.05$, continuous dividend yields $q_1 = q_2 = 0$, volatilities $\sigma_1 = \sigma_2 = \sigma_3 = 0.3$ correlation between the assets $\rho_{ij} = 0.5, i \neq j$, and strike price $E = 20$.

for other numerical methods for high dimensional options, or as an approximation to other, actively traded, financial options.

Hedging these options may be an interesting problem. In the case of a single underlying asset they have the same hedging difficulties as standard single-asset barrier options. Investigating how this problem applies in the multi-asset case may be worthwhile.

Future research could involve further modification of the boundary conditions to define more realistic financial options. This approach can be extended to options defined within a barrier. It may also be possible to use this style of barrier options to model defaults in credit derivatives.

These results are closely related to the work of Göing-Jaeschke and Yor (2003) on Bessel processes, as discussed in the next chapter.

Chapter 5

Bessel Process Hitting Times

5.1 Introduction

The inverse Laplace transforms obtained in the previous chapter describe the hitting time of a Bessel process started in x to a level y. There is a large body of research on Bessel processes. The standard reference is Revuz and Yor (1999) (see also Rogers and Williams (2000a) and Rogers and Williams (2000b)). Recent reviews focusing on Bessel processes include Göing-Jaeschke and Yor (2003) and Biane et al. (2001). The standard definition of the Bessel process $R_t^{(\nu)}$ for $\nu > -1$ has been extended to all $\nu \in \mathbb{R}$ in Göing-Jaeschke and Yor (2003). Bessel processes of negative order are also used in Davydov and Linetsky (2001a).

Much of the recent interest in Bessel processes has stemmed from the results in pricing Asian options in Geman and Yor (1993). The result is based on the Lamperti relationship (Lamperti, 1972) which links the exponential of a Brownian motion, B_t, with drift to a time-changed Bessel process.

$$\exp(B_t + \nu t) = R^{(\nu)} \left(\int_0^t \exp\left(2(B_s + \nu s)\right) ds \right) .$$

Since then numerous papers have applied Bessel processes to Asian option pricing (Carr and Schroder, 2004; Chesney et al., 1997; Linetsky, 2002).

Bessel processes with additional drift terms have been used implicitly in finance for longer, see Linetsky (2004c) for definitions and a review. They have been used in interest rate modelling as the CIR process (Cox and Rubinstein, 1985) in equity modelling as CEV processes (Cox and Ross, 1976) and in credit modelling (Duffie and Singleton, 1999).

Simulation and numerical work focusing on Bessel processes in their own right is limited. Notable is the spectral expansion approach of Linetsky (2004b,c). Simulation of Bessel processes can be done for integer dimensions by simulating the underlying Brownian motions, but this is inefficient. The SDE describing the Bessel process can be simulated using a standard Euler discretization, however, this can lead to bias in the vicinity of zero (Glasserman, 2004). Exact simulation can be performed by sampling from a non-central chi-squared distribution.

In Section 5.2 we present definitions for Bessel processes, and the existing results giving the hitting times of Bessel processes as Laplace transforms. We apply the Laplace inversion techniques of Chapter 4 in Section 5.3. In Section 5.4 we present some asymptotic results for small values of time. Numerical results are presented in Section 5.5 and conclusions drawn in Section 5.6.

5.2 Definitions and existing results

We follow the definitions and notation for Bessel processes in Borodin and Salminen (1996). The generator, $\mathcal{G}^{(\nu)}$, of a Bessel process of order ν is

$$\mathcal{G}^{(\nu)} = \tfrac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}$$

with the parameter δ defined as $\delta = 2\nu+2$. If $B_{i,t}$ are independent Brownian motions then

$$\sum_{i=1}^{\delta} B_{i,t}^2 = \left(R_t^{(\nu)}\right)^2 .$$

The process $R_t^{(\nu)}$ has a physical interpretation as the radial distance from the origin of a δ dimensional Brownian motion. The process $R_t^{(\nu)}$ is defined on $(0,\infty)$ or $[0,\infty)$. The boundary condition at 0 is

- $\nu \le -1$: exit-not-entrance
- $-1 < \nu < 0$: non-singular
- $0 \le \nu$: entrance-not-exit

We define $T_y := \inf\{t : R_t^{(\nu)} = y\}$ to be the hitting time of a Bessel process to a level y. The process $R_t^{(\nu)}$ is the unique strong solution of

$$\begin{cases} dR_t^{(\nu)} &= dB_t + \dfrac{2\nu+1}{2dR_t^{(\nu)}}\,dt \qquad t > 0 \\ dR_0^{(\nu)} &= x > 0 \end{cases}$$

for $\nu \geq 0$ and B_t a standard Brownian motion. No classical SDE exists for $-1 < \nu < 0$. For $\nu \leq -1$ there is a unique solution up to the "explosion time" $T_0 := \inf\{t : R_t^{(\nu)} = 0\}$ (Göing-Jaeschke and Yor, 2003).

5.2.1 Bessel process hitting times

The Laplace transform for the hitting time T_y down to a level y of a Bessel process started in x for $0 < y < x < \infty$ is known for all $\nu \in \mathbb{R}$ (Borodin and Salminen, 1996; Getoor, 1979; Getoor and Sharpe, 1979; Göing-Jaeschke and Yor, 2003; Kent, 1978, 1982),

$$\mathbb{E}_x\left[\exp(-\lambda T_y)\right] = \left(\frac{x}{y}\right)^{-\nu} \frac{K_\nu(x\sqrt{2\lambda})}{K_\nu(y\sqrt{2\lambda})}. \tag{5.1}$$

The Laplace transform for the hitting time up to a level y of a Bessel process started in x, where $0 < x < y < \infty$ is more difficult. The case $\nu > -1$ is known (Borodin and Salminen, 1996; Kent, 1978) to be

$$\mathbb{E}_x\left[\exp(-\lambda T_y)\right] = \left(\frac{x}{y}\right)^{-\nu} \frac{I_\nu(x\sqrt{2\lambda})}{I_\nu(y\sqrt{2\lambda})}.$$

Göing-Jaeschke and Yor (2003) investigate Bessel processes with $\nu < 0$, however no results on the hitting time up were presented. However, Davydov and Linetsky (2001a) give the Laplace transform for $\nu \leq -1$ as,

$$\mathbb{E}_x\left[\exp(-\lambda T_y)\right] = \left(\frac{x}{y}\right)^{-\nu} \frac{I_{|\nu|}(x\sqrt{2\lambda})}{I_{|\nu|}(y\sqrt{2\lambda})}. \tag{5.2}$$

The hitting time of a process $R_t^{(\nu)}$ is related to its running maximum by

$$\mathbb{P}_x\left(T_y < t\right) = \mathbb{P}_x\left(\sup_{0 \leq s \leq t} R_s^{(\nu)} > y\right).$$

5.2.2 Existing results

Many existing results are included in Borodin and Salminen (1996). For reflecting Brownian motion, Bessel process $\nu = -1/2$, the density of the hitting time down is

$$\text{(3) 2.02} \qquad \mathbb{P}_x\left(T_y \in dt\right) = \frac{(x-y)}{\sqrt{2\pi}\, t^{3/2}} \exp\left(-\frac{(x-y)^2}{2t}\right) dt \qquad y \leq x\,.$$

For Bessel process $\nu = 1/2$ we have

$$\text{(5) 2.02} \qquad \mathbb{P}_x\left(T_y \in dt\right) = \frac{y}{x} \frac{(x-y)}{\sqrt{2\pi}\, t^{3/2}} \exp\left(-\frac{(x-y)^2}{2t}\right) dt \qquad y \leq x\,.$$

The density for the hitting time up for $\nu > 0$ where $0 \leq x \leq y$ is well known (Borodin and Salminen, 1996, p. 269 Eq (1.1.4)),

$$\mathbb{P}_x\left(\sup_{0\leq s\leq t} R_s^{(\nu)} \geq y\right) = 1 - 2\left(\frac{x}{y}\right)^{-\nu} \sum_{k=1}^{\infty} \frac{J_\nu(\frac{x}{y} j_{\nu,k})}{j_{\nu,k} J_{\nu+1}(j_{\nu,k})}\, e^{-j_{\nu,k}^2 t/(2y^2)}\,. \tag{5.3}$$

This is usually obtained by assuming a Fourier-Bessel series expansion.

These results are closely related to results concerning the maximum of a Bessel bridge (Pitman and Yor, 1998). Many identities between Brownian motion and Bessel processes are discussed in Biane et al. (2001). Other related results can be found in the theory of heat conduction. For example Carslaw and Jaeger (1959, 13.5(I)) corresponds to the hitting time down of a two dimensional Bessel process.

5.2.2.1 Spectral expansion

Building on the work of Kent (1982) a number of results have recently been published by Linetsky (Linetsky, 2004a,b,c). The hitting time density can be written as an eigenfunction expansion.

Hitting down Linetsky's solution for the hitting time down, which does not need killing, where $0 < y < x < \infty$ for $\nu < -1$ is

$$\begin{aligned} &\mathbb{P}_x\left(T_y \leq t\right) = 1 - \\ &\frac{2}{\pi}\left(\frac{x}{y}\right)^{-\nu} \int_0^\infty e^{-s^2 t/2}\, \frac{[Y_{-\nu}(xs)J_{-\nu}(ys) - Y_{-\nu}(ys)J_{-\nu}(xs)]}{[J_{-\nu}^2(ys) + Y_{-\nu}^2(ys)]} \frac{ds}{s}\,. \end{aligned} \tag{5.4}$$

The speed density is not integrable for $\nu > -1$ so the spectral expansion is not valid. This problem can be avoided by introducing an upper barrier at a high level.

Hitting up Linetsky (2004b) finds the density of the hitting time up for the Bessel process killed at 0 where $0 < x < y < \infty$ with $\nu < 0$,

$$\mathbb{P}_x\left(T_y \le t\right) = \left(\frac{x}{y}\right)^{-2\nu} - 2\left(\frac{x}{y}\right)^{-\nu} \sum_{n=1}^{\infty} \exp\left(-\frac{j_{-\nu,n}^2 t}{2y^2}\right) \frac{J_{-\nu}(\frac{x}{y} j_{-\nu,n})}{j_{-\nu,n} J_{-\nu+1}(j_{-\nu,n})}\,. \tag{5.5}$$

5.3 Inversion of Laplace transforms

It is possible to invert these Laplace transforms analytically, though the results are not available in closed form in all cases. We first present a simple example performing the inversion in the case $\nu = 1/2$. We then present the inversion for the hitting time down, and then the hitting time up.

5.3.1 Example

Take the example case of a three-dimensional Bessel process, where $\nu = 1/2$. Let $0 < y < x < \infty$ so the Laplace transform in time, of the hitting time down to the level y is given by (5.1) for $\nu = 1/2$,

$$\mathbb{E}_x\left[\exp(-\lambda T_y)\right] = \left(\frac{x}{y}\right)^{-1/2} \frac{K_{1/2}(x\sqrt{2\lambda})}{K_{1/2}(y\sqrt{2\lambda})} = \frac{x}{y}\exp\left((y-x)\sqrt{2\lambda}\right)\,,$$

using the identities for spherical Bessel functions. If we now invert this Laplace transform with respect to the transform parameter λ we obtain the density

$$\mathbb{P}_x\left(T_y \in (t, t+dt)\right) = \frac{y}{x}\frac{(x-y)}{t^{3/2}\sqrt{2\pi}} \exp\left(-(x-y)^2/2t\right)\,dt\,,$$

which we then integrate with respect to t over $(0, T)$ to find the distribution

$$\mathbb{P}_x\left(T_y \le T\right) = \frac{y}{x}\operatorname{erfc}\left(\frac{x-y}{\sqrt{2T}}\right)\,.$$

However, it is easier to use the properties of Laplace transforms to do the integration in transform space. Let v be the cumulative distribution function. We know the Laplace transform of the density $\mathcal{L}\left\{\partial v/\partial t\right\}$. Therefor the distribution is given by

$$v = v_0 + \mathcal{L}^{-1}\left\{\frac{1}{\lambda}\mathcal{L}\left\{\frac{\partial v}{\partial t}\right\}\right\}\,,$$

and we know that $v_0 = 0$. If we divide through by the transform variable λ and invert we get directly to the result

$$\mathbb{P}_x\left(T_y \le T\right) = \mathcal{L}^{-1}\left\{\frac{1}{\lambda}\frac{x}{y}\exp\left((y-x)\sqrt{2\lambda}\right)\right\} = \frac{y}{x}\operatorname{erfc}\left(\frac{x-y}{\sqrt{2T}}\right)\,.$$

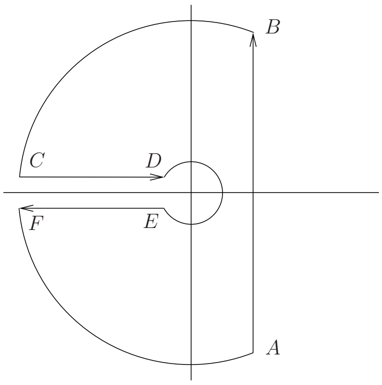


Figure 5.1: The keyhole contour for inverting the Laplace transform

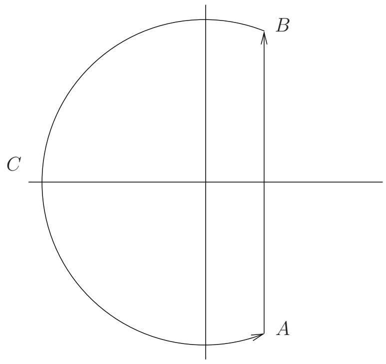


Figure 5.2: The contour for inverting the Laplace transform when there is no branch cut in the integrand.

5.3.2 Hitting down

We need to invert the Laplace transform (5.1). First divide through by λ so that we obtain the distribution. Invert the solution using the Laplace inversion theorem,

$$\mathbb{P}_x\left(T_y \le t\right) = \left(\frac{x}{y}\right)^{-\nu} \frac{1}{2\pi\, i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{K_\nu(x\sqrt{2\lambda})}{K_\nu(y\sqrt{2\lambda})} \frac{d\lambda}{\lambda}\,. \tag{5.6}$$

As $K_{-\nu}(z) = K_\nu(z)$, we can see that the $\delta = 1$ and $\delta = 3$ cases will differ only in factors of (x/y), which is indeed the case.

The integrand has a branch point at $\lambda = 0$, so we compute the integral using the contour in Figure 5.1. The details follow through as for radial barrier options and we obtain the general solution, $\forall \nu \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}_x\left(T_y \le t\right) = {} & \left(\frac{x}{y}\right)^{-\nu-|\nu|} \\ & -\frac{2}{\pi}\left(\frac{x}{y}\right)^{-\nu} \int_0^\infty e^{-\zeta^2 t/2} \frac{J_\nu(x\,\zeta)Y_\nu(y\,\zeta) - J_\nu(y\,\zeta)Y_\nu(x\,\zeta)}{J_\nu(y\,\zeta)^2 + Y_\nu(y\,\zeta)^2} \frac{d\zeta}{\zeta}\,.\end{aligned} \tag{5.7}$$

Compare with the recent result of Linetsky (2004b) reproduced as (5.4).

5.3.3 Hitting up

For the hitting time up we have to invert (5.2). Recall that $0 < x < y < \infty$. First divide through by λ to obtain the distribution. The Laplace inversion theorem says

$$\mathbb{P}_x\left(T_y \le t\right) = \left(\frac{x}{y}\right)^{-\nu} \frac{1}{2\pi\, i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_{|\nu|}(x\sqrt{2\lambda})}{I_{|\nu|}(y\sqrt{2\lambda})} \frac{d\lambda}{\lambda}\,. \tag{5.8}$$

The integrand has no branch cut, and we invert the transform as in radial barrier options using the contour in Figure 5.2 to obtain, $\forall \nu \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}_x\left(T_y \le t\right) = &\left(\frac{x}{y}\right)^{-\nu+|\nu|} \\ &-2\left(\frac{x}{y}\right)^{-\nu} \sum_{m=1}^{\infty} \exp\left(\frac{-j_{|\nu|,m}^2 t}{2y^2}\right) \frac{J_{|\nu|}(\frac{x}{y} j_{|\nu|,m})}{j_{|\nu|,m}\, J_{|\nu|+1}(j_{|\nu|,m})}\end{aligned} \tag{5.9}$$

where $j_{|\nu|,m}$ are the zeros $J_{|\nu|}(j_{|\nu|,m}) = 0$ on the positive real line. This is a generalisation of the result for $\nu > 0$ from Borodin and Salminen (1996, p. 269 Eq (1.1.4)) reproduced as (5.3), and the recent result of Linetsky (2004b) reproduced as (5.5).

5.4 Asymptotics solutions for small t

5.4.1 Hitting down

Unfortunately we can see that (5.7) does not converge well for small values of t. To find solutions for small t we expand the Laplace transform (5.1) as an asymptotic series for large values of the transform parameter λ. We follow the approach in Carslaw and Jaeger (1959, 13.5) and use Abramowitz and Stegun (1974, 9.7.2). For the down hitting time we have $0 < y < x < \infty$. We have, using $p = \lambda$, $q = \sqrt{2\lambda}$ and $\mu = 4\nu^2$

$$\left(\frac{x}{y}\right)^{-\nu-\frac{1}{2}} \frac{e^{-q\,(x-y)}}{p} \left\{1 + \frac{\mu-1}{8xq} + \frac{(\mu-1)(\mu-9)}{2(8xq)^2} + \mathcal{O}\left(\frac{1}{q^3}\right)\right\} \Big/ \left\{1 + \frac{(\mu-1)}{8yq} + \frac{(\mu-1)(\mu-9)}{2(8yq)^2} + \mathcal{O}\left(\frac{1}{q^3}\right)\right\}.$$

We can expand this, using the binomial theorem, as

$$\left(\frac{x}{y}\right)^{-\nu-\frac{1}{2}} \frac{e^{-q\,(x-y)}}{p} \left\{1 - \frac{(\mu-1)(x-y)}{8xyq} + \frac{(\mu-1)(x-y)(x(\mu+7) - y(\mu-9))}{2(8xy)^2 q^2} + \mathcal{O}\left(\frac{1}{q^3}\right)\right\}.$$

Inverting the Laplace transform term by term using Abramowitz and Stegun (1974, 29.3.86) gives

$$\begin{aligned}\mathbb{P}_x\left(T_y \le t\right) = \left(\frac{x}{y}\right)^{-\nu-\frac{1}{2}} \Bigg\{ &\operatorname{erfc}\left(\frac{x-y}{\sqrt{2t}}\right) \\ &-\frac{\sqrt{2t}\,(\mu-1)(x-y)}{8}\,\mathrm{i}^1\operatorname{erfc}\left(\frac{x-y}{\sqrt{2t}}\right) \\ &+\frac{t\,(\mu-1)(x-y)(x(\mu+7)-y(\mu-9))}{8^2}\,\mathrm{i}^2\operatorname{erfc}\left(\frac{x-y}{\sqrt{2t}}\right)+\dots\Bigg\},\end{aligned} \tag{5.10}$$

where $\mathrm{i}^n\operatorname{erfc}(.)$ is the iterated error function (Abramowitz and Stegun, 1974).

5.4.2 Hitting up

The solution (5.9) does not converge well for small t either. As above we expand the Laplace transform (5.2) as an asymptotic series for large values of the transform parameter λ. We then invert term by term to obtain a series for the small time behaviour. We follow the approach in Carslaw and Jaeger (1959, 13.3) and use Abramowitz and Stegun (1974, 9.7.1). For the up hitting time we have $0 < x < y < \infty$. Set $q = \sqrt{2\lambda}$ and $p = \lambda$ and $\mu = 4\nu^2$ so

$$\begin{aligned}&\left(\frac{x}{y}\right)^{-\nu-\frac{1}{2}} e^{-q(y-x)} \left\{1-\frac{\mu-1}{8xq}+\frac{(\mu-1)(\mu-9)}{2(8xq)^2}+\mathcal{O}\left(\frac{1}{q^3}\right)\right\} \\ &\qquad \Bigg/ \left\{1-\frac{\mu-1}{8yq}+\frac{(\mu-1)(\mu-9)}{2(8yq)^2}+.\mathcal{O}\left(\frac{1}{q^3}\right)\right\}.\end{aligned}$$

Using the binomial expansion gives

$$\begin{aligned}&\left(\frac{x}{y}\right)^{-\nu-\frac{1}{2}} \frac{e^{-q(y-x)}}{p}\Bigg\{1+\frac{(\mu-1)(x-y)}{8xyq}+ \\ &\qquad \frac{(\mu-1)(x-y)(x(\mu+7)-y(\mu-9))}{2(8xy)^2q^2}+\mathcal{O}\left(\frac{1}{q^3}\right)\Bigg\}.\end{aligned}$$

Inverting term by term using Abramowitz and Stegun (1974, 29.3.86) gives

$$\begin{aligned}\mathbb{P}_x\left(T_y \le t\right) = \left(\frac{x}{y}\right)^{-\nu-\frac{1}{2}} \Bigg\{ &\operatorname{erfc}\left(\frac{y-x}{\sqrt{2t}}\right) \\ &+\frac{\sqrt{2t}\,(\mu-1)(x-y)}{8}\,\mathrm{i}^1\operatorname{erfc}\left(\frac{y-x}{\sqrt{2t}}\right) \\ &+\frac{t\,(\mu-1)(x-y)(x(\mu+7)-y(\mu-9))}{8^2}\,\mathrm{i}^2\operatorname{erfc}\left(\frac{y-x}{\sqrt{2t}}\right)+\dots\Bigg\}.\end{aligned} \tag{5.11}$$

5.5 Numerical results

Tables 5.1 and 5.2 show results for the hitting times down and up respectively. We compare the numerical results for a number of cases, Bessel processes with different ν and for different time windows. (L) refers to the results obtained by inversion of the Laplace transform in this thesis (or, equivalently, the results of the eigenfunction expansion of Linetsky (2004b)). (SDE) results were calculated by discretizing the SDE using an Euler scheme. Though numerical results seem good there is no classical SDE in the case $\nu \in (-1, 0)$.

In Table 5.1 we show results for Bessel processes starting in 1.5 hitting down to a level of 1. Note that the Monte Carlo results (SDE) are more accurate for the case $t = 1/12$ when the barrier is unlikely to be hit. All simulations are for 10000 paths with 10000 timesteps. This means that the size of the timestep is greater for the $t = 5$ case and this introduces bias. Nearly all the probabilities are underestimated by the Monte Carlo simulation. This bias can be corrected by using a bridge decomposition. This is done for the case of geometric Brownian motion in Beaglehole et al. (1997) and Andersen and Brotherton-Ratcliffe (1996). For Bessel processes this can be achieved using the results in Pitman and Yor (1998).

Results for the probability of hitting up to a barrier are shown in Table 5.2. The Bessel processes are started at 0.5, and the barrier level is again 1. The process is killed if the simulated path hits zero. Again an Euler discretization of the SDE is used. As discussed in Glasserman (2004) significant bias can be introduced for paths near to zero under the Euler discretization. Simulating either the underlying Brownian motions, or sampling from the non-central chi-square distribution directly would improve results.

We verified these results by using numerical inversion of the Laplace transform, This was done by integrating along the line AB in Figures 5.1 and 5.2. The `NIntegrate` method in Mathematica was used, care has to be taken, as the integrand is oscillatory. The numerical inversion in Davydov and Linetsky (2001a), is performed using the method of Abate and Whitt (1995).

Table 5.3 shows the accuracy of the asymptotic approximations using a single term in the expansion. The expansion (5.11) is used for the hitting time up, and (5.10) for the hitting time down. The numbers in square brackets show the number of terms required in the eigenfunction expansion series to match the single term asymptotic

δ	ν	Method	1/12	1	5
	-1.3	L	0.11222	0.76676	0.94643
		SDE	0.1113 (.003)	0.7665 (.004)	0.943 (.002)
	-0.7	L	0.08991	0.65711	0.86357
		SDE	0.0876 (.003)	0.6586 (.005)	0.8611 (.003)
$1 = \vert B_t\vert$	-1/2	L	0.08326	0.61708	0.82306
		SDE	0.0825 (.003)	0.6162 (.005)	0.8192 (.004)
2	0	L	0.06829	0.51358	0.69633
		SDE	0.0641 (.002)	0.5055 (.005)	0.6919 (.005)
3	1/2	L	0.05551	0.41138	0.54871
		SDE	0.0583 (.002)	0.4046 (.005)	0.5415 (.005)
	1.3	L	0.039106	0.26719	0.32980
		SDE	0.0372 (.002)	0.2642 (.004)	0.3026 (.005)
5	3/2	L	0.03570	0.23665	0.28525
		SDE	0.0342 (.002)	0.2439 (.004)	0.2747 (.004)

Table 5.1: Hitting down: The probability, $\mathbb{P}_{1.5}(T_1 \leq t)$, of the process, $R_t^{(\nu)}$, started at 1.5 hitting 1 before a particular time $t \in \{1/12, 1, 5\}$. SDE is results of a simulation using an Euler discretization of the SDE with 10000 timesteps and 10000 cycles. Standard errors are shown in parentheses. L is the result of the Laplace inversion.

δ	ν	Method	1/12	1	5
	-1.3	L	0.043214	0.16490	0.16494
		SDE	0.0442 (.002)	0.1658 (.004)	0.164 (.004)
	-0.7	L	0.07126	0.37745	0.37893
		SDE	0.0684 (.003)	0.3807 (.005)	0.3708 (.005)
$1 = \vert B_t\vert$	-1/2	L	0.08326	0.49542	0.5
		SDE	0.0842 (.003)	0.4968 (.005)	0.4953 (.005)
2	0	L	0.11987	0.94045	1
		SDE	0.116 (.003)	0.902 (.003)	0.9301 (.003)
3	1/2	L	0.16653	0.99084	1
		SDE	0.1693 (.004)	0.9894 (.001)	0.9985 (.000)
	1.3	L	0.262	0.99979	1
		SDE	0.2594 (.004)	0.9998 (.000)	1 (-)
5	3/2	L	0.28946	0.99993	1
		SDE	0.295 (.005)	0.9999 (.000)	1 (-)

Table 5.2: Hitting up: The probability, $\mathbb{P}_{0.5}(T_1 \leq t)$, of the process $R_t^{(\nu)}$, started at 0.5, hitting 1 before a particular time $t \in \{1/12, 1, 5\}$. SDE is simulation using an Euler discretization of the SDE with 10000 timesteps and 10000 runs. Standard errors are shown in parentheses. L indicates the Laplace transform inversion results.

x	ν	Method	1/100	1/12	1
0.5	0	L	8.1272e-7	0.11987	0.94045
		A	8.1077e-7 [16]	0.11775 [2]	0.87267
	1.3	L	1.9690e-6	0.262	0.9979
		A	1.9964e-6 [17]	0.28994 [2]	2.1488
1.5	0	L	-	0.06829	0.51358
		A	4.681e-7	0.06799	0.50384
	1.3	L	-	0.03911	0.26719
		A	2.763e-7	0.04013	0.29742

Table 5.3: Small t asymptotics: The probability, $\mathbb{P}_x(T_1 \leq t)$, of the process $R_t^{(\nu)}$ started at x, hitting 1 before a particular time t. (A) represents asymptotic results using a single term from (5.11) or (5.10). Numbers in square brackets show the number of terms required in the eigenfunction expansion series to match a single term asymptotic expansion. (L) represents results obtained by Laplace inversion in this chapter, or eigenfunction expansion. Where no results are returned Mathematica failed to compute the numerical integral.

expansion. The asymptotic solution is very accurate for small values of t and it does not require that the zeros of Bessel functions to be found. However, the full series is not as easy to generate as the spectral representation.

For the hitting down problem the Laplace inversion has a continuous spectrum and results for $t = 1/100$ in Table 5.3 failed to compute in Mathematica. This can be avoided by imposing a second barrier far above the first, causing a discrete spectrum of eigenvalues, which avoids the need for numerical integration (Linetsky, 2004b).

5.6 Conclusions

We have calculated the hitting times of Bessel processes both for hitting up and hitting down. Inverting the Laplace transform using contour integration gives a series which is useful for large values of time. We also performed an expansion for large values of the transform parameter, which, when inverted term by term, gives a series for the hitting probability useful for small values of time.

These results can be adapted and extended to give option pricing formulae for barrier dependent options under the CEV model. This would give formula for options close to expiry that would complement those in Linetsky (2004b).

Working directly with Bessel processes, rather than the many related stochastic processes allows the development of numerical schemes and analytic solutions that can be more easily transferred to other research fields, while still being applicable to a number of different processes in finance.

Sampling from a Bessel bridge, would remove bias from the simulation, and allow faster simulation of probabilities dependent on barrier crossings. This has been applied in finance to option pricing under geometric Brownian motion by Beaglehole et al. (1997) and others.

Faster generation of Bessel random variates would also be useful. Current methods are adapted from methods for sampling from other distributions, and can probably be improved (Glasserman, 2004).

Chapter 6

An Asymptotic Analysis of an American Call Option with Small Volatility

6.1 Introduction

In Chapter 4 we obtained closed form solutions for multi–asset options. However, radial barrier options are not directly related to the problem of pricing high dimensional American options. In this chapter we tackle the problem of pricing American options more directly.

As we saw in Chapter 2, previous asymptotic work has performed integral transforms of the BSM equation before investigating asymptotic solutions. In this chapter we apply singular perturbation techniques directly to the BSM equation[1]. We make the key assumption that $\epsilon^2 = \frac{\sigma^2}{|r-q|} \ll 1$ is a small parameter. We use this small parameter in an asymptotic analysis of the free boundary formulation for the American call option in the presence of dividends. The details of the analysis and numerical results can be found in Appendix C.

In Section 6.2 we present an asymptotic analysis of the European call option, which we then extend in Section 6.3 to the American call option.

6.1.1 Problem formulation

The Black–Scholes–Merton equation for the value, $V(S,t)$, of a European call option is,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{6.1}$$

[1]Recent independent work by Widdicks et al. (2005) has followed a similar approach

with the final (or payoff) condition at the option's expiry date T

$$V(S,T) = \max(S - E, 0), \tag{6.2}$$

where S is the price of the underlying risky asset, σ is the constant volatility of S, r is the constant risk-free interest rate, $q \neq r$ is the constant, continuous dividend yield on S and t denotes time.

An American option can be exercised at any time up to and including the expiry date T. The option value problem can be formulated as a free boundary problem where $S^*(t)$ represents the location of the free boundary at time t. The problem for the American option is the same as that defined by (6.1) and (6.2), with the additional (free boundary) conditions that

$$V(S^*(t), t) = S^*(t) - E \tag{6.3}$$

$$\frac{\partial}{\partial S} V(S^*(t), t) = 1. \tag{6.4}$$

6.1.2 Non-dimensionalisation

We assume $r \neq q$ throughout and introduce the following transformations,

$$V = E\bar{V}\ , \quad S = E\bar{S} \quad \text{and} \quad \tau = (T-t)|r-q|\ , \tag{6.5}$$

and parameters,

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \quad \text{and} \quad k = \frac{r}{|r-q|}, \tag{6.6}$$

to write the European problem (6.1)–(6.2) in dimensionless form as

$$\bar{V}_\tau = \tfrac{1}{2}\epsilon^2 \bar{S}^2 \bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} - k\bar{V}$$
$$\bar{V}(\bar{S}, 0) = \max(\bar{S} - 1, 0),$$

where we have used subscripts to indicate partial derivatives, that is $\bar{V}_\tau$ means $\partial\bar{V}/\partial\tau$. The American problem will be transformed to dimensionless parameters using the same transformations in Section 6.3.

6.2 European call option

We use the transformation

$$\bar{V} = e^{-k\tau} u$$

and expand the solution using

$$u \sim u_0 + \epsilon^2 u_1 + \dots$$

At leading order, we obtain the first order hyperbolic equation

$$u_{0,\tau} - \bar{S} u_{0,S} = 0$$

with Cauchy data $u_0(\bar{S}, 0) = \max(\bar{S} - 1, 0)$. The method of characteristics implies that

$$u_0 = \max(\bar{S} e^{\tau} - 1, 0) \ .$$

Note that u_0's first derivative with respect to $\bar{S}$ is not continuous along the characteristic $\bar{S} = e^{-\tau}$, and hence that its second $\bar{S}$ derivative involves a delta-function along this characteristic. Specifically, in an $O(\epsilon)$ region about this characteristic, the second $\bar{S}$ partial derivative of $\epsilon^2 u_0$ is $O(1)$ rather than $O(\epsilon^2)$.

Writing this in terms of $\bar{V}$ gives

$$V_0 = e^{-k\tau} \max(\bar{S} e^{\tau} - 1, 0). \tag{6.7}$$

The assumption that the second derivative is small is not valid along the discounted strike value. Therefore we look for a solution that is valid in the region of $\bar{S} = e^{-\tau}$, we call this the inner solution.

6.2.1 Inner solution

We transform to an inner variable, keeping our time-like τ, but use a space-like inner variable x, defined by

$$\bar{S} = e^{-\tau} + \epsilon x \ .$$

If we then use an expansion of the form

$$u \sim \epsilon u_0 + \epsilon^2 u_1 + \dots$$

equations for the leading and first order equations reduce to

$$\begin{aligned} u_{0,\bar{\tau}} &= u_{0,\zeta\zeta} & \qquad u_0(\zeta, 0) &= \max(\zeta, 0) \\ u_{1,\bar{\tau}} &= u_{1,\zeta\zeta} + 2\zeta u_{0,\zeta\zeta} & \qquad u_1(\zeta, 0) &= 0, \end{aligned}$$

where ζ and $\bar{\tau}$ are defined by

$$\zeta = e^{\tau} x \quad \text{and} \quad \bar{\tau} = \tfrac{1}{2}\tau \ .$$

Solving the equations for u_0 and u_1 and expressing the solution in terms of the original non-dimensional variables we find that

$$\begin{aligned}\bar{V} \sim& \left(\bar{S}e^{\tau(1-k)} - e^{-k\tau}\right) \mathrm{N}\left[\frac{1}{\epsilon\sqrt{\tau}}(\bar{S}e^{\tau} - 1)\right] \\ &+\frac{\epsilon}{2}\sqrt{\frac{\tau}{2\pi}}\left(e^{-k\tau} + \bar{S}e^{(1-k)\tau}\right)\exp\left(-\frac{(\bar{S}e^{\tau}-1)^2}{2\epsilon^2\tau}\right) ,\end{aligned} \tag{6.8}$$

where N[.] is the cumulative normal distribution function. The first term matches the outer solution automatically and the second term is always exponentially small away from the inner region, so we can leave it in the solution always and recover (6.7) in the outer region.

Verifying that this solution coincides with a suitable Taylor expansion of the exact Black–Scholes–Merton solution is a worthwhile exercise. This also enables us to find an expansion valid when $r = q$.

6.3 American call option

6.3.1 Case $r > q$

When we non-dimensionalize the American call option problem formulated in (6.1)–(6.4), using the transformations (6.5) and parameters (6.6) we get,

$$\begin{aligned}\bar{V}_{\tau} &= \tfrac{1}{2}\epsilon^2\bar{S}^2\bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} - k\bar{V} \\ \bar{V}(\bar{S},0) &= \max(\bar{S}-1,0) \\ \bar{V}(S^*,\tau) &= S^* - 1 \\ \bar{V}_{\bar{S}}(S^*,\tau) &= 1.\end{aligned}$$

In the limit $\epsilon \ll 1$, when we expand in a regular asymptotic expansion in powers of ϵ and consider the leading order (first order hyperbolic) term, we find that there are two distinct regions, dependent on the boundary conditions involved. Let us denote the boundary between these regions by $\hat{S}(\tau)$. We call $0 < \bar{S} < \hat{S}$ the *lower* region and denote the option value by $\bar{V}^{\text{lower}}(\bar{S},\tau)$, the solution found is the same as in the European case for $r > q$, as given in (6.8).

We call $\hat{S} < \bar{S} < S^*$ the *upper* region, where the option value is given by $\bar{V}^{\text{upper}}(\bar{S},\tau)$. In this region the problem given after the asymptotic expansion is solved using conditions (6.3) and (6.4) from the free boundary rather than the terminal condition (6.2) used in the lower region.

6.3.1.1 Asymptotic expansion

We make an asymptotic expansion, assuming that $0 < \epsilon^2 \ll 1$. We look for the generally valid outer solution. Let us expand as before

$$\bar{V}^{\text{upper}} \sim \bar{V}_0^{\text{upper}} + \epsilon^2 \bar{V}_1^{\text{upper}} + \dots ,$$

and also expand the free boundary as

$$S^* \sim S_0^* + \epsilon^2 S_1^* + \dots .$$

To leading order we have the first order hyperbolic equation

$$\bar{V}_{0,\tau}^{\text{upper}} - \bar{S}\bar{V}_{0,\bar{S}}^{\text{upper}} = -k\bar{V}_0^{\text{upper}},$$

which we solve using non-dimensionalized boundary conditions on the free boundary,

$$\bar{V}_0^{\text{upper}}(S^*, \tau) = S^* - 1 \quad \text{and} \quad \bar{V}_{0,\bar{S}}^{\text{upper}}(S^*, \tau) = 1 .$$

The first term in the expansion for the free boundary is time independent,

$$S_0^* = \frac{k-1}{k} , \tag{6.9}$$

and the value of the option is approximated by

$$\bar{V}_0^{\text{upper}}(\bar{S}, \tau) = \frac{1}{k-1}\left(\frac{\bar{S}}{S_0^*}\right)^k \qquad \text{for} \qquad \hat{S} < \bar{S} < S_0^* ,$$

where the critical characteristic dividing the two regions is given by $\hat{S} = S_0^* e^{-\tau}$. Note that there is no time dependence in the value of the option in the upper region.

$\bar{V}_0^{\text{lower}}$ and $\bar{V}_0^{\text{upper}}$, and their first derivatives with respect to $\bar{S}$, are equal when evaluated on the critical characteristic $\hat{S}$ where the two regions meet.

If we analyse the ϵ^2 terms in the asymptotic expansions we find that the $\bar{V}^{\text{upper}}$ is independent of τ. In the original variables we have

$$V^{\text{upper}}(S, t) \sim E\left[\frac{r}{q} - 1 - \tfrac{1}{2}\sigma^2 \frac{Er}{(r-q)^2} \log\left(\frac{qS}{rE}\right)\right]\left(\frac{qS}{rE}\right)^{\frac{r}{r-q}} , \tag{6.10}$$

and the location of the free boundary is given by

$$S^* \sim \frac{Er}{q}\left(1 + \frac{\sigma^2}{2(r-q)}\right) . \tag{6.11}$$

6.3.2 Case $r < q$

We non-dimensionalize the American call option problem formulated in (6.1)–(6.4) using same transformations and non-dimensional parameters as in the European case and the American $r > q$ case. Making an asymptotic expansion in powers of ϵ we find that there is only one region within which the leading order solution is $V_0 \equiv 0$ and the free boundary is equal to the strike of the option $S_0^* = 1$. If we use the inner variable $y = (\bar{S} - 1)/\epsilon^2$ and also expand the boundary conditions as Taylor series we can find the asymptotic solution to $O(\epsilon^2)$. The value of the option is given by

$$V \sim \frac{\sigma^2 E}{2(q-r)} \exp\left(\frac{2(q-r)}{\sigma^2 E}(S - E) - 1 \right) , \tag{6.12}$$

and the free boundary is flat at the level

$$S^* \sim E\left(1 + \frac{\sigma^2}{2(q-r)}\right) . \tag{6.13}$$

6.3.3 Perpetual American call option

The perpetual American call option has no expiry date, so has no time dependence in the problem formulation, or problem solution. Let $V_\infty(S)$ indicate the value of the perpetual option. In dimensional variables, we have the ordinary differential equation

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2 V_\infty}{dS^2} + (r-q)S\frac{dV_\infty}{dS} - rV_\infty = 0$$

with boundary conditions

$$V_\infty(0) = 0, \quad V_\infty(S^*) = S^* - E \quad \text{and} \quad \frac{dV_\infty}{dS}(S^*) = 1.$$

When $r > q$ and $0 < \epsilon^2 \ll 1$ and we take the limit as $\epsilon^2 \to 0$ we find that the option value is equal to (6.10) and the boundary matches (6.11).

When $r < q$, in the limit as $\epsilon^2 \to 0$ we can confirm that $V_\infty(S)$ tends to (6.12) and the expression for the free boundary tends to (6.13).

6.4 Conclusions

We have found time independent asymptotic expansions for the location of the free boundary of the American call option using the small parameter $\epsilon^2 = \frac{\sigma^2}{|r-q|} \ll 1$.

We have shown that in the limit as $\epsilon^2 \to 0$ these equations match the limit of the solution to the perpetual American call problem (except in a boundary layer about the option's expiry date).

Numerical results for the accuracy of the approximation (Appendix C) show that the leading order approximation for the American option price is poor. However, Widdicks et al. (2005) recently found that using more terms and solving the equations recursively gives very accurate approximations. This suggests that extending this analysis to multi–asset options would be worthwhile.

Chapter 7

Future Work

In previous chapters we have indicated how some of our research may merit further investigation. We review the suggestions highlighted thus far, and identify other research opportunities. The following sections use the same order as the body of the thesis:

- High Dimensional American Options (Section 7.1)
- High Dimensional Radial Barrier Options (Section 7.2)
- Bessel Process Hitting Times (Section 7.3)
- An Asymptotic Analysis of an American Call Option with Small Volatility (Section 7.4)

We discuss pricing American options for different asset classes in the following sections:

- Hybrid products (Section 7.5)
- Credit derivatives with American style features (Section 7.6)

7.1 High dimensional American options

As we have seen in Chapters 2 and 3 Monte Carlo methods are well suited to high dimensional problems. They are easy to implement, popular in practice, and can be extended in a number of different ways.

7.1.1 Martingale basis functions in Monte Carlo methods

A systematic comparison of the duality methods of Andersen and Broadie (2004), Glasserman and Yu (2004), and Kogan and Haugh (2004), using QuantLib for the implementation, would allow an informed evaluation of which method is the fastest and most accurate. Using martingale basis functions during regression avoids the need to do simulation on simulation to obtain an upper bound. Hence, with an appropriate choice of basis functions, we expect it to be the fastest method.

Andersen and Broadie (2004) produce an upper bound by generating an estimate for the duality gap, and adding that to an estimate of the lower bound. As their method attempts to estimate a value that should be small it may be possible to find an estimator with less variance than that in Theorem 2 in Chapter 3 and hence, find a tighter upper bound.

Numerical tests on the utility of quasi Monte Carlo in the computation of upper bounds could be made relatively easily using the Brownian bridge construction and the low–discrepancy sequences implemented in QuantLib. Recent research has been started in this area by Chaudhary (2005) and Lemieux (2004).

7.1.2 Exploiting structure

In Chapter 2 we have seen that the free boundary moves quickly only near to the option expiry time. This implies that using more time steps close to expiry, and fewer where the boundary is static, may significantly improve the speed of algorithms such as Longstaff and Schwartz (2001). Care should be taken to understand the bias introduced by the discretisation. As studied by Beaglehole et al. (1997) discrete simulation has an effect the valuation of barrier style options. However, this effect seems to be less noticeable in the case of American options because their valuation is a smoother problem than that of barrier option valuation. Many studies have commented that the option price seems relatively insensitive to the specification of the optimal exercise boundary.

7.1.3 American options under other stochastic processes

The Variance Gamma model (Madan et al., 1998) for asset price processes is becoming more popular, as can be seen from the number of industrial training courses being run on the subject. Monte Carlo methods should be able to value multi–asset American options under more general stochastic processes. There is little published research in this area.

7.2 High dimensional radial barrier options

Radial barrier options can be enhanced by finding barriers with more natural payoffs when the barrier is hit, such as paying a constant amount, rather than an amount dependent on the barrier location. The case of a barrier that moves through time also needs further investigation.

Currently radial barrier options expire worthless if the barrier is not hit before expiry. This can be modified so that a payoff that depends on the radial distance is made at expiry. Both call and put options could be investigated.

It would also be interesting to see whether radial barrier options can be used to approximate any other financial instrument, and whether they have any utility as a variance reduction techniques.

7.3 Bessel process hitting times

An obvious extension is to invert analytically the Laplace transforms inverted numerically in Davydov and Linetsky (2001a). This will give expressions such as those obtained by the eigenfunction expansion approach. However, it is also possible to investigate the small time behaviour of options on the CEV process. Pricing European capped call options is a good way to obtain a reasonable approximation to American option prices (Detemple and Tian, 2002).

7.4 An asymptotic analysis of an American call option with small volatility

Work on analytic approximations for American options can be extended to cases of more than one underlying asset. This can be done both for the small time approximation, close to option expiry, and for the case of low volatility. This will allow the fast approximation of basket option structures, for example. Analytic solutions to the two asset European, perpetual American, and European capped call basket options (Chapter 2) may well provide a good approximation to American options.

Integral equation approaches can be extended to other types of option, such as American barrier options and lookback options. Various asymptotic analyses could then be performed to better understand the valuation equation.

Multi–asset options can also be decomposed into their European value and early exercise premium. This may be a profitable approach for asymptotic analysis on American basket options.

7.5 Hybrid products

In financial markets the main trading areas in investment banks are Equities, Foreign Exchange (FX), Fixed Income, and increasingly Credit Derivatives (see Section 7.6). These product areas have always overlapped, such as with quanto products that depend on equity and FX factors. However, there is increasing interest in hybrid products that involve payoffs dependent on more factors, such as long dated foreign equity baskets that require expertise in equity, FX and fixed income. Increasingly these products cannot be handled by tree or finite difference models due to the dimensionality. As we have seen Monte Carlo methods have difficulty dealing with American style features. A final practical point is that even if a numerical model were able to price these products effectively it would still be prohibitively expensive to hedge the exposures to every underlying product, and every correlation involved.

7.6 Credit derivatives with American style features

There has been amazing growth in the size of the market for various type of credit derivatives. There are now deep markets in 5 year Credit Default Swaps (CDS), markets in European style CDS options are emerging, and it would not be surprising to see American style instruments develop. Derivatives on portfolios of defaultable instruments, such as Collateralized Debt Obligations (CDOs) have also proved immensely popular, and occasionally deals include some callable features (Schönbucher, 2003).

7.7 Conclusions

We have identified the use of singular perturbation asymptotic methods to find analytic approximations for multi-asset American options. Radial Barrier Options may also provide a fast method to approximate American options and efficiency improvements for numerical methods. Analytic approximations will also be needed for more realistic models of asset price movements, such as Lévy processes.

Primal-dual Monte Carlo option pricing methods can be applied to more realistic general diffusion and Lévy models for asset price returns. Monte Carlo methods can also be applied to cross asset financial products (e.g. hybrids) which are becoming more popular with investors.

Due to recent growth in the credit derivatives market there will be a need for methods to price American options on credit instruments. There has been little research in this area thus far.

Chapter 8

Conclusions

Throughout this thesis we have seen that pricing multi–asset American style options is mathematically challenging. Options on a few assets can be approximated using grid based methods, but for many underlying assets Monte Carlo simulation is the only currently established method. We have investigated a method for pricing American options using regression methods with martingale basis functions, and shown that it improves the speed of convergence.

Motivated by the lack of closed form solutions for multi–asset options we have introduced a new class of option called "Radial Barrier Options" for which we have derived analytic valuation formulae. The Laplace transform inversions performed are applicable to the problem of finding hitting times of Bessel processes. We have also investigated the utility of asymptotic analysis to American option pricing, assuming that the volatility is small relative to the risk neutral drift.

In this chapter we draw conclusions about this work, highlighting the original contributions (Section 8.1) and limitations (Section 8.2).

8.1 Contributions

In the following subsections we detail our four main contributions, which are:

- An extension and implementation of the method of Glasserman and Yu (2004), by suggesting basis functions that are martingales under geometric Brownian motion (Section 8.1.1).
- A new type of multi–asset option, that we call the "Radial Barrier Option", for which we give analytic solutions (Section 8.1.2).

- We invert the Laplace transform for the hitting time of a Bessel process to a level. The results are valid for all $\nu \in \mathbb{R}$ (Section 8.1.3).
- A new asymptotic analysis for the single asset American call option in the presence of a constant dividend yield (Section 8.1.4).

Full details of presentations and publications based on these contributions are given in Appendix A.

8.1.1 Monte Carlo methods for high dimensional American options

We have implemented and evaluated the theoretical paper by Glasserman and Yu (2004). We have provided concrete basis functions that are martingales under geometric Brownian motion both for single asset processes and multi–asset correlated processes. We present, for the first time, numerical results for Monte Carlo regression methods using regression at the end of each time step, rather than the beginning (Chapter 3). Our results support the hypothesis of Glasserman and Yu (2004), as we obtain more accurate price estimates than methods such as Longstaff and Schwartz (2001) that use regression on non-martingale basis functions at the start of each time[1]. We have shown, for the first time, that this method provides lower and upper bounds with much less computational effort than previous "simulation on simulation" methods. This is a substantial improvement to a well established methodology for pricing multi–asset American options.

8.1.2 Radial barrier options

We have proposed a new class of options, called "Radial Barrier Options" which are a type of multi–asset cash-at-hit digital option. In Chapter 4 we presented analytic expressions for the option value[2]. For the outer style option numerical computation of an integral may be required. For the inner style option a series of Bessel function zeros is required. Having analytic solutions to non-trivial path-dependent multi–asset option pricing problems is a useful, and hard, test case for Monte Carlo implementations. Also, as there are very few closed form solutions for multi–asset options this is an interesting result in itself.

[1] available from Firth (2004b), presented at Stochastic Finance 2004, see Appendix A

[2] published as Firth and Dewynne (2004), presented at the Bachelier Finance Society Third World Congress 2004, see Appendix A

8.1.3 Bessel process hitting times

The Laplace transform inversions performed in Chapter 4 are in fact those required to find the probability of a Bessel process started in x hitting a level y. We have applied these results in Chapter 5 and verified them using Monte Carlo for processes with $\nu \in \mathbb{R}$. Previous results were only available for $\nu > -1$. Alternate methods use eigenfunction expansions (Linetsky, 2004b), which are good for long time scales, but weak for short time scales. We have presented results for the hitting probability for small values of time.

8.1.4 An asymptotic analysis for the American call option

We have contributed a new analytic approximation to the American call option price for case of low volatility relative to the risk neutral drift[3]. The free boundary quickly attains its final level, and that level is the same as a perpetual American option with the same parameters. This explains the success of Bjerksund and Stensland (1993) as a fast numerical approximation to the American option price. Our analysis improves our understanding of the relation between European, Perpetual and American style options. Previous investigations have used integral transforms or integral equations before performing an asymptotic analysis. Recent independent work by Widdicks et al. (2005) has followed the same approach as ours. This approach can be extended to multi–asset options.

8.2 Limitations

Our asymptotic analysis (Chapter 6) gives more understanding of the relationship between European, Perpetual and American options. Using these results directly, at leading order, does not give particularly accurate estimates for American option prices. However, recent independent work by Widdicks et al. (2005) has shown that the equations found using this methodology can solved recursively to give accurate single asset American option values. This approach can be extended to multi–asset options.

The introduction of "Radial Barrier Options" gives a multi–asset path dependent option that can be valued in closed form and provides a useful, and hard, test case

[3]published as Firth et al. (2004), presented at ECMI 2002, see Appendix A

for Monte Carlo engines. However, “Radial Barrier Options” are not currently traded products. Also, it is not clear how these barrier options could be used to approximate multi–asset American options. A particularly interesting feature of the option specification is that the effective barrier is sensitive to the correlations between assets.

The contribution of martingale basis functions to the work of Glasserman and Yu (2004) is of more immediate utility. However, as mentioned in Chapter 7, finding basis functions that are martingales under more realistic models for asset prices, such as stochastic volatility, will require further research.

8.3 Concluding remarks

Asymptotic analysis is a useful tool for understanding the behaviour of solutions to partial differential equations. The technique will continue to be applied in mathematical finance, and may well enable fast approximations to be made for multi–asset American options.

We have shown that martingale basis functions do improve the performance of regression techniques for pricing multi–asset American options. Having a confidence interval for the option value enables us to assess the quality of the regression basis functions used.

As closed form solutions are not available for multi–asset American options numerical techniques have to be verified for single asset options, multi–asset European options, or geometric average options. The introduction of “Radial Barrier Options”, for which we have found a closed form valuation formula, gives an alternative method for benchmarking numerical techniques in high dimensions.

The analytic work done on the Laplace transform inversion can be used for valuing options on other processes that can be written as Bessel processes, such as the CEV process. This will enable capped call approximations to American options to be made.

Appendix A

Papers and Conferences

Upper Bounds for American Option Prices using Regression with Martingale Basis Functions

- presented at Stochastic Finance 2004 26-30 September 2004. Lisbon, Portugal.

High dimensional American options have no analytic solution and are difficult to price numerically. Progress has been made in using Monte Carlo simulation to give both lower and upper bounds on the price. Building on an idea of Glasserman and Yu we investigate the utility of martingale basis functions in regression based approximation methods. Regression methods are known to give lower bounds easily, however upper bounds are usually computationally expensive. Martingale basis functions enable fast calculation of upper bounds on the price. These algorithms are implemented in the open-source derivatives pricing library, QuantLib.

High Dimensional Radial Barrier Options

- Oxford University Mathematical Finance Working Paper 2004-MF-02
- presented at Bachelier Finance Society Third World Congress 21-24 July 2004. Chicago, USA.

Pricing high dimensional American options is a difficult problem in mathematical finance. Many simulation methods have been proposed, but Monte Carlo is numerically intensive, and therefore slow. We derive an analytic expression for a new type of multi-asset barrier option using Laplace transform methods. The solution is assumed to be radially symmetric in the normalized non dimensional variables, hence the name “Radial Barrier Options”. In the single-asset case our results reduce to published results for American binary barrier options.

QuantLib: Option Pricing and Clusters

- Invited speaker at ClusterWorld Conference 5-8 April 2004. San Jose, California.

QuantLib is an open-source library for pricing financial derivatives. It is released under the BSD license, so can be used commercially. During this talk I will describe the history and aims of the project, and compare it to other open-source projects in the area. I will discuss the use of computing clusters in derivatives pricing and risk management. Finally I will talk about the use of Monte Carlo simulation for pricing financial derivatives and look at recent advances in the valuation of multi-asset American options.

Why Use QuantLib?

- see `http://www.maths.ox.ac.uk/ firth/research/quantlib.pdf`
- presented at EPSRC / LMS Graduate School in Mathematical Finance. 18 March 2004. Oxford, U.K.

As the open–source movement gains momentum more projects are emerging in the domain of mathematical finance. There are a number of pricing libraries available for financial derivatives, including QuantLib. QuantLib has an effective project structure and is achieving a critical mass of users and developers. Many user groups would benefit by adopting the project.

An Asymptotic Analysis of an American Call Option with Small Volatility

- Oxford University Mathematical Finance Working Paper 2004-MF-03

- presented at The European Consortium for Mathematics in Industry. 12th ECMI Conference 11-14 September 2002. Jurmala, Latvia.
- also presented at Junior Applied Mathematics Seminar. 22 November 2002. Oxford, U.K.

In this paper we present an asymptotic analysis of an American call option where the diffusion term (volatility) is small compared to the drift terms (interest rate and continuous dividend yield). We show that in the limit where diffusion is negligible, relative to drift, then, at leading order, the American call's behaviour is the same as a perpetual American call option (except in a boundary layer about the option's expiry date).

Appendix B

Details of Laplace inversion

B.1 Outer problem

We need to show that the integral goes to zero along the contours AF and CB in Figure B.1.

- AF: We write $s = Re^{i\theta}$ so the integral along the arc AF, from $-\pi/2$ to $-\pi$ in the negative θ direction, becomes
$$I_{AF} = \int_{-\pi}^{-\pi/2} e^{R\tau e^{i\theta}} \frac{K_\nu(\sqrt{R}\, e^{i\theta/2}\rho)}{K_\nu(\sqrt{R}\, e^{i\theta/2})}\, i\, d\theta\, .$$
We wish to prove that this integral goes to zero as $\rho \to \infty$. This is a standard exercise in integration in the complex plane. We consider the modulus of the integral I_{AF}, then we move the modulus inside the integral and consider the modulus of each term
$$|I_{AF}| \le \int_{-\pi}^{-\pi/2} \left|e^{R\tau e^{i\theta}}\right| \left|\frac{K_\nu(\sqrt{R}\, e^{i\theta/2}\rho)}{K_\nu(\sqrt{R}\, e^{i\theta/2})}\right| d\theta\, .$$
We write $e^{R\tau e^{i\theta}}$ as $e^{R\tau(\cos\theta + i\sin\theta)}$ and notice that the imaginary part has a modulus of 1, which gives
$$|I_{AF}| \le \int_{-\pi}^{-\pi/2} \left|e^{R\tau\cos\theta}\right| \left|\frac{K_\nu(\sqrt{R}\, e^{i\theta/2}\rho)}{K_\nu(\sqrt{R}\, e^{i\theta/2})}\right| d\theta\, .$$
Expanding both Bessel functions as series for large argument and using the binomial theorem we have
$$|I_{AF}| \le \int_{-\pi}^{-\pi/2} \left|e^{R\tau\cos\theta}\right| \left|\rho^{-1/2}\right| \left|e^{-(\rho-1)\sqrt{R}\cos(\theta/2)}\right| \left|1 + \mathcal{O}\left(\frac{1}{\sqrt{R}}\right)\right|\, .$$

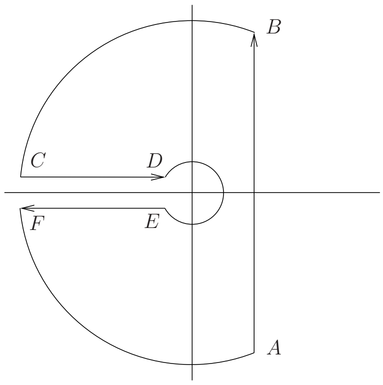


Figure B.1: The keyhole contour for inverting the Laplace transform

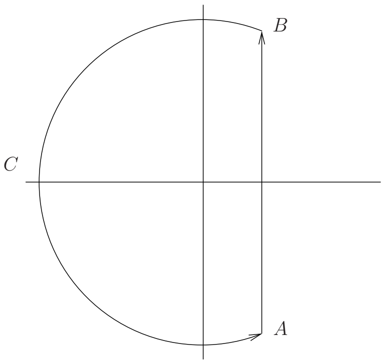


Figure B.2: The contour for inverting the Laplace transform when there is no branch cut in the integrand.

On the interval $-\pi/2 < \theta < -\pi$ in the first term $\cos\theta$ is negative and as $\tau > 0$ when $R \to \infty$ the term goes to zero. The second term is bounded. The $\cos\theta$ in the third term has $-\pi/4 < \theta < -\pi/2$ and so is positive. $\rho - 1 > 0$ so the third term tends to zero as $R \to \infty$. The forth term tends to 1 as $R \to \infty$. Hence $I_{AF} \to 0$ for $\tau > 0$, as required.

- CB: This integral goes to zero. The argument is similar to the case AF. hence we find that as $R \to \infty$, $I_{CB} \to 0$ for $\tau > 0$.

B.2 Inner problem

Details of the inner problem. We need to show that the integral along the contour ACB in Figure B.2 goes to zero.

- ACB: We fix $0 < \rho < 1$ and show that the integral along this contour goes to zero. We follow a similar argument to that in the exterior problem. We write $s = Re^{i\theta}$ so the integral along the arc ACB, from $3\pi/2$ to $\pi/2$ in the negative θ direction, becomes

$$I_{ACB} = \int_{\pi/2}^{3\pi/2} e^{R\tau e^{i\theta}} \frac{I_\nu(\rho\sqrt{R}\,e^{i\theta/2})}{I_\nu(\sqrt{R}\,e^{i\theta/2})}\, i\, d\theta\,.$$

We wish to prove that this integral converges to zero as $R \to \infty$. Consider the modulus of the integral I_{ACB}. If we move the modulus inside the integral and remember that $e^x > 0$ for all x, we have

$$|I_{ACB}| \leq \int_{\pi/2}^{3\pi/2} e^{R\tau\cos\theta} \left| \frac{I_\nu(\rho\sqrt{R}\,e^{i\theta/2})}{I_\nu(\sqrt{R}\,e^{i\theta/2})} \right| d\theta\,.$$

Consider the modulus of the second term in the integral. We can write this as an expansion of $I_\nu(z)$ for large z using (Abramowitz and Stegun, 1974, 9.7.1),

$$\left| \frac{I_\nu(\rho\sqrt{R}\,e^{i\theta/2})}{I_\nu(\sqrt{R}\,e^{i\theta/2})} \right| = \left| \sqrt{\rho}\, e^{\sqrt{R}\,e^{i\theta/2}(\rho-1)} \left\{ \frac{1 - (\frac{4\nu^2-1}{8\rho\sqrt{R}\,e^{i\theta/2}}) + \mathcal{O}(\frac{1}{\rho^2 R})}{1 - (\frac{4\nu^2-1}{8\sqrt{R}\,e^{i\theta/2}}) + \mathcal{O}(\frac{1}{R})} \right\} \right|.$$

If the dimension n is not too great we can expand the fraction on the right hand side using a Taylor series to get,

$$= \left|\sqrt{\rho}\right| \left| e^{\sqrt{R}\,e^{i\theta/2}(\rho-1)} \right| \times$$
$$\left| \left(1 - (\tfrac{4\nu^2-1}{8\rho\sqrt{R}\,e^{i\theta/2}}) + \mathcal{O}(\tfrac{1}{R})\right) \left(1 + (\tfrac{4\nu^2-1}{8\sqrt{R}\,e^{i\theta/2}}) + \mathcal{O}(\tfrac{1}{R})\right) \right|$$

Now, $0 < \rho < 1$ so letting $0 < M_1 = \sqrt{\rho} < 1$ and $0 < M_2 = 1 - \rho$ we have

$$\leq M_1 e^{-\sqrt{R}\,M_2\cos(\theta/2)} \left| 1 + \mathcal{O}\left(\frac{1}{\sqrt{R}}\right) \right|.$$

So $|I_{ACB}|$ is bounded above by

$$|I_{ACB}| \leq M_1 \int_{\pi/2}^{3\pi/2} e^{\sqrt{R}\,(\sqrt{R}\,\tau\cos\theta - M_2\cos(\theta/2))} \left| 1 + \mathcal{O}\left(\frac{1}{\sqrt{R}}\right) \right| d\theta\,.$$

The first term in the exponential, $\cos\theta$, is negative for $\pi/2 < \theta < 3\pi/2$. In the second term $\cos(\theta/2)$, while positive for $\pi < \theta < 3\pi/2$, it is negative for $\pi/2 < \theta < \pi$. However, in the limit as $R \to \infty$ the first term dominates the second and the exponential converges to zero. Hence as $R \to \infty$ we have that $|I_{ACB}| \to 0$ as required.

Appendix C

One Dimensional Asymptotics of American Options

This appendix contains the details of the asymptotic analysis of the partial differential equation describing the price of a single asset American call option (presented in Chapter 6). We approximate the solution in various parameter regions to obtain solutions that can be computed more quickly. We observe that in most markets volatility is approximately 0.2 therefore $\sigma^2 \approx 0.04$. We can use this observation to suggest a small parameter to use in an asymptotic analysis to find approximate solutions to the equation for the free boundary.

We present an asymptotic analysis of an American call option where the diffusion term (volatility) is small compared to the drift terms (interest rate and continuous dividend yield). We show that in the limit where diffusion is negligible, relative to drift, then, at leading order, the American call's behaviour is the same as a perpetual American call option (except in a boundary layer about the option's expiry date).

First we calculate the characteristics for the European case. Next we perform an asymptotic analysis using the same small parameter for the perpetual American call option. Finally we perform the same analysis on the vanilla American call option[1].

C.1 European option

We use the standard Black–Scholes–Merton formulation for a European call option paying a continuous dividend yield given in Let the option value be given by $V(S,t)$,

[1]Slides of a presentation on this subject are available from Dewynne and Firth (2002).

then we have

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0\,,$$

with payoff condition, for a call,

$$V(S,T) = \max(S-E,0)\,.$$

We are going to assume that

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \ll 1$$

and perform a regular asymptotic expansion in powers of ϵ.

C.1.1 Non-dimensionalisation

First, however, we non-dimensionalise the equation using the following transformations,

$$V = E\bar{V}\,, \quad S = E\bar{S} \quad \text{and} \quad \tau = (T-t)|r-q|\,. \tag{C.1}$$

We assume that $r > q$ as this is the only case where the following analysis is interesting for the full American option case. We use the the subscript notation for partial derivatives, where $\bar{V}_{\bar{S}}$ means $\partial\bar{V}/\partial\bar{S}$, and obtain,

$$\bar{V}_\tau = \tfrac{1}{2}\epsilon^2\bar{S}^2\bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} - k\bar{V}$$
$$\bar{V}(\bar{S},0) = \max(\bar{S}-1,0)$$

where

$$k = \frac{r}{|r-q|}\,.$$

We also remove the discount term by making the following transformation

$$\bar{V} = e^{-k\tau}u \tag{C.2}$$

and get

$$u_\tau = \tfrac{1}{2}\epsilon^2\bar{S}^2 u_{\bar{S}\bar{S}} + \bar{S}u_{\bar{S}}$$
$$u(\bar{S},0) = \max(\bar{S}-1,0)\,.$$

C.1.2 Leading order solution

We now expand u to find an asymptotic expression for the solution in the outer region and to discover the location of any inner regions,

$$u \sim u_0 + \epsilon^2 u_1 + \dots$$

We assume that the second derivative, $u_{\bar{S}\bar{S}}$, is of $\mathcal{O}(1)$, and so when we make an asymptotic expansion in ϵ^2 we can drop the diffusion term. This gives, to leading order

$$u_{0,\tau} - \bar{S}u_{0,S} = 0$$
$$u_0(\bar{S}, 0) = \max(\bar{S} - 1, 0)\,.$$

Now we calculate the characteristics of the equation. Using p as the parameter along the characteristic, and $\bar{S}_{(0)}$, the terminal stock price, as the parameter along the boundary condition, here the final stock price. Letting prime represent differentiation with respect to p we get

$$\frac{\tau'}{1} = \frac{\bar{S}'}{-\bar{S}} = \frac{u_0'}{u}\,,$$

with boundary data at time $\tau = 0$ and $p = 0$,

$$\tau(0) = T \qquad \bar{S}(0) = \bar{S}_{(0)} \qquad u_0 = \max(\bar{S}_{(0)} - 1, 0)\,,$$

therefore

$$\begin{array}{lll} p & = \tau & \\ \log \bar{S} & = -p + a(\bar{S}_{(0)}) & \Rightarrow \bar{S} \quad = \bar{S}_{(0)}e^{-\tau} \\ \log u_0 & = b(\bar{S}_{(0)}) & \Rightarrow u_0 \quad = \max(\bar{S}_{(0)} - 1, 0)\,. \end{array}$$

Eliminating p we get the leading order solution

$$u_0 = \max(\bar{S}e^{\tau} - 1, 0)\,.$$

We can see the characteristics in Figure C.1. Writing the leading order solution in terms of $\bar{V}$ gives

$$\bar{V} \approx e^{-k\tau}\max(\bar{S}e^{\tau} - 1, 0)\,.$$

If we look at the second derivative of this approximation we find that

$$u_{0,\bar{S}\bar{S}} = e^{\tau}\delta(e^{-\tau} - \bar{S})\,,$$

therefore the assumption that the second derivative is going to be small is not valid along the discounted strike value, $\bar{S} = e^{-\tau}$. In fact it involves a delta function. Therefore we will need to look for another solution in this inner region.

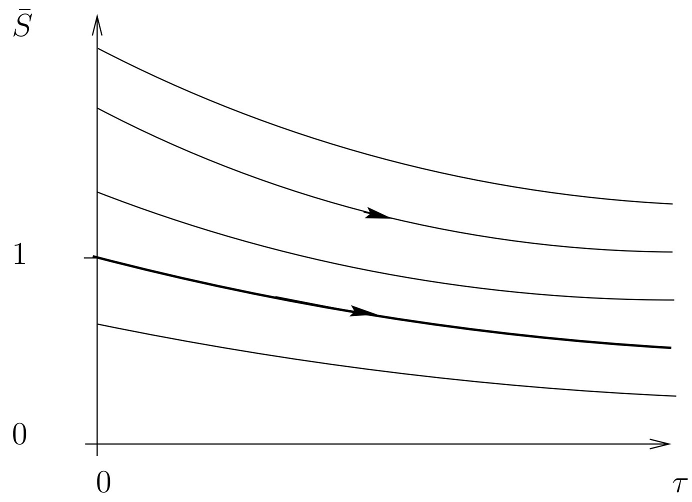


Figure C.1: Characteristic for the European option with $r > q$

C.1.3 Inner solution

We have noted that the outer solution's second $\bar{S}$ derivative involves a delta function along the critical characteristic

$$\bar{S} = e^{-\tau}\,.$$

We anticipate a boundary layer around the critical characteristic, as we cannot ignore the second derivative in that area. Therefore let us transform variables, keeping our time-like τ, but use a space-like inner variable x, defined by

$$\bar{S} = e^{-\tau} + \epsilon x\,. \tag{C.3}$$

The Black-Scholes equation and payoff conditions, become

$$u_\tau = \tfrac{1}{2}(e^{-\tau} + \epsilon x)^2 u_{xx} + x u_x$$
$$u(x,0) = \max(x,0)\,.$$

C.1.3.1 Inner solution, asymptotic expansion

Now look for the inner solution approximation using the expansion

$$u \sim \epsilon u_0 + \epsilon^2 u_1 + \ldots$$

We need to keep the first and second order terms in ϵ, and these give

$$\begin{aligned}
\epsilon^0 : \quad & u_{0,\tau} = \tfrac{1}{2}e^{-2\tau}u_{0,xx} + x u_{0,x} && u_0(x,0) = \max(x,0) \\
\epsilon^1 : \quad & u_{1,\tau} = \tfrac{1}{2}e^{-2\tau}u_{1,xx} + e^{-\tau}x u_{0,xx} + x u_{1,x} && u_1(x,0) = 0\,.
\end{aligned}$$

A series of transformations will allow us to reduce the first equation to the heat equation. First we transform using two functions $\alpha(\tau)$ and $\gamma(\tau)$ that will be chosen such that the equation for ϵ^0 reduces to the heat equation,

$$\zeta = \alpha(\tau)x \quad \text{and} \quad \bar{\tau} = \gamma(\tau)\,,$$

which gives

$$\begin{aligned}
\epsilon^0: \qquad \dot{\gamma}u_{0,\bar{\tau}} &= \tfrac{1}{2}e^{-2\tau}\alpha^2 u_{0,\zeta\zeta} + \zeta\Big(1-\frac{\dot{\alpha}}{\alpha}\Big)U_{0,\zeta} \\
\epsilon^1: \qquad \dot{\gamma}u_{1,\bar{\tau}} &= \tfrac{1}{2}e^{-2\tau}\alpha^2 u_{1,\zeta\zeta} + e^{-\tau}\frac{\zeta}{\alpha}u_{0,\zeta\zeta} + \zeta\Big(1-\frac{\dot{\alpha}}{\alpha}\Big)U_{1,\zeta}\,.
\end{aligned}$$

Now let us choose α to eliminate the term containing the first derivative. Also, let us choose γ to balance the remaining equation. Choosing

$$\alpha = e^{\tau} \quad \text{and} \quad \gamma = \tfrac{1}{2}\tau$$

reduces our two equations to

$$\begin{aligned}
u_{0,\bar{\tau}} &= u_{0,\zeta\zeta} & \qquad u_0(\zeta,0) &= \max(\zeta,0) \\
u_{1,\bar{\tau}} &= u_{1,\zeta\zeta} + 2\zeta u_{0,\zeta\zeta} & \qquad u_1(\zeta,0) &= 0\,.
\end{aligned}$$

We can write down the usual solution to the first equation

$$u_0(\zeta,\bar{\tau}) = \frac{1}{2\sqrt{\pi\bar{\tau}}}\int_{-\infty}^{\infty} u_0(y,0)e^{-\frac{(\zeta-y)^2}{4\bar{\tau}}}dy\,.$$

We know that at the payoff the value is determined by a maximum function, therefore we know that function u_0 is zero over the negative integrand range. Hence,

$$u_0 = \frac{1}{2\sqrt{\pi\bar{\tau}}}\int_0^{\infty} ye^{-\frac{(\zeta-y)^2}{4\bar{\tau}}}dy\,.$$

Now change variables

$$z = \frac{y-\zeta}{\sqrt{2\bar{\tau}}} \qquad dz = \frac{dy}{\sqrt{2\bar{\tau}}}\,,$$

to get

$$u_0 = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\zeta}{\sqrt{2\bar{\tau}}}}^{\infty} (\zeta + \sqrt{2\bar{\tau}}z)e^{-\frac{1}{2}z^2}dz$$

$$u_0 = \frac{\zeta}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\zeta}{\sqrt{2\bar{\tau}}}} e^{-\frac{1}{2}z^2}dz + \sqrt{\frac{\bar{\tau}}{\pi}}\int_{-\infty}^{\frac{\zeta}{\sqrt{2\bar{\tau}}}} ze^{-\frac{1}{2}z^2}dz\,.$$

Using $\mathrm{N}(d)$ to represent the cumulative normal distribution function

$$\mathrm{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds$$

the solution is

$$u_0(\zeta, \bar{\tau}) = \zeta \mathrm{N}\left(\frac{\zeta}{\sqrt{2\bar{\tau}}}\right) + \sqrt{\frac{\bar{\tau}}{\pi}} e^{-\frac{\zeta^2}{4\bar{\tau}}}.$$

Now we have a solution for u_0 we can proceed and solve the ϵ^1 term.

The general solution for an equation of the form

$$V_t - V_{xx} = f(x,t) \quad \text{where} \quad V(x,0) = 0$$

is

$$V = \int_0^t \frac{1}{2\sqrt{\pi}\sqrt{t-t'}} \int_{-\infty}^{\infty} f(y,t') \exp\left(-\frac{(x-y)^2}{4(t-t')}\right) dy\, dt'.$$

We have $2\zeta u_{0,\zeta\zeta}$ as the function f. We know that $u_{0,\zeta\zeta}$ is the source solution

$$u_{0,\zeta\zeta} = \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-\frac{\zeta^2}{4\bar{\tau}}}.$$

Therefore we can substitute this into the solution and perform the integration

$$u_1 = \int_0^{\bar{\tau}} \frac{1}{2\sqrt{\pi}\sqrt{\bar{\tau}-\bar{\tau}'}} \int_{-\infty}^{\infty} 2y \frac{1}{2\sqrt{\pi\bar{\tau}'}} \exp\left(-\frac{y^2}{4\bar{\tau}'}\right) \exp\left(-\frac{(\zeta-y)^2}{4(\bar{\tau}-\bar{\tau}')}\right) dy\, d\bar{\tau}'.$$

Let us focus on the inner integral first

$$\int_{-\infty}^{\infty} y \frac{1}{\sqrt{\pi\bar{\tau}'}} \exp\left(-\frac{y^2}{4\bar{\tau}'} - \frac{(\zeta-y)^2}{4(\bar{\tau}-\bar{\tau}')}\right) dy.$$

Complete the square in the exponential to get

$$\int_{-\infty}^{\infty} y \frac{1}{\sqrt{\pi\bar{\tau}'}} \exp\left(-\frac{\bar{\tau}(y-\frac{\zeta\bar{\tau}'}{\bar{\tau}})^2}{4(\bar{\tau}-\bar{\tau}')\bar{\tau}'} - \frac{\zeta^2}{4\bar{\tau}}\right) dy$$

$$\frac{1}{\sqrt{\pi\bar{\tau}'}} e^{-\frac{\zeta^2}{4\bar{\tau}}} \int_{-\infty}^{\infty} y \exp\left(-\frac{\bar{\tau}(y-\frac{\zeta\bar{\tau}'}{\bar{\tau}})^2}{4(\bar{\tau}-\bar{\tau}')\bar{\tau}'}\right) dy.$$

Now change variables using

$$\xi = \frac{\sqrt{\bar{\tau}}}{2\sqrt{(\bar{\tau}-\bar{\tau}')\bar{\tau}'}} \left(y - \frac{\zeta\bar{\tau}'}{\bar{\tau}}\right),$$

to get

$$\frac{1}{\sqrt{\pi\bar{\tau}'}}e^{-\frac{\zeta^2}{4\bar{\tau}}}\int_{-\infty}^{\infty}\left(\frac{2\sqrt{(\bar{\tau}-\bar{\tau}')\bar{\tau}'}}{\sqrt{\bar{\tau}}}\xi+\frac{\zeta\bar{\tau}'}{\bar{\tau}}\right)\,e^{-\xi^2}\,\frac{2\sqrt{(\bar{\tau}-\bar{\tau}')\bar{\tau}'}}{\sqrt{\bar{\tau}}}d\xi\,.$$

This is now two integrals, the first of which can be seen to be equal to zero because it is an odd function integrated over $-\infty$ to $+\infty$. Therefore we have

$$\frac{1}{\sqrt{\pi\bar{\tau}'}}e^{-\frac{\zeta^2}{4\bar{\tau}}}\frac{\zeta\bar{\tau}'}{\bar{\tau}}\frac{2\sqrt{(\bar{\tau}-\bar{\tau}')\bar{\tau}'}}{\sqrt{\bar{\tau}}}\int_{-\infty}^{\infty}e^{-\xi^2}\,d\xi\,,$$

and we can see that the integral is equal to $\sqrt{\pi}$, so we get

$$\frac{2\bar{\tau}'\sqrt{\bar{\tau}-\bar{\tau}'}}{\bar{\tau}^{\frac{3}{2}}}e^{-\frac{\zeta^2}{4\bar{\tau}}}\,.$$

We then substitute this result back into our outer integral in $\bar{\tau}'$

$$u_1=\frac{\zeta}{2\sqrt{\pi}\bar{\tau}^{\frac{3}{2}}}e^{-\frac{\zeta^2}{4\bar{\tau}}}\int_0^{\bar{\tau}}2\bar{\tau}'\,d\bar{\tau}'\,.$$

This final integral is trivial to evaluate, so we have

$$u_1(\zeta,\bar{\tau})=\frac{1}{2}\sqrt{\frac{\bar{\tau}}{\pi}}\zeta e^{-\frac{\zeta^2}{4\bar{\tau}}},$$

and combining our two results

$$u(\zeta,\bar{\tau})\approx\zeta\epsilon\mathrm{N}\left(\frac{\zeta}{\sqrt{2\bar{\tau}}}\right)+\epsilon\sqrt{\frac{\bar{\tau}}{\pi}}e^{-\frac{\zeta^2}{4\bar{\tau}}}+\frac{1}{2}\epsilon^2\sqrt{\frac{\bar{\tau}}{\pi}}\zeta e^{-\frac{\zeta^2}{4\bar{\tau}}}\,.$$

This can be written as

$$u(\zeta,\bar{\tau})\approx\zeta\epsilon\mathrm{N}\left(\frac{\zeta}{\sqrt{2\bar{\tau}}}\right)+\epsilon\sqrt{\frac{\bar{\tau}}{\pi}}e^{-\frac{\zeta^2}{4\bar{\tau}}}\left(1+\frac{\epsilon\zeta}{2}\right)\,.$$

This inner solution automatically matches the outer solution.

C.1.3.2 Inner solution, reverse transformations

Now, we have to reverse the variable transformations we made earlier

$$\zeta=e^{\tau}x\qquad\bar{\tau}=\tfrac{1}{2}\tau\qquad\bar{V}=e^{-k\tau}u$$

so we have

$$\bar{V}\approx\epsilon\,x\,e^{\tau(1-k)}\mathrm{N}\left(\frac{xe^{\tau}}{\sqrt{\tau}}\right)+\epsilon\sqrt{\frac{\tau}{2\pi}}e^{-k\tau}\left(1+\frac{\epsilon\,x\,e^{\tau}}{2}\right)\exp\left(-\frac{(e^{\tau}x)^2}{2\tau}\right)\,.$$

Now recall, from (C.3), that

$$x = \frac{1}{\epsilon}(\bar{S} - e^{-\tau}),$$

giving the non-dimensional result

$$\bar{V} \approx \left(\bar{S}e^{\tau(1-k)} - e^{-k\tau}\right) \mathrm{N}\left(\frac{1}{\epsilon\sqrt{\tau}}(\bar{S}e^{\tau} - 1)\right)$$
$$+\frac{\epsilon}{2}\sqrt{\frac{\tau}{2\pi}}\left(e^{-k\tau} + \bar{S}e^{(1-k)\tau}\right)\exp\left(-\frac{(\bar{S}e^{\tau} - 1)^2}{2\epsilon^2\tau}\right). \tag{C.4}$$

The first term matches the outer solution automatically and the second term is always exponentially small away from the inner region, so we can leave it in the solution always.

Away from the critical characteristic the cumulative normal in the first term gives one if we are above the critical characteristic, and zero if we are below, so we can see that we have recovered our result for the outer region,

$$\bar{V} \approx e^{-k\tau}\max(\bar{S}e^{\tau} - 1, 0).$$

C.1.4 Original variables

We can re-dimensionalise our solution so that we can compare it to the exact Black-Scholes solution that we already know. To do this we reverse the transformations, (C.1),

$$V = E\bar{V} \qquad S = E\bar{S} \qquad \tau = (T-t)(r-D)$$

and also the transformation for the constants

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \quad \text{and} \quad k = \frac{r}{|r-q|}$$

to obtain the result in the original variables,

$$V \approx \left(Se^{-q(T-t)} - Ee^{-r(T-t)}\right)\mathrm{N}\left(\frac{1}{\sigma\sqrt{T-t}}\left(\frac{Se^{-q(T-t)}}{Ee^{-r(T-t)}} - 1\right)\right)$$
$$+\sigma\sqrt{\frac{T-t}{2\pi}}\left(Se^{-q(T-t)} + Ee^{-r(T-t)}\right)\exp\left(\frac{-1}{2\sigma^2(T-t)}\left(\frac{Se^{-q(T-t)}}{Ee^{-r(T-t)}} - 1\right)^2\right).$$

C.1.5 Comparison with exact solution

We can check our calculations by expanding the exact solution and comparing it with our leading order approximation. We do this by writing the two cumulative normal functions in a symmetrical manner and expanding in a Taylor series. We can then rewrite the argument of the upper integration limit of the cumulative normal distribution by expanding in a series. We find that the leading order matches with the leading order solution of the previous section.

We know that the exact Black-Scholes solution is

$$V = Se^{-q(T-t)}\mathrm{N}\left(d_+\right) - Ee^{-r(T-t)}\mathrm{N}\left(d_-\right)$$

where

$$d_\pm = \frac{\log(\frac{S}{E}) + (r - q \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Now, rewrite these as

$$d_+ = A + \tfrac{1}{2}\sigma\sqrt{T-t},$$

and

$$d_- = A - \tfrac{1}{2}\sigma\sqrt{T-t},$$

where

$$A = \frac{1}{\sigma\sqrt{T-t}}\log\left(\frac{Se^{-q(T-t)}}{Ee^{-r(T-t)}}\right).$$

The exact solution can now be written as follows,

$$V = Se^{-q(T-t)}\mathrm{N}\left(A + \tfrac{1}{2}\sigma\sqrt{T-t}\right) - Ee^{-r(T-t)}\mathrm{N}\left(A - \tfrac{1}{2}\sigma\sqrt{T-t}\right).$$

A Taylor expansion of this equation about A yields

$$\begin{aligned} V \approx\ & (Se^{-q(T-t)} - Ee^{-r(T-t)})\mathrm{N}\left(A\right) \\ & + \tfrac{\sigma}{2}\sqrt{\tfrac{T-t}{2\pi}}(Se^{-q(T-t)} + Ee^{-r(T-t)})\exp\left(-\tfrac{A^2}{2}\right). \end{aligned}$$

Rewriting A and expanding the logarithm about 1 gives

$$\begin{aligned} A &= \tfrac{1}{\sigma\sqrt{T-t}}\log\left(\tfrac{S}{E}e^{(r-q)(T-t)}\right) \\ &= \tfrac{1}{\sigma\sqrt{T-t}}\log\left(1 + (\tfrac{S}{E}e^{(r-q)(T-t)} - 1)\right) \\ &\approx \tfrac{1}{\sigma\sqrt{T-t}}\left((\tfrac{S}{E}e^{(r-q)(T-t)} - 1) - \tfrac{1}{2}(\tfrac{S}{E}e^{(r-q)(T-t)} - 1)^2 + \tfrac{1}{3}(\ldots)^3 - \ldots\right). \end{aligned}$$

Our choice of inner variable was such that the first term in the expansion of the logarithm is of order 1. Thus we have that the exact solution, expanded in orders of σ, is

$$V \approx (Se^{-q(T-t)} - Ee^{-r(T-t)})\mathrm{N}\left(\frac{1}{\sigma\sqrt{T-t}}(\frac{S}{E}e^{(r-q)(T-t)} - 1)\right) + \mathcal{O}(\sigma),$$

which is the same as our approximation, to leading order. The inner solution is valid over all parameter values.

C.2 Perpetual American call option

We can find a solution for the long term behaviour of the full American call option by considering the perpetual option, where the option never expires and therefore has no time dependence. Let $V_\infty(S)$ indicate the value of the perpetual option. We modify the free boundary formulation for the American call, (2.15) – (2.18), removing all time dependence to obtain the ordinary differential equation,

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2V_\infty}{dS^2} + (r-q)S\frac{dV_\infty}{dS} - rV_\infty = 0\,,$$

with boundary conditions

$$V_\infty(0) = 0\,, \quad V_\infty(S^*) = S^* - E \quad \text{and} \quad \frac{dV_\infty}{dS}(S^*) = 1\,.$$

We assume that $S^* > E$. Much of the analysis of the perpetual American option is common between the $r > q$ and $r < q$ cases. We consider the $r > q$ case first.

C.2.1 Perpetual American call solution when $r > q$

Let us non-dimensionalise the equation as we did in the European case using (C.1). We do not need to use a transformation for time, as there is no time dependence. We use

$$V_\infty = E\bar{V}_\infty \quad \text{and} \quad S = E\bar{S}\,.$$

We now have the ordinary differential equation

$$\tfrac{1}{2}\epsilon^2 \bar{S}^2 \frac{d^2\bar{V}_\infty}{d\bar{S}^2} + S\frac{d\bar{V}_\infty}{dS} - k\bar{V}_\infty = 0\,,$$

where, as before, we have

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \quad \text{and} \quad k = \frac{r}{|r-q|}$$

with boundary conditions

$$\bar{V}_\infty(0) = 0, \quad \bar{V}_\infty(\bar{S}^*) = \bar{S}^* - 1 \quad \text{and} \quad \frac{d\bar{V}_\infty}{d\bar{S}}(\bar{S}^*) = 1\,.$$

The second order ordinary differential equation has solutions of the form $\bar{V}_\infty(\bar{S}) = \bar{S}^m$. Therefore the general solution will be of the form

$$\bar{V}_\infty(\bar{S}) = A\bar{S}^{m_+} + B\bar{S}^{m_-}\,,$$

where m_+ and m_- are solutions of the quadratic equation

$$\tfrac{1}{2}\epsilon^2 m(m-1) + m - k = 0\,.$$

When we consider the $r < q$ case we find that the only change is the sign of the middle m term. The solution to the quadratic equation is

$$m_\pm = \frac{1}{\epsilon^2}\left[-(1-\tfrac{1}{2}\epsilon^2) \pm \sqrt{(1-\tfrac{1}{2}\epsilon^2)^2 + 2\epsilon^2 k}\right]. \tag{C.5}$$

From the properties of quadratic equations we have that

$$m_+ m_- = -\frac{2k}{\epsilon^2}\,.$$

We are assuming that $r > 0$, so $k > 0$ and we can say that $m_+ > 0$ and that $m_- < 0$. As the product of the roots does not depend on the m term in the quadratic equation this is true in the $r < q$ case also. Now let us consider the boundary condition $\bar{V}_\infty(0) = 0$, which implies that $B = 0$ so that

$$\bar{V}_\infty(\bar{S}) = A\bar{S}^{m_+}.$$

The condition $\bar{V}_\infty(\bar{S}^*) = \bar{S}^* - 1$ gives

$$\bar{V}_\infty(\bar{S}) = (\bar{S}^* - 1)\left(\frac{\bar{S}}{\bar{S}^*}\right)^{m_+}. \tag{C.6}$$

We then use the smooth pasting condition on the derivative of $\bar{V}_\infty$ to see that

$$\frac{m_+}{\bar{S}^*}(\bar{S}^* - 1)\left(\frac{\bar{S}}{\bar{S}^*}\right)^{m_+ - 1}\Bigg|_{\bar{S}=\bar{S}^*} = 1\,,$$

equivalently
$$m_+ \frac{(\bar{S}^* - 1)}{\bar{S}^*} = 1\,.$$
We rearrange this to find an analytic solution for the free boundary in the case of the perpetual American call,
$$\bar{S}^* = \frac{m_+}{m_+ - 1}\,. \tag{C.7}$$

C.2.2 Limit as $\epsilon^2 \to 0$ with $r > q$

Now assume that $0 < \epsilon^2 \ll 1$ and look at the limit as $\epsilon^2 \to 0$. Recall from (C.5) that
$$m_+ = \frac{1}{\epsilon^2}\left[-(1 - \tfrac{1}{2}\epsilon^2) + \sqrt{(1 - \tfrac{1}{2}\epsilon^2)^2 + 2\epsilon^2 k}\right]\,.$$
After some algebra, keeping terms $\mathcal{O}(\epsilon^2)$ or greater we find
$$m_+ \sim k\left(1 + \tfrac{1}{2}\epsilon^2(1-k)\right)\,.$$
Substituting this into the equation for $\bar{S}^*$, (C.7), we have that
$$\bar{S}^* \sim \frac{k}{k-1}\left(1 + \tfrac{1}{2}\epsilon^2\right)\,. \tag{C.8}$$
It will be informative to write this as an asymptotic expansion in ϵ^2,
$$\bar{S}^* \sim \bar{S}^*_0 + \epsilon^2 \bar{S}^*_1 + \mathcal{O}(\epsilon^4)\,,$$
so
$$\bar{S}^*_0 = \frac{k}{k-1} \quad \text{and} \quad \bar{S}^*_1 = \tfrac{1}{2}\bar{S}^*_0\,. \tag{C.9}$$
Putting the results for m_+ and $\bar{S}^*$ into the equation for the value of the option, (C.6), we obtain the leading order expression,
$$\bar{V}_\infty(\bar{S}) \sim \left(\frac{1}{k-1}\right)\left(\frac{\bar{S}(k-1)}{k}\right)^k\,. \tag{C.10}$$
Including the $\mathcal{O}(\epsilon^2)$ terms we find that the option value is given by,
$$\bar{V}_\infty(\bar{S}) \sim \left(\frac{\bar{S}(k-1)}{k}\right)^k\left(\frac{1}{k-1} - \tfrac{1}{2}\epsilon^2 \log\left(\frac{\bar{S}(k-1)}{k}\right)^k\right)\,. \tag{C.11}$$

C.2.2.1 Original variables

Taking the solution, (C.10), and reversing the transformations (C.1) gives the option value to leading order in the original variables,

$$V_\infty(S) \sim E\left(\frac{r}{q}-1\right)\left(\frac{qS}{rE}\right)^{\frac{r}{r-q}} . \tag{C.12}$$

Including $\mathcal{O}(\epsilon^2)$ terms, (C.11) becomes,

$$V(S,t) \sim E\left(\frac{r}{q}-1-\tfrac{1}{2}\sigma^2\frac{E}{(r-q)}\log\left(\frac{S}{S_0^*}\right)^{\frac{r}{r-q}}\right)\left(\frac{S}{S_0^*}\right)^{\frac{r}{r-q}} , \tag{C.13}$$

and the location of the free boundary, (C.8) is given by

$$S^* \sim \frac{r}{q}E\left(1+\tfrac{1}{2}\frac{\sigma^2}{(r-q)}\right) . \tag{C.14}$$

C.2.3 Limit as $\epsilon^2 \longrightarrow 0$ with $r < q$

The solutions of the boundary value problem for the perpetual American call option when $r < q$ and when $r > q$ are very similar. When $r < q$ we find that (C.5) becomes,

$$m_+ = \frac{1}{\epsilon^2}\left[(1+\tfrac{1}{2}\epsilon^2)+\sqrt{(1+\tfrac{1}{2}\epsilon^2)^2+2\epsilon^2 k}\right] . \tag{C.15}$$

The results for the location of the free boundary and the option value remain unchanged. Let us now consider the small volatility limit as $\epsilon^2 \to 0$. With repeated application of Taylor's theorem we obtain

$$m_+ \sim \frac{2}{\epsilon^2}+1-k ,$$

and so

$$\bar{S}^* \sim 1+\tfrac{1}{2}\epsilon^2 . \tag{C.16}$$

To find an expression for the option value we take (C.6) and write it as

$$\bar{V}_\infty = (\bar{S}^*-1)\exp\left[m_+\log\left(\frac{\bar{S}}{\bar{S}^*}\right)\right] .$$

We use our approximation (C.16), and split the product in the argument of the logarithm into the addition of two logarithms and expand the second logarithm as a Taylor series, to obtain

$$\bar{V}_\infty = \tfrac{1}{2}\epsilon^2\exp\left[\left(\frac{2}{\epsilon^2}+1+k\right)\left(\log\bar{S}-\tfrac{1}{2}\epsilon^2\right)\right] ,$$

which can be simplified to

$$\bar{V} \sim \tfrac{1}{2}\epsilon^2 \exp\left[\frac{2}{\epsilon^2}\left(\bar{S}-1\right)-1\right]. \tag{C.17}$$

C.2.3.1 Original variables

If we reverse the transformations we find that the dimensional result for the value of the option, (C.17), as $\epsilon^2 \to 0$ is given by

$$V_\infty(S) \sim \frac{\sigma^2 E}{2(q-r)} \exp\left[\frac{2(q-r)}{\sigma^2}\left(\frac{S-E}{E}\right)-1\right], \tag{C.18}$$

and the location of the free boundary, (C.16), tends to

$$S^* \sim E\left(1+\frac{\sigma^2}{2(q-r)}\right). \tag{C.19}$$

C.2.4 Case $r = q$

In this case the original equation reduces and there is no difference, on average, between holding the asset and holding money in the bank. Only the money made by the asset appreciating in price is important

We find that the expression for m_+ becomes,

$$m_+ = \tfrac{1}{2} + \sqrt{\frac{1}{4}+\frac{2r}{\sigma^2}}\,,$$

and the limit of the free boundary as $\sigma^2/r \to 0$ is

$$\bar{S}^* \sim 1\,.$$

C.2.5 Case $q = 0$

For the case $q = 0$ we can calculate that $m_+ = 1$. The free boundary tends toward infinity, indicating that it is never optimal to exercise an American call early when the underlying stock pays no dividends. In accordance with what we already know.

C.3 American call option

Let us now move on to the one dimensional American call option. We use the smooth pasting formulation, (2.15)–(2.18), of the problem for the value of an option, $V(S,t)$, on a risky asset that pays a continuous dividend yield, q,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0,$$

with boundary conditions

$$\begin{aligned} V(S,T) &= \max(S-E,0) \\ V(S^*(t),t) &= S^*(t) - E \\ \frac{\partial V}{\partial S}(S^*(t),t) &= 1\,, \end{aligned}$$

and where $S < E$ and $t < T$. $S^*(t)$ represents the location of the free boundary at time t.

This problem has different solutions when $r > q$ and when $r < q$. We will look at each of these in turn to get a complete understanding of the problem.

C.3.1 Case $r > q$

First we consider the case when $r > q$.

C.3.1.1 Non-dimensionalisation

We non-dimensionalise the equation using the following transformations, as in the European case (C.1),

$$V = E\bar{V} \qquad S = E\bar{S} \qquad \tau = (T-t)|r-q|\,,$$

to obtain, using the subscript notation for partial derivatives,

$$\begin{aligned} \bar{V}_\tau &= \tfrac{1}{2}\epsilon^2 \bar{S}^2 \bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} - k\bar{V} \\ \bar{V}(\bar{S},0) &= \max(\bar{S}-1,0) \\ \bar{V}(\bar{S}^*,\tau) &= \bar{S}^* - 1 \\ \bar{V}_{\bar{S}}(\bar{S}^*,\tau) &= 1 \end{aligned}$$

where

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \quad \text{and} \quad k = \frac{r}{|r-q|}.$$

C.3.1.2 Outer solution

We make an asymptotic expansion, assuming that $0 < \epsilon^2 \ll 1$. We look for the generally valid outer solution. Let us expand as before, in the European case,

$$\bar{V} \sim \bar{V}_0 + \epsilon^2 \bar{V}_1 + \dots .$$

To leading order we have that

$$\bar{V}_{0,\tau} - \bar{S}\bar{V}_{0,\bar{S}} = -k\bar{V}_0 \,, \tag{C.20}$$

with boundary conditions

$$\bar{V}_0(\bar{S},0) = \max(\bar{S}-1,0) \tag{C.21}$$

$$\bar{V}_0(\bar{S}^*,\tau) = \bar{S}^* - 1 \tag{C.22}$$

$$\bar{V}_{0,\bar{S}}(\bar{S}^*,\tau) = 1 \,. \tag{C.23}$$

Let us also expand the expression for the free boundary

$$\bar{S}^* \sim \bar{S}_0^* + \epsilon^2 \bar{S}_1^* + \dots .$$

We find that there are two solution regions, depending on which boundary conditions are used in the asymptotic solution. One solution, using (C.21), is valid for small values of $\bar{S}$. The other solution, using (C.22) and (C.23), is valid for larger values of $\bar{S}$, bounded above by the free boundary. We call the first region the *lower* region, and the second region the *upper* region.

C.3.1.3 Lower region

In the lower region we can solve (C.20) using (C.21) as we did for the European call option. We get the same solution, (C.4), valid in the region where the characteristics propagate out from the terminal boundary condition, $\bar{V}(\bar{S},0) = \max(\bar{S}-1,0)$. The solution is

$$\begin{aligned}\bar{V}^{lower}(\bar{S},\tau) \sim{}& \left(\bar{S}e^{\tau(1-k)} - e^{-k\tau}\right) \mathrm{N}\left(\frac{1}{\epsilon\sqrt{\tau}}(\bar{S}e^{\tau}-1)\right) \\ &+\frac{\epsilon}{2}\sqrt{\frac{\tau}{2\pi}}\left(e^{-k\tau} + \bar{S}e^{(1-k)\tau}\right)\exp\left(-\frac{(\bar{S}e^{\tau}-1)^2}{2\epsilon^2\tau}\right) .\end{aligned} \tag{C.24}$$

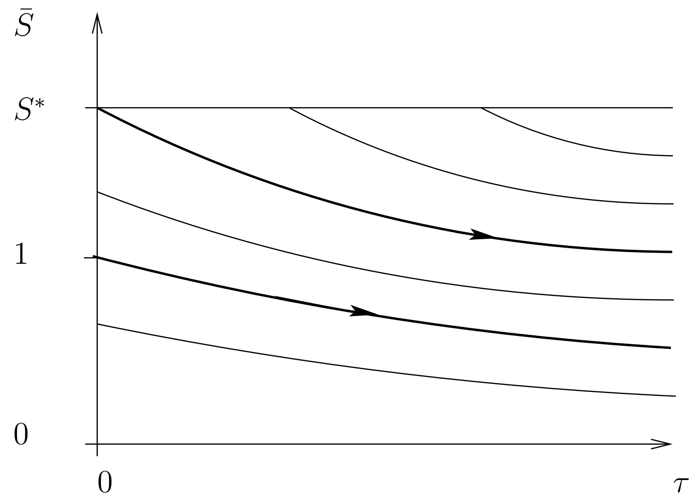


Figure C.2: Characteristics for the American option with $r > q$

C.3.1.4 Upper region

The leading order solution in the upper region is different to the European case, as we have to take into consideration the free boundary. We no longer use the boundary condition propagating from the terminal payoff at time $\tau = 0$ (see Figure C.2). We now use characteristics propagating from the free boundary and the smooth pasting condition, $\bar{V}_{\bar{S}} = 1$.

We find that the relation defining the characteristic equations is

$$\frac{\tau'}{1} = \frac{\bar{S}'}{-\bar{S}} = \frac{\bar{V}_0'}{-k\bar{V}_0}.$$

We solve these equations. First we find that

$$\bar{S} = Ae^{-\tau} \quad \text{and} \quad \bar{V}_0 = Be^{-k\tau}$$

so, for some function g we have

$$\bar{V}_0 = g\left(\bar{S}e^{\tau}\right) e^{-k\tau} \tag{C.25}$$

We use the two boundary conditions

$$\bar{V}_0(\bar{S}_0^*, \tau) = \bar{S}_0^* - 1 \tag{C.26}$$

$$\bar{V}_{0,\bar{S}}(\bar{S}_0^*, \tau) = 1 \tag{C.27}$$

Applying boundary condition (C.26) and rearranging gives

$$g\left(\bar{S}_0^* e^{\tau}\right) = \left(\bar{S}_0^* - 1\right) e^{k\tau} \,. \tag{C.28}$$

Differentiating (C.25), our expression for $\bar{V}$, with respect to $\bar{S}$ and applying (C.27), the second boundary condition, gives

$$e^{\tau} g'\left(\bar{S}_0^* e^{\tau}\right) = e^{k\tau} \,. \tag{C.29}$$

Differentiating our expression found using the first boundary condition, (C.28), with respect to τ gives

$$g'\left(\bar{S}_0^* e^{\tau}\right) e^{\tau} \left(\frac{d\bar{S}_0^*}{d\tau} + \bar{S}_0^*\right) = \left(\frac{d\bar{S}_0^*}{d\tau} + k(\bar{S}_0^* - 1)\right) e^{k\tau} \,. \tag{C.30}$$

Substituting Equation (C.29) into (C.30) causes the cancellation of differential terms and exponential terms leaving

$$\bar{S}_0^* = \frac{k}{k-1} \,. \tag{C.31}$$

Hence we have in the original variables

$$S_0^* = \frac{Er}{q} \,. \tag{C.32}$$

Therefore, to leading order the free boundary is constant, and has the same value as the small σ limit for the free boundary for the perpetual option (C.14). Substituting this into (C.28) allows us to find an expression for the unknown function g

$$g\left(\bar{S}_0^* e^{\tau}\right) = \frac{1}{k-1} e^{k\tau} \,. \tag{C.33}$$

If we write

$$\xi = \bar{S}_0^* e^{\tau} \,,$$

then we have that

$$\tau = \log\left(\frac{\xi}{\bar{S}_0^*}\right) \,.$$

We can use this and (C.33) to find $g(\xi)$

$$g(\xi) = -\frac{1}{k}\left(\frac{\xi}{\bar{S}_0^*}\right)^k \,.$$

We now use this function definition in our expression for $\bar{V}_0$, (C.25), to obtain,

$$\bar{V}_0 = \frac{1}{k-1}\left(\frac{\bar{S}e^{\tau}}{\bar{S}_0^*}\right)^k e^{-k\tau} \,,$$

simplifying gives

$$\bar{V}_0 = \frac{1}{k-1}\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k \,. \tag{C.34}$$

Note that there is no time dependence in the leading order solution for the value of the option and the expression, in the small σ limit, is the same as for the perpetual option (C.10).

C.3.1.5 Leading order solution

To summarise, the leading order solution to the outer problem is (from (C.24)),

$$\begin{aligned}\bar{V}^{lower}(\bar{S},\tau) \sim{}& \left(\bar{S}e^{\tau(1-k)} - e^{-k\tau}\right) \mathrm{N}\left(\frac{1}{\epsilon\sqrt{\tau}}(\bar{S}e^{\tau}-1)\right) \\ &+\frac{\epsilon}{2}\sqrt{\frac{\tau}{2\pi}}\left(e^{-k\tau} + \bar{S}e^{(1-k)\tau}\right)\exp\left(-\frac{(\bar{S}e^{\tau}-1)^2}{2\epsilon^2\tau}\right) \\ &\qquad\qquad \text{for} \quad 0 < \bar{S} < \bar{S}_0^* e^{-\tau} \,. \end{aligned} \tag{C.35}$$

and (from (C.34))

$$\bar{V}_0^{upper}(\bar{S},\tau) = \frac{1}{k-1}\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k \qquad \text{for} \qquad \hat{S} < \bar{S} < \bar{S}_0^* \,,$$

where

$$\hat{S} = \bar{S}_0^* e^{-\tau} \,.$$

Note that there is no time dependence in the value of the option in the upper region. The value is the same as the low volatility limit of the perpetual option for $r > q$.

We can see the shape of the value function in Figure C.3. The two functions fit together smoothly, so we cannot see where the two solutions join.

C.3.1.6 Boundary layers

We need to check how the solution behaves across the boundary between the two solutions. To leading order the lower solution across the discounted strike is smooth.

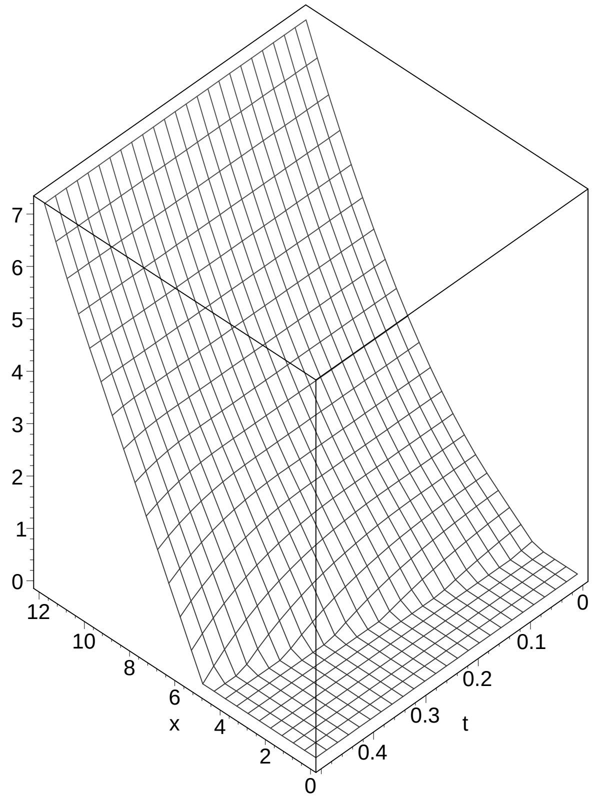


Figure C.3: The leading order solution to the value of the call with $r > q$

We find that the upper and lower solutions match smoothly, and that there is no boundary layer.

First let us check what happens near

$$\bar{S} = \hat{S} = \bar{S}_0^* e^{-\tau}.$$

If we evaluate the two solutions, $\bar{V}_0^{lower}$ and $\bar{V}_0^{upper}$ at the boundary, we find that

$$\bar{V}_0^{lower}(\hat{S}) = \left(\bar{S}_0^* - 1\right) e^{-k\tau},$$

and

$$\bar{V}_0^{upper}(\hat{S}) = \frac{e^{-k\tau}}{k-1}.$$

As we know that $\bar{S}_0^* = \frac{k}{k-1}$ we can see that the leading order solution is continuous across the characteristic , $\hat{S}$, propagating from $\bar{S} = \bar{S}_0^*$ at time $\tau = 0$.

We can also check the values for the derivatives across this boundary. The lower solution's derivative does not depend on $\bar{S}$

$$\bar{V}_{0,\bar{S}}^{lower}(\bar{S},\tau) = e^{(1-k)\tau}.$$

The upper solution's derivative is

$$\bar{V}_{0,\bar{S}}^{upper}(\bar{S},\tau) = \left(\frac{\bar{S}}{S_0^*}\right)^{k-1},$$

which, evaluated on $\hat{S}$ is also

$$\bar{V}_{0,\bar{S}}^{upper}(\hat{S},\tau) = e^{(1-k)\tau}.$$

Hence the derivative is continuous as well.

C.3.1.7 Higher order terms

We can now look at the expansions that we have made and increase the accuracy of our approximation by including more terms in the expansion,

$$\bar{V} \sim \bar{V}_0 + \epsilon^2 \bar{V}_1 + \dots$$

The $\mathcal{O}(\epsilon^2)$ equation is now

$$\bar{V}_{1,\tau} - \bar{S}\bar{V}_{1,\bar{S}} = \tfrac{1}{2}\bar{S}^2 \bar{V}_{0,\bar{S}\bar{S}} - k\bar{V}_1$$

with the boundary conditions for the full problem in the upper region being

$$\begin{aligned} \bar{V}^{upper}(S^*,\tau) &= S^* - 1, \\ \bar{V}_{\bar{S}}^{upper}(S^*,\tau) &= 1. \end{aligned}$$

The equations which determine the characteristics are

$$\frac{\tau'}{1} = \frac{\bar{S}'}{-\bar{S}} = \frac{\bar{V}_1'}{\frac{1}{2}\bar{S}^2\bar{V}_{0,\bar{S}\bar{S}} - k\bar{V}_1}. \tag{C.36}$$

The second partial derivatives of $\bar{V}_0$ with respect to $\bar{S}$ are

$$\begin{aligned} \bar{V}_{0,\bar{S}\bar{S}}^{lower}(\bar{S},\tau) &= 0 \\ \bar{V}_{0,\bar{S}\bar{S}}^{upper}(\bar{S},\tau) &= \tfrac{k-1}{S_0^*}\left(\tfrac{\bar{S}}{S_0^*}\right)^{k-2}. \end{aligned}$$

We can see that the characteristic equations are unchanged in the lower region. All the boundary data has been taken care of in the $\bar{V}_0$ case for the lower region, therefore

$$\bar{V}_1^{lower} \equiv 0\,.$$

Hence we shall drop the *upper* and *lower* superscripts and consider the higher order terms only the *upper* region. Substituting the equation for $\bar{V}_{0,\bar{S}\bar{S}}^{upper}$ and rearranging (C.36) we can write

$$\frac{d\bar{V}_1}{d\tau} = k\bar{V}_1 - \tfrac{1}{2}k\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k.$$

If we substitute

$$\bar{S} = A(x)e^{\tau}$$

into this equation we obtain

$$\frac{d\bar{V}_1}{d\tau} - k\bar{V}_1 = -\tfrac{1}{2}k\left(\frac{A(x)}{\bar{S}_0^*}\right)^k e^{k\tau}.$$

We can solve this ordinary differential equation to get a solution of the form

$$\bar{V}_1 = B(x)e^{k\tau} + C(x)\tau e^{k\tau}.$$

We find that

$$C(x) = -\tfrac{1}{2}k\left(\frac{A(x)}{\bar{S}_0^*}\right)^k.$$

Hence, so far, we have

$$\bar{V}_1 = B(x)e^{k\tau} - \tfrac{1}{2}k\left(\frac{A(x)}{\bar{S}_0^*}\right)^k \tau e^{k\tau}. \tag{C.37}$$

Let us now expand the boundary conditions in terms of the expansion of the expression for the free boundary. Remember that in the upper region we only use boundary conditions on the free boundary. In general we use a Taylor expansion

$$f(\bar{S}_0^* + \epsilon^2\bar{S}_1^* + \ldots) = f(\bar{S}_0^*) + \epsilon^2\bar{S}_1^* f'(\bar{S}_0^*) + \ldots.$$

We have already solved the $\mathcal{O}(\epsilon^0)$ case, so we now consider the $\mathcal{O}(\epsilon^2)$ case. Using this Taylor expansion we find that the expansion for the boundary conditions gives

$$\begin{aligned} \bar{V}_1(\bar{S}_0^*,\tau) \quad &+\bar{S}_1^*\bar{V}_{0,\bar{S}}(\bar{S}_0^*,\tau) &&= \bar{S}_1^* \\ \bar{V}_{1,\bar{S}}(\bar{S}_0^*,\tau) \quad &+\bar{S}_1^*\bar{V}_{0,\bar{S}\bar{S}}(\bar{S}_0^*,\tau) &&= 0. \end{aligned}$$

Remembering from (C.31) that $\bar{S}_0^* = k/(k-1)$ we can evaluate

$$\bar{V}_{0,\bar{S}}(\bar{S}_0^*, \tau) = 1$$

and

$$\bar{V}_{0,\bar{S}\bar{S}}(\bar{S}_0^*, \tau) = \frac{k-1}{\bar{S}_0^*}.$$

Hence,

$$\bar{V}_1(\bar{S}_0^*, \tau) = 0 \tag{C.38}$$

and

$$\bar{V}_{1,\bar{S}}(\bar{S}_0^*, \tau) = -\bar{S}_1^* \frac{k-1}{\bar{S}_0^*}. \tag{C.39}$$

Now, on $\tau = y$ we have $\bar{S} = \bar{S}_0^*$. Thus,

$$\bar{S}_0^* = A(x)e^y$$

and so

$$\bar{S} = \bar{S}_0^* e^{(\tau - y)}. \tag{C.40}$$

Substituting this into the expression for $\bar{V}_1$, (C.37), and using the boundary condition (C.38) we get

$$\bar{V}_1(\bar{S}_0^*, \tau) = 0 = B(y)e^{ky} - \tfrac{1}{2}kye^{ky}$$

and so

$$B(y) = \tfrac{1}{2}kye^{-ky}.$$

Substituting into (C.37), the solution for $\bar{V}_1$, gives

$$\bar{V}_1(\bar{S}, \tau) = -\tfrac{1}{2}k(\tau - y)e^{k(\tau - y)}.$$

Next we eliminate the dependence on y by using (C.40) to note that

$$(\tau - y) = \log \frac{\bar{S}}{\bar{S}_0^*}.$$

Therefore, we have the result

$$\bar{V}_1(\bar{S}, \tau) = -\tfrac{1}{2}k \log\left(\frac{\bar{S}}{\bar{S}_0^*}\right)\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k. \tag{C.41}$$

Substituting this into our asymptotic expansion for $\bar{V}$ with $\bar{V}_0$ gives

$$\bar{V}(\bar{S}, \tau) \sim \left(\frac{1}{k-1} - \tfrac{1}{2}\epsilon^2 \log\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k\right)\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k. \tag{C.42}$$

This agrees with the dimensionless result for the perpetual American call option given in (C.11).

Finally we need to determine the adjusted location of the free boundary. We do this by using the remaining boundary condition, (C.39). Differentiating (C.41) with respect to $\bar{S}$ gives

$$\bar{V}_{1,\bar{S}}(\bar{S},\tau) = -\tfrac{1}{2}\frac{k}{\bar{S}}\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k\left(1+\log\left(\frac{\bar{S}}{\bar{S}_0^*}\right)^k\right).$$

Evaluating this at $\bar{S} = \bar{S}_0^*$ and using the boundary condition (C.39) allows us to find the relation between $\bar{S}_0^*$ and $\bar{S}_1^*$

$$V_{1,S}(\bar{S}_0^*,\tau) = -\tfrac{1}{2}\frac{k}{\bar{S}_0^*} = -\bar{S}_1^*\frac{k-1}{\bar{S}_0^*}.$$

Simplifying, the expression for the next term in the expansion for the free boundary is

$$\bar{S}_1^* = \tfrac{1}{2}\bar{S}_0^*,$$

and so our asymptotic expansion for the boundary at $\mathcal{O}(\epsilon^2)$ is

$$\bar{S}^*(\tau) \sim \frac{k}{k-1}\left(1+\tfrac{1}{2}\epsilon^2\right). \tag{C.43}$$

Again this agrees with the free boundary for the perpetual American call option given in (C.8).

Note that even at the second order there is no time dependence in the expression for the free boundary. However, we know that there must be a boundary layer as we approach $\tau = 0$ as the free boundary must come in to meet the strike at 1.

C.3.1.8 Original variables

In the original variables, when $r > q$, the value of the option (C.42) is

$$V^{upper}(S,t) \sim E\left(\frac{r}{q} - 1 - \tfrac{1}{2}\sigma^2\frac{E}{(r-q)}\log\left(\frac{S}{S_0^*}\right)^{\frac{r}{r-q}}\right)\left(\frac{S}{S_0^*}\right)^{\frac{r}{r-q}}, \tag{C.44}$$

which agrees with the small volatility limit of the perpetual option, (C.13). The location of the free boundary for the perpetual option, (C.14), also agrees with the location of the free boundary, (C.43), in the original variables

$$S^*(t) \sim \frac{Er}{q}\left(1+\frac{\sigma^2}{2(r-q)}\right). \tag{C.45}$$

C.3.2 Case $r < q$

We non-dimensionalise the equation using the following transformations, as in the European case,

$$V = E\bar{V} \qquad S = E\bar{S} \qquad \tau = (T-t)|r-q|$$

to obtain, using the subscript notation for partial derivatives ,

$$\begin{aligned}
-\bar{V}_\tau &= -\tfrac{1}{2}\epsilon^2 \bar{S}^2 \bar{V}_{\bar{S}\bar{S}} + \bar{S}\bar{V}_{\bar{S}} + k\bar{V} \\
\bar{V}(\bar{S},0) &= \max(\bar{S}-1,0) \\
\bar{V}(\bar{S}^*,\tau) &= \bar{S}^* - 1 \\
\bar{V}_{\bar{S}}(\bar{S}^*,\tau) &= 1
\end{aligned}$$

where

$$\epsilon^2 = \frac{\sigma^2}{|r-q|} \quad \text{and} \quad k = \frac{r}{|r-q|}.$$

Note that as we have used the modulus in the non-dimensionalisation the signs of some of the terms have changed.

C.3.2.1 Outer expansion

Again, we just find where to make our inner expansion. We expand the function for the option value using

$$\bar{V} \sim \bar{V}_0 + \epsilon^2 \bar{V}_1 + \dots$$

and the free boundary using

$$\bar{S}^* \sim \bar{S}_0^* + \epsilon^2 \bar{S}_1^* + \dots$$

In Figure C.4 we can see that information propagating from the terminal condition, when the option fails to pay out any money, crosses the whole space and reaches the free boundary. Therefore, to first order

$$\bar{V}_0 \equiv 0,$$

which means that we must have,

$$\bar{S}_0^* = 1.$$

We need to look for an inner solution near to 1 to get a more interesting expansion.

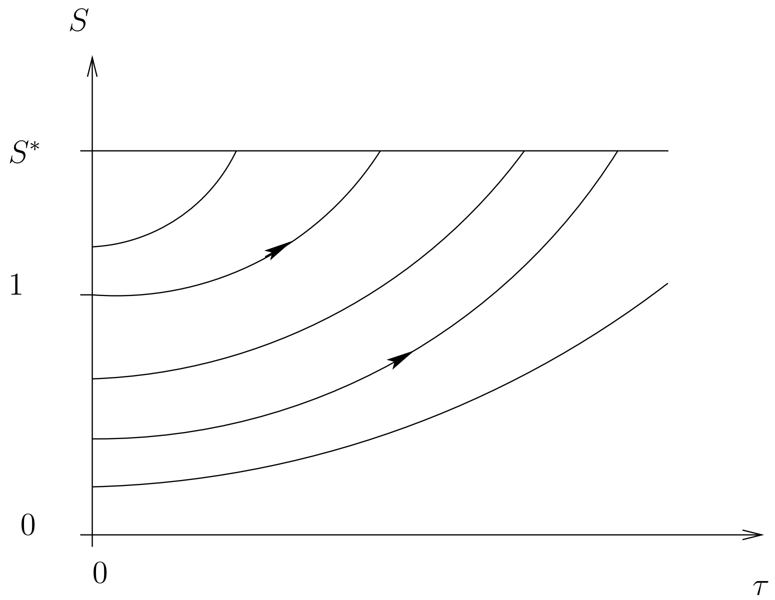


Figure C.4: Characteristics for the American option with $q > r$

C.3.2.2 Inner expansion

We have the Black-Scholes equation

$$-\bar{V}_\tau + \tfrac{1}{2}\epsilon^2 \bar{S}^2 \bar{V}_{\bar{S}\bar{S}} - \bar{S}\bar{V}_{\bar{S}} - k\bar{V} = 0,$$

with boundary conditions

$$\begin{aligned} \bar{V}(\bar{S}, 0) &= 0 \\ \bar{V}(\bar{S}^*(\tau), \tau) &= \bar{S}^*(\tau) - 1 \\ \bar{V}_{\bar{S}}(\bar{S}^*(\tau), \tau) &= 1\,. \end{aligned}$$

We know the first term in the expansion for the free boundary, so we now have,

$$\bar{S}^* \sim 1 + \epsilon^2 \bar{S}_1^* + \dots$$

We use an inner variable

$$s = \frac{\bar{S} - 1}{\epsilon^2},$$

and note that in the new variables the next term in the expansion for the free boundary is now

$$\beta_1 = \bar{S}_1^*.$$

Substitute into the dimensionless Black-Scholes equation to get

$$-\bar{V}_\tau + \frac{1}{2\epsilon^2}\left[1+\epsilon^2 s\right]^2 \bar{V}_{ss} - \frac{1}{\epsilon^2}\left[1+\epsilon^2 s\right]\bar{V}_s - k\bar{V} = 0.$$

Now expand the inner value function

$$\bar{V} \sim \epsilon^2 \bar{V}_0 + \epsilon^4 \bar{V}_1 + \dots$$

to get

$$\begin{aligned}-\left[\epsilon^2\bar{V}_{0,\tau} + \epsilon^4\bar{V}_{1,\tau} + \dots\right] + \left[\tfrac{1}{2} + \epsilon^2 s + \tfrac{1}{2}\epsilon^4 s^2\right]\left[\bar{V}_{0,ss} + \epsilon^2\bar{V}_{1,ss} + \dots\right]\\ -\left[1+\epsilon^2 s\right]\left[\bar{V}_{0,s} + \epsilon^2\bar{V}_{1,s} + \dots\right] - k\left[\epsilon^2\bar{V}_0 + \epsilon^4\bar{V}_1\right] = 0.\end{aligned}$$

Picking out the ϵ^0 terms we obtain

$$\tfrac{1}{2}\bar{V}_{0,ss} - \bar{V}_{0,s} = 0.$$

Note that this equation has no time dependence, so we expect to be able to relate the solution to results for the perpetual American call.

Again we can expand the boundary conditions on the free boundary as a Taylor series. This enables us to match the inner solution to the outer solution. The $\mathcal{O}(\epsilon^0)$ terms are

$$\bar{V}_0 \to 0 \quad \text{as} \quad s \to -\infty, \tag{C.46}$$

$$\bar{V}_0(\beta_1, \tau) = \beta_1, \tag{C.47}$$

$$\bar{V}_{0,\bar{S}}(S_1^*, \tau) = 1. \tag{C.48}$$

Rearrange our equation to get the second derivative on the left,

$$\bar{V}_{0,ss} = 2\bar{V}_{0,s}.$$

We can integrate to find

$$\bar{V}_{0,s} = 2\bar{V}_0 + a(s).$$

and integrate again to find

$$\bar{V}_0 = C(\tau)e^{2s} + D(\tau).$$

As $\bar{V} \to 0$ as $s \to -\infty$, from (C.46), we must have that $D = 0$. Now we consider what happens on the free boundary. We have, using (C.47),

$$\bar{V}_0(\beta_1, \tau) = C(\tau)e^{2\beta_1} = \beta_1,$$

therefore we have that

$$C(\tau) = \beta_1 e^{-2\beta_1}$$

and

$$\bar{V}_0(s,\tau) = \beta_1 e^{2(s-\beta_1)}$$

We can then use (C.48), the boundary condition on the derivative, to find

$$\bar{V}_{0,s}(\beta_1,\tau) = 2\beta_1 = 1.$$

Hence we have

$$\beta_1 = \frac{1}{2},$$

and so

$$\bar{V}_0(s,\tau) = \tfrac{1}{2}\exp\left(2s-1\right).$$

In the outer variables we have that the option value is

$$\bar{V}(\bar{S},\tau) \sim \tfrac{1}{2}\epsilon^2 \exp\left(2\left(\frac{\bar{S}-1}{\epsilon^2}\right)-1\right), \tag{C.49}$$

and that the expression for the free boundary is

$$\bar{S}^*(\tau) \sim 1 + \tfrac{1}{2}\epsilon^2 . \tag{C.50}$$

C.3.2.3 Original variables

If we reverse the non-dimensionalisation we find that the expression for the option value, (C.49), becomes

$$V(S,t) \sim \frac{\sigma^2 E}{2(q-r)} \exp\left(\frac{2(q-r)}{\sigma^2 E}(S-E)-1\right), \tag{C.51}$$

which matches the small volatility limit of the perpetual American option value given in (C.18). The small volatility limit of the free boundary for the perpetual option, (C.19), also agrees with (C.50) in the original variables,

$$S^*(t) \sim E\left(1+\frac{\sigma^2}{2(q-r)}\right). \tag{C.52}$$

C.4 Numerical results

We have found an asymptotic expression for the level of the free boundary for an American call option in the presence of a constant continuous dividend yield. We can use this barrier level in the closed form valuation formula for an up–and–out barrier call option. This is essentially the insight in Bjerksund and Stensland (1993).

In this chapter we investigate the accuracy of this approach. A good check for any numerical method for American options is to verify that in the case of no dividends the American price is the same as the European price. We then investigate the behaviour of our approximation for the cases of $r > q$ and $r < q$.

C.4.1 Implementation

We detail the implementation of methods derived in chapter C that were used to obtain the results in this chapter. A number of approximations for single asset American options are available in QuantLib. The following methods were implemented during this research:

- `juquadraticengine.cpp` implements the modified quadratic method of Ju (1999), discussed in chapter 2. The implementation was tested (to an absolute accuracy of 10^{-3}) against the values in Table 3 of Ju (1999) using `americanoption.cpp` in the QuantLib test suite.

The following methods were already available in QuantLib:

- `baroneadesiwhaleyengine.cpp` implements the approximation method of Barone-Adesi and Whaley (1987). The implementation is tested (to an absolute accuracy of 10^{-2}) against values in Haug (1997, page 24) using `americanoption.cpp` in the QuantLib test suite.

- `bjerksundstenslandengine.cpp` implements the flat barrier approximation of Bjerksund and Stensland (1993). The implementation is tested (to an absolute accuracy of 10^{-4}) against values in Haug (1997, page 27) using the file `americanoption.cpp` in the QuantLib test suite.

- `binomialengine.cpp` implements the binomial tree algorithm (Cox et al., 1979). These values are verified against the analytic European option values, and greeks, using `europeanoption.cpp` in the QuantLib test suite.

As a closed form expression is available for the value of the American perpetual option (Wilmott, 1998), values were calculated using Mathematica (2004). European option values were calculated using `analyticeuropeanengine.cpp`, the QuantLib pricing engine.

C.4.2 Case $q = 0$

When the continuous dividend yield is zero it is never optimal to exercise an American call option early. In our approximation the free boundary tends towards infinity, and so makes no contribution to the option value. Therefore the option value is just the European value.

C.4.3 Perpetual American call option

Transforming (C.6) into financial variables we have an expression for the value of the perpetual call option

$$V_\infty = (S^* - E)\left(\frac{S}{S^*}\right)^{m_+},$$

where, from (C.5),

$$m_+ = \frac{1}{\sigma^2}\left[-(r - q - \tfrac{1}{2}\sigma^2) + \sqrt{(r - q - \tfrac{1}{2}\sigma^2)^2 + 2r\sigma^2}\right].$$

We can also find an analytic solution for the free boundary, (C.7), in the case of the perpetual call, in the original variables,

$$S^* = \frac{m_+}{m_+ - 1}E.$$

C.4.4 Boundary approximation

We recall our expressions for the location of the boundary. We use American put call symmetry to write put options as call options. It is well known that the optimal exercise boundary at the expiry date satisfies

$$S_0^* = \max\left(E, \frac{Er}{q}\right).$$

Hence we can unify our expressions for the optimal exercise boundary, (C.45) and (C.52), in terms of our small parameter, $\epsilon^2 = \sigma^2/|r - q|$ as,

$$S^* \sim S_0^*\left(1 + \tfrac{1}{2}\epsilon^2\right).$$

S_0/E	European	Perpetual	American (binomial)	American approx	Absolute Error
0.8	0.4449	24.0335	0.4448	0.4449	0.0001
0.9	2.2914	29.0466	2.2914	2.2914	-
1.0	6.7187	34.4109	6.7185	6.7187	0.0002
1.1	13.7279	40.1122	13.7279	13.7279	-
1.2	22.3858	46.1380	22.3858	22.3858	-

Table C.1: Call option approximation with $r > q$; $E = 100$, $r = 0.1$, $q = 0.05$, $\sigma = 0.2$, $T = 0.5$, $\epsilon^2 = 0.8$, $S^* = 280$. The binomial tree has 10000 steps.

C.4.5 Numerical results

We use this boundary in the valuation of an up-and-out European barrier call option. The option pays a rebate of $S^* - E$ if the barrier at S^* is hit before the option expires. Otherwise the option pays the standard European payoff. This barrier option can be valued analytically. This is a very simple approximation, yet can be surprisingly accurate, as can be seen from table C.1. The level of the barrier is 280, so as the strike is 100 the rebate paid if the barrier is hit is 180. The option expiry is $T = 0.5$, we can see that the values are very different to the perpetual American call option values. These parameters give a value of $\epsilon^2 = 0.8$.

The European value quoted is that calculated from the solution to the Black-Scholes equation with a continuous dividend yield, (2.8)–(2.9).

We see, in figure C.5, that the approximate American valuation formula and American call option value are very close to the European value for small and medium time scales. However, as the time to expiry increases the American option values converge to the perpetual option value, while the European option value does not. The effect of the discounting term on the value of the contingent payout at expiry time drives the European value back towards zero.

We evaluate the accuracy of this approximation by considering the 27 short term puts as in Exhibit 3 in Ju (1999). Figure C.6 shows that, although using the perpetual boundary approximation gives an answer very quickly, it is not very accurate, and corrections have to be made, as in Bjerksund and Stensland (1993).

The numerical comparisons in figure C.6 were made using approximations implemented in QuantLib. As this is an object-oriented library written in C++ there are

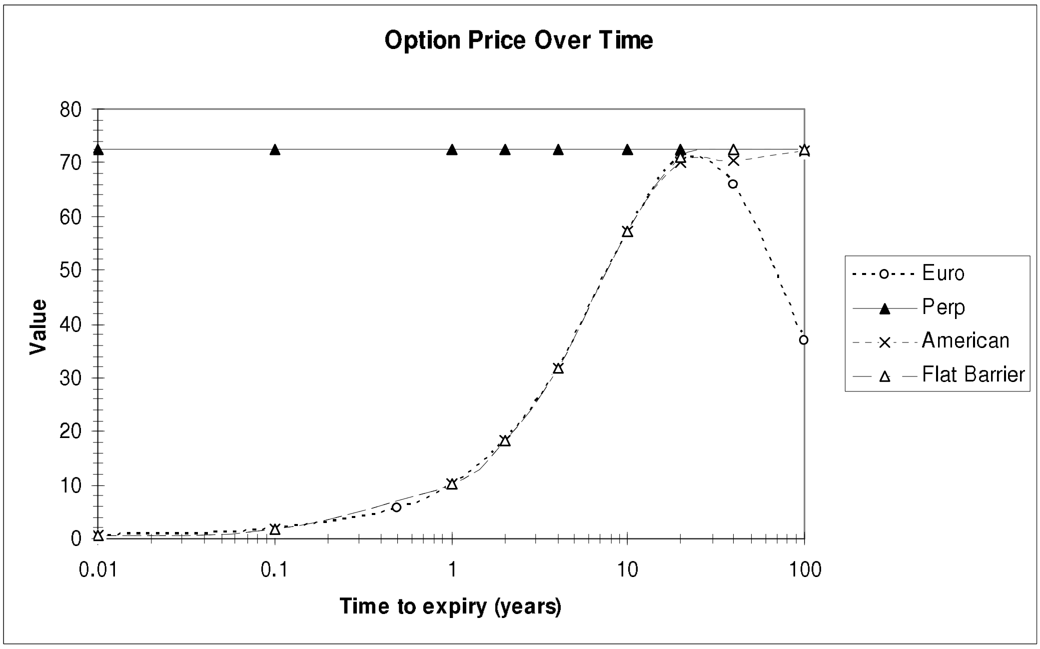


Figure C.5: Comparison of perpetual American call options, European options, vanilla American options, and the up-and-out barrier call option approximation. The initial stock price is $S_0 = 100$, the strike is $E = 100$, the risk free rate is $r = 0.11$, the dividend yield is $q = 0.01$, and the volatility of asset returns is $\sigma = 0.1$.

overheads for object creation. The implementations have not been optimised for this test.

For each results table we include the value of the small parameter, ϵ^2, used in the asymptotic expansion. In Table C.2 we show the effect of increasing the small parameter ϵ^2. For expiry $T = 0.5$ there is hardly any effect. This is encouraging, as we can apply our technique even when ϵ^2 rather large.

C.4.6 Evaluation

The dependence on the nondimensional time is through $\tau = (T-t)|r-q|$. For typical parameter values, such as those in table C.2, we have $\tau = 0.02$. As other studies have shown, the exercise boundary moves rapidly when approaching the expiry date of the option. The standard, alternative, nondimensionalisation is $\tau = \frac{1}{2}\sigma^2(T-t)$. Typical option values of $\sigma = 0.2$ and $T = 0.5$ give value of $\tau = 0.01$.

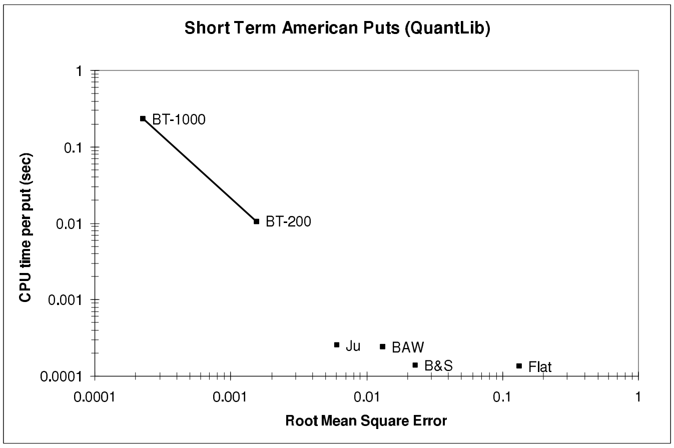


Figure C.6: Comparison of approximation methods for the portfolio of short term American put options in Ju (1999), using algorithms implemented in QuantLib. We use the same legend as in figures 2.2 and 2.3, the only difference being that B&S is the method of Bjerksund and Stensland (1993), and Flat is the up-and-out barrier option with the barrier at the level of the perpetual barrier.

ϵ^2	σ	European	American (binomial)	American approx	relative error (%)
0.000625	0.005	4.8527	4.8527	4.854	0.02834
0.0025	0.01	7.6519	7.6519	7.641	0.1367
0.25	0.1	5.8211	5.8203	5.803	0.3692
1	0.2	6.4312	6.4299	6.416	0.8736
2.25	0.3	2.9347	2.9365	2.940	0.1906
25	1.0	8.5944	8.6850	8.559	2.036

Table C.2: Effect on American call option prices of changing the value of the small parameter ϵ^2. There parameters are $S_0 = 100$, $E = 100$, $r = 0.08$, $q = 0.04$, $T = 0.5$.

C.5 Conclusions

We have presented an asymptotic analysis of an American call option where the diffusion term (volatility) is small compared to the drift terms (interest rate and continuous dividend yield). We have shown that in the limit where diffusion is negligible, relative to drift, then, at leading order, the American call's behaviour is the same as a perpetual American call option (except in a boundary layer about the option's expiry date).

Appendix D

QuantLib

For most people working in the field of mathematical finance it becomes necessary to code up derivatives pricing algorithms. While it is possible to design and write new code independently it is easier to use and build upon existing projects, if these projects are well implemented.

There are a number of libraries of varying quality for pricing financial derivatives available on the internet. We look at the aims of these projects, and at the aims of open–source projects in general. We consider only open–source libraries, as they allow implementations of algorithms to be peer reviewed, and do not require any licensing fees. At this time open–source libraries for pricing financial derivatives do not cover as many products, or implement as many models as closed–source packages. For a survey of closed–source software providers see Davidson (2004). Most real derivative pricing systems are implemented in C++ however, software tools such as Mathematica (2004) or Matlab (2004) are used by some companies for particular applications. These softwares are more useful for prototyping and are proprietary and therefore, beyond the scope of this paper.

In section D.1 we discuss some of the terms and issues involved in open–source software development, and survey existing projects for financial derivatives. In section D.2 we look at the different motivations of participants in open–source projects, and which are relevant to financial mathematics. In the following section (section D.3) we relate these issues to the QuantLib project. We look at potential user groups and briefly discuss the practicalities of contributing to QuantLib. Conclusions are given in section D.4.

D.1 Open–source derivatives libraries

Open–source software can be developed under one of several licenses and hosted in various ways. The choice of license and project structure has an influence on the success of the project. Existing derivatives libraries have differing project structures.

D.1.1 Free software

The profile of free software has increased greatly in recent years, mainly due to the emergence of GNU[1]/Linux as a viable alternative operating systems to Microsoft Windows. The GNU project for free software was started in 1984 by Richard Stallman at MIT (GNU, 2004; Stallman, 2002). The definition of free software is given on the GNU website[2]:

> "Free software" is a matter of liberty, not price. To understand the concept, you should think of "free" as in "free speech," not as in "free beer."
>
> Free software is a matter of the users' freedom to run, copy, distribute, study, change and improve the software. More precisely, it refers to four kinds of freedom, for the users of the software:
>
> - The freedom to run the program, for any purpose (freedom 0).
> - The freedom to study how the program works, and adapt it to your needs (freedom 1). Access to the source code is a precondition for this.
> - The freedom to redistribute copies so you can help your neighbor (freedom 2).
> - The freedom to improve the program, and release your improvements to the public, so that the whole community benefits (freedom 3). Access to the source code is a precondition for this.
>
> A program is free software if users have all of these freedoms. Thus, you should be free to redistribute copies, either with or without modifications,

[1]GNU is a recursive acronym for "GNU's Not Unix"; it is pronounced"guh-no" (GNU, 2004)
[2]For the full GNU definition of free software see `http://www.gnu.org/philosophy/free-sw.html`

> either gratis or charging a fee for distribution, to anyone anywhere. Being free to do these things means (among other things) that you do not have to ask or pay for permission.

A crucial part of Stallman's project was the development of legal licenses to ensure the ongoing freedom of software.

D.1.2 Open–source software

More recently, the Open Source Initiative (2004) (OSI) has put a more business friendly face on free software. The term was decided upon in 1998[3],

> ... it was time to dump the confrontational attitude that has been associated with "free software" in the past and sell the idea strictly on [...] pragmatic, business-case grounds ...

The precise definition used by the Open Source Initiative (OSI) runs to ten clauses[4]. Raymond (2000a,b,c, 2001) has written an accessible account of the open–source development process. The OSI maintains a list of software licenses which they approve as satisfying the definition, and which are allowed to carry the OSI certification mark[5].

D.1.3 Developing, debugging and code maintenance

Brooks (1995) explains the difficulties faced in large software projects in his book *The Mythical Man Month*. If a project falls behind schedule, adding more developers makes the situation worse as the cost of communication explodes. Boehm (1987) found that 40 to 50 percent of time in most projects is spent on testing and debugging.

There have been many project management solutions suggested, such as the waterfall model (Royce, 1970), the spiral model (Boehm, 1988) and more recently extreme programming (Beck, 1999). However, (Sommerville, 1992) states that, in fact, ongoing maintenance costs are typically two to four times as much as total development

[3] A history of the Open Source Initiative is available at `http://www.opensource.org/docs/history.php`

[4] see `http://www.opensource.org/docs/definition.php`

[5] see `http://www.opensource.org/docs/certification_mark.php`

costs. The time spent initially coding a solution is a small fraction of the effort involved in a successful software system.

Raymond (2001) discusses how some of these software development problems are overcome by the network effects present in a successful open–source project. Bugs are found quickly when the project has many users, but the communication burden between developers need not explode as you have few core developers and many more users (Raymond, 2000a, 2001). As many users are technically literate the quality of bug reports is increased, and fixes found more easily. The accelerated release cycle brings bugs to light more quickly and allows fixes to be disseminated rapidly.

Dependent libraries of projects are forwardly compatible and code is maintained after particular developers lose interest (Raymond, 2001). This means that it is possible to leave a project temporarily and return later to find that it works with the latest versions of other related projects.

However, Schach et al. (2003) find that the instances of global coupling within the 17 modules of the Linux kernel has been growing exponentially with kernel version number. If this trend continues Linux may become increasingly hard to maintain. Future studies will show whether this effect applies to all open–source development or just Linux.

D.1.4 Licensing

Careful choice of license is critical to the survival and success of open–source projects. For example, if a license is not deemed open–source by the Open Source Initiative (2004) then the project will have difficulty attracting developers as in the case of Netscape in 1998. Netscape released the source code of their browser under the Netscape Public License, this was not an open–source compatible license, according to the Open Source Initiative (2004), and few developers took up the project. Netscape changed the license to the Mozilla Public License, which was acceptable to the Open Source Initiative (2004), and the project is now successful. Choice of licensing by companies using open–source software is discussed in Bonaccorsi and Rossi (2003b).

We need to define the idea of *Copyleft*,

Copyleft is a general method for making a program free software and requiring all modified and extended versions of the program to be free software as well[6].

[6] see `http://www.gnu.org/copyleft/copyleft.html`

The three most popular free licenses are, as described on the GNU website[7]:

GNU General Public License or GNU GPL for short. This is a free software license, and a copyleft license. We recommend it for most software packages.

GNU Lesser General Public License or GNU LGPL for short.
This is a free software license, but not a strong copyleft license, because it permits linking with non-free modules. It is compatible with the GNU GPL. We recommend it for special circumstances only.

Modified Berkeley Software Distribution License or modified
BSD license for short. This is the original BSD license, modified by removal of the advertising clause. It is a simple, permissive non-copyleft free software license, compatible with the GNU GPL. If you want a simple, permissive non-copyleft free software license, the modified BSD license is a reasonable choice.

A full list of the OSI certified licenses is available from Open Source Initiative (2004)[8].

D.1.5 Hosting

The code of an open–source project has to be hosted on a server, somewhere on the internet. The GNU project hosts around 2,000 projects at a site called Savannah (2004). The largest such software development website is SourgeForge.net (2004). Of the approximately 50,000 open–source projects hosted at SourgeForge.net (2004):

- 71% are licensed under the GPL.
- 11% under the LGPL.
- 7% under the modified BSD license (SourgeForge.net, 2004).

In comparison there are just 2345 projects that are not deemed open–source by Open Source Initiative (2004). Non open–source projects can have great difficulty attracting and retaining members as illustrated by the Netscape example above (Bonaccorsi and Rossi, 2003b; Raymond, 2001).

[7] see `http://www.gnu.org/licenses/license-list.html`
[8] see `http://www.opensource.org/licenses/index.php`

Hosting a project at a dedicated site such as SourgeForge.net (2004) makes the project more accessible and it is easier to follow open–source management practices as described in Raymond (2000b). Setting up a project at SourgeForge.net (2004) indicates a commitment to the open–source community and an acceptance of 'open–source values' (Raymond, 2000b). This helps to attract developers and users (Bonaccorsi and Rossi, 2003b).

D.1.6 Existing projects

There are four main projects for derivatives pricing on the internet:

- Financial Numerical Recipes – Introductory collection of algorithms written in C++ (Ødegaard, 2004). Released under the GPL license.

- Premia – Implementation of many algorithms in C (Premia, 2004). Owned by the Institut National de Recherche en Informatique et en Automatique (INRIA) and the Centre d'Etude et de Recherche en Mathématiques Informatique et Calcul Scientifique, Laboratory of the Ecole Nationale des Ponts et Chausses (CERMICS, ENPC) in France and released under a restrictive license incompatible with the GPL. Commercial users pay a license fee.

- Project Martingale – Aimed at mainly academic experimentation with mathematical finance models from a probabilistic point of view. Monte Carlo models in C++ and Java (Martingale, 2004). Released under the GPL license.

- QuantLib — QuantLib is a free/open–source library for modelling, trading, and risk management in real-life (QuantLib, 2004). Written in C++ and released under the modified BSD license.

Financial Numerical Recipes is a good introduction to derivatives pricing, and is very accessible. However, it is not aimed at producing a real-life derivatives system, neither is Project Martingale. Both projects lack a clear open–source project management framework, and do not appear to have a critical mass of users.

The Premia project has a restrictive license, and is a development project of INRIA and ENPC, so is only useful as a reference implementation, or as a comparison. Contributions are not sought, and if they were, the fact that the license is not classified as open–source by the Open Source Initiative (2004) would make it difficult to attract developers.

QuantLib has an open–source license, is accessible to users, and the project administrators welcome developers. Currently QuantLib has 14 registered developers and around 400 users subscribed to the mailing list(Ametrano, 2004). It is achieving the critical mass of users needed to ensure the success of the project.

QuantLib is also in use at a number of financial institutions (Ametrano, 2004; Ballabio, 2004). While these institutions use the library they will ensure its quality for the financial products in their portfolios.

D.2 Motivations of participants

Individuals and companies involved in open–source projects have differing motivations. Ideas from anthropology have been used to describe the processes involved in academic research and open–source software. These ideas are extended to the field of mathematical finance.

D.2.1 Gift economy

The term *gift economy* was first coined in 1925 by Mauss (1925), when discussing the 'potlatch' gift–giving ceremonies of the American Northwest coast native tribes. Tribes that had sufficient food, and other goods, gave away surpluses to neighbouring tribes. After a time this practice became a custom and the gift–givers came to gain social reputation by the giving of ever more costly gifts. Gifts carry the implicit obligation to reciprocate. This contrasts with the idea of an *exchange economy* where we exchange money for goods and services. Arrow (1972) gives an illuminating discussion of the merits of the two systems in public policies for blood donation.

Hyde (1983) discusses academic research as a gift economy. Scientists were originally wealthy men who had no need to work for a living. Some felt driven to investigate their environment and to further knowledge by publishing their results. Note that we always talk of "giving" an academic paper at a conference. A brilliant scientist is thought to waste her effort if she does not publish her research. Einstein is respected because he gave the world the theory of relativity.

Raymond (2001) applies these ideas to the open–source software movement. In developed economies programmers have spare time to code and have come to contribute code to open–source projects. These contributers often need or desire a particular solution, they code it themselves, and release the code to a project. Coders receive other contributed code in return. Coders gain reputation from the quality and utility of the software they contribute. The most famous open–source developer is no doubt Linus Torvalds the creator of the Linux kernel used in the GNU/Linux operating system.

These informal accounts are now being backed up by empirical research. Lakhani and Wolf (2003) find that creativity is the greatest motivating factor in participation, followed by creating a public good. Hertel et al. (2003) suggests that motivations are similar to other social movements, such as the civil rights movement. In particular Hertel et al. (2003) find that developers believe code should be free, they benefit from code reciprocation within the community, they improve their coding skills and gain a reputation within the community. Both studies find that lack of time prevents participation in open–source projects.

D.2.2 Academic research

In the previous section we discussed research as a gift economy. Researchers do not extend the bounds of knowledge unless they disseminate what they have discovered and allow others to verify their results. What is the best method of dissemination? For many years research has been published in paper journals and knowledge shared verbally at conferences. Knowledge is also disseminated through funding reports.

Today the most efficient way of disseminating research is through the internet. Lawrence (2001) finds that online articles are around three times more likely to be cited than articles of similar quality not available online. Articles are also made easier to find by online citation indexes such as CiteSeer (2004) (a description of how CiteSeer works is given in Lawrence et al. (1999)). The point of research is reduced if results are not published on the internet.

As well as being available research also needs to be peer reviewed. Poor research is as easily disseminated as high quality research. Henneberg (1997) discusses the history and some of the failings of the peer review system.

Level of electronic availability of research	Number of researchers	Percentage of researchers
No website	3	10%
Website only	6	21%
Papers online	19	66%
Papers and code	1	3%

Table D.1: Dissemination of research in Mathematical Finance

D.2.3 Open–science

The Open Science Project (Open Science, 2004) is a group of scientists, mathematicians and engineers who recognise the importance of computer software to scientific development in their field. The project aim is that computer software developed to obtain or process scientific results should be released under an open–source license. This enables the peer review of results by other researchers, and allows results to be verified by other researchers (Kiernan, 1999). The publication of any visualisation techniques and data sets in the public domain has already been recommended by Brown et al. (1995).

Academic papers that are available online are more highly cited (Lawrence, 2001). It is easy to imagine that similar trends will appear between citations and participation in open–science projects. The author has experienced a marked increase in requests for preprints and speaking invitations since contributing to QuantLib.

Clearly purely theoretical papers are still vital to progress, and should be available on the internet. However, mathematical finance as a research area is driven by practical needs so ideas must be implemented in order to be used. It is at this computational stage that open–science ideas should be adopted.

D.2.3.1 Survey

About two thirds of 29 Mathematical Finance researchers' websites (from the United States and the European Union) informally surveyed have some of their publications available in pdf format on their website (see Table D.1). However, only in a single case was any code at all made available. Clearly the benefits of disseminating research over the internet have not yet been recognised by all researchers in the field.

D.2.4 Motivations for companies

Possible business models for companies using open–source software are discussed in Raymond (2000c). The actual motivations of companies involved in open–source projects have been investigated empirically by Bonaccorsi and Rossi (2003a) in a survey of 146 Italian companies. They found that the top three motivating reasons, using their exact wording, were:

1. Because Open Source software allows small enterprises to afford innovation.
2. Because contributions and feedback from the Free Software community are very useful to fix bugs and improve our software.
3. Because of the reliability and quality of Open Source software.

The first of these reasons probably does not apply in the financial sector, as banks are generally early adopters of technology and have large budgets for information technology. However, smaller institutions, such as asset management firms, or firms in developing countries, may benefit from the innovation in open–source software.

The second reason is certainly not altruistic. Firms participate in open–source projects so they can get free labour.

The third most popular reason is valid for firms of all sizes. The most mature open–source projects are stable and reliable.

Overall Bonaccorsi and Rossi (2003a) found that firms in general took more from the projects than they contributed. However, they also argue that successful projects are robust to firms being 'free–riders'.

D.2.5 Potential disadvantages

Raymond (2000b) notes that there is only 'room' for one project satisfying a particular need. Once a project achieves a critical mass it is difficult for a new project with similar aims to get started. If a developer picks the wrong project his work may not be used, and the project may become inactive.

A project becoming inactive does not necessarily mean that it has failed. The project may be finished, the program does what it was designed to do and can be used without further development.

If a developer is unable to recruit fellow developers to a project she is still free to develop the project herself, as she would have done before thinking about open–source solutions. However, the project will be backed up, and be under proper version control, and someone may become interested in it in the future.

There is a danger that an open–source project deteriorates because of poor code quality. This is probably due to a weak developer community and weak project administration. If the project administration neglects the project or mishandles the management the project may split into two competing projects, causing a great deal of disruption (Raymond, 2000b).

D.3 QuantLib

QuantLib is a derivatives pricing library implemented in C++ . As the project website (QuantLib, 2004) says,

> The QuantLib project is aimed at providing a comprehensive software framework for quantitative finance. QuantLib is a free/open–source library for modeling, trading, and risk management in real-life.

QuantLib is released under the modified BSD license (QuantLib, 2004). This enables the code to be modified and redistributed by banks or software houses without having to the release code for their proprietary models. A copyleft license (such as the GPL) would only permit the modification to take place within the company, if any code was redistributed all modifications to the source would also have to be released under the copyleft license. As in all open–source projects there is a conflict between the interests of individuals and corporations, and between contributers and free–riders. Individuals may take the code, but are not forced to contribute code in return. QuantLib is not noticeably suffering from the presence of free–riders.

There are not currently any competing projects to QuantLib with similar aims. It is the only pricing library which aims to be, and is, used in practice. The current project administrators are managing the project well, so the danger of a fork (when an existing project splits into two competing projects), or other conflict is low. The project administrators are also maintaining a high level of code quality.

One danger is the emergence of a competing project of higher quality, with a restrictive, non-free, license, which could divert effort from QuantLib. However, the emergence of such a project is unlikely at the moment, and as we have seen, a non-free license would discourage developers from contributing.

D.3.1 Why use QuantLib?

Various user groups have different reasons for using QuantLib. For example, academics would use QuantLib to disseminate research and allow results to be verified. Whereas companies may use QuantLib in practice to run their business. The advantages and disadvantages for such user groups using QuantLib are discussed in the following sections.

D.3.1.1 MSc or PhD students

QuantLib is an ideal tool for the technically competent student. Student coding is thrown away and never used again, albeit, most of it is not worth keeping. However, if the quality code is contributed to a viable open–source project it takes on a life of its own. The student avoids having to implement basic facilities such as date calculations. As discussed in section D.1.3 the student also benefits from the free debugging and code maintenance that is part of the open–source development process (Raymond, 2000a, 2001).

The student also gains an audience for their research and algorithms through the open–science process. As already mentioned, the author has noticed a marked increase in speaking invitations and requests for preprints since contributing to QuantLib.

Another attraction of the QuantLib project is exposure to and use of practical pricing models in C++ . These are essential skills for any student hoping to obtain a quantitative job in a financial institution. In the future QuantLib may well be used in projects for Mathematical Finance Masters degrees. Good C++ skills increase employability, and improved employability is a major factor in the choice of Masters students returning to study after some time working. Improved employability increases the reputation of the institution offering the professional course, therefore attracting the best students in a virtuous circle.

D.3.1.2 Academics

Academic faculty would also gain from the code debugging and software maintenance benefits. They would be able to build models on an established base and be able to focus on interesting research problems. They will be able to continue computational work after their PhD students have left the university.

Working code will be verified, tested, and adopted more quickly than academic papers alone. Academics researching in computational areas such as mathematical finance can greatly increase the impact of their work if both their papers and software implementations are available on the internet.

As noted in the previous section, QuantLib may also become a useful tool in Masters courses in computational finance, attracting students paying large fees to the university.

D.3.1.3 Financial institutions

Buy–side firms and sell–side firms have different requirements and interests, so are considered separately. Obviously financial institutions have the option of using commercial software solutions for their needs. Institutions, particularly in developing countries, may find that the fact that QuantLib is free influences their decision to adopt it as their pricing library.

Sell–side institutions Sell-side institutions will not want to release proprietary pricing techniques into the code base. They may also benefit from using QuantLib as a benchmark.

QuantLib is not a zero sum game. Just because sell–side institutions may not contribute greatly to the project, that does not detract from QuantLib's usefulness to others. Also, just being a user and submitting bug reports is very helpful.

QuantLib is useful for training potential recruits, and enabling those recruits to have some quality coding experience before starting work, thereby saving on training costs. A bank involved in QuantLib also gains access to the most technically competent students, ready filtered, without going through recruitment agents. Developers who already know the library also require less training, which is another saving.

Buy–side institutions There is more motivation for a buy–side institution to contribute code for pricing algorithms than a sell-side institution. QuantLib is already used in production at a number of institutions (Ametrano, 2004; Ballabio, 2004). Each individual asset management house does not have as many resources to devote to model development as an Investment Bank, so pooling resources makes sense. Asset managers are able to tell whether they are being quoted good prices by sell–side firms.

D.3.1.4 Software and consultancy firms

Profitable businesses can be run installing, extending, or customising QuantLib for particular financial institutions. Various service based business models for open–source companies are discussed in Raymond (2000c).

D.3.1.5 Financial regulation

A possible future use for QuantLib is in financial regulation. Capital requirements for portfolios of derivatives could be specified using QuantLib pricing algorithms. A reference portfolio could utilize the QuantLib pricing library.

D.3.2 Contributing to QuantLib

There are many different levels of contribution to open–source projects, though essentially there is one main distinction, between users and developers.

A successful open–source library needs users testing the library in different environments, and for many uses. A good bug report is a valid contribution to the project in itself.

The process for developing and contributing code is more involved. There are technical prerequisites that have to be met before contributing code. Learning the existing code in the library takes time and effort. A developer must be able to write C++ and configure the necessary libraries. To get involved in a project there must also be the desire to contribute quality code, as a developer is unlikely to contribute code that reflects badly on his programming ability. Therefore, though code contributions are vetted by the project manager they are welcomed and are usually worth including.

D.4 Conclusions

The impact, and scientific utility, of computational research in mathematical finance is much greater if it is implemented in a financial derivatives project such as QuantLib, than if it were not available on the internet. Research available in QuantLib is easier to verify, as anyone can download, read, and run the code. It is also simpler for future researchers to improve code, and test new algorithms against existing implementations in a systematic manner. The Open Science Project aids the transfer of knowledge from universities to commerce.

Companies benefit from a stable code base. They are able to take advantage of the latest developments from academic research and improve the speed of implementations as needed. Financial institutions are not restricted in what they can do with the library and can add or change features as required. Such an approach is not possible with closed–source packages.

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