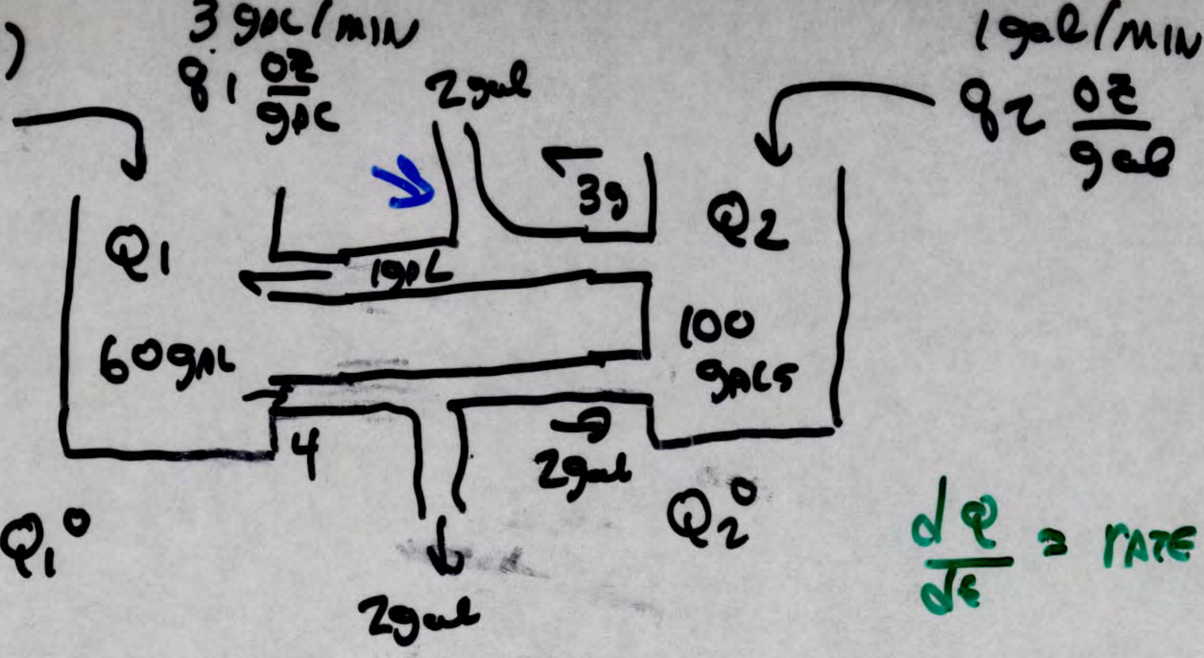
(1)



$$\frac{dQ}{dt} = \text{RATE IN} - \text{RATE OUT}$$

TANK 1 $$\boxed{\frac{dQ_1}{dt} = 3q_1 + \frac{Q_2}{100} - \frac{Q_1}{15}}$$

TANK 2 $$\boxed{\frac{dQ_2}{dt} = q_2 + \frac{Q_1}{30} - \frac{Q_2}{100}(3)}$$

(b)

E.Q

$$0 = 3q_1 + \frac{Q_2}{100} - \frac{Q_1}{15}$$

$$0 = q_2 + \frac{Q_1}{30} - 3\frac{Q_2}{100}$$

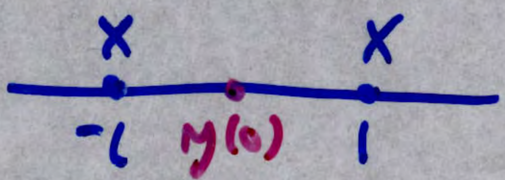
$$\begin{bmatrix} -\frac{1}{15} & \frac{1}{100} \\ \frac{1}{30} & -\frac{3}{100} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -3q_1 \\ -q_2 \end{bmatrix}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -18 & -6 \\ -20 & -40 \end{bmatrix} \begin{bmatrix} -3q_1 \\ -q_2 \end{bmatrix}$$

$$Q_{1_E} = 54q_1 + 6q_2$$

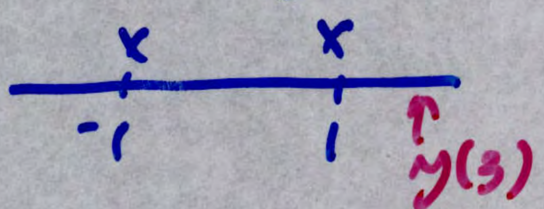
$$Q_{2_E} = 60q_1 + 40q_2$$

a) $\dot{y} + \frac{1}{\sqrt{1-t^2}}\, y = \sin(t) \qquad y(0) = 2$



$(-1\ ,\ 1)$

b) $y(3) = 2$



$\cancel{(1, \infty)}$

SAME EQ

NOT possible
SQUART - NEG

$(-\infty\ -1)$ IC < -1 other case

3a $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x$ $\quad \overbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}^{A-\lambda I} = \begin{bmatrix} -\lambda & 1 \\ -2 & -(3+\lambda) \end{bmatrix}$

$$|A - \lambda I| = \lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) \qquad \lambda_1 = -1$$
$$\lambda_2 = -2$$

$(A - \lambda_i I) x_i = 0$

$\lambda = -1$ $\quad \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 + x_2 = 0$$
$$x_1 = -x_2$$

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad x_2 = 1$$

$\lambda = -2$ $\quad \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad 2x_1 + x_2 = 0$$
$$x_1 = -\frac{1}{2} x_2$$

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad x_2 = 2$$

3b $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x$ $\quad Ax = \lambda x$

a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ NOT

b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ EV λ_1

c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$ NO

d) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ EV λ_2

$\lambda_2 = -2$

$\lambda_1 = -1$

4a $\ddot{y} + 13\dot{y} + 30y = 0 \quad y(0) = 15 \quad \dot{y}(0) = -28$

$$\boxed{x_1 = y \qquad x_2 = \dot{y}}$$

$$\boxed{\dot{x}_1 = \dot{y} = x_2} \qquad \underline{\dot{x}_2 = \ddot{y}}$$

$$\dot{x}_2 = \ddot{y} = -13\dot{y} - 30y$$

$$\dot{x}_2 = -13x_2 - 30x_1$$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -30 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 15 \\ -28 \end{bmatrix}}$$

4 b $\ddot{y} + 4\dot{y} + y^2 = \sin(t)$ $y(0)=1$ $\dot{y}(0)=2$

$x_1 = y$ $x_2 = \dot{y}$

$\boxed{\dot{x}_1 = \dot{y} = x_2}$ $\dot{x}_2 = \ddot{y} = \sin(t) - 4\dot{y} - y^2$

$\dot{x}_2 = \sin(t) - 4x_2 - x_1^2$

$$\boxed{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(t) - 4x_2 - x_1^2 \end{bmatrix} \quad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Suppose $\dot{x}_1 = x_2$
$\dot{x}_2 = \sin(t) - 4x_2 - x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

Prob 5 $\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -12 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40 \end{bmatrix}$

a) $0 = q_2$ $\qquad q(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$0 = -100 q_1 - 12 q_2 + 40$

$q_1 = 0.4 \qquad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$

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b) CONVERT TO HOMOGENEOUS

* $\boxed{\text{Let } \tilde{x}_1 = q_1 - 0.4 \qquad \tilde{x}_2 = q_2 - 0}$

$q_1 = \tilde{x}_1 + 0.4 \qquad q_2 = \tilde{x}_2$

$\dot{q}_1 = \dot{\tilde{x}}_1 \qquad \dot{q}_2 = \dot{\tilde{x}}_2$

$\dot{\tilde{x}}_1 = \tilde{x}_2$

$\dot{\tilde{x}}_2 = -100(\tilde{x}_1 + .4) - 12(\tilde{x}_2) + 40$

$-100\tilde{x}_1 \underbrace{- 40 - 12\tilde{x}_2 + 40}$

$\boxed{\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -100 & -12 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}}$

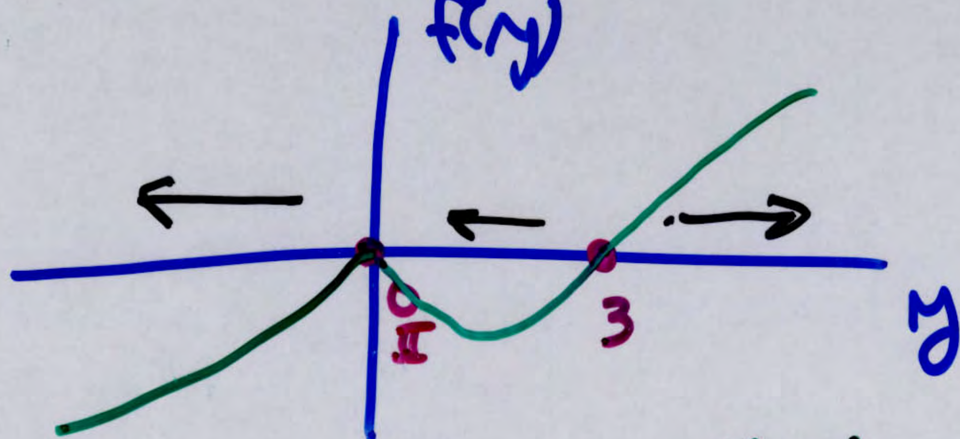
c) $\begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{bmatrix} = \begin{bmatrix} .6 \\ 5 \end{bmatrix}$

d E.P $\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \tilde{x}_1 = q_1 - .4$

$\tilde{x}_2 = q_2$

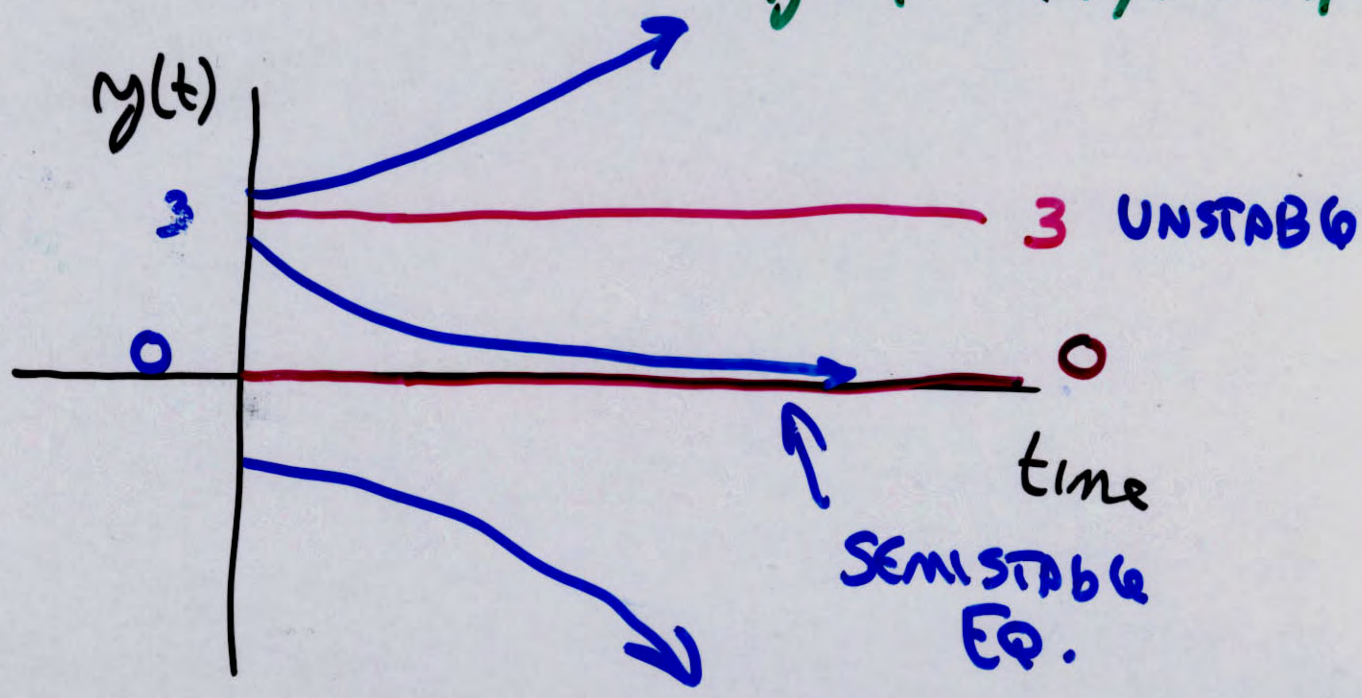
(6a) $\dot{y} = y^2(y-3) = f(y)$

E.P $0, 0 \quad 3$



$y = 1 \quad f(y) = -2$

$y = -1 \quad f(y) = -4$

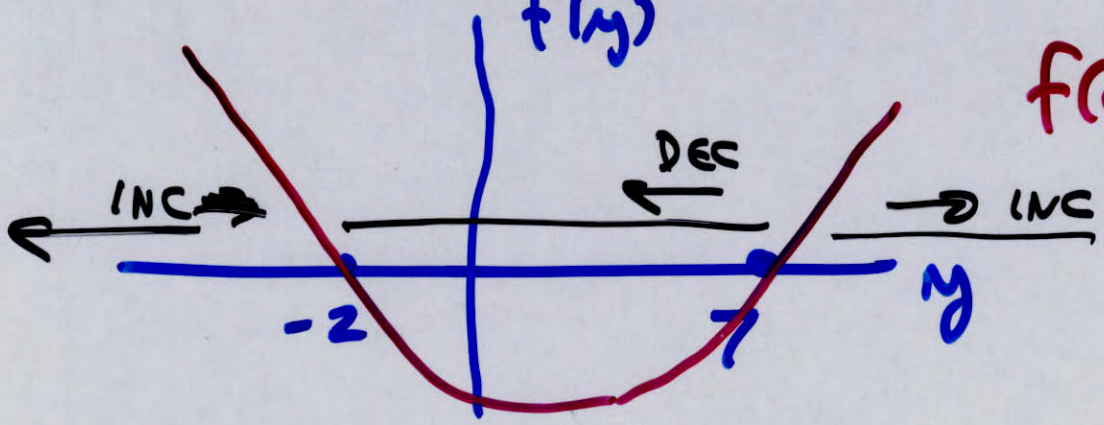


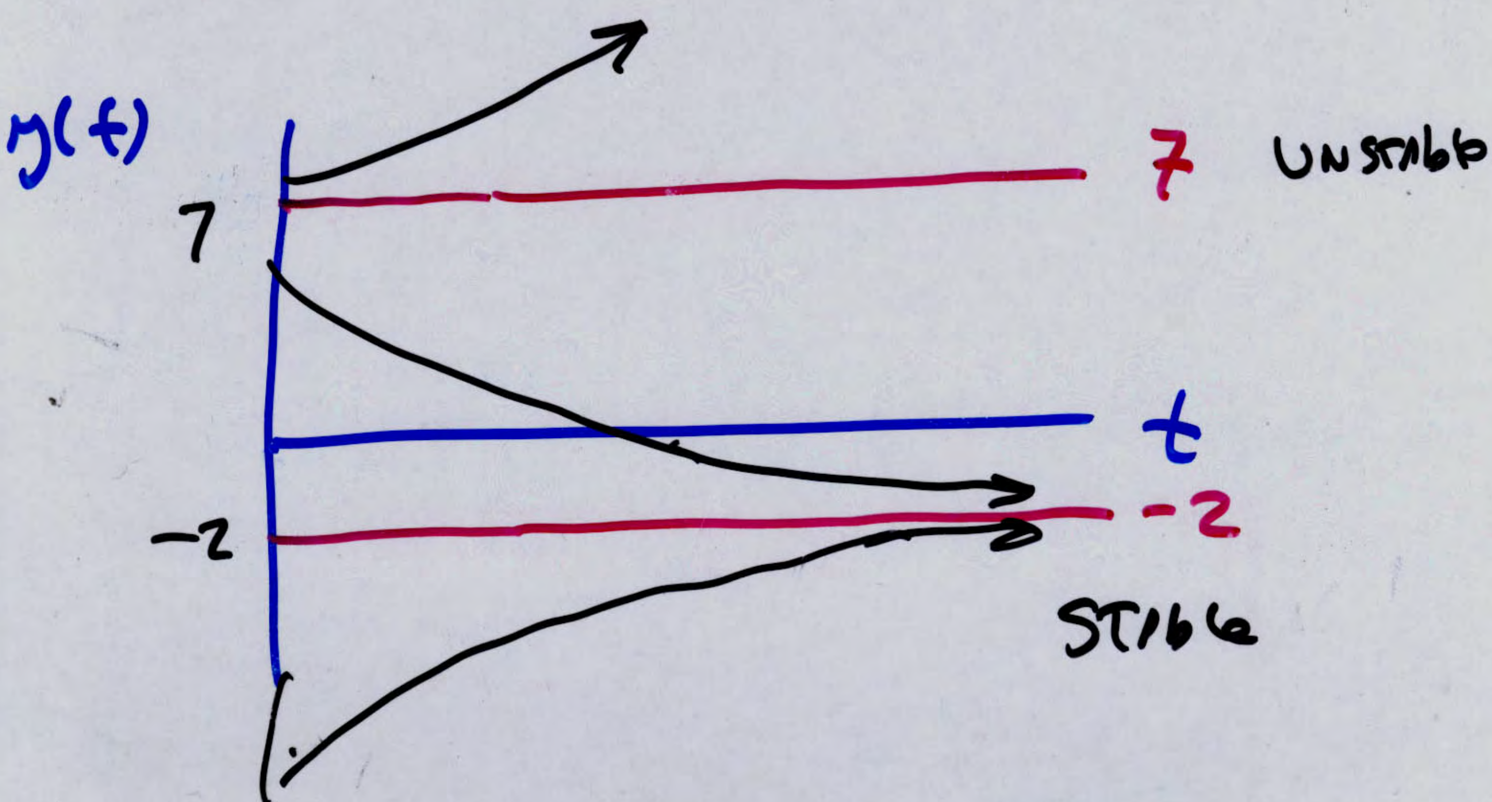
SKETCH

6b $\dot{y} = (y-7)(y+2) = f(y)$

E.P ~~?~~ 7, −2

$f(0) = -14$





$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} x \qquad \lambda_1 = -1 \quad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -5 \quad x_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-5t} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$1 = c_1 + c_2$$
$$1 = -c_1 - 5c_2$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad c_1 = 3/2 \quad c_2 = -1/2$$

$$x(t) = \frac{3}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} - \frac{1}{2} \begin{bmatrix} 1 \\ -5 \end{bmatrix} e^{-5t}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7b

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} x \qquad \lambda_{1,2} = -1 \pm 2i \qquad v_{1,2} = \begin{bmatrix} -1 \\ 1 \mp 2i \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}}$$

$$x(t) = c_1 \begin{bmatrix} -1 \\ 1-2i \end{bmatrix} e^{(-1+2i)t} + c_2 \begin{bmatrix} -1 \\ 1+2i \end{bmatrix} e^{(-1-2i)t}$$

$\boxed{@\ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ldots}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1-2i \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1+2i \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ (1-2i) & (1+2i) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solve by MATLAB or calculator through matrix inverse $\longrightarrow$ $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.5 + 0.25i \\ -0.5 - 0.25i \end{bmatrix}$

so

$$x(t) = (-0.5+0.25i)\begin{bmatrix} -1 \\ 1-2i \end{bmatrix}(e^{-t})\underbrace{(e^{(2i)t})}_{=\cos(2t)+i\sin(2t)}$$

$$+\ (-0.5-0.25i)\begin{bmatrix} -1 \\ 1+2i \end{bmatrix}(e^{-t})\underbrace{(e^{(-2i)t})}_{=\cos(2t)-i\sin(2t)}$$

(*) The first term simplifies to:

$$(-0.5+0.25i)\begin{bmatrix} -1 \\ 1-2i \end{bmatrix}(\cos(2t)+i\sin(2t))(e^{-t})$$

$$\Downarrow$$

$$\begin{bmatrix} (0.5-0.25i)(\cos(2t)+i\sin(2t)) \\ (-0.5+0.25i)(1-2i)(\cos(2t)+i\sin(2t)) \end{bmatrix} e^{-t}$$

$$\Downarrow$$

$$\boxed{\begin{bmatrix} [0.5\cos(2t)+0.25\sin(2t)] + i[-0.25\cos(2t)+0.5\sin(2t)] \\ [-1.25\sin(2t)] + i[1.25\cos(2t)] \end{bmatrix} e^{-t}}$$

(*) The second term simplifies to:

$$(-0.5 - 0.25i)\begin{bmatrix} -1 \\ 1+2i \end{bmatrix}(\cos(2t) - i\sin(2t))(e^{-t})$$

$$\Downarrow$$

$$\begin{bmatrix} (0.5 + 0.25i)(\cos(2t) - i\sin(2t)) \\ (-0.5 - 0.25i)(1+2i)(\cos(2t) - i\sin(2t)) \end{bmatrix} e^{-t}$$

$$\Downarrow$$

$$\boxed{\begin{bmatrix} [0.5\cos(2t) + 0.25\sin(2t)] + i[0.25\cos(2t) - 0.5\sin(2t)] \\ [-1.25\sin(2t)] + i[-1.25\cos(2t)] \end{bmatrix} e^{-t}}$$

(*) Notice how the imaginary parts cancel out! A real system w/ real inputs will always have a real solution.

Therefore:

$$\boxed{x(t) = \begin{bmatrix} \cos(2t) + 0.5\sin(2t) \\ -2.5\sin(2t) \end{bmatrix} e^{-t}}$$

7c

$$\frac{dx}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_{1,2} = \pm i$$

$$v_{1,2} = \begin{bmatrix} 1 \\ \mp i \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$* \quad \boxed{x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}}$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-it}$$

$$@ \ \boxed{x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\hookrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\boxed{\text{Matrix Inversion}} \longrightarrow \boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.5 + 0.5i \\ 0.5 - 0.5i \end{bmatrix}}$$

15

<u>SO</u>

$$x(t) = (0.5 + 0.5i)\begin{bmatrix} 1 \\ -i \end{bmatrix} \overbrace{e^{it}}^{= \cos(t) + i\sin(t)} + (0.5 - 0.5i)\begin{bmatrix} 1 \\ i \end{bmatrix} \underbrace{e^{-it}}_{= \cos(t) - i\sin(t)}$$

(*) The first term yields:

$$(0.5 + 0.5i)\begin{bmatrix} 1 \\ -i \end{bmatrix} \left(\cos(t) + i\sin(t)\right)$$

$$\Downarrow$$

$$\begin{bmatrix} (0.5 + 0.5i)(\cos(t) + i\sin(t)) \\ (0.5 - 0.5i)(\cos(t) + i\sin(t)) \end{bmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{bmatrix} \left[0.5\cos(t) - 0.5\sin(t)\right] + i\left[0.5\cos(t) + 0.5\sin(t)\right] \\ \left[0.5\cos(t) + 0.5\sin(t)\right] + i\left[0.5\cos(t) + 0.5\sin(t)\right] \end{bmatrix}} \; ?$$

(*) The second term yields:

$$(0.5-0.5i)\begin{bmatrix} 1 \\ i \end{bmatrix}(\cos(t) - i\sin(t))$$

$$\Downarrow$$

$$\begin{bmatrix} (0.5-0.5i)(\cos(t) - i\sin(t)) \\ (0.5+0.5i)(\cos(t) - i\sin(t)) \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} [0.5\cos(t) - 0.5\sin(t)] + i[-0.5\cos(t) - 0.5\sin(t)] \\ [0.5\cos(t) + 0.5\sin(t)] + i[0.5\cos(t) - 0.5\sin(t)] \end{bmatrix}$$

Therefore

$$\boxed{x(t) = \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}}$$

(8) $$\frac{dx}{dt} = \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix} x \qquad \lambda_1 = \lambda_2 = -0.5$$

$$v_1 = v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

* Generalized Eigen vector $\underline{(A - \lambda I)w = v}$

$$\begin{bmatrix} -0.5+0.5 & 1 \\ 0 & -0.5+0.5 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\left.\begin{aligned} 0w_1 + w_2 &= 1 \\ 0w_1 + 0w_2 &= 0 \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} &\underline{w_2 = 1} \\ &w_1 \Rightarrow \text{free.} \end{aligned}}$$

So for convenience, set $v_1 = 0$. IF

$$\boxed{w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

(*) The general solution is

$$X(t) = c_1 V e^{\lambda t} + c_2 \left(V t e^{\lambda t} + W e^{\lambda t} \right)$$

$$X(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-0.5t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t e^{-0.5t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-0.5t} \right)$$

(*) @ $\boxed{x(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}}$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1) + \cancel{c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0)(1)} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

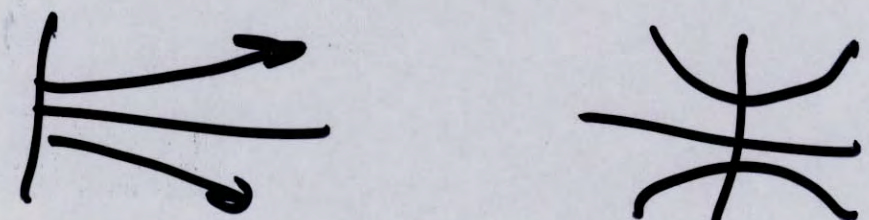
So $\boxed{c_1 = c_2 = 3}$

$$\boxed{X(t) = 3e^{-0.5t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}$$

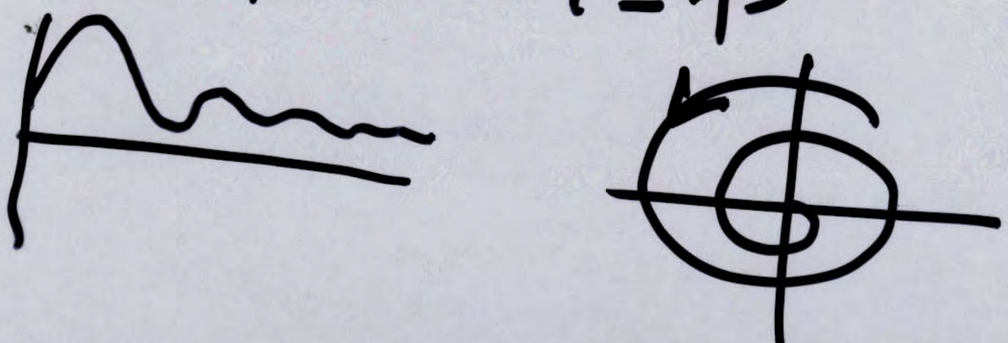
Stable Node $\lambda_1 = -5$ $\lambda_2 = -7$

Unstable $\lambda_1 = 4$ $\lambda_2 = 937$

Saddle $\lambda_1 = +2$ $\lambda_2 = -38$



Stable spiral $-1 \pm j5$



Unstable spr

$+5 \pm j3$