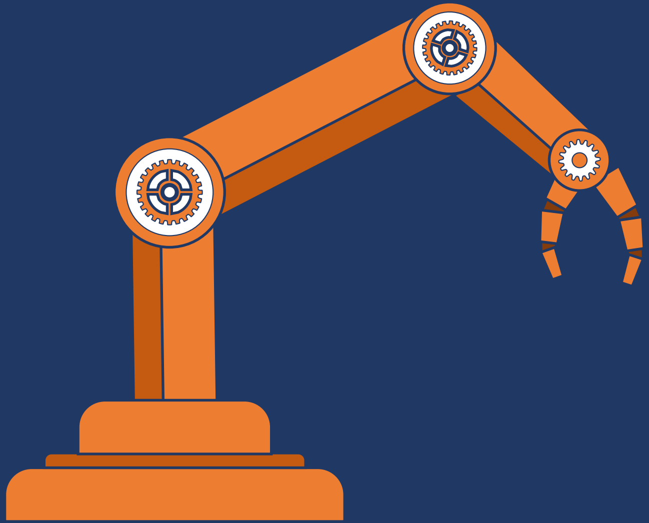

Kinematics Analysis and Modeling of 6 Degree of Freedom Robotic Arm from DFROBOT

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Contents

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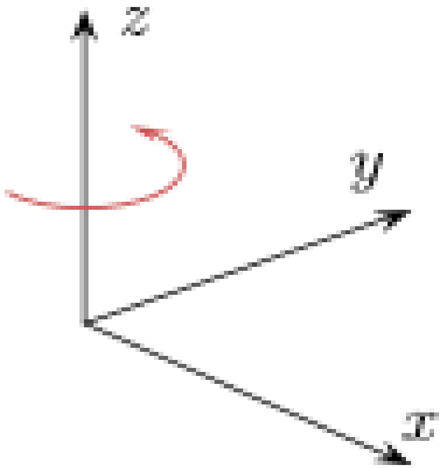
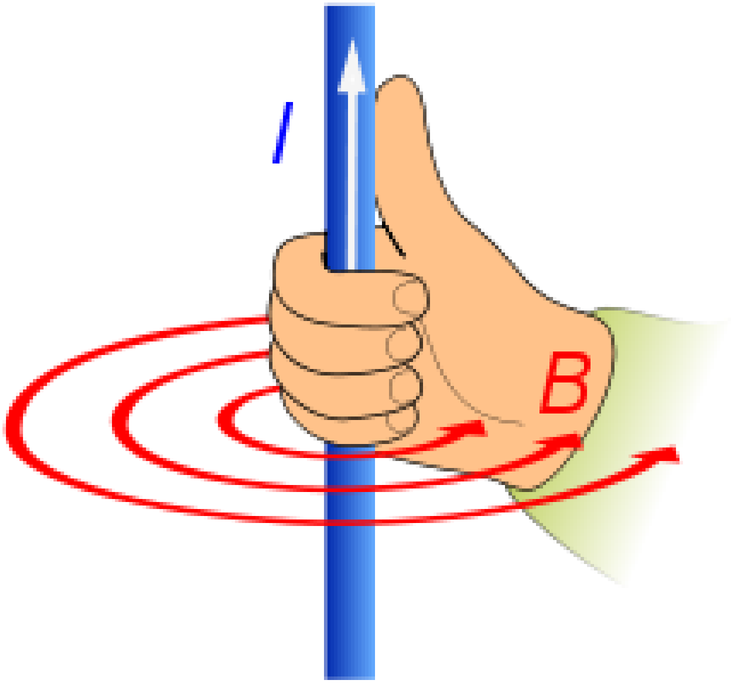
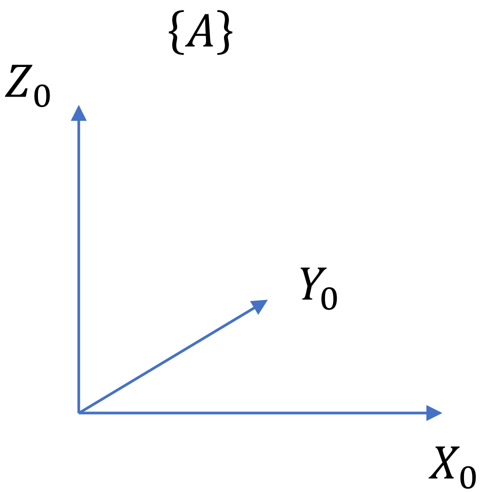
Right-hand rule





Frames

Translation



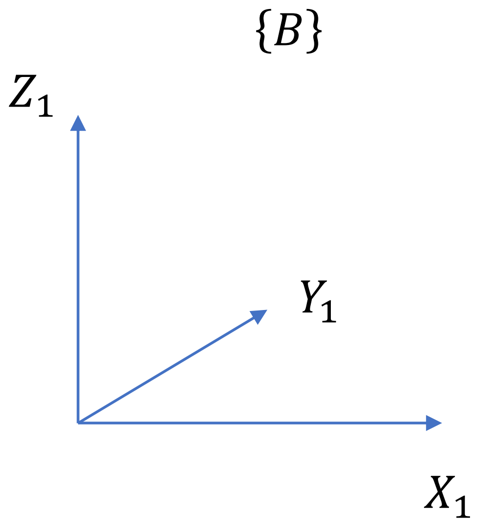
$$Z_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Y_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix

$$P^A = R_B^A * P^B$$

Rotation



Frames

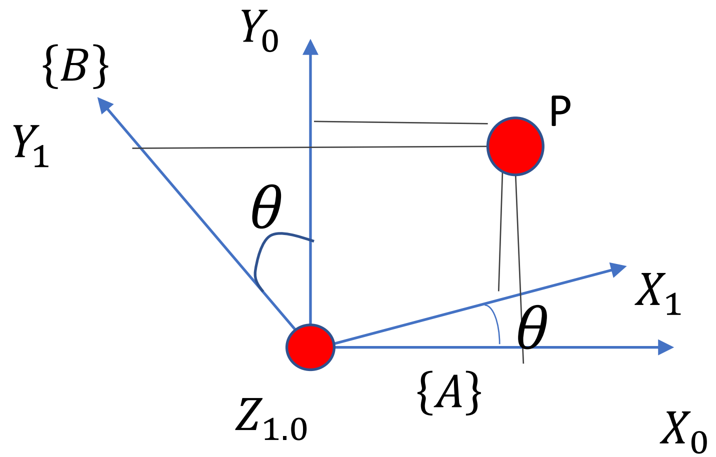
$$P^{A} = R_{B}^{A} * P^{B}$$

Rotation Matrix about Z axis

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} Cos\theta & -Sin\theta & 0 \\ Sin\theta & Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

Rotation Matrix about X axis

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Cos\theta & -Sin\theta \\ 0 & Sin\theta & Cos\theta \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$



$$Cos\theta = \frac{\text{مجاور}}{\text{وتر}}$$

$$Sin\theta = \frac{\text{مقابل}}{\text{وتر}}$$

Denavit–Hartenberg convention

In order to solve by DH convention you should follow these steps:

1. Put all the frames, Z _ X _ Y

Z axis cases:

- Parallel
- Skew
- Intersecting

2. Identify DH parameters: { a - α – d - θ }

a = The distance from Z to Z+1 measured along X (Translation).
α = The angle from Z to Z+1 measured about X (Rotation).

d = The distance from X to X+1 measured along Z (Translation).
θ = The angle from X to X+1 measured about Z (Rotation).

Denavit–Hartenberg convention

Ai =[Rotation – Z (θ) * Translation - Z (d)] * [Rotation - X (α) * Translation (a)]

$$= \begin{bmatrix} Cos\theta & -Sin\theta & 0 & 0 \\ Sin\theta & Cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos\alpha & -Sin\alpha & 0 \\ 0 & Sin\alpha & Cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Ai} = \begin{bmatrix} C\theta i & -S\theta i C\alpha i & S\theta i S\alpha i & a i C\theta i \\ S\theta i & C\theta i C\alpha i & -C\theta i S\alpha i & a i S\theta i \\ 0 & S\alpha i & C\alpha i & d i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematics

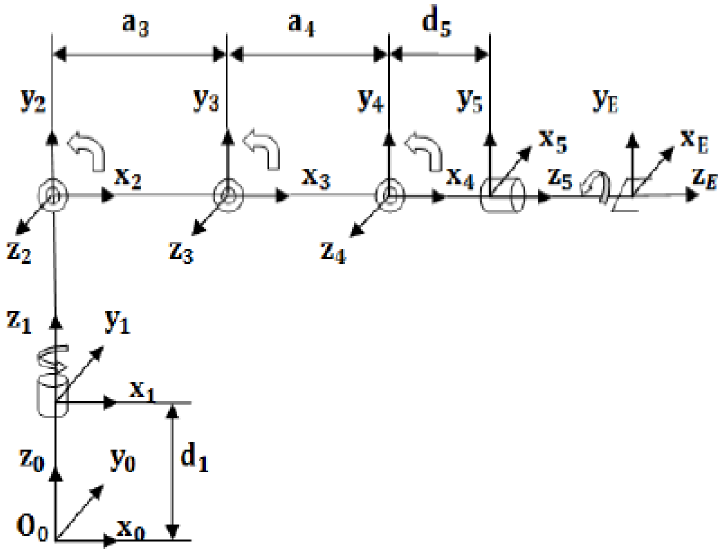
$$A_i = \begin{bmatrix} C\theta i & -S\theta i C\alpha i & S\theta i S\alpha i & a_i C\theta i \\ S\theta i & C\theta i C\alpha i & -C\theta i S\alpha i & a_i S\theta i \\ 0 & S\alpha i & C\alpha i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a = The distance from Z to Z+1 measured along X (Translation).

α = The angle from Z to Z+1 measured about X (Rotation).

d = The distance from X to X+1 measured along Z (Translation).

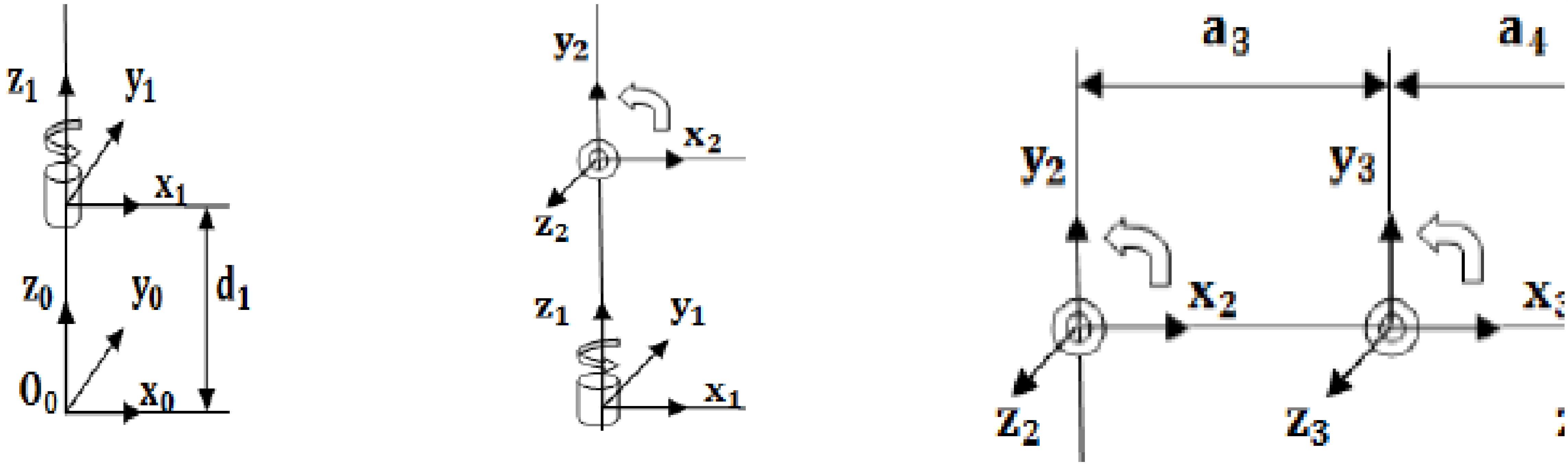
θ = The angle from X to X+1 measured about Z (Rotation).



Link	a	α	d	θ
1	0	0	d1	$\theta 1$
2	0	90	0	$\theta 2$
3	a3	0	0	$\theta 3$
4	a4	0	0	$\theta 4$
5	0	-90	d5	$\theta 5$
6	0	0	0	$\theta 6$

a = The distance from Z to Z+1 measured along X (Translation).
α = The angle from Z to Z+1 measured about X (Rotation).

d = The distance from X to X+1 measured along Z (Translation).
θ = The angle from X to X+1 measured about Z (Rotation).



a = The distance from Z to Z+1 measured along X (Translation).
α = The angle from Z to Z+1 measured about X (Rotation).

d = The distance from X to X+1 measured along Z (Translation).
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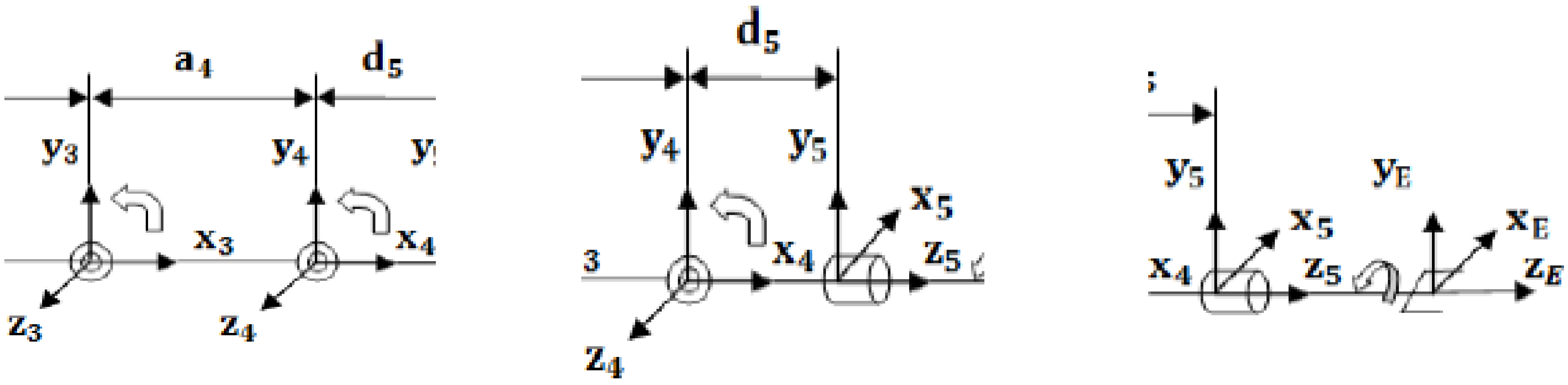

Forward kinematics

Link	a	α	d	θ
1	0	0	d1	$\theta 1$
2	0	90	0	$\theta 2$
3	a3	0	0	$\theta 3$

$$Ai = \begin{bmatrix} C\theta i & -S\theta i C\alpha i & S\theta i S\alpha i & aiC\theta i \\ S\theta i & C\theta i C\alpha i & -C\theta i S\alpha i & aiS\theta i \\ 0 & S\alpha i & C\alpha i & di \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1^0 = A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2^1 = A_2 = \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^2 = A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4^3 = A_4 = \begin{bmatrix} C_4 & -S_4 & 0 & a_4C_4 \\ S_4 & C_4 & 0 & a_4S_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5^4 = A_5 = \begin{bmatrix} C_5 & 0 & -S_5 & 0 \\ S_5 & 0 & C_5 & 0 \\ 0 & -1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^5 = A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematics

$$A_6^0 = A_1^0 * A_2^1 * A_3^2 * A_4^3 * A_5^4 * A_6^5$$

مفهوم أساسي متطابقات المجموع والفرق

متطابقات المجموع

- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

متطابقات الفرق

- $\sin(A-B) = \sin A \cos B - \cos A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Forward kinematics

$$A_6^0 = A_1^0 * A_2^1 * A_3^2 * A_4^3 * A_5^4 * A_6^5$$
$$= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$n_x = C_6 C_{12} C_{345} - S_6 S_{12}$$
$$n_y = C_6 S_{12} C_{345} + C_{12} S_6$$
$$n_z = C_6 S_{345}$$
$$o_x = -C_{12} S_6 C_{345} - S_{12} C_6$$
$$o_y = -S_6 S_{12} C_{345} + C_{12} C_6$$
$$o_z = -S_6 S_{345}$$
$$a_x = -C_{12} S_{345}$$
$$a_y = -S_{12} S_{345}$$
$$a_z = C_{345}$$

$$p_x = a_4 C_{12} C_3 C_4 - a_4 C_{12} S_3 S_4 + S_{12} d_5 + a_3 C_{12} C_3$$
$$p_y = a_4 S_{12} C_3 C_4 - a_4 S_{12} S_3 S_4 - C_{12} d_5 + a_3 S_{12} C_3$$
$$p_z = a_4 S_3 C_4 + a_4 C_3 S_4 + a_3 S_3 + d_1 \tag{8}$$

where,

$C_{23} = \cos(\theta_2 + \theta_3), S_{23} = \sin(\theta_2 + \theta_3)$, $C_{234} = \cos(\theta_2 + \theta_3 + \theta_4)$ and $S_{234} = \sin(\theta_2 + \theta_3 + \theta_4)$

Making use of some trigonometric equations helps for easy solutions:

$$C_{12} = C_1 C_2 - S_1 S_2$$
$$S_{12} = C_1 S_2 + S_1 C_2$$
$$C_{234} = C_2(C_3 C_4 - S_3 S_4) - S_2(C_4 S_3 + C_3 S_4)$$
$$S_{234} = S_2(C_3 C_4 - S_3 S_4) + C_2(S_3 C_4 + C_3 S_4).$$

Inverse kinematics

$$A_1^{-1} = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^{-1} = \begin{bmatrix} C_1 & S_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_2 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^{-1} = \begin{bmatrix} C_3 & S_3 & 0 & -a_3 \\ -S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4^{-1} = \begin{bmatrix} C_4 & S_4 & 0 & -a_4 \\ -S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5^{-1} = \begin{bmatrix} C_5 & S_5 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ -S_5 & C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^{-1} = \begin{bmatrix} C_6 & S_6 & 0 & 0 \\ -S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

INVERSE KINEMATIC SOLUTIONS

To solve the matrix in Eq. (7) it is easy to use the algebraic solution technique for:

$$A_6^0 = A_1^0 A_2^1 A_3^2 A_4^3 A_5^4 A_6^5 \tag{15}$$

To solve for θi when A_6^0 is given as numeric values, multiply each side by A_1^{-1}:

$$A_1^{-1} * \left[\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\right] = A_1^{-1} * A_1^0 * A_2^1 * A_3^2 * A_4^3 * A_5^4 * A_6^5 \tag{16}$$

The matrix manipulations has resulted the following matrix solutions:

$$\begin{bmatrix} . & . & C_1 a_x + S_1 a_y & C_1 p_x + S_1 p_y \\ . & . & -S_1 ax + C_1 a_y & -S_1 p_x + C_1 p_y \\ . & . & a_z & p_z - d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} . & . & -C_2 S_{345} & a_4 C_2 C_{34} + a_3 C_2 C_3 + S_2 d_5 \\ . & . & -S_2 S_{345} & a_4 S_2 C_{34} + a_3 S_2 C_3 - C_2 d_5 \\ . & . & C_{345} & a_4 S_{34} + a_3 S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Both matrix elements in Eq. (17) are equated to each other and the resultant θ values are extracted.
By taking (1, 4) (2, 4):

$$C_1 p_x + S_1 p_y = a_4 C_2 C_{34} + a_3 C_2 C_3 + S_2 d_5 \tag{18}$$

$$-S_1 p_x + C_1 p_y = a_4 S_2 C_{34} + a_3 S_2 C_3 - C_2 d_5 \tag{19}$$

Squaring and adding the two Eq. (18) and (19):

Squaring and adding the two Eq. (18) and (19):

$$C_3 = \mathrm{Cos}\ \theta_3 = \frac{\sqrt{p_x^2 + p_y^2 - d_5^2} - a_4 C_{34}}{a_3} = n$$

$$\theta_3 = \mathrm{Cos}^{-1} n = \mathrm{Atan2}\left(\mp\sqrt{1 - n^2}, n\right) \tag{20}$$

Eq. (3, 4):

$$p_z - d_1 = a_4 S_{34} + a_3 S_3$$

$$S_{34} = \frac{a_3 S_3 - p_z + d_1}{a_4}$$

$$\theta_{34} = \mathrm{Atan2}\left[\frac{a_3 S_3 - p_z + d_1}{a_4}, \mp\sqrt{1 - \left(\frac{a_3 S_3 - p_z + d_1}{a_4}\right)^2}\right] \tag{21}$$

$$\theta_4 = \theta_{34} - \theta_3 \tag{22}$$

Multiplying each side of Eq. (15) with $A_1^{-1} A_2^{-1}$:

$$A_1^{-1} * A_2^{-1} * \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_3^2 * A_4^3 * A_5^4 * A_6^5 \tag{23}$$

$$\begin{bmatrix} . & . & . & C_1 C_2 p_x + C_1 S_2 p_y + S_1 p_z \\ . & . & . & -S_1 C_2 p_x - S_1 S_2 p_y + C_1 p_z \\ S_2 n_x - C_2 n_y & S_2 o_x - C_2 o_y & . & S_2 p_x - C_2 p_y - d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_6 C_{345} & -S_6 C_{345} & -S_{345} & a_4 C_{34} + a_3 C_3 \\ C_6 S_{345} & -S_6 S_{345} & C_{345} & a_4 S_{34} + a_3 S_3 \\ -S_6 & -C_6 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

Equating elements (3, 4) of the right hand side matrix and the left hand side matrix of Eq. (24):

$$S_2p_x - C_2p_y - d_1 = d_5$$
$$S_2p_x - C_2p_y = d_1 + d_5$$
$$\theta_2 = \text{atan2}(p_x, -p_y) \mp \text{atan2}\left[\sqrt{p_x^2 + p_y^2 - (d_1 + d_5)^2}, (d_1 + d_5)\right] \quad (25)$$

From Eq. (8) we can obtain:

$$a_x = -C_{12}S_{345}$$

$$a_y = -S_{12}S_{345}$$

Dividing the two equations:

$$\frac{S_{12}}{C_{12}} = \frac{a_y}{a_x}\theta_{12} = \text{atan2}\,(a_y, a_x) \quad (26)$$

And then we find:

$$\theta_1 = \theta_{12} - \theta_2 \quad (27)$$

Then also equating elements (3, 1) and (3, 2) of the two sides of the matrices in Eq. (24):

$$-S_6 = S_2n_x - C_2n_y \text{ Or} S_6 = C_2n_y - S_2n_x$$
$$-C_6 = S_2o_x - C_2o_y \text{Or } C_6 = C_2o_y - S_2o_x\theta_6 = \text{Atan2}[(C_2n_y - S_2n_x), (C_2o_y - S_2o_x)] \quad (28)$$

Or alternatively:

$$\theta_6 = \text{Atan2}\left[\mp\sqrt{1 - (C_{12}o_y - S_{12}o_x)^2}, (C_{12}o_y - S_{12}o_x)\right] \quad (29)$$

Now multiply each side of Eq. (15) by:

Now multiply each side of Eq. (15) by:

$$A_1^{-1} * A_2^{-1} * A_3^{-1} * \left[\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\right] = A_4^3 * A_5^4 * A_6^5 \tag{30}$$

$$\begin{bmatrix} C_1C_{23} & C_1S_{23} & S_1 & -a_3C_1C_2 \\ -S_1C_{23} & -S_1S_{23} & C_1 & a_3S_1C_2 \\ S_{23} & -C_{23} & 0 & -a_3S_2 - d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{RHS} \begin{bmatrix} C_6C_{45} & -S_6C_{45} & -S_{45} & a_4C_4 \\ C_6S_{45} & -S_6S_{45} & C_{45} & a_4S_4 \\ -S_6 & -C_6 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{LHS} \tag{31}$$

Equating elements (3, 4) from the two sides of Eq. (31):

$$S_{23}p_x - C_{23}p_y - a_3S_2 - d_1 = d_5$$
$$S_{23}p_x - C_{23}p_y = a_3S_2 + d_1 + d_5$$
$$\theta_{23} = \operatorname{atan2}\left(p_x, -p_y\right) \mp \operatorname{atan2}\left[\sqrt{p_x^2 + p_y^2 - \left(a_3S_2 + d_1 + d_5\right)^2}, (a_3S_2 + d_1 + d_5)\right] \tag{32}$$

$$\theta_3 = \theta_{23} - \theta_2 \tag{33}$$

From the Eq. in (8) we can also obtain:

$$C_{345} = a_z\theta_{345} = \operatorname{atan2}\left(\mp\sqrt{1 - a_z^2}, a_z\right) \ldots \tag{34}$$

$$\theta_5 = \theta_{345} - \theta_3 - \theta_4 \tag{35}$$

$$\theta_2 = \operatorname{atan2}\left(p_x, -p_y\right) \mp \operatorname{atan2}\left[\sqrt{p_x^2 + p_y^2 - (d_1 + d_5)^2}, (d_1 + d_5)\right]$$

$$\theta_{12} = \operatorname{atan2}\,(a_y, a_x)$$

$$\theta_1 = \theta_{12} - \theta_2$$

$$\theta_{23} = \operatorname{atan2}\left(p_x, -p_y\right) \mp \operatorname{atan2}\left[\sqrt{p_x^2 + p_y^2 - (a_3 S_2 + d_1 + d_5)^2}, (a_3 S_2 + d_1 + d_5)\right]$$

$$\theta_3 = \theta_{23} - \theta_2$$

$$\theta_3 = \cos^{-1} n = \operatorname{Atan2}\,(\mp\sqrt{1 - n^2}, n),$$

where,

$$n = \cos\theta_3 = \frac{\sqrt{p_x^2 + p_y^2 - d_5^2} - a_4 C_{34}}{a_3}$$

$$\theta_{34} = \operatorname{Atan2}\left[\frac{a_3 S_3 - p_z + d_1}{a_4}, \mp\sqrt{1 - \left(\frac{a_3 S_3 - p_z + d_1}{a_4}\right)^2}\right]$$

$$\theta_4 = \theta_{34} - \theta_3$$

$$\theta_{345} = \operatorname{atan2}\left(\mp\sqrt{1 - a_z^2}, a_z\right)$$

$$\theta_5 = \theta_{345} - \theta_3 - \theta_4$$

$$\theta_6 = \operatorname{Atan2}\left[(C_2 n_y - S_2 n_x), (C_2 o_y - S_2 o_x)\right]$$

References

- https://www.youtube.com/channel/UC5ZQinPsJ4C8YiauoT8xZUg
- https://www.youtube.com/channel/UCw3eyA3edUq3AJkh8HFLD2Q
- https://www.youtube.com/channel/UCgwcC5Kb7_CuU99aVjWo8GQ