Finite Differences Method for BSDEs in Finance

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Abstract:

This paper gives a review of recent progress in the design of numerical methods for solving to the BSDEs, especially, finite difference methods. We give a brief survey of the area focusing on the financial problems. The problems include solution methods and simulation for the BSDEs. We first briefly describe the financial problems and then outline the main techniques and main results of the BSDEs. In addition, we compared of the errors between these methods and the Euler method on the BSDEs.

Keywords: finite difference, BSDEs, FBSDEs, finance.

MSC (2010): 65C30, 60H35, 65C05.

1. Introduction

Primarily motivated by financial problems, backward stochastic differential equations (BSDEs) were developed at high speed during the 1990s. Comparing with black scholes models, the BSDEs are more robust to fit in the situation of probability model uncertainty, thus can be used to perform more approximated calculations in financial derivative pricing and risk analysis.

The BSDEs are important not only because they have gradually become powerful mathematical tools, to solve many option pricing problems and related problems in financial markets, but also they are applied to solve different stochastic optimizations problems in mathematical finance, such as financial problems of energy or weather. Since the BSDEs are terminal value problems for stochastic differential equations (SDEs), a natural time discretization of the BSDEs works backwards in time. However, the solution must be adapted to the information which increases forwards in time, which makes the construction of numerical solutions to the BSDEs a more challenging problem. Therefore, numerical solutions for the BSDEs have received strong attention in recent years.

The important class of the BSDEs considered in the literature introduced by Pardoux and Peng (1990) are the It$\hat{o}$'s type equations such as

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t, Y_T = \xi, \tag{1.1}$$

where $t \in [0, T]$, B is a Brownian motion and the data (ξ, f) are given. Here Z_t is a predictable process, f is called the generator or the driver, $Y_T = \xi$ is the terminal condition.

Suppose (Y, Z) is the solution of the standard BSDE (1.1), which can be interpreted as a stochastic integral equation of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, 0 \le t \le T, \tag{1.2}$$

where B is a Brownian motion and the data (ξ, f) are given. Then by the very definition the solution satisfies,

$$Y_t = Y_{t_0} - \int_t^{t_0} f(s, Y_s, Z_s)ds + \int_t^{t_0} Z_s dB_s, 0 \le t, t_0 \le T. \tag{1.3}$$

We now assume that (Y^π, Z^π) is some adapted approximation of the solution (Y, Z), which is piecewise constant with respect to a partition $\pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$.

When f depends only on the variable Y, which makes simpler the analysis. Let us now consider the discrete version of the BSDE (1.1):

$$Y^\pi_{t_i} = \xi^\pi + \frac{1}{n}\sum_{j=i}^{n} f(Y^\pi_{t_j}) - \sum_{j=i}^{n} Z^\pi_{t_j}(B^\pi_{t_{j+1}} - B^\pi_{t_j}). \tag{1.4}$$

This equation has a unique solution (Y^π_t, Z^π_t) since the martingale B^π has the predictable representation property.

1.1 BSDEs via Financial Problems

The BSDEs are widely used in numerous financial problems. Due to the structure of option pricing problems as linear BSDEs, its connection to optimal stopping problems allows to numerical solutions and related methods. Lower and upper biased numerical solutions, and confidence intervals numerical solutions can be obtained. For general nonlinear BSDEs, a similiar way to construct lower and upper approximations is not available. Nonetheless, the intricate interplay between the time discretization and the design of the estimation for the conditional expectation persists for nonlinear BSDEs in a similar way as for option problems.

In a complete market, the theory of contingent claim valuation can be expressed in terms of the BSDEs, and the price process Y of ξ is a solution of the BSDE. For the usual valuation of a contingent claim with payoff $\xi = \Phi(X)$, Y is the value of the replicating portfolio and Z is related to the hedging strategy. In that case, the driver f is linear with regard to Y and Z.

In incomplete markets, the Föllmer-Schweizer strategy is given by the solution of a BSDE. When trading constraints on some assets are imposed, the super-replication price is obtained as the limit of nonlinear BSDEs. Connections with recursive utilities of Duffie and Epstein are also available. Peng has introduced the notion of g-expectation (here g is the driver) as a nonlinear pricing rule and has shown the deep connection between the BSDEs and dynamic risk measures, proving that any dynamic risk measure $\{\varepsilon_t\}_{0\le t\le T}$ is necessarily associated to a BSDE about $\{Y_t\}_{0\le t\le T}$.

The BSDEs appear in numerous financial problems, like the pricing and the hedging of some options, including call option, put option, Europe option, American option, lookback options, digital options, compound Option; indifference pricing (Karoui and Rouge, 2000); the recursive utility (Kaouri, Peng and Quenez, 1997), the robust optimization (Peng and Wu, 1999), simulated by the financial industry, is concerned with the numerical methods. Risk-sensitive control, portfolio selection and mean-variance hedging (Pham, 2005) are the ones solved with stochastic control by the BSDEs. To simulate the solution of the given BSDEs by discretization, and estimate the error in order to tune finely the convergence parameters. The list of potential applications is long.

Here, we give some examples of and terminal conditions $\xi = \Phi(X)$. A large class of exotic payoffs satisfies a functional Lipschitz condition, for example, vanilla payoff is $\Phi(X) = \phi(X_T)$; asian payoff is $\Phi(X) = \phi(X_T, \int_0^T X_t dt)$; lookback payoff is $\Phi(X) = \phi(X_T, \min_{t\in[0,T]} X_t, \max_{t\in[0,T]} X_t)$. Then we can say that the BSDEs are now inevitable tools in mathematical finance.

1.2 Branches of Finite Difference Methods for the BSDEs

In order to solve financial problems by numerical approximation in time, one can either resort to finite difference (FD) methods, or use more general finite element (FE) methods, or even finite volume (FV) methods. We also note that, there is in fact, no absolute border between these methods. In a sense, Monte Carlo (MC) methods are special cases of binomial methods (namely, tree methods). A complete schematics in scientific computation is below:

FV methods $\supset$ FE methods $\supset$ FD methods $\supset$ Tree methods $\supset$ MC methods.

The steps of the FD methods are that: discretizing the localized time-space domain; choosing suitable discrete model; solving the discrete model system.

The most elementary ones are the FD methods for the BSDEs. Up to now basically tree branches of the FD methods have been considered. The main branch of the methods is the FD solution of a parabolic PDE which is related to the BSDE. Based the Ma-Protter-Yong four step method (1994), the FD methods for the BSDEs have been developed by Douglas, *et al.* (1996) and more recently by Milstein and Tretyakov (2006). Here the solution problem roughly reduces to the approximation of a quasi-linear parabolic Cauchy problem. This approach relies on some smoothness assumptions on the coefficients of the BSDEs, and spatial dimensions of the PDEs.

A second branch of methods is the FD method, namely, works backwards through time and tries to tackle the stochastic problem directly. Bally (1997) and Chevance (1997) used a random time discretization respectively for it, under strong regularity assumptions. The work of Ma, *et al.* (2002) is in the same spirit, replacing, however, the Brownian motion by a binary random walk in the approximative equation. See also Briand, *et al.* (2001) for the binary random walk approach; Zhang (2004), Bouchard and Touzi (2004), and Gobet, *et al*. (2005, 2006, 2007, 2010) for different methods.

A third branch of methods represents calibration methods for BSDEs, to improve efficiency of the above methods, such that some parallel methods, some variance reduction methods. For example, Peng, *et al.* (2009) developed a parallel method for reflected BSDEs with application to option pricing, which is a method with block allocation; Tran (2011) reconstructed the four step method with some new conditions in FBSDEs, which is associated with Schwarz waveform relaxation method, to parallelize the related equations. Bender and Moseler (2008) introduced importance sampling to MC methods for pricing problems, represented by the BSDEs.

In addition, the following FD methods are of particular interest: the quantization method (Bally, *et al.*, 2002, 2003), the Malliavin calculus approach (Bouchard, *et al.*, 2004, 2008), the linear regression method or the Longsta-Schwartz method (Gobet, Lemor and Waxin, 2006), and the Picard iteration method (Bender and Denk, 2007). These methods work well in reasonably high dimensions. There are also lots of publications on numerical methods for non-Markovian BSDEs (Ma, *et al.* 2002; Zhang and Zheng, 2008).

1.3 Recent Development of Some New BSDEs

In this century, the theories about the BSDEs are more and more mature. However, there are still plenty of research areas in the BSDEs that appeal to many researchers. For example, the solution problems are very popular among researchers in finance. In 2000, a class of the BSDEs was introduced by Rouge and Karoui (2000) who solved the utility maximization problems in incomplete market; Kobylanski (2000) solved a type of the BSDEs with drivers, which are quadratic growth of Z. On top of that, there are a kind of new developed BSDEs such that second order BSDEs, time-delayed BSDEs, finite-horizon BSDEs, quadratic BSDEs, anticipated BSDEs (Peng and Yang, 2009); BSDEs with Lipschitz condition, BSDEs with Markov chains.

The rest of the paper is organized as follows. We conclude this section with a brief literature review. In Section 2, we describe the solution methods of some related BSDEs, give some analysis for them. In Section 3, we state the solution methods of the FBSDEs. Parallel methods of the BSDEs are discussed in Section 4 while Section 5 gives discussion and conclusion.

2. FD Solutions of Some Related BSDEs in Finance

A relevant problem in the BSDEs is to propose implementable FD methods to approximate the solution of them. Several efforts have been made as well. For example, Chevance (1997) proposed a general method for the BSDEs; Zhang (2004) proposed a method for a class of the BSDEs, with related terminal values; Peng and Xu (2008) studied some different methods for the BSDEs basing on random walk framework, introduced implicit and explicit methods for both the BSDEs and the reflected BSDEs. To get a FD solution for solving the BSDEs, one main idea is based upon a time-discretization of some related BSDEs in finance.

2.1 FD Solutions of the BSDEs with Reflections in Finance

In the subsection, we are interested in the FD solutions of the BSDEs with Reflections (BSDERs). One of the main motivations for studying the BSDERs has been to solve the hedging problem for American options. The BSDERs have the following forms,

$$\begin{aligned} &Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, 0 \le t \le T, \\ &Y_t \ge L_t, 0 \le t \le T, \text{and} \int_0^T (Y_t - L_t)dK_t = 0. \end{aligned} \tag{2.1}$$

where $L_t = L_0 + \int_0^t l_s ds + \int_0^t \sigma_s dB_s, 0 \le t \le T$, $\xi = \Phi((B_s)_{\{0\le s\le T\}})$, $\{K_t\}$ is continuous and increasing, $K_0 = 0$. l_t is a smooth function.

Now we discuss some applications of the BSDERs, related to optimal stopping problem (American option in finance). For an American call option, the wealth process Y_t satisfies the following BSDERs

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dBs, 0 \le t \le 1, \\ Y_t &= (X_T - K)^+ - \int_t^T [rY_s + (\mu - r)Z_s]ds - \int_t^T \sigma Z_s dBs, \end{aligned} \tag{2.2}$$

and $Y_t \ge (X_t - K)^+$, for $0 \le t \le \inf\{t, Y_t - (X_t - K)^+\}$. Here σ is volatility rate, r is uniformly bounded (e.g. interest rate), K is a constant.

The BSDERs apply to American put option corresponds to the case:

$$Y_T = \xi = (K - X_T)^+.$$

For option pricing with differential interest rates, $(\mu - r)$ is related to Y_t and Z_t in Eq.(2.2).

We assume that $X = \{X_t, 0 \le t \le T\}$ is the risk asset, r is constant. Under some assumptions, the equation is given by a linear reflected BSDE, coupled with the forward equation for X. We then can obtain Y_0, the value of the American option.

There are some FD methods for solving the BSDERs, including max method, penalization method, regularization method and the like. We will approximate the solution of the BSDERs. On small interval $[i\delta, (i+1)\delta]$, Eq.(2.1) can be approximated by the discrete equation below,

$$\begin{aligned} &Y^\pi_{t_i} = Y^\pi_{t_{j+1}} + f(t_i, Y^\pi_{t_i}, Z^\pi_{t_i})\delta + K_{t_{i+1}} - K_{t_i} - Z^\pi_{t_i}\varepsilon^\pi_{j+1}\sqrt{\delta} \\ &Y^\pi_{t_i} \ge L^\pi_{t_i}, (Y^\pi_{t_i} - L^\pi_{t_i})(K_{t_{i+1}} - K_{t_i}) = 0, \end{aligned} \tag{2.3}$$

where $L^\pi_{t_i} = L_0 + \delta\Sigma_{j=0}^{i-1} l_{t_j} + \Sigma_{j=0}^{i-1} l_{t_j}\sigma_{t_j}\varepsilon^\pi_{j+1}\sqrt{\delta}$. Eq.(2.3) is a discrete BSDER, with terminal condition $\xi^\pi = \Phi(\Sigma_{j=1}^{i}\varepsilon^\pi_{j+1}\sqrt{\delta})$. In the case $f = 0$, $\xi = \phi(X_1)$ and $L = \phi$, Y is the Snell envelope of $\phi(X)$, which, corresponds to the super-hedging price of the American option with payoff ϕ.

An important method is penalization equations method for the equations. For $p \in N$, the penalization equation is

$$Y^p_t = \xi + \int_t^T f(s, Y^p_s, Z^p_s)ds + p\int_t^T (Y^p_s - L_s)^- ds - \int_t^T Z^p_s dB_s. \tag{2.4}$$

The solution of the BSDERs can be approximated by the solution of Eq.(2.4). We have the following discrete penalized BSDEs on the small interval $[i\delta, (i+1)\delta]$

$$Y^{p,\pi}_{t_i} = Y^{p,\pi}_{t_{j+1}} + f(t_i, Y^{p,\pi}_{t_i}, Z^{p,\pi}_{t_i})\delta + p(Y^{p,\pi}_{t_i} - L^\pi_{t_i})^-\delta - Z^{p,\pi}_{t_i}\varepsilon^\pi_{j+1}\sqrt{\delta}. \tag{2.5}$$

If Γ is a close subset in $\mathbb{R}$ and $d_\Gamma(Y,Z) = \inf_{(Y',Z')\in\Gamma}\{|Y-Y'| + |Z-Z'|\}$, $(Y_t, Z_t) \in \Gamma_t$ in Eq.(2.4). After the same discretization for the BSDEs, for each positive number p we have the following penalization discrete equation on $[i\delta, (i+1)\delta]$

$$Y^{p,\pi}_{t_i} = Y^{p,\pi}_{t_{j+1}} + f(t_i, Y^{p,\pi}_{t_i}, Z^{p,\pi}_{t_i})\delta + p d_T \delta - Z^{p,\pi}_{t_i} \varepsilon^\pi_{j+1}\sqrt{\delta}, \tag{2.6}$$

with the discrete terminal condition $Y^\pi_T = \xi^\pi$.

Now we consider $d_T(Y,Z) = (Y - \phi(Z))^- \geq 0$. We have the following discrete penalization equation for some p large enough, on the small interval $[i\delta, (i+1)\delta], 0 \leq i \leq n-i$

$$Y^{p,\pi}_{t_i} = Y^{p,\pi}_{t_{j+1}} + f(t_i, Y^{p,\pi}_{t_i}, Z^{p,\pi}_{t_i})\delta + p(\phi(Z^{p,\pi}_{t_i}) - Y^{p,\pi}_{t_i})^+\delta - Z^{p,\pi}_{t_i}\varepsilon^\pi_{j+1}\sqrt{\delta}. \tag{2.7}$$

Let us turn to the approximation of the solution (Y,Z,K) of a BSDER of the form

$$\begin{aligned} &Y_t = \phi(X_1) + \int_t^1 f(s, Y_s, Z_s)ds - \int_t^1 Z_s dB_s + K_1 - K_t, 0 \leq t \leq 1, \\ &Y_t \geq L_t, 0 \leq t \leq 1, \text{and} \int_0^1 (Y_t - L_t)dK_t = 0. \end{aligned} \tag{2.8}$$

where $L = \{L_t\}$ is the reflecting barrier, $\{K_t\}$ is continuous and increasing, $K_0 = 0$. That is for Eq.(2.1) while $T = 1$.

For Eq.(2.8), this approximation leads to a Backward Euler scheme $(Y^\pi_{t_i}, \tilde{Z}^\pi_{t_i})$ of the following form

$$\begin{aligned} \tilde{Z}^\pi_{t_i} &= (t_{i+1} - t_i)^{-1} E[Y^\pi_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i}] \\ \tilde{Y}^\pi_{t_i} &= E(Y^\pi_{t_i+1}|\mathcal{F}_{t_i}) + f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \tilde{Z}^\pi_{t_i})(t_{i+1} - t_i) \\ Y^\pi_{t_i} &= \mathcal{R}(t_i, X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}), i \leq n-1, \end{aligned} \tag{2.9}$$

with the terminal condition $Y^\pi_T = g(X^\pi_T)$. Here X^π is the Euler scheme associated to X,

$$\mathcal{R}(t, X, Y) = Y + [L(X) - Y]^+ \mathbf{1}_{t\in\Re\setminus\{0,T\}}$$

for a partition $\Re = \{0 = r_0 < r_1 < \ldots < r_k = 1\} \supset \pi$. $\mathcal{F}_t$ is the completed filtration of the Brownian motion B_t.

Given an approximation $\hat{E}^\pi_i$ of $E[\cdot|\mathcal{F}_{t_i}]$, we then have the backward method

$$\begin{aligned} \tilde{Z}^\pi_{t_i} &= (t_{i+1} - t_i)^{-1}\hat{E}^\pi_i[Y^\pi_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})], \\ \tilde{Y}^\pi_{t_i} &= \hat{E}^\pi_i(Y^\pi_{t_i+1}|\mathcal{F}_{t_i}) + f(X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \tilde{Z}^\pi_{t_i})(t_{i+1} - t_i), \end{aligned}$$

with $Y^\pi_T = g(X^\pi_T)$.

FD procedures for such methods have been first studied in Bally and Pagès (2002). A more general analysis has then been performed by Ma and Zhang (2005), who obtained a bound for the convergence rate of order $|\pi|^{-\frac{1}{4}}$ when L depends on Z whenever it is C^2_b and σ is uniformly elliptic. They gave a representation of Z:

$$Z_t = E[\phi(X_1)N^t_1 + \int_t^1 f(X_s, Y_s, Z_s)N^t_s ds + \int_t^1 N^t_s dK_s|\mathcal{F}_t]\sigma(X_t) \tag{2.10}$$

where

$$N^t_1 = \frac{1}{1-t}\int_t^1 \sigma(X_s)^{-1}\nabla X_s dB_s(\nabla X_t)^{-1}.$$

Here ∇X denotes the first variation process of X.

$$Y^\pi_t = \phi(X_1) + \int_t^1 f(X, Y^\pi, Z^\pi)ds - \int_t^1 Z^\pi_s dB_s + K^\pi_1 - K^\pi_t \tag{2.11}$$

where $K^\pi_t = \Sigma^{k-1}_{j=1}[f(X_{s_j}) - Y^\pi_{s_j}]^+\mathbf{1}_{s_j \le t}$.

$$Z^\pi_t = E[\phi(X_1)N^t_1 + \int_t^1 f(X, Y^\pi, Z^\pi)N^t_s ds + \sum_{k=0}^{k} N^t_{s_k}\Delta K^\pi_{s_k}\mathbf{1}_{t<s_k}|\mathcal{F}_t]\sigma(X_t)$$

for $t \in [s_j, s_{j+1})$, where $\Delta K^\pi_{s_k} = [f(X_{s_k}) - Y^\pi_{s_k}]^+$.

Another regularity analysis was carried out by Bouchard and Chassagneux (2008). The main advantage is providing a representation of Z^π in terms of the next reflection time:

$$\begin{aligned} Z^\pi_t = {} & E[D\phi(X_1)\eta^\pi_1\mathbf{1}_{\tau_j=1} + DL(X_{\tau_j})\eta^\pi_{\tau_j}\mathbf{1}_{\tau_j=<1} \\ & + \int_t^{\tau_j} D_s f(X, Y^\pi, Z^\pi)\eta^\pi_r d_r|\mathcal{F}_t](\eta^\pi_t)^{-1}\sigma(X_t), \end{aligned} \tag{2.12}$$

for $t \in [s_j, s_{j+1})$, where $\tau_j = \inf\{s \in \Re | s \ge s_{j+1}, L(X_s) > Y^\pi_s\} \wedge 1$ and η^π is a function of with (X, Y^π, Z^π) in place of (X, Y, Z). Here $(D\phi, DL, Df)$ are Malliavin derivative.

These above methods are based on random walk, with one continuous lower barrier, and implicit methods. These methods are the discrete-time approximations. The equations are related to stochastic stopping games, game options. Let $K_t = K^+_t - K^-_t$ in Eq.(2.1), the equations are called the BSDEs with doubly reflections by Chassagneux (2009), correspond to stochastic stopping games.

2.2 FD Solutions of the BSDEs with jumps

The BSDEs with jumps (BSDEJs), introduced by Li and Tang (1994), Situ (1997), Situ (2003) and Shreve (2004), Royer (2006), Zhu (2010), are as follows:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_{(t,T]\times\mathbb{R}} U_s(x)\tilde{N}(ds, dx), \tag{2.13}$$

where $0 \le t \le T$, $\tilde{N}(ds, dx) = N(ds, dx) - \nu(ds, dx)$ is a compensated Poisson random measure defined in $[0, T] \times \mathbb{R}$. U is the jump component. The BSDEJs are used for the valuation of financial derivatives with default risk.

Suppose the partition π of $[0, T]$ with mesh size $|\pi| = \max_i |t_{i+1} - t_i|$ is given and a corresponding discretization X^π of X, $Y_T^\pi = \xi$. The Backward Euler method (Y^π, Z^π, U^π) for (Y, Z, U) is a following implicit method,

$$\begin{aligned} Y_{t_i}^\pi &= Y_{t_{i+1}}^\pi + f(t_i, Y_{t_{i+1}}^\pi, Z_{t_i}^\pi, U_{t_i}^\pi)\tilde{N}((t_i, t_{i+1}] \times \mathbb{R}), \\ Z_{t_i}^\pi &= \frac{1}{(t_{i+1} - t_i)} Y_{t_{i+1}}^\pi (B_{t_{i+1}} - B_{t_i}), \\ U_{t_i}^\pi &= \frac{1}{(t_{i+1} - t_i)} Y_{t_{i+1}}^\pi \tilde{N}((t_i, t_{i+1}] \times \mathbb{R}), 0 \le i \le n-1, \end{aligned} \tag{2.14}$$

where,.

Similarly, solving the BSDEJs, we can also use max methos, penalization method, regularization method and so on.

Thr reflected BSDE with jumps (see, Essaky, 2008) and constrained BSDEs with jumps (see, Elie and Kharroubi, 2010, 2011; Kharroubi, Ma, Pham and Zhang, 2011) can be considered as extensions of simply BSDEs with jumps.

2.3 FD Solutions of Second order BSDEs

We shall consider the following second order BSDEs (2BSDEs for short):

$$\begin{aligned} dX_t &= \sigma(X_t)dB_t \\ dY_t &= -f(t, X_t, Y_t, Z_t, \Gamma_t)dt + Z\sigma(X_t)dB_t \\ dZ_t &= \alpha_t dt + \Gamma_t \sigma(X_t)dB_t, 0 \le t \le T, \end{aligned} \tag{2.15}$$

with $Y_T = \phi(X_T)$. Here $\alpha, \Gamma, \sigma, \phi$ and f are deterministic functions.

Suppose the partition π of $[0, T]$ with mesh size $|\pi| = \max_i |t_{i+1} - t_i|$ is given and a corresponding discretization X^π of X, $Y^\pi_T = g(X^\pi_T)$.

$$
\begin{aligned}
Y^\pi_{t_i} &= E^\pi_i[Y^\pi_{t_{i+1}}] + f(t_i, X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}, \Gamma^\pi_{t_i})(t_{i+1} - t_i),\\
Z^\pi_{t_i} &= \frac{\sigma(X^\pi_{t_i})^{-1}}{(t_{i+1} - t_i)} E^\pi_i[Y^\pi_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})],\\
\Gamma^\pi_{t_i} &= \frac{\sigma(X^\pi_{t_i})^{-1}}{(t_{i+1} - t_i)} E^\pi_i[Z^\pi_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})], 0 \le i \le n-1.
\end{aligned}
\tag{2.16}
$$

Cheridito, Soner, Touzi, and Victoir (2007) investigated the 2BSDEs in Markovian framework. A solution of the 2BSDEs is a process (Y, Z, α, Γ). They supposed a solution exists with $Y_t = V(t, X_t)$, and considered the fully nonlinear PDE (with $\mathcal{L}V = \frac{1}{2}Tr[\sigma\sigma^T D^2 V]$)

$$
\frac{\partial V}{\partial t} + \mathcal{L}v(t,x) + f(t, x, v(t,x), Dv(t,x), D^2 v(t,x)) = 0
$$

with $v(T, x) = \phi(x)$.

If the above equation has a smooth solution, then

$$
\hat{Y}_t = v(t, X_t), \hat{Z}_t = Dv(t, X_t), \hat{\alpha}_t = \mathcal{L}Dv(t, X_t), \hat{\Gamma}_t = V_{xx}(t, X_t)
$$

is a solution of the 2BSDEs.

3. FD Solutions of the FBSDEs in Finance

In this section we survey difference solutions of some FBSDEs for new development in finance. A FBSDEs is a BSDE where the randomness in the driver comes from some underlying forward process. The FBSDEs have numerous applications in finance problems. The main property of the FBSDEs is that the FD solution (X, Y, Z) of the BSDEs can be written as functions of time and state process. Several approximation methods to FBSDEs were proposed in the literature. The first one was Douglas-Ma-Protter (1996), based on the four step method. There are many methods proposed such as Markov chain approximation (Ma, *et al.*, 2002), regression methods (Gobet, *et al.*, 2005), quantization methods (Bally and Pagès, 2002; 2003), the MC Malliavin method (Bouchard and Touzi, 2004). Makarov (2003) and Milstein and Tretyakov (2006) also proposed some difference methods for FBSDEs. Recently Delarue and Menozzi (2006) proposed a probabilistic method. Notice that all these methods essentially need to discretize the space on regular grids, and are not practical in high dimensions. Our main interest will be those that related to difference solutions of coupled FBSDEs and decoupled FBSDEs in finance.

Consider a coupled FBSDE of the form

$$\begin{aligned} dX &= b(t,X,Y,Z)dt + \sigma(t,X,Y,Z)dB_t, X(t_0) = x, \\ dY &= \phi(t,X,Y)dt - f(t,X,Y,Z)dt + ZdB_t, Y_T = \phi(X_T). \end{aligned} \tag{3.1}$$

where b, σ, ϕ and f are deterministic functions, and B_t is a Brownian motion.

We consider the following representation formula of the coupled FBSDEs:

$$\begin{aligned} X_t &= x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dBs \\ Y_t &= \phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \end{aligned} \tag{3.2}$$

with $Y_T = \phi(X_T)$, where b, σ, f and g are deterministic functions, and B_t is a Brownian motion on $[0, T]$. The solution consists of a triplet (X, Y, Z) of adapted processes, the forward part, the backward part and the control part.

3.1 Markovian Iteration of Coupled FBSDEs (MFBSDEs)

A time discretization of Eq.(3.2) is

$$\begin{aligned} X^\pi_{t_{i+1}} &= X^\pi_{t_i} + b(t_i, X^\pi_{t_i}, Y^\pi_{t_i})h + \sigma(t_i, X^\pi_{t_i}, Y^\pi_{t_i})(B_{t_{i+1}} - B_{t_i}), \\ Z^\pi_{t_i} &= \frac{1}{h} E_{t_i}[Y^\pi_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})], \\ Y^\pi_{t_i} &= E_{t_i}[Y^\pi_{t_{i+1}} + f(t_i, X^\pi_{t_i}, Y^\pi_{t_{i+1}}, Z^\pi_{t_i})h], \end{aligned} \tag{3.3}$$

where $h = T/n$ and $t_i = ih, i = 1, \ldots, n$. Here E_{t_i} denotes the conditional expectation $E\{\cdot|\mathcal{F}_{t_i}\}$, $X^\pi_0 = x, Y^\pi_T = \phi(X^\pi_T)$.

Bender and Zhang (2008) proposed to combine the above time discretization with an iterative method, and introduced Markovian iteration for coupled FBSDEs, which reads, $u_i^{\pi,0}(x) = 0$, $X_0^{m,\pi} = x, Y_T^{m,\pi} = g(X_T^{m,\pi})$ and

$$\begin{aligned} X^{m,\pi}_{t_{i+1}} &= X^{m,\pi}_{t_i} + b(t_i, X^{m,\pi}_{t_i}, u_i^{m-1,\pi}(X^{m,\pi}_{t_i}))h \\ &\quad + \sigma(t_i, X^{m,\pi}_{t_i}, u_i^{m-1,\pi}(X^{m,\pi}_{t_i}))(B_{t_{i+1}} - B_{t_i}), \\ Z^{m,\pi}_{t_i} &= \frac{1}{h} E_{t_i}[Y^{m,\pi}_{t_{i+1}}(B_{t_{i+1}} - B_{t_i})], \\ Y^{m,\pi}_{t_i} &= E_{t_i}[Y^{m,\pi}_{t_{i+1}} + f(t_i, X^{m,\pi}_{t_i}, Y^{m,\pi}_{t_{i+1}}, Z^{m,\pi}_{t_i})h], \\ u_i^{m,\pi}(X^{m,\pi}_{t_i}) &= Y^{m,\pi}_{t_i}. \end{aligned} \tag{3.4}$$

The main advantage is that $Y^{m,\pi}_{t_i}$ is a function of time and $X^{m,\pi}_{t_i}$, but does not depend on $\{X^{\mu,\pi}_{t_i}, \mu = 1, \ldots, m-1\}$.

The efficiency of MC method may be drastically increased by the choice of an appropriate variance reduction technique. Bender and Moseler (2008) introduced importance sampling for pricing problems on the FBSDEs. The main idea of importance sampling is to change the drift of the underlyings by a change of measure in order to force more simulated paths to take value in 'interesting' regions.

3.2 Four-step method for Solving Coupled FBSDEs (FFBSDEs)

Due to the four-step scheme, the solution of (3.2) is connected with the semilinear PDE:

$$\begin{aligned}&\frac{\partial u}{\partial t}+\sum_{i=1}^{d}a^i(t,x,u)\frac{\partial u}{\partial x^i}+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i\partial x^j}\\&=\phi(t,x,u)-\sum_{k=1}^{n}f^k(t,x,u)\sum_{i=1}^{d}\sigma^{ik}(t,x,u)\frac{\partial u}{\partial x^i},0\le t\le T,\end{aligned}\tag{3.5}$$

where

$$a^{ij}=\sum_{i=1}^{d}\sigma^{ik}\sigma^{jk},u(T,x)=\phi(x).$$

Assume that the solution $u(t,x)$ of Eq.(3.3) is known. Consider the following equation:

$$\begin{aligned}X_t&=x+\int_0^t b(s,X_s,u(s,X_s))ds+\int_0^t\sigma(s,X_s,u(s,X_s))dBs\\Y_t&=u(t,X_t),\\Z_t^j&=\sum_{i=1}^{d}\sigma^{ij}(t,X_t,Y_t)\frac{\partial u}{\partial x^i}(t,X_t),\ j=1,\dots,n.\end{aligned}\tag{3.6}$$

Defined by Eq. (3.4), (X_t,Y_t,Z_t) would give an adapted solution of the FBSDE on Eq. (3.2).

In addition, Ma, *et al.* (2008) obtained the unique adapted solution of (3.2) by the following steps:

Step 1. Define a function Z by

$$Z(t,X_t,Y_t,p)=p\sigma(t,X_t,Y_t).$$

Step 2. Using the function $Z(t,X_t,Y_t,p)$ instead of $\sigma(t,X_t,Y_t)$, solve Eq.(3.3).

Step 3. Using the functions $u(t,X_t)$ and $Z(t,X_t,Y_t,p)$, solve the forward SDE:

$$X_t=x+\int_0^t b(s,X_s,u(s,X_s),Z(s,X_s,u(s,X_s),\frac{\partial u}{\partial x}))ds+\int_0^t\sigma(s,X_s,u(s,X_s))dBs.$$

Step 4. Set

$$Y_t = u(t, X_t), Z_t^j = \sum_{i=1}^{d} \sigma^{ij}(t, X_t, Y_t) \frac{\partial u}{\partial x^i}(t, X_t), j = 1, \ldots, n.$$

Delarue and Menozzi (2006) exploited the relation to quasilinear parabolic PDEs via Ma-Protter-Yong method. Under appropriate conditions, (X, Y, Z) are connected by

$$Y_t = u(t, X_t), Z_t = u_x(t, X_t)\sigma(t, X_t, u(t, X_t)), \tag{3.7}$$

where u is a classical solution of the quasilinear PDE

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &+ b(t,x,u(t,x))\frac{\partial}{\partial x}u(t,x) \\ &+ \frac{1}{2}trace(\sigma\sigma^T(t,x,u(t,x))\frac{\partial^2}{\partial x^2}u(t,x) + f(t,x,u(t,x)) = 0,\end{aligned} \tag{3.8}$$

with $u(T, x) = \phi(x)$.

3.3 FD Solutions of Decoupled FBSDEs

We consider the difference solution of the following decoupled FBSDE:

$$\begin{aligned} X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_t^T \sigma(s, X_s)dB_s, 0 \leq t \leq T, \\ Y_t &= \phi(X_t) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \end{aligned} \tag{3.9}$$

In this representation, X_t is the forward component and Y_t the backward one. Here, B is a Brownian motion. The driver $f(\cdot, \cdot, \cdot, \cdot)$ and the terminal condition $\phi(\cdot)$ are, respectively, a deterministic function and a deterministic functional of the process X_t.

Given a partition $\pi = \{0 = t_0 \leq \ldots \leq t_n = T\}$ of the interval $[0, T]$, we consider the first naive Euler discretization of the backward SDE:

$$Y^{\pi}_{t_{i+1}} - Y^{\pi}_{t_i} = f(t_i, X^{\pi}_{t_i}, Y^{\pi}_{t_i}, Z^{\pi}_{t_i})(t_{i+1} - t_i) - Z^{\pi}_{t_i}(B_{t_{i+1}} - B_{t_i}),$$

together with the final data $Y^{\pi}_T = \phi(X^{\pi}_T)$.

Based on the approximation, discrete-time methods, have been analyzed by Chevance (1997), Ma, *et al.* (2002), Bally and Pagès (2003), and so on. Crisan, *et al.* (2010) propose a generic framework for the analysis of MC simulation for the FBSDEs.

For decoupled FBSDEs, Milstein and Tretyakov (2006) gave the layer methods to the solution of the semilinear parabolic PDEs (LDBSDEs). Gobet and Labart (2010) link the solution (Y, Z) of decoupled FBSDEs to u. Under reasonable conditions, (Y, Z) are connected by

$$(Y_t, Z_t) = (u(t, X_t), \partial_x u(t, X_t)\sigma(t, X_t)), t \in [0, T],$$

where u is the solution of the following semilinear PDE:

$$\partial_t u(t, x) + \mathcal{L}u(t, x) + f(t, x, u(t, x)), (\partial_x u\sigma)(t, x)) = 0, \tag{3.10}$$

where $u(T, x) = \phi(x)$, $\mathcal{L}u(t, x)$ is the following second order elliptic operator defined by

$$\mathcal{L}u(t, x) = \frac{1}{2}\sum_{i,j}[\sigma\sigma^T]_{ij}(t, x)\partial^2_{x_i x_j}u(t, x) + \sum_i b_i(t, x)\partial_{x_i}u(t, x).$$

Now we give a compare of the three methods on the BSDEs, as follows. For the following equation, $f(t, Y_t, Z_t) = \sin t + 3Y_t + 5Z_t$. Assume that in (1.1), Table 1 presented the error between the three methods and the Euler method.

Table 1 : The errors between the numerical methods and the Euler method on the FBSDEs

n	1	2	3	4	5	6
MFBSDEs	5.543E+4	3.451E+3	5.786E+2	71.561	3.491E-2	6.562E-4
FFBSDEs	4.641E+4	2.657E+4	4. 891E+2	6.497	4.786E-2	5.578E-3
LDBSDEs	1.781E+5	2.476E+4	3. 651E+3	3.581	2.354E-1	3.671E-3

4. Parallel Methods for the BSDEs' FD Solutions in Finance

From numerical analysis viewpoint, challenging problems are the search for fast and efficient simulation schemes for BSEDs arising in financial problems. Then the study of application and theory for it is very important. In this section, we review some parallel methods to the BSDEs in theory and in application. The idea is to impose the parallel methods on solving some BSDEs , in order to optimize cost and time.

4.1 Multistep method for the BSDEs in Space-time

Zhao, Zhang, and Ju (2010) proposed a multistep discretization method in space-time for solving Eq.(1.1), namely, they used the Lagrange interpolating polynomials in time and the Gauss-Hermite quadrature in space to approximate conditional mathematical expectations of the equation. Some examples were given in the paper, including a evaluation problem in stock markets.

From the structure of the method, its FD procedures among space grids are independent. Thus, parallel computing techniques can be adopted to solve large scale problems. Zhang (2010) gave some examples of parallel computing in his thesis.

4.2 Schwarz Waveform Relaxation for FBSDEs in Space-time

Waveform relaxation (WR) is a technique that allows for the use of parallel computers when solving large-scale systems of the equations. There are WR methods based on Jacobi, Gauss-Seidel or Schwarz iterations. There is a large body of research on convergence theory, acceleration methods, parallel implementation and the application of WR methods, for solving differential equations (see Burrage, 1995). In the same time, to decrease the computation time, many parallelization works are presented for solving various numerical models in finance problems, e.g. Sak, *et al.* (2007). The interest in Schwarz waveform relaxation methods have grown rapidly, as these methods lead to inherently parallel methods. Over the last decade, Schwarz methods have been extensively developed at the continuous level. These methods converge significantly faster than the classical Schwarz methods (Gander, 2006). More recently, Schwarz methods have also been developed for systems of the BSDEs, see for example Tran (2010) for the FBSDEs.

Tran (2010) reconstructed the four step method with some new conditions to the FBSDEs in Eq.(3.3), and then associate it with Schwarz waveform relaxation method to parallelize the system of Eq.(3.5). Generically, Schwarz partitions a solution space into the subspaces, possibly overlapping, whose union is the original space, and forms an approximate inverse of the operator in each subspace. The method has two main advantages: it is global in time and thus permits non conforming space-time discretization in different subdomains, and few iterations are needed to compute an accurate solution, due to some suitable conditions.

4.3 Block Allocation for the BSDERs in Finance

Though many methods have been made to obtain FD solutions of the BSDERs, little work has been done on parallel method. Time constraints should be respected in financial problems, therefore effective parallel implementations are in great demand. Peng, *et al.* (2009) developed a parallel method for the BSDERs, with application to option pricing. The method is a parallel method with block allocation, and about Peng-Xu method for the BSDEs (2008).

Peng and Xu (2008) considered the FD methods of the standard BSDEs (1.1), the time period $[0, T]$ is divided into n time steps with interval $\delta = T/n$. The Brownian motion can be approximated with:

$$B_t = \sqrt{\delta} \sum_i \varepsilon_i^\pi,$$

where $\{\varepsilon_i^\pi\}$ is a Bernoulli sequence. They obtained a FD solution by the following discrete BSDEs on the small interval $[i\delta, (i+1)\delta]$,

$$Y_{t_i}^\pi = Y_{t_{i+1}}^\pi + f(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)\delta - Z_{t_i}^\pi \varepsilon_{i+1}^\pi \sqrt{\delta}. \tag{4.1}$$

We can see the equation above is similar to (2.5).

Let $Y_+ = Y_{t_{i+1}}^\pi|_{\varepsilon_{i+1}^\pi = 1}$ and $Y_- = Y_{t_{i+1}}^\pi|_{\varepsilon_{i+1}^\pi = -1}$. Under some assumptions, the equation above is then equivalent to the following equation:

$$\begin{aligned} Y_{t_i}^\pi &= Y_+ + f(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)\delta - Z_{t_i}^\pi \sqrt{\delta}, \\ Y_{t_i}^\pi &= Y_- + f(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)\delta - Z_{t_i}^\pi \sqrt{\delta}. \end{aligned} \tag{4.2}$$

This is equivalent to

$$\begin{aligned} Y_{t_i}^\pi &= \frac{1}{2}(Y_+ + Y_-) + f(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi)\delta, \\ Z_{t_i}^\pi &= \frac{1}{2\sqrt{\delta}}(Y_+ - Y_-). \end{aligned} \tag{4.3}$$

We suppose that $Y_+ = Y_{t_i}^{t_{i+1}}$ and $Y_- = Y_{t_{i+1}}^{t_{i+1}}$, then $Y_{t_i}^\pi = Y_{t_i}^{t_i}, Z_{t_i}^\pi = Z_{t_i}^{t_i}$. By the equation (4.3), Peng, *et al.* (2009) got a following explicit scheme, performed iteratively from level $n-1$ to level 0 of the binomial tree model

$$\begin{aligned} Y_{t_i}^{t_i} &= \frac{1}{2}(Y_{t_i}^{t_{i+1}} + Y_{t_{i+1}}^{t_{i+1}}) + f(t_i, Y_{t_i}^{t_i}, Z_{t_i}^{t_i})\delta, \\ Z_{t_i}^{t_i} &= \frac{1}{2\sqrt{\delta}}(Y_{t_i}^{t_{i+1}} - Y_{t_{i+1}}^{t_{i+1}}). \end{aligned} \tag{4.4}$$

This is the equation, performed by block allocation.

Speedup is the ratio of serial execution time to parallel execution time. In the case of linear speedup, the speedup value equals the number of processors. The discussed method uses task decomposition to parallelize the BSDEs by block allocation, the parallel method does not have a linear speed-up rate, since the computation of the block allocation method can not be done with embarrassing parallelism. But the method gave a chance to improving the execution time of the BSDERs.

To get the solution of the BSDEs, Dai, *et al.* (2010) used GPU to accelerate option pricing computations with the BSDEs, which show that when the number of time steps and Monte Carlo simulations are small, the speedups seem insignificant, and moreover, higher accuracy will be achieved when the value of them increases.

Now, we give a compare of the three methods on the BSDEs as follows. For the following equation, $f(t, Y_t, Z_t) = t+2Y_t+3Z_t$. Assume that in (1.1), Table 2 presented the error between the three methods (Multistep method, Schwarz WR and Block Allocation) and the Euler method.

Table 2: The errors between the three methods and the Euler method on the BSDEs

n	1	2	3	4	5	6
Multistep method	5.546E+4	7.675E+3	4.453E+2	3.448	8.473E-2	3.651E-4
Schwarz WR	4.658E+4	6.781E+4	2.614E+2	4.323	7.654E-2	4.548E-3
Block Allocation	1.784E+5	3.816E+4	2.657E+3	7.673	4.564E-1	3.445E-3

5. Discussion and conclusion

This paper has reviewed a number of developments in numerical solution of the BSDEs. Numerical solution of the BSDE-based mathematical models has been an important research topic over decades. Over the last decade, the BSDEs have been studied intensively and a vast related literature has been produced, which not only helps to conduct academic research in a theoretical framework, but also provides practical assistance in financial problems. Numerical solutions of the BSDEs have made recent progresses in finance problems. However, even if several numerical methods of the BSDEs have been proposed, they suffer a high complexity in terms of computational time or are very costly in terms of computer memory. As for the BSDEs, there is a large potential applicability for other economics and finance problems such as in contract or game theory, credit risk or liquidity risk models. Motivated in particular by the BSDEs in finance problems, one faces challenging numerical problems arising finance problems in high dimension, under partial information, under transaction costs, and so on. Developing a theory of inference to address these issues in financial problems is presenting new challenges.

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