# Theoretical Astrophysics

Moodle:

Exam: 5. 2. 2018 14.00 h

2. Exam:

Exercises start in second week

~~[illegible]~~

Summarize, explain exercise, discuss a problem or answer

a comprehension question ⇒ get ~~[illegible]~~ 0 – 2 points

Minimum for exam: 5 points

Handing in exercises in group of max. 3 people

Handing in maximum 2 exercises per sheet ?

# Summary of Electrodynamics

(i) Units (Gaussian cgs) Mass [g] ; length [cm] ; time [s]

$$10^{-5}[N] = [dyn] = \left[\frac{g\,cm}{s^2}\right] = F_C = \frac{q^2}{r^2}$$

$$= \frac{\cancel{[g]}}{\cancel{[s]}} \frac{q^2}{[cm]^2}$$

$$\Rightarrow \quad q = \left[\left(\frac{g\,cm^3}{s^2}\right)^{1/2}\right] = esu$$

$$\Rightarrow \quad 1\,C = 2{,}9979 \cdot 10^9 \; esu$$

$$\cancel{\text{and}} \quad 1\,V = 1\,\frac{J}{C} = \frac{10^7\,erg}{2{,}9979 \cdot 10^9\,esu} = \frac{1}{299{,}79} \frac{\sqrt{g}\sqrt{cm}}{s}$$

$$\text{with } 1\,erg = \frac{g\,cm^2}{s^2} = 10^{-7}\,J \qquad = 1\ \text{Gauss}$$

$$1\,eV = 1{,}6 \cdot 10^{-19}\,C \cdot 1\,V$$

$$= 1{,}6 \cdot 10^{-19} \cdot 2{,}9979 \cdot 10^9 \,\frac{1}{299{,}79}\ esu\ \text{Gauss}$$

$$= 1{,}6 \cdot 10^{-12}\ erg$$

$\cancel{A^\mu = \begin{pmatrix}\phi \\ \bar{A}\end{pmatrix}}$ four vector: $x^\mu = \begin{pmatrix} x^0 \\ \bar{x} \end{pmatrix}$

metric: $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$

dual vector: $x_\mu = \begin{pmatrix} -x^0 \\ \bar{x} \end{pmatrix}$ (with Minkowski Metric)

$$\boxed{\begin{gathered} F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \\ \text{with } \partial_\mu = \frac{\partial}{\partial x^\mu} \text{ and } \partial^\mu = \frac{\partial}{\partial x_\mu} \\ \bar{E} = \frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \nabla \phi \\ \bar{B} = \nabla \times \bar{A} \end{gathered}}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & \bar{E}^T \\ -\bar{E} & \mathbb{B} \end{pmatrix} \qquad \text{with } \mathbb{B}_{ij} = \varepsilon_{ijk} B^k$$

Current Density: $j^\mu = \begin{pmatrix} \rho c \\ \rho \bar{v} \end{pmatrix}$

Four vector potential: $A^\mu = \begin{pmatrix} \phi \\ \bar{A} \end{pmatrix}$

Field tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Lorentz Scalar: $F^{\mu\nu}F_{\mu\nu}$ or $A^\mu j_\mu$ or ($A^\mu A_\mu$)
↳ gives photon a mass

Lagrangian: $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu} - \frac{1}{c} A^\mu j_\mu$

Euler Lagrange: $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$

$\Rightarrow \quad \partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu$

only four equations, so one other equation is Jacobi Identity:

$$\partial_\alpha F_{\beta\gamma} - \partial_\gamma F_{\alpha\beta} - \partial_\beta F_{\gamma\alpha} = 0$$ oder so

We get: $c\,\bar{\nabla} \times \bar{B} - \dot{\bar{E}} = 4\pi j$ $\qquad c\,\bar{\nabla}\times\bar{E} + \dot{\bar{B}} = 0$

$\bar{\nabla}\bar{E} = 4\pi\rho$ $\qquad \bar{\nabla}\bar{B} = 0$

$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu$

$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = -\frac{4\pi}{c} j^\nu$

Lorentz gauge: $\partial_\mu A^\mu = 0$

$\Rightarrow \quad \Box A^\nu = -\frac{4\pi}{c} j^\nu$

Lorentz gauge: $\partial_\mu A^\mu = \partial_\mu A'^\mu + \partial^\mu \partial_\mu f = 0$

based on $F^{\mu\nu}$ stays the same with

$A^\mu = A'^\mu + \partial^\mu f$

with $\partial^\mu\partial^\nu f - \partial^\nu\partial^\mu f = 0$

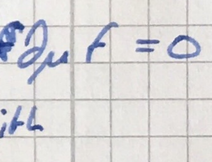
$\Rightarrow \quad \partial_\mu A'^\mu = -\partial^\mu \partial_\mu f$

$\Box G(t-t', x-x') = -4\pi\,\delta_D(t-t', x-x')$

$\Rightarrow \quad G(t-t', x-x') = \frac{1}{|x-x'|}\,\delta_D\left(t - t' - \frac{|x-x'|}{c}\right)$

$A^\nu(t,\bar{x}) = \frac{1}{c}\int G(t-t', x-x')\, j^\nu(t', x')\, dt'\, d\bar{x}'$

Particle moving: $j^\nu(\bar{x}, t) = q\begin{pmatrix} c \\ v \end{pmatrix} \delta_D(\bar{x} - \bar{x}_0(t))$




inserting $A^\nu$: $\phi = \frac{q}{R(1-\bar{e}\bar{\beta})}\Big|_{ret}$ $\qquad \bar{A} = \bar{\beta}\phi\big|_{ret}$

(Liénard-Wiechert) $R = |\bar{x} - \bar{x}_0(t)|$, $\bar{e}$ = unit vector from charge to us

Hamiltonian: $H = \dot{q} p - L$ with $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$

now: $\dot{q} = \partial_\mu A_\alpha$ $p = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)}$

Energy Momentum Tensor: $T^\nu{}_\mu = \partial_\mu A_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)} - \delta^\nu_\mu \mathcal{L}$

$$T^\nu{}_\mu = \frac{1}{4\pi} \begin{pmatrix} (\bar{E}^2+\bar{B}^2)/2 & (\bar{E}\times\bar{B})^T \\ \bar{E}\times\bar{B} & T_{ij} \end{pmatrix}$$

$\mathcal{E} = \frac{1}{8\pi}(\bar{E}^2 + \bar{B}^2)$ ⇒ Energy Density

$\bar{S} = \frac{c}{4\pi}(\bar{E}\times\bar{B})$ ⇒ pointing vector

Lienard-Wiechert Potentials:

$\phi = \frac{q}{R(1-\bar{e}\bar{\beta})}$ $\bar{A} = \phi\bar{\beta}$ with $R = |\bar{x}-\bar{x}'|$, $\bar{e}$ to observer

$\bar{\beta} = \frac{\bar{v}}{c}$

$t = t' + \frac{R}{c}$ $t' = t - \frac{R}{c}$ with $t'$... time of charge

$t$... time of observer

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c}\frac{\partial}{\partial t}\sqrt{R_i R^i} = 1 - \frac{1}{c}\frac{R_i}{R}\frac{\partial R^i}{\partial t}$$

$$= 1 - \frac{1}{c}\underbrace{\frac{R_i}{R}}_{-e_i}\underbrace{\frac{\partial R^i}{\partial t'}}_{\text{velocity}}\frac{\partial t'}{\partial t}$$

$$= 1 + \bar{e}\bar{\beta}\frac{\partial t'}{\partial t}$$

$$\Rightarrow (1-\bar{e}\bar{\beta})\frac{\partial t'}{\partial t} = 1 \quad \Rightarrow \quad \frac{\partial t'}{\partial t} = \frac{1}{1-\bar{e}\bar{\beta}}$$

$$dt = (1-\bar{e}\bar{\beta})\,dt'$$

Energy momentum tensor: $T^\mu{}_\nu = \frac{1}{4\pi}\begin{pmatrix} \frac{1}{2}(E^2+B^2) & (\bar{E}\times\bar{B})^T \\ (\bar{E}\times\bar{B}) & T_{ij} \end{pmatrix}$

$$\bar{S} = \frac{c}{4\pi}(\bar{E}\times\bar{B})$$

We could calculate: $\frac{dP}{d\Omega} = \frac{q^2}{4\pi c(1-\bar{e}\bar{\beta})^5}\left|\bar{e}\times\left((\bar{e}-\bar{\beta})\times\dot{\bar{\beta}}\right)\right|^2$

per unit retarded time (time of particle)

$P = \frac{2q^2}{3c}\gamma^6\left(\dot{\bar{\beta}}^2 - (\bar{\beta}\times\dot{\bar{\beta}})^2\right)$

↳ integrating over all angles

non relativistic limit: $\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} \left|\bar{e} \times (\bar{e} \times \dot{\bar{\beta}})\right|^2 = \frac{q^2}{4\pi c} \left|\bar{e} \times \dot{\bar{\beta}}\right|^2$

$$= \frac{q^2}{4\pi c} \dot{\beta}_\perp^2$$

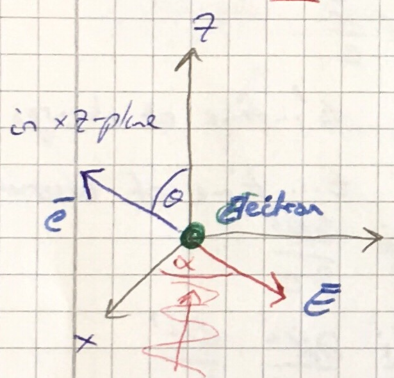
$$P = \frac{2q^2}{3c} \dot{\bar{\beta}}^2$$

$$S = -mc^2 \int d\tau - \frac{q}{c} \int A^\mu u_\mu \, d\tau$$

with Variation: $\Rightarrow m \frac{du^\mu}{d\tau} = \frac{q}{c} F^\mu{}_\nu u^\nu$ with $\tau$... proper time of particle

$$\Rightarrow \boxed{\frac{d(\gamma m c^2)}{dt} = q \bar{E} \bar{v}}$$

$$\boxed{\frac{d}{dt}(\gamma m \bar{v}) = q(\bar{E} + \bar{\beta} \times \bar{B})}$$




Lorentz: $m \ddot{\bar{x}} = -e(\bar{E} + \underbrace{\bar{\beta} \times \bar{B}}_{=0})$

for non relativistic:

$$\dot{\bar{\beta}} = -\frac{e}{m_e c} \bar{E}$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} (\bar{e} \times \bar{E})^2 \frac{e^2}{m_e^2 c^2}$$ out of non relativistic

introduce $r_e$: $m_e c^2 = \frac{e^2}{r_e}$ potential energy of electron in a field of an electron

$$r_e = \frac{e^2}{m_e c^2} \approx 3 \cdot 10^{-13}\,\text{cm}$$

$$\frac{dP}{d\Omega} = \frac{r_e^2 c}{4\pi} (\bar{e} \times \bar{E})^2$$

Poynting vector: $\bar{S} = \frac{c}{4\pi} (\bar{E} \times \bar{B}) = \frac{c}{4\pi} |\bar{E}|^2 \cdot \bar{e}_z$

$$\frac{d\sigma_T}{d\Omega} = \frac{c}{|\bar{S}|} \frac{dP}{d\Omega} = r_e^2 \frac{|\bar{e} \times \bar{E}|^2}{|\bar{E}|^2}$$

$$\bar{e} \times \bar{E} = \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} E = E \begin{pmatrix} -\cos\theta \sin\alpha \\ \cos\theta \cos\alpha \\ \sin\theta \sin\alpha \end{pmatrix}$$

$$(\bar{e} \times \bar{E})^2 = E^2 (\cos^2\theta + \sin^2\theta \sin^2\alpha)$$

$$= E^2 (1 - \sin^2\theta \cos^2\alpha)$$

$$\boxed{\frac{d\sigma_T}{d\Omega} = r_e^2 (1 - \sin^2\theta \cos^2\alpha)}$$

$$\langle \frac{d\sigma_T}{d\Omega} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \; r_e^2 (1 - \sin^2\theta \cos^2\alpha)$$

$$= \frac{r_e^2}{2\pi} (2\pi - \sin^2\theta \, \pi)$$

$$= \frac{r_e^2}{2} (2 - \sin^2\theta) = \frac{r_e^2}{2} (1 + \cos^2\theta)$$

$$\boxed{\langle \frac{d\sigma_T}{d\Omega} \rangle = \frac{r_e^2}{2} (1 + \cos^2\theta)}$$

$$\sigma_T = \int \langle \frac{d\sigma_T}{d\Omega} \rangle d\Omega = \frac{r_e^2}{2} 2\pi \int_{-1}^{1} d(\cos\theta) (1 + \cos^2\theta)$$

$$= \frac{8\pi r_e^2}{3}$$

$$\boxed{\sigma_T = \frac{8\pi r_e^2}{3} \approx 6{,}6 \cdot 10^{-25} \text{ cm}^2}$$

<u>Eddington Luminosity:</u>

$\frac{\vec{S}}{c}$ ... momentum current density

$\frac{\vec{S}}{c} \cdot \sigma_T$ ... electromagnetic force per electron

$L = \int \vec{S} \, d\vec{A} = 4\pi R^2 S \quad \Rightarrow S = \frac{L}{4\pi R^2}$

for const. $\vec{S}$

$\Rightarrow \quad \frac{\vec{S}}{c} = \frac{L}{4\pi R^2 c} \sigma_T \vec{e}_r \qquad \frac{L}{4\pi R^2 c} \sigma_T = \frac{GMm}{R^2}$

$\Leftrightarrow$ force on electron equal to gravitation

$\Leftrightarrow$ Strahlungsdruck gleich Gravitationsdruck

$\Rightarrow \quad L_{Edd} = \frac{4\pi c G M m}{\sigma_T}$

Spectra: $\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \frac{d^2E}{d\Omega \, d\omega} d\omega$

with $E$... Energy and $f(t) \in \mathbb{R}$ then Fourier Trafo: $\tilde{F}(-\omega) = \tilde{F}^*(\omega)$

Fourier Trafo: $\int_{-\infty}^{\infty} |F(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{F}(\omega)|^2 d\omega$ (Parseval Plancherel)

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} dt \; \frac{e^2}{4\pi c} \underbrace{\left( \frac{\vec{e} \times ((\vec{e} - \vec{\beta}) \times \dot{\vec{\beta}})}{(1 - \vec{e}\vec{\beta})^3} \right)^2}_{F(t)}$$

$$= \frac{e^2}{4\pi c} \int_{-\infty}^{\infty} |F(t)|^2 = \frac{e^2}{4\pi c} \int_{-\infty}^{\infty} d\omega \, |\tilde{F}(\omega)|^2$$

small change because now $t$.. time of observer but $P$ is given with proper time

$$= \int \frac{d\varepsilon}{d\Omega d\omega} d\omega \Rightarrow \frac{d\varepsilon}{d\Omega d\omega} = \frac{e^2}{4\pi c} |\tilde{F}(\omega)|^2$$

$$f(t) = \frac{\bar{e} \times ((\bar{e}-\bar{\beta}) \times \dot{\bar{\beta}})}{(1-\bar{e}\bar{\beta})^3}$$

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} dt \, \frac{\bar{e} \times ((\bar{e}-\bar{\beta}) \times \dot{\bar{\beta}})}{(1-\bar{e}\bar{\beta})^3} e^{-i\omega t}$$

$t \to t'$

$$= \int_{-\infty}^{\infty} dt' \, \frac{\bar{e} \times ((\bar{e}-\bar{\beta}) \times \dot{\bar{\beta}})}{(1-\bar{e}\bar{\beta})^2} e^{-i\omega(t'+\frac{R}{c})}$$

$$\frac{\bar{e} \times ((\bar{e}-\bar{\beta}) \times \dot{\bar{\beta}})}{(1-\bar{e}\bar{\beta})^2} = \frac{d}{dt'}\left(\frac{\bar{e} \times (\bar{e} \times \bar{\beta})}{1-\bar{e}\bar{\beta}}\right)$$

$$= -\int_{-\infty}^{\infty} dt' \, \frac{\bar{e} \times (\bar{e} \times \bar{\beta})}{1-\bar{e}\bar{\beta}} \frac{d}{dt'} e^{-i\omega(t'+\frac{R}{c})}$$

with boundary terms vanish

$$\frac{d}{dt'} e^{-i\omega(t'+\frac{R}{c})} = -i\omega\left(1 + \frac{R_i}{Rc} \partial_{t'} R^i\right) e^{-i\omega(t'+\frac{R}{c})} = -i\omega(1-\bar{e}\bar{\beta}) e^{-i\omega(t'+\frac{R}{c})}$$

$$\tilde{F}(\omega) = i\omega \int_{-\infty}^{\infty} \bar{e} \times (\bar{e} \times \bar{\beta}) \exp\left(-i\omega\left(t'+\frac{R}{c}\right)\right) dt'$$

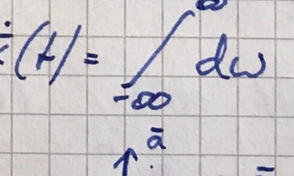
$$\boxed{\frac{d\varepsilon}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi c} \left| \int_{-\infty}^{\infty} (\bar{e} \times (\bar{e} \times \bar{\beta})) e^{-i\omega(t'+R/c)} dt' \right|^2}$$

non relativistic: $\varepsilon = \int_{-\infty}^{\infty} dt \, \frac{2q^2}{3c^3} |\bar{a}|^2 = \frac{2q^2}{3c^3} \int_{-\infty}^{\infty} |\tilde{a}(\omega)|^2 d\omega$

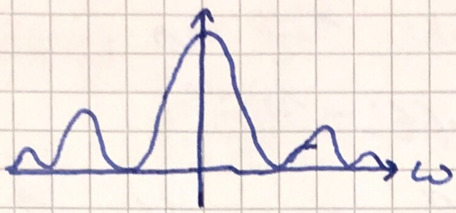
$$\Rightarrow \boxed{\frac{d\varepsilon}{d\omega} = \frac{2q^2}{3c^3} |\tilde{a}(\omega)|^2 = \frac{2q^2\omega^4}{3c^3} |\tilde{x}(\omega)|^2}$$

with: $\bar{a}(t) = \ddot{x}(t) = \int_{-\infty}^{\infty} d\omega \, \tilde{x}(\omega) e^{i\omega t} \omega^2$

const. acceleration:




Spectra: $a_0 \bar{e}_x \int_{-T/2}^{T/2} dt \exp(-i\omega t) = \frac{a_0 \bar{e}_x}{-i\omega} \exp(-i\omega t) \Big|_{-T/2}^{T/2} = \frac{2 a_0 \bar{e}_x}{\omega} \sin\left(\omega \frac{T}{2}\right)$




## Synchrotron Radiation:

Larmir frequency: $\frac{d(\gamma m c^2)}{dt} = -e\bar{E}\bar{v}$

$$\frac{d(\gamma\bar{v})}{dt} = \frac{-e}{m}\bar{\beta}\times\bar{B}$$

$$\frac{d\bar{v}}{dt} = -\frac{e}{\gamma m c}(\bar{v}\times\bar{B})$$

for $\bar{B}$ in $\bar{e}_z$ direction: $\frac{d\bar{v}}{dt} = -\frac{e}{\gamma m c}B(\bar{v}\times e_z)$

$\Rightarrow v_z = \text{const.}$ $\quad \frac{dv_y}{dt} = \frac{e}{\gamma m c}v_x B \quad \frac{dv_x}{dt} = -\frac{e}{\gamma m c}v_y B$

$\ddot{v}_y = -\left(\frac{eB}{\gamma m c}\right)^2 v_y$ $\quad \Rightarrow$ harmonic oscillator

$$\boxed{\omega_L = \frac{eB}{\gamma m c}}$$

relativistic beaming: $\frac{1}{1-\bar{e}\bar{\beta}} = \frac{1}{1-\beta\cos\theta} \overset{!}{=} \frac{1}{2(1-\beta)}$ $\quad \left(\frac{1}{1-\beta}\text{ at max.}\right)$

$\Rightarrow 1-\beta\left(1-\frac{\theta^2}{2}\right) = 2(1-\beta)$ $\quad \rightarrow$ Taylor of $\cos\theta$

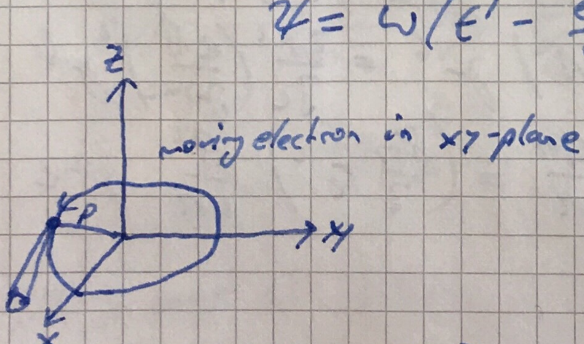
$\Rightarrow \frac{\theta^2}{2} = 1-\beta = \frac{1-\beta^2}{1+\beta} = \frac{1}{2\gamma^2}$ $\quad$ with $\beta\approx 1$

$\Rightarrow \theta = \frac{1}{\gamma}$

$t' + \frac{R}{c} \qquad \bar{R} = \bar{R}_0 + \bar{x} \qquad R = -\bar{e}\bar{R} = -\bar{e}\bar{R}_0 - \bar{e}\bar{x}$

$$\frac{d^2\varepsilon}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt'\,\bar{e}\times(\bar{e}\times\bar{\beta})\,e^{i\psi}\right|^2$$

$$\psi = \omega\left(t' - \frac{\bar{e}\bar{x}}{c}\right)$$




$\Rightarrow \bar{e} = \bar{e}_x$

$\bar{x} = x\begin{pmatrix}\sin\psi\\ \cos\psi\\ 0\end{pmatrix}$

$\bar{\beta}(t) = x\frac{\omega_L}{c}\begin{pmatrix}\cos\psi\\ -\sin\psi\\ 0\end{pmatrix}$

$\Rightarrow \bar{e}\times\bar{\beta} = \beta_y\bar{e}_z \qquad \Rightarrow \bar{e}\times(\bar{e}\times\bar{\beta}) = -\beta_y e_y$

$\Rightarrow = \sin\psi\, e_y \approx \psi\, e_y = \omega_L t' \bar{e}_y$

with ultra relativistic limit: $\frac{x\omega_L}{c} \approx 1$

$$\psi = \omega\left(t' - \frac{\vec{e}\,\vec{x}}{c}\right) = \omega\left(t' - \frac{x\sin\varphi}{c}\right)$$

$$\frac{x\omega_L}{c} = \beta \Rightarrow x = \frac{c\beta}{\omega_L} \Rightarrow = \omega t'\left(1 - \frac{\beta}{\omega_L t'}\left(\varphi - \frac{\varphi^3}{6}\right)\right)$$

with Taylor of $\sin()$

$$= \omega t'\left(1 - \beta\left(1 - \frac{\varphi^2}{6}\right)\right) \quad \text{with } \omega_L t' = \varphi$$

$$= \omega t'\left(1 - \beta + \beta\frac{\varphi^2}{6}\right)$$

$$= \frac{\omega t'}{2\gamma^2}\left(1 + \frac{\varphi^2\gamma^2}{3}\right) = \frac{\omega}{2\gamma^2}\frac{\tau}{\gamma\omega_L}\left(1 + \frac{\tau^2}{3}\right)$$

$$\varphi = \omega_L t' \quad \varphi\gamma = \omega_L\gamma t' = \tau \quad \xi = \frac{\omega}{3\gamma^3\omega_L} = \frac{\omega}{\omega_c}$$

$$= \frac{3}{2}\xi\tau\left(1 + \frac{\tau^2}{3}\right)$$

$$\frac{d^2\varepsilon}{d\Omega d\omega} = \frac{e^2\omega^2}{4\pi c}\left|\int_{-\infty}^{\infty} dt'\,\omega_L t'\,e^{i\psi}\right|^2$$

$$= \frac{e^2}{4\pi c\gamma^4}\left(\frac{\omega}{\omega_L}\right)^2\left|\int_{-\infty}^{\infty}\tau\,d\tau\,e^{i\psi}\right|^2$$

$$= \frac{e^2}{3\pi c\gamma^4}\left(\frac{\omega}{\omega_L}\right)^2 K^2_{2/3}(\xi)$$

$$= \frac{3}{\pi}\frac{e^2}{c}\gamma\,\xi^2 K_{2/3}^2(\xi)$$

Synchrotron Power:

$$P = \frac{2e^2}{3c}\left(\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right)\gamma^6 = \frac{2e^2}{3c}\left(\omega_L^2\beta^2 - (\beta^2\omega_L)^2\right)$$

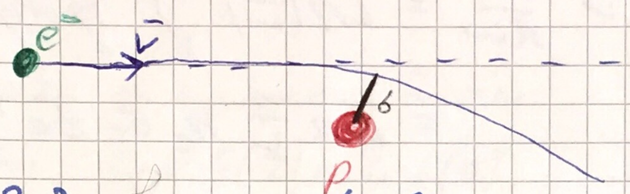
$$\text{with } \vec{x} = x\begin{pmatrix}\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} \quad \vec{\beta} = \frac{x\omega_L}{c}\begin{pmatrix}\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix} \quad \dot{\vec{\beta}} = \frac{x\omega_L^2}{c}\begin{pmatrix}-\sin\varphi\\ -\cos\varphi\\ 0\end{pmatrix}$$

$$= \frac{2e^2}{3c}\omega_L^2\beta^2(1-\beta^2)\gamma^6 = \frac{2e^2}{3c}\left(\frac{eB}{\gamma mc}\right)^2\gamma^4$$

$$= \frac{2}{3}B^2\gamma^2 c\,r_e^2\frac{4\pi}{4\pi} = \left(\frac{8\pi}{3}r_e^2\right)\frac{B^2}{4\pi}\gamma^2 c$$

$$= \sigma_T c\gamma^2\varepsilon_B = P$$

## Scattering:




$$\bar{F}_C = -\frac{Ze^2}{r^3}\,\bar{r} \qquad \bar{r} = \begin{pmatrix} vt \\ b \\ 0 \end{pmatrix} \text{ straight line}$$

$$\mathcal{E} = \int_{-\infty}^{\infty} dt\, \frac{2e^2}{3c^3}|\bar{a}|^2 = \frac{2e^2}{3c^3}\int_{-\infty}^{\infty} d\omega\, |\tilde{\bar{a}}|^2$$

$$\boxed{\frac{d\mathcal{E}}{d\omega} = \frac{2e^2}{3c^3}|\tilde{\bar{a}}|^2}$$

$$a_x = -\frac{Ze^2}{m}\frac{vt}{\sqrt{(vt)^2+b^2}^{\,3}} \qquad a_y = -\frac{Ze^2}{m}\frac{b}{\sqrt{(vt)^2+b^2}^{\,3}}$$

$$\tilde{a}_x(\omega) = -\frac{Ze^2}{m}\int_{-\infty}^{\infty} dt\, \frac{\cancel{\omega}(vt)e^{-i\omega t}}{\sqrt{(vt)^2+b^2}^{\,3}} \cancel{= -\frac{Ze^2}{m}\int_{-\infty}^{\infty}\frac{v}{\omega^2}\tau\, d\tau \frac{e^{-i\tau}\omega^3}{v^3\sqrt{\omega t}}}$$

$$= -\frac{Ze^2}{m}\int_{-\infty}^{\infty}\frac{v}{\omega^2}\tau\, d\tau\, \frac{e^{-i\tau}\omega^3}{v^3\sqrt{(\omega t)^2+\left(\frac{\omega b}{v}\right)^2}^{\,3}}$$

$$= -\frac{Ze^2}{m}\frac{\omega}{v^2}\int_{-\infty}^{\infty}\frac{\tau e^{-i\tau}}{\sqrt{\tau^2+\eta^2}}\, d\tau = -\frac{Ze^2}{m}\frac{\omega}{v^2}\, 2iK_0(\eta)$$

with $\omega t = \tau$ and $\frac{\omega b}{v} = \eta$ → dimensionless const.

$$\tilde{a}_y(\omega) = -\frac{Ze^2}{m}\int_{-\infty}^{\infty} dt\, \frac{b\, e^{-i\omega t}}{\sqrt{(vt)^2+b^2}^{\,3}} = -\frac{Ze^2}{m}\frac{b}{\omega}\left(\frac{\omega}{v}\right)^3\underbrace{\int_{-\infty}^{\infty}\frac{e^{-i\tau}}{\sqrt{\tau^2+\eta^2}^{\,3}}\, d\tau}_{\text{factor}}$$

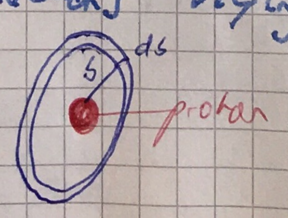
$$= -\frac{Ze^2}{m}\frac{b\omega^2}{v^3}\frac{2}{\eta}K_1(\eta) = -\frac{Ze^2}{m}\frac{\omega}{v^2}\, 2K_1(\eta)$$

Es gilt: $K_\nu(\eta) = \sqrt{\frac{\pi}{2\eta}}\, e^{-\eta}$

$$|\tilde{a}|^2 = \frac{Z^2e^4}{m^2}\frac{\omega^2}{v^4}\frac{2\pi}{\eta}e^{-2\eta} \qquad \cancel{\Rightarrow \frac{d\mathcal{E}}{d\omega} = \frac{2e^2}{3c^3}\, 2\pi}$$

$$\boxed{\frac{d\mathcal{E}}{d\omega} = \frac{4e^2}{3c^3}\, 2\pi\, \frac{Z^2e^4c}{m^2c}\, \frac{\omega^2}{v^4}\, \frac{e^{-2\eta}}{\eta} = \sigma_T Z^2e^2c\, \frac{\omega^2}{v^4}\, \frac{e^{-2\eta}}{\eta}}$$

Integrating over all Electrons flying through a ring $db\, 2\pi b$ with radius $b$:



$$2\pi \int b\, ds\, n_e v \, \frac{d\mathcal{E}}{d\omega} = 2\pi \left(\frac{v}{\omega}\right)^2 \int_0^\infty \eta\, d\eta\, n_e v \sigma_T Z^2 e^2 \frac{\omega^2}{v^4} \frac{e^{-2\eta}}{\eta}$$

$$= 2\pi \frac{1}{v} n_e \sigma_T Z^2 e^2 \int_0^\infty d\eta\, e^{-2\eta}$$

$$\frac{d^2\mathcal{E}}{d\omega\, dt} = \pi n_e \sigma_T Z^2 e^2 \left\langle \frac{1}{v} \right\rangle$$

Maxwell Distribution: $p(v) = A v^2 \exp(-a v^2)$

$$a = \frac{m}{2 k_B T}$$

$$\Rightarrow 1 = \int p(v)\, dv \Rightarrow A^{-1} = \int_0^\infty v^2 \exp(-a v^2)\, dv$$

$$= \frac{\sqrt{\pi}}{4 a^{3/2}}$$

$$\langle v \rangle = \int_0^\infty \frac{1}{v} p(v)\, dv = \int_0^\infty A v \exp(-a v^2)\, dv =$$

$$= \frac{A}{2a} \int_0^\infty e^{-x}\, dx = \frac{A}{2a} = \frac{4 a^{3/2}}{\sqrt{\pi}\, 2a} = \frac{2\sqrt{a}}{\sqrt{\pi}}$$

$$= \frac{2}{\sqrt{\pi}} \sqrt{\frac{m}{2 k_B T}}$$

$$\Rightarrow \frac{d^2\mathcal{E}}{d\omega\, dt} = \pi n_e \sigma_T Z^2 e^2 \frac{2}{\sqrt{\pi}} \sqrt{\frac{m}{2 k_B T}}$$ $\Rightarrow$ constant over all $\omega$ ↯

$\Rightarrow$ makes no sense

A electron only can emitt a photon, which is smaller than his own energy:

$$\cancel{\frac{d^2\mathcal{E}}{d\omega\, dt}} \quad \frac{m}{2} v_{min}^2 = \hbar\omega \Rightarrow v_{min} = \sqrt{\frac{2\hbar\omega}{m}}$$

$$\left\langle \frac{1}{v} \right\rangle = \int_{v_{min}}^\infty v\, dv\, p(v) = -\frac{A}{2a} \int_{x_{min}}^\infty e^{-x}\, dx = \frac{A}{2a} e^{-x_{min}}$$

$$\Rightarrow x_{min} = a \frac{2\hbar\omega}{m} \qquad \cancel{\Rightarrow \left\langle \frac{1}{v} \right\rangle = \frac{2}{\sqrt{\pi}} \sqrt{\frac{m}{2 k_B T}}}$$

$$\Rightarrow \left\langle \frac{1}{v} \right\rangle = \frac{2}{\sqrt{\pi}} \sqrt{\frac{m}{2 k_B T}} \exp\left(-\frac{2\hbar\omega}{m} \frac{m}{2 k_B T}\right)$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{m}{k_B T}} \exp\left(-\frac{\hbar\omega}{k_B T}\right)$$

$$\frac{d^2\mathcal{E}}{d\omega\, dt\, dV} = j_\omega = \sqrt{2\pi}\, Z^2 e^2 \sigma_T n_i n_e \sqrt{\frac{m}{k_B T}} \exp\left(-\frac{\hbar\omega}{k_B T}\right)$$

## Radiation back-reaction:

Euler's theorem on homogenous functions: $f(\lambda \bar{x}) = \lambda^k f(\bar{x})$

$$\Rightarrow \quad \bar{x}\,\bar{\nabla} f(\bar{x}) = k f(\bar{x})$$

Evidence: $\dfrac{\partial f(\lambda\bar{x})}{\partial(\lambda\bar{x})}\dfrac{\partial(\lambda\bar{x})}{\partial\lambda} = \bar{x}\dfrac{\partial f}{\partial(\lambda\bar{x})}(\lambda\bar{x}) = k\lambda^{k-1} f(\bar{x})$

with $\lambda = 1$ the problem follows

$P$ (Larmour) homogenous $k=2$ in $\dot{\bar{\beta}}$

$$\Rightarrow \quad \dot{\bar{\beta}}\frac{\partial P}{\partial\dot{\bar{\beta}}} = 2P$$

$$\mathcal{E} = \int P\,dt = -\int \bar{F}_{rad}\,\bar{v}\,dt$$

$$= \frac{1}{2}\int \dot{\bar{\beta}}\frac{\partial P}{\partial\dot{\bar{\beta}}}dt = -\frac{1}{2}\int \bar{\beta}\frac{d}{dt}\frac{\partial P}{\partial\dot{\bar{\beta}}}dt$$

$$\Rightarrow \boxed{\bar{F}_{rad} = \frac{1}{2c}\frac{d}{dt}\frac{\partial P}{\partial\dot{\bar{\beta}}}}$$

Nonrelativistic: $P = \dfrac{2e^2}{3c}\dot{\bar{\beta}}^2 \Rightarrow \boxed{\bar{F}_{rad} = \dfrac{2e^2}{3c^2}\ddot{\bar{\beta}}}$

Example: harmonically bound $e^-$ (Näherungsweise jedes Elektron)

$$\ddot{\bar{x}} + \omega_0^2\bar{x} = 0 \qquad \bar{x} = \bar{x}_0 e^{i\omega_0 t}$$

$$\Rightarrow \ddot{\bar{\beta}} = -\omega_0^2\bar{\beta} \qquad \Rightarrow \bar{F}_{rad} = -\frac{2e^2}{3c^2}\omega_0^2\bar{\beta}$$

$$= -\frac{2e^2}{3c^3}\omega_0^2\dot{\bar{x}}$$

New differential equation: $\ddot{\bar{x}} + \gamma\dot{\bar{x}} + \omega_0^2\bar{x} = 0$

$$\gamma = \frac{2e^2}{3mc^3}\omega_0^2$$

incoming electromagnetic wave: $\ddot{\bar{x}} + \gamma\dot{\bar{x}} + \omega_0^2\bar{x} = -\frac{e}{m}\bar{E}_0 e^{i\omega t}$

$$\bar{x} = \bar{x}_0 e^{i\omega t} \qquad (-\omega^2 + i\omega\gamma + \omega_0^2)\bar{x}_0 = -\frac{e}{m}\bar{E}_0$$

with: $\bar{x}(t) = -\dfrac{e}{m}\bar{E}_0 e^{i\omega t}\dfrac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}$

$$P = \frac{2e^2}{3c}\dot{\bar{\beta}}^2 = \frac{2e^4 E_0^2}{3c^3 m^2}\frac{\omega^4}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$

Crosssection: $\sigma(\omega) = \dfrac{P}{\frac{c}{4\pi}|E_0|^2} = \dfrac{8\pi}{3}\dfrac{e^4}{(mc^2)^2}\dfrac{\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$ total

incoming power $\qquad = \sigma_T\dfrac{\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$

$\omega \gg \omega_0 \gg \gamma: \quad \sigma \to \sigma_T$

$\omega_0 \gg \omega \gg \gamma: \quad \sigma \to \sigma_T \left(\frac{\omega}{\omega_0}\right)^4$ Rayleigh scattering

$\omega \approx \omega_0 \gg \gamma: \quad \sigma \to \sigma_T \frac{\omega_0^4}{4\omega_0^2(\omega_0-\omega)^2 + \omega_0^2\gamma^2}$

$= \sigma_T \frac{\omega_0^2}{4(\omega_0-\omega)^2+\gamma^2}$ → Lorentz profile

with $\omega_0^2 - \omega^2 = (\omega_0-\omega)(\omega_0+\omega) = 2\omega_0(\omega_0-\omega)$

## Random Electromagnetic Field:

$\langle E_i \rangle = 0 = \langle B_j \rangle$

$\langle E_i E_j \rangle = 0 = \langle B_i B_j \rangle$ for $i \neq j$

$\langle E_i B_j \rangle = 0$ for $i \neq j$

But $\langle E_i^2 \rangle = \frac{4\pi U}{3} = \langle B_j^2 \rangle$ with $U$... Energy density

because: $\frac{1}{4\pi}\langle E_i^2 \rangle = U$

rest frame of E.M. cloud

Rest frame of $e^-$: $F \to \Lambda F \Lambda^T$

$$\left.\begin{aligned} E_x' &= \gamma(E_x + \beta B_y) \\ E_y' &= \gamma(E_y - \beta B_x) \\ E_z' &= E_z \end{aligned}\right\} \ddot{x}' = -\frac{e}{m} E'$$

Larmor Power ~~when~~ non relativistic:

$$P' = \frac{2e^2}{3c^3}\langle |\ddot{x}'|^2 \rangle = \frac{2e^4}{3c^3 m^2}\langle (E_x')^2 + (E_y')^2 + (E_z')^2 \rangle$$

$$= \frac{2e^4}{3m^2c^3}\gamma^2 \langle E_x^2 + \underbrace{2\beta E_x B_y + \beta^2 B_y^2 + E_y^2 - 2\beta E_y B_x}_{=0 \text{ because } \langle E_i B_j \rangle} + \beta^2 B_x^2 + \left(\frac{E_z}{\gamma}\right)^2 \rangle$$

$$= \frac{8\pi}{3}\frac{e^4}{m^2c^4} c\gamma^2 U \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3\gamma^2} + \frac{\beta^2}{3} + \frac{\beta^2}{3}\right)$$

$$= \sigma_T c U \gamma^2 \left(1 + \frac{\beta^2}{3}\right)$$

Power is Lorentz invariant:

$$\boxed{P_{em} = \sigma_T c U \gamma^2 \left(1 + \frac{\beta^2}{3}\right)}$$

absorbed Power: $P_{ab} = \sigma_T |\bar{S}| = \sigma_T c \frac{|E|^2}{4\pi} = c \sigma_T U$

net Power: $P = P_{em} - P_{ab} = c \sigma_T U \gamma^2 \left(1 + \frac{\beta^2}{3} - \frac{1}{\gamma^2}\right)$

$$\boxed{P = \frac{4}{3} c \sigma_T U \beta^2 \gamma^2}$$

$E^2 = (mc^2)^2 + (pc)^2$

$p_0 = E/c$

$p = m\gamma\beta c$

Energy loss by $e^-$: $\frac{dE}{dt} = -P = -\frac{4}{3} c\sigma_T U(\gamma^2 - 1)$

$= mc^2 \frac{d\gamma}{dt}$ with relativistic Energy: $E = m\gamma c^2$

$\Rightarrow \frac{d\gamma}{\gamma^2 - 1} = -\frac{4}{3} \frac{c\sigma_T U}{mc^2} dt = -\frac{dt}{\tau}$

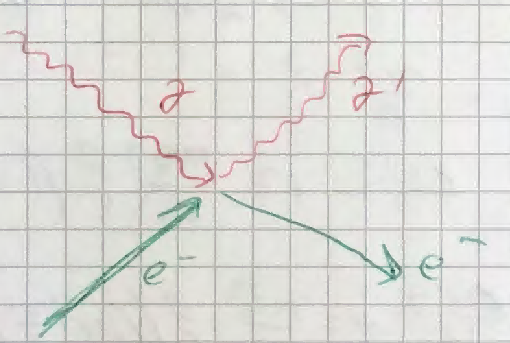
with $\tau = \frac{3mc}{4\sigma_T U}$ → definition of characteristic time

$\int \frac{1}{2}\left(\frac{1}{\gamma - 1} - \frac{1}{\gamma + 1}\right) d\gamma = -\frac{t}{\tau} = \frac{1}{2} \ln\left(\frac{\gamma - 1}{\gamma + 1}\right)$

$\Rightarrow \frac{\gamma - 1}{\gamma + 1} = \exp\left(-\frac{2t}{\tau}\right) \rightsquigarrow \boxed{\beta(t) = \frac{2\exp(-\frac{t}{\tau})}{1 + \exp(-\frac{2t}{\tau})}}$

characteristic length: $\lambda = c\tau = \frac{3mc^2}{4\sigma_T U}$

## Compton scattering




$p^\mu + \hbar k^\mu = (p^\mu)' + \hbar (k^\mu)'$

$k^\mu = \frac{\omega}{c}\begin{pmatrix} 1 \\ \bar{e} \end{pmatrix} \quad |\bar{k}| = \frac{\omega}{c}$

$k^\mu k_\mu = 0 = \left(\frac{\omega}{c}\right)^2 (-1 + \bar{e}^2)$

$p^\mu = \gamma m \begin{pmatrix} c \\ \bar{v} \end{pmatrix} = \begin{pmatrix} E/c \\ \bar{p} \end{pmatrix}$

$p^\mu p_\mu = \bar{p}^2 - \frac{E^2}{c^2} = (\gamma m)^2 (v^2 - c^2)$

$\Rightarrow E^2 = c^2 p^2 + m^2 c^4$

first equation: $\mu = 0$: $E + \hbar\omega = E' + \hbar\omega'$

$\mu = i$: $\bar{p} + \frac{\hbar\omega}{c} \bar{e} = \bar{p}' + \frac{\hbar\omega'}{c} \bar{e}'$

$\Rightarrow p'^2 = p^2 + \frac{2\hbar}{c}(\omega\bar{e} - \omega'\bar{e}')\bar{p} + \left(\frac{\hbar}{c}\right)^2 (\omega\bar{e} - \omega'\bar{e}')^2$

→ with relativistic energy: $E'^2 = E^2 + 2\hbar c(\omega\bar{e} - \omega'\bar{e}')\bar{p} + \hbar^2(\omega\bar{e} - \omega'\bar{e}')^2$

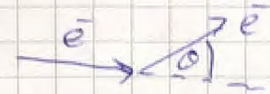
with $E' = E + \hbar(\omega - \omega')$

$E^2 + \hbar^2(\omega - \omega')^2 + 2E\hbar(\omega - \omega') = E^2 + 2\hbar c(\omega\bar{e} - \omega'\bar{e}')\bar{p} + \hbar^2(\omega\bar{e} - \omega'\bar{e}')^2$

$$\Rightarrow 2E\hbar(\omega-\omega') = 2\hbar c(\omega\bar{e}-\omega'\bar{e}')\bar{p} + 2\hbar^2(\omega\omega' - \omega\omega'\bar{e}\bar{e}')$$

$$\boxed{E(\omega-\omega') = c(\omega\bar{e}-\omega'\bar{e}')\bar{p} + \hbar\omega\omega'(1-\cos\theta)}$$

with:




rest frame of $e^-$ prior to scattering: $\bar{p}=0 \quad E=mc^2$

$$\Rightarrow \frac{\omega-\omega'}{\omega'} = \frac{\hbar\omega}{mc^2}(1-\cos\theta)$$

$$\Rightarrow \boxed{\frac{\omega'}{\omega} = \frac{1}{1+\varepsilon(1-\cos\theta)} \quad \text{with } \varepsilon = \frac{\hbar\omega}{mc^2}}$$

Average frequency change: $\langle \frac{\omega'}{\omega} \rangle_\theta = \frac{r_e^2}{2\sigma_T}\int (1+\cos^2\theta)\frac{1}{1+\varepsilon(1-\cos\theta)}d\Omega$

$$= \frac{\pi r_e^2}{\sigma_T}\int_{-1}^{+1}\frac{(1+\mu^2)}{1+\varepsilon(1-\mu)}d\mu$$

$$\int_{-1}^{1}\frac{(1+\mu^2)}{1+\varepsilon(1-\mu)}d\mu = \frac{1}{\varepsilon}\int_{\alpha-1}^{\alpha+1}\frac{1+(\alpha-x)^2}{x}dx = \frac{1}{\varepsilon}\int_{\alpha-1}^{\alpha+1}\frac{(1+\alpha^2) - 2\alpha x + x^2}{x}d\mu$$

$$= \frac{1}{\varepsilon}\left((1+\alpha^2)\ln\left(\frac{\alpha+1}{\alpha-1}\right) - 4\alpha + 2\alpha\right)$$

$$= \frac{1}{\varepsilon}\left((1+\alpha^2)\ln\left(\frac{\alpha+1}{\alpha-1}\right) - 2\alpha\right)$$

with $\mu = \alpha - x \quad \alpha = \frac{1+\varepsilon}{\varepsilon}$

$$\frac{\langle \Delta E_\gamma \rangle}{E_\gamma} = \frac{\hbar\langle\omega'-\omega\rangle}{\hbar\omega} = \langle\frac{\omega'}{\omega}\rangle - 1 = \frac{\pi r_e^2}{\sigma_T}\frac{1}{\varepsilon}\left((1+\alpha^2)\ln\left(\frac{\alpha+1}{\alpha-1}\right) - 2\alpha\right) - 1$$

$$\frac{\alpha+1}{\alpha-1} = 1+2\varepsilon \rightarrow \langle\frac{\omega'}{\omega}\rangle - 1 = \frac{\pi r_e^2}{\sigma_T}\frac{1}{\varepsilon}\left(\left(1+\left(\frac{1+\varepsilon}{\varepsilon}\right)^2\right)\ln(1+2\varepsilon) - \frac{2(1+\varepsilon)}{\varepsilon}\right)$$

You have to expand $\ln(1+2\varepsilon)$ to the order of 4:

$$\ln(1+2\varepsilon) = 2\varepsilon - \frac{(2\varepsilon)^2}{2} + \frac{(2\varepsilon)^3}{3} - \frac{(2\varepsilon)^4}{4} + O(\varepsilon^5)$$

because then you get a linear term because of $\sim \frac{1}{\varepsilon}\left(1+\left(\frac{1}{\varepsilon}+1\right)^2\right)\ln(\ldots)$

$$\sim \frac{1}{\varepsilon^3}\ln(\ldots)$$

We get $\frac{\langle \Delta E_\gamma \rangle}{E_\gamma} = -\varepsilon = -\frac{\hbar\omega}{mc^2}$

We go to electrons: $P_\omega = \frac{4}{3}\beta^2\gamma^2 c\sigma_T u_\omega \quad u_\omega = \frac{dU_\omega}{dx}$

with $U = \int u_\omega d\omega \rightarrow u_\omega$ inner energy by one $\omega$

$$\frac{1}{n_\gamma}\frac{dU_\omega^+}{dt} = \frac{4}{3}\beta^2\gamma^2 c\sigma_T \hbar\omega$$

with $n_\gamma = \frac{U_\omega}{\hbar\omega}$ → energy gain per photon

$$\int \frac{4}{3}\beta^2\gamma^2 c\sigma_T\hbar\omega\, n_e(\beta)\, d\beta = \frac{4}{3}\langle\beta^2\gamma^2\rangle c\sigma_T\hbar\omega n_e = \frac{dE_\gamma^+}{dt}$$

$$\frac{dE_\gamma^-}{dt} = \langle\Delta E_\gamma\rangle n_e c\sigma_T = -\frac{(\hbar\omega)^2}{mc^2} n_e c\sigma_T$$

with $\langle\beta^2\gamma^2\rangle = \frac{1}{n_e}\int \beta^2\gamma^2 n_e(\beta)\, d\beta$

net energy gain per photon with freq. $\omega$ per unit volume per unit time:

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{4}{3}\langle\beta^2\gamma^2\rangle c\sigma_T\hbar\omega n_e - \frac{(\hbar\omega)^2}{mc^2} n_e c\sigma_T$$

$$= c\sigma_T\hbar\omega n_e\left(\frac{4}{3}\langle\beta^2\gamma^2\rangle - \frac{\hbar\omega}{mc^2}\right)$$

photons $T_\gamma$ $\quad n_\gamma(T_\gamma) = \frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \frac{x^2}{e^x-1}\, dx$

$$x = \frac{\hbar\omega}{k_B T_\gamma} \qquad \int_0^\infty \frac{x^n}{e^x-1}\, dx = n!\,\zeta(n+1)$$

$\langle\hbar\omega\rangle$, $\langle(\hbar\omega)^2\rangle$

$$\langle\hbar\omega\rangle = k_B T_\gamma \frac{6}{2}\frac{\zeta(4)}{\zeta(3)} \qquad \langle(\hbar\omega)^2\rangle = (k_B T_\gamma)^2 \frac{24\,\zeta(5)}{2\,\zeta(3)}$$

$$\langle\beta^2\gamma^2\rangle \approx \langle\beta^2\rangle = \frac{\langle v^2\rangle}{c^2} = \frac{\langle E_{kin}\rangle}{mc^2}2$$

$$= \frac{2}{mc^2}\frac{3}{2}k_B T_e = \frac{3k_B T}{mc^2}$$

$$\left\langle\frac{dE}{dt}\right\rangle = c\sigma_T n_e\left(\frac{4}{\cancel{3}}\frac{\cancel{3}k_B T}{mc^2}\frac{3\zeta(4)}{\zeta(3)} - \frac{(k_B T_\gamma)^2}{mc^2}12\frac{\zeta(5)}{\zeta(3)}\right)$$

$$= c\sigma_T n_e 12\frac{(k_B T_e)(k_B T_\gamma)}{mc^2}\frac{\zeta(4)}{\zeta(3)}\left(1 - \frac{T_\gamma}{T_e}\frac{\zeta(5)}{\zeta(4)}\right)$$

Because $\frac{\zeta(5)}{\zeta(4)} \neq 1$ $\left\langle\frac{dE}{dt}\right\rangle = 0$ for $T_\gamma \neq T_e$

⇒ is against the energy equality

⇒ we have to ~~[illegible]~~ respect the chemical potential

for photons: $n(\omega) = \frac{1}{e^{\frac{\hbar\omega}{k_B T}}-1} \longrightarrow \frac{1}{e^{\frac{1}{k_B T}(\hbar\omega-\mu)}-1}$

we can calculate $\mu$ out of $\left\langle\frac{dE}{dt}\right\rangle = 0$ for $T_\gamma = T_e$

# Spectral lines

$E \sim \alpha m c^2$

$|\bar{p}| \sim \alpha m c$ with $\alpha = \frac{1}{137} = \frac{e^2}{\hbar c}$ } non-relativistic !

Quantumechanical Perturbation Theory:

$P_{fi}$ … Transition probability

$$P_{fi} = \frac{1}{\hbar^2} \left| \int dt \, \langle f | H_1 | i \rangle e^{i\omega_{fi} t} \right|^2$$

$\hat{H} = \hat{H}_0 + \hat{H}_1 + \dots$ with $|i\rangle$ and $|f\rangle$ are Eigenstates to $\hat{H}_0$

$$\omega_{fi} = \frac{E_f - E_i}{\hbar}$$

with $[\hat{\vec{x}}, \hat{\vec{p}}^2] = \hat{\vec{p}} [\hat{\vec{x}}, \hat{\vec{p}}] + [\hat{\vec{x}}, \hat{\vec{p}}] \hat{\vec{p}} = 2 i \hbar \hat{\vec{p}}$

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{x}})$$

$$[\hat{\vec{x}}, \hat{H}_0] = \frac{1}{2m} [\hat{\vec{x}}, \hat{\vec{p}}^2] = \frac{i\hbar}{m} \hat{\vec{p}}$$

$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \Rightarrow \vec{E}_\omega = -\frac{1}{c}(-i\omega) \vec{A}_\omega$ Fouriertrafo

$\Rightarrow |\vec{A}_\omega| = \frac{c}{\omega} |\vec{E}_\omega| \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

$$= \frac{\lambda |\vec{E}_\omega|}{2\pi}$$

relativistic Lagrangian: $\mathcal{L} = -mc^2 \sqrt{1-\beta^2}$

$$\bar{p} = \frac{\partial \mathcal{L}}{\partial v} = \frac{mc^2}{\sqrt{1-\beta^2}} \frac{\vec{\beta}}{c} = \gamma m \bar{v}$$

for electron in a field: $\mathcal{L} = \sqrt{1-\beta^2} \left( \frac{q}{c} A^\mu u_\mu - mc^2 \right)$

$$A^\mu = \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix} \quad u^\mu = \gamma \begin{pmatrix} c \\ v \end{pmatrix} \quad A^\mu u_\mu = \gamma (\vec{A}\bar{v} - c\phi)$$

$$\Rightarrow \mathcal{L} = -\sqrt{1-\beta^2}\, mc^2 - q\phi + \frac{q}{c} \vec{A} \bar{v}$$

$$\Rightarrow \bar{p} = \frac{\partial \mathcal{L}}{\partial v} = \bar{p} + \frac{q}{c} \vec{A}$$

$$H = \bar{p}\bar{v} - L = \bar{p} \frac{1}{\gamma m} \left( \bar{p} - \frac{q}{c} \vec{A} \right) + \frac{mc^2}{\gamma} + q\phi$$

$$- \frac{q}{c} \vec{A} \left( \bar{p} - \frac{q}{c} \vec{A} \right) \frac{1}{\gamma m}$$

$$= \underbrace{\frac{1}{\gamma m} \left( \bar{p} - \frac{q}{c} \vec{A} \right)^2}_{} + \frac{mc^2}{\gamma} + q\phi$$

$\frac{1}{\gamma} = 1 - \frac{1}{2} \bar{\beta}^2$ → we can estimate $\gamma \approx 1$ because the order of $\beta^2$ would be

$\gamma \sim \beta^4$ → very small

$$= \frac{1}{m}\left(\bar{p} - \frac{q}{c}\bar{A}\right)^2 + mc^2\left(1 - \frac{1}{2}\bar{\beta}^2\right) + q\phi$$

$$= \frac{1}{2m}\left(\bar{p} - \frac{q}{c}\bar{A}\right)^2 + q\phi + mc^2 \quad \text{with } \bar{v}^2 = \frac{1}{m^2}\left(\bar{p} - \frac{q}{c}\bar{A}\right)$$

$$\hat{H} = \frac{1}{2m}\underbrace{\left(\bar{p} - \frac{q}{c}\bar{A}\right)^2}_{\bar{p}^2 - \frac{q}{2mc}(\bar{p}\bar{A} + \bar{A}\bar{p}) + \frac{q^2}{c^2}\bar{A}^2} + mc^2 + q\phi \quad \text{with gauge } \phi = 0$$

$$\bar{p} \longrightarrow -i\hbar\bar{\nabla}$$

$$\hat{H} = \frac{\bar{p}^2}{2m} - \frac{q}{2mc}\left(\overbrace{(-i\hbar\bar{\nabla}\bar{A})}^{=0 \text{ out of gauge}} + 2\bar{A}\bar{p}\right) + \frac{q^2}{c^2}\bar{A}^2\frac{1}{2m}$$

because $\bar{p}\bar{A}|\psi\rangle = \bar{p}\bar{A}|\psi\rangle + \bar{A}\bar{p}|\psi\rangle$

$$\hat{H}_1 = -\frac{q}{2mc}\bar{A}\bar{p} + \frac{q^2}{2mc^2}\bar{A}^2$$

$$\Rightarrow \eta = \frac{\frac{e^2}{2mc}|\bar{A}^2|}{\frac{e}{mc}\bar{A}\bar{p}} \sim \frac{e}{2c}\frac{A}{p} = \frac{e}{2c}\left(\frac{\lambda E}{2\pi}\right)\frac{1}{\alpha mc} = \frac{\lambda e E}{4\pi\alpha mc^2} \ll$$

$$\Rightarrow \boxed{\hat{H}_1 = \frac{e}{mc}\bar{A}\bar{p}} \quad \text{perturbing Hamilton operator}$$

$$\bar{A} = A(t)\bar{e}\,e^{i\bar{k}\bar{x}}$$

$$P_{fi} = \frac{1}{\hbar^2}\left|\int dt\, \langle f|\hat{H}_1|i\rangle e^{i\omega_{fi}t}\right|^2$$

$$= \frac{1}{\hbar^2}\left(\frac{e}{mc}\right)^2 \underbrace{\left|\int dt\, A(t)e^{i\omega_{fi}t}\right|^2}_{|A(\omega_{fi})|^2} |\langle f|\underbrace{e^{i\bar{k}\bar{x}}}\bar{e}\hat{p}|i\rangle|^2$$

$\approx 1 + i\bar{k}\bar{x} + \dots$

$\approx 1$ because $\bar{x}$ is small to $\bar{k} \sim \frac{2\pi}{\lambda}$

$\Rightarrow$ dipole approximation

$$\langle f|\bar{e}\hat{p}|i\rangle = \langle f|\left(\frac{m}{i\hbar}\right)[\bar{e}\bar{x}, \hat{H}_0]|i\rangle = \frac{m}{i\hbar}(E_i - E_f)\langle f|\bar{e}\hat{\bar{x}}|i\rangle$$

with $2m[\bar{x}, \hat{H}_0] = [\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$

with $\langle f|\bar{e}\bar{x}\hat{H}_0 - \hat{H}_0\bar{e}\bar{x}|i\rangle$

$$P_{fi} = \frac{e^2}{\hbar^2 m^2 c^2}|A(\omega_{fi})|^2 m^2\omega_{fi}^2|\langle f|\bar{e}\bar{x}|i\rangle|^2 = \frac{e^2}{\hbar^2}E_\omega^2(\omega_{fi})|\langle f|\bar{e}\hat{x}|i\rangle|^2$$

with $A_\omega^2 = \left(\frac{c}{\omega}\right)^2 E_\omega^2$

$$|S| = \frac{c}{4\pi}E^2 \qquad \int \psi^* \bar{d}\psi\, dV$$

$$P_{fi} = \frac{4\pi}{c}I_{\omega_{fi}}\frac{1}{\hbar^2}|\langle f|\bar{e}\bar{d}|i\rangle|^2$$

# Cross Sections

$I_{\omega_{fi}} \rightarrow I \cdot \phi(\omega - \omega_{fi})\, d\omega$ with $\int \phi\, d\omega = 1$

number of photons per $cm^2$ and s at $\omega$ $\dfrac{I_\omega d\omega}{\hbar\omega_{fi}}$

crosssection: $\sigma_{fi} = \dfrac{P_{fi}}{\left(\frac{I_\omega d\omega}{\hbar\omega}\right)} = 4\pi \dfrac{\hbar\omega_{fi}}{\hbar^2 c} \phi(\omega - \omega_{fi}) |\underbrace{\langle f|\bar{e}\bar{d}|i\rangle}_{\propto \cos^2\theta}|^2$

$$\Rightarrow \sigma_{fi} = \frac{4\pi}{3} \frac{\omega_{fi}}{\hbar c} \phi(\omega - \omega_{fi}) |d_{fi}|^2$$

dimensionless definition: $\boxed{\dfrac{m\omega_{fi}}{\hbar e^2} |d_{fi}|^2 \dfrac{2}{3} = f_{fi}}$ → oscillator strength

$$\Rightarrow \boxed{\sigma_{fi} = 2\pi r_e c f_{fi} \phi(\omega - \omega_{fi})}$$

$r_e = \dfrac{e^2}{mc^2}$ → electron radius

Number of final states for $e^-$: $N_e = \dfrac{V 4\pi p_f^2 dp_f}{(2\pi\hbar)^3}$

$\hbar\omega = \epsilon_1 + \dfrac{p_f^2}{2m} \Rightarrow \hbar d\omega = \dfrac{p_f}{m} dp_f$

$\Rightarrow N_e = \dfrac{V 4\pi p_f m\hbar d\omega}{(2\pi\hbar)^3}$ number of transition:

$$\boxed{\sigma_{if} = N_e p_f \frac{\hbar\omega_{fi}}{I_\omega d\omega} = \frac{2}{\pi} \frac{r_e c}{\hbar} p_f \frac{V}{\omega} |\langle f| e^{i\bar{k}\bar{x}} \bar{e}\bar{\nabla}|i\rangle|}$$

with $I_{\omega_{fi}} \rightarrow I \phi(\omega - \omega_{fi}) d\omega$

$|i\rangle$: $\psi_i(x) = \dfrac{1}{\sqrt{\pi a_0^3}} \exp(-r/a_0)$ $a_0$... Bohr radius

$|f\rangle$: $\psi_f(x) = \dfrac{1}{\sqrt{V}} \exp(-i\bar{k}\bar{x})$ $\hbar\bar{k} = \bar{p}_f$

$$\langle f| e^{i\bar{k}\bar{x}} \bar{e}\bar{\nabla} |i\rangle^* = \int d^3x \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} (\bar{e}\bar{\nabla}) \frac{1}{\sqrt{V}} e^{-i\bar{k}\bar{x}}$$

$$= \frac{-i(\bar{k}\bar{e})}{\sqrt{\pi a_0^3 V}} \int_0^{2\pi} d\varphi \int_0^\infty r^2 dr \int_{-1}^{1} d\mu\, e^{-r/a_0} e^{-ir\mu k}$$

with $\bar{k} = k\bar{e}_z$

$$= \frac{-4\pi i(\bar{k}\bar{e})}{\sqrt{\pi a_0^3 V}} \int_0^\infty r^2 dr\, e^{-r/a_0} j_0(kr) = \frac{-i 4\pi}{\sqrt{\pi a_0^3 V}} \frac{2a_0^3}{(1+k^2a_0^2)^2} (\bar{k}\bar{e})$$

$$\approx \frac{-8\pi i}{\sqrt{\pi a_0^3 V}} \frac{\bar{k}\bar{e}}{k^4 a_0}$$

$$\Rightarrow \sigma_{bf} = 128 \frac{1}{a_0^3} \frac{ec}{\hbar} \rho_f \frac{1}{\omega} \left(\frac{\hbar e}{k^4 a_0}\right)^2$$

$$= 128 \frac{1}{a_0^5} ec \frac{k}{\omega} \left(\frac{\hbar e}{k^8}\right)^2$$

$$\sim \frac{128}{3} \frac{ec}{a_0^5 k^5} \frac{1}{\omega}$$

SDGL: $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$ with $\hat{H} = \hat{H}_0 + \hat{H}_1$

and $|\psi\rangle = \sum_k a_k |k, t\rangle$ $\quad |k,t\rangle = e^{-iE_k t/\hbar} |k, 0\rangle$

↳ Time evolution

$H_0 |k, 0\rangle = E_k |k, 0\rangle$

$$i\hbar \frac{\partial}{\partial t} \sum_k a_k e^{-iE_k t/\hbar} |k,0\rangle = \sum_k (\hat{H}_0 + \hat{H}_1) a_k e^{-iE_k t/\hbar} |k, 0\rangle$$

$$\Rightarrow i\hbar \sum_k \left(\dot{a}_k e^{-iE_k t/\hbar} |k,0\rangle + a_k \left(-\frac{iE_k}{\hbar}\right) e^{-iE_k t/\hbar}\right) |k,0\rangle$$

$$= \sum_k a_k e^{-iE_k t/\hbar} E_k |k,0\rangle + \sum_k a_k e^{-iE_k t/\hbar} H_1 |k,0\rangle$$

$$\Rightarrow \sum_k i\hbar \dot{a}_k e^{-iE_k t/\hbar} |k,0\rangle = \sum_k a_k e^{-iE_k t/\hbar} H_1 |k,0\rangle$$

use orthonormality:

project $\langle j, 0|$ $\quad i\hbar \dot{a}_j e^{-iE_j t/\hbar} = \sum_k a_k e^{-iE_k t/\hbar} \langle j,0| H_1 |k,0\rangle$

$$\hat{H}_1(t) = \hat{H}_{10} e^{-i\omega t}$$

$$i\hbar \dot{a}_j = \sum_k a_k \langle j,0| \hat{H}_{10} |k,0\rangle e^{-i(\omega - \omega_{jk})t}$$

$$\Rightarrow \dot{a}_j = -\frac{i}{\hbar} \sum_k a_k \langle j,0| \hat{H}_{10} |k,0\rangle e^{-i(\omega-\omega_{jk})t}$$

$$\dot{a}_2 = -\frac{i}{\hbar} a_1 \langle 2,0| \hat{H}_{10} |1,0\rangle e^{-i(\omega - \omega_{21})t}$$

for almost all particles in ground state: $a_1 = 1$

$$\dot{a}_2 = -\frac{i}{\hbar} H_{21} e^{-i(\omega-\omega_{21})t} \qquad H_{21} = \langle 2,0| \hat{H}_{10} |1,0\rangle$$

out of QED: spontaneous decay $\dot{a}_2' = \dot{a}_2 - \frac{\Gamma}{2} a_2$

$$\Rightarrow \dot{a}_2' = -\frac{i}{\hbar} H_{21} e^{-i(\omega-\omega_{21})t} - \frac{\Gamma}{2} a_2$$

with decay rate $\Gamma$

$$e^{\Gamma/2\, t}\left(\dot a_2 + \frac{\Gamma}{2} a_2\right) = -\frac{i}{\hbar} H_{21}\, e^{-i(\omega-\omega_{21})t + \Gamma/2\, t}$$

$$\left(a_2\, e^{\Gamma/2\, t}\right)^{\cdot} = -\frac{i}{\hbar} H_{21}\, e^{-i(\omega-\omega_{21})t + \Gamma/2\, t}$$

$$a_2\, e^{\Gamma/2\, t} = -\frac{i}{\hbar} H_{21} \left.\frac{e^{-i(\omega-\omega_{12})t + \Gamma/2\, t}}{-i(\omega-\omega_{12}) + \Gamma/2}\right|_0^t$$

$$= \frac{1}{\hbar} H_{21} \frac{\exp(-i(\omega-\omega_{12})t + \Gamma/2\, t) - 1}{(\omega-\omega_{12}) + \Gamma i/2}$$

$$a_2(t) = \frac{H_{21}}{\hbar} \frac{\exp(-i(\omega-\omega_{21})t) - \exp(-\Gamma t/2)}{(\omega-\omega_{12}) + \Gamma i/2}$$

for $t \gg \Gamma^{-1}$: $|a_2|^2 = \left(\frac{H_{21}}{\hbar}\right)^2 \frac{1}{(\omega-\omega_{12})^2 + \Gamma^2/4}$

Lorentz profile: $\frac{1}{2\pi} \frac{\Gamma}{(\omega-\omega_{12})^2 + \Gamma^2/4} = \phi(\omega-\omega_{12})$

with collisions: $\Gamma \to \underbrace{\Gamma}_{\text{spontaneous decay}} + \underbrace{\Gamma_c}_{\text{collisions}}$

<u>Doppler broadening</u>: $\omega_{12} \to \omega_{12}\left(1 + \frac{v_\parallel}{c}\right)$

$$\int_{-\infty}^{\infty} \underbrace{\delta\left(\omega - \omega_{12}\left(1+\frac{v_\parallel}{c}\right)\right)}_{\text{intensity per line}} \underbrace{\frac{dv_\parallel}{\sqrt{2\pi\sigma_v^2}} \exp\left(-\frac{v_\parallel^2}{2\sigma_v^2}\right)}_{\text{probability of this velocity } v_\parallel}$$

$\delta(ax) = \frac{1}{a}\delta_D(x)$

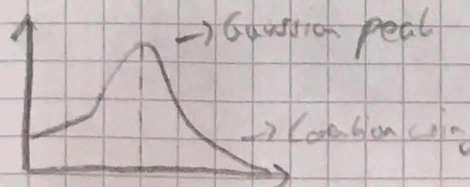
$$= \frac{1}{\sqrt{2\pi\sigma_v^2}} \frac{c}{\omega_{12}} \int_{-\infty}^{\infty} dv_\parallel\, \delta_D\left(\frac{\omega-\omega_{12}}{\omega_{12}} c - v_\parallel\right) \exp\left(-\frac{v_\parallel^2}{2\sigma_v^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_v^2}} \frac{c}{\omega_{12}} \exp\left(-\frac{c^2}{\sigma_v^2}\left(\frac{\omega-\omega_{12}}{\omega_{12}}\right)^2\right)$$

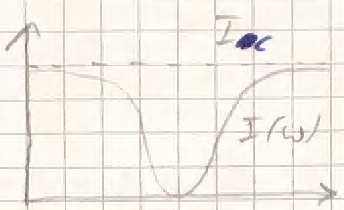
Now the same with solution above:

$$\int \frac{dv_\parallel}{\sqrt{2\pi\sigma_v^2}} \exp\left(-\frac{v_\parallel^2}{2\sigma_v^2}\right) \frac{\Gamma}{2\pi\left(\left(\omega - \omega_{12}\left(1+\frac{v_\parallel}{c}\right)\right) + \Gamma^2/4\right)}$$

$$= A \int_{-\infty}^{\infty} dq \frac{e^{-q^2}}{(\omega-q)^2 + a^2} \rightsquigarrow \text{Voigt function profile}$$



## Absorption:



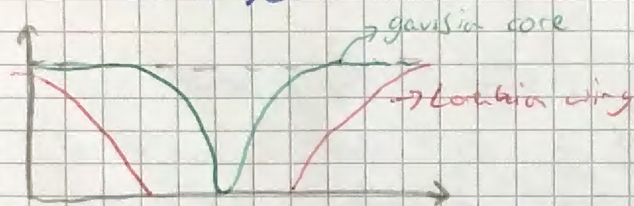

equivalent width:

$$W = \int \frac{I_c - I(\omega)}{I_c}\, d\omega$$
$$= \int \left(1 - \frac{I}{I_c}\right) d\omega$$

optical depth: $I = I_0 e^{-\tau}$

$$\tau = \underbrace{N}_{\text{column density}} \sigma = L \sigma n \rightarrow \text{optical depth}$$

$$\tau = N\sigma_0 \phi(\omega - \omega_0)$$
$$W = \int (1 - e^{-\tau})\, d\omega \qquad \text{for } \tau \ll 1: \quad W = \int \tau\, d\omega = N\sigma_0$$
$$W \propto N$$




Case 1: Gaussian core

$$1 = \tau \sim N\sigma_0 \exp\left(-\frac{\Delta^2}{2\sigma^2}\right) \Rightarrow W = 2\Delta$$
$$\Rightarrow \sqrt{-\ln \frac{1}{N\sigma_0}} \propto \Delta \Rightarrow W \propto \sqrt{\ln N}$$

Case 2: Lorentian wings:

$$1 - \tau \sim N\sigma_0 \frac{1}{\Delta^2} \Rightarrow \sqrt{N} \propto W$$

## Radiation Quantities:

### specific intensity:

$$dE = \frac{dV p^2 dp\, d\Omega}{(2\pi\hbar)^3} n_{p\alpha} \hbar\omega = \frac{p^2 dp\, d\Omega}{(2\pi\hbar)^3} n_{p\alpha} \hbar\omega\, dA \cos\theta\, c\, dt$$

with $p = \frac{\hbar\omega}{c}$ : $= \frac{\hbar\omega^3 d\omega}{(2\pi)^3 c^2} n_{p\alpha}\, d\Omega\, dA\, dt \cos\theta$

$$\Rightarrow \frac{dE}{dt\, dA\, d\omega\, d\Omega} = \underbrace{\frac{\hbar\omega^3}{(2\pi)^3 c^2} n_{p\alpha}}_{I_{\omega\alpha}} \cos\theta$$

Sum over spin-states $\alpha$ : $I_\omega = \frac{\hbar\omega^3}{4\pi^3 c^2} n_p \rightarrow$ specific intensity

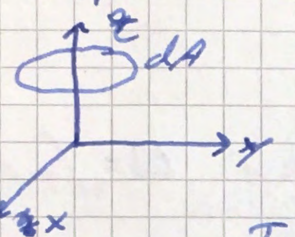
Poynting vector: $\vec{I} = \frac{c}{4\pi} \vec{E}^2 \vec{e}_k = c\, u\, \vec{e}_k$

$\frac{|S|}{4\pi} = I = \frac{cu}{4\pi}$ ; $\int \frac{I}{c}\, d\Omega = u$

$$\int_{\partial V} \bar{S}\, d\bar{A} = \int c u \underbrace{\bar{e}_k\, \bar{e}_r}_{\cos\theta} R^2\, d\Omega = \cancel{4\pi}\, c \int u \cos\theta\, d\Omega\, R^2$$

$$\Rightarrow \quad F = c \int u \cos\theta\, d\Omega = 4\pi \int I \cos\theta\, d\Omega$$

maybe Factor $\frac{1}{4\pi}$ ?




$dF_i = T_{ij}\, dA^j$

$dF_3 = T_{3j}\, dA\, \delta_3^{\,j}$ → specific $dA$

$T_{ij} = \frac{1}{2\pi}\left(\frac{E^2}{2}\delta_{ij} - E_i E_j\right)$

$dF_3 = \frac{1}{2\pi}\left(\frac{E^2}{2} - E_3^2\right) dA$

$= \frac{1}{4\pi} E_3^2\, dA$ with $E_1^2 = E_2^2 = E_3^2$

$dP = \frac{1}{4\pi} E_3^2 = \frac{E^2 \cos^2\theta}{4\pi} = u \cos^2\theta$

↳ radiation pressure $\Rightarrow P = \int u \cos^2\theta\, d\Omega$