ENGR 232: Dynamic Engineering Systems

Summer 2010

Midterm Exam 2

Name: Solutions

Lab Section: ____________

INSTRUCTIONS: You will have 50 minutes to complete this exam. Students are NOT allowed to use textbooks, notes, photocopies, and laptop computers. Make sure that you write your name and section number in the space provided above. Correct answers without adequate work will not receive full credit. **You MUST show all work and place your final answers in the boxes provided.** *Calculators are allowed!*

TABLE OF IMPORTANT RELATIONSHIPS:

$$\sec^2(t) = 1 + \tan^2(t)$$

Quadratic Equation: $\lambda = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ for $A\lambda^2 + B\lambda + C = 0$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Problem 1 20 points	
Problem 2 20 points	
Problem 3 10 points	
Problem 4 10 points	
Problem 5 20 points	
Problem 6 20 points	
FINAL GRADE **Max 100**	

1. **Solution Validity:** Fill in the following table, specifying the maximum interval of ***t*** in which the solution of the initial value problem is valid. Show your answer using interval notation, for example **(-6,65)** for greater than -6 and less than 65 (20 points).

SYSTEM	**LONGEST RANGE?** **4 POINTS EACH**
$y'+\frac{1}{\sqrt{1-t^2}}y=\cos(t)$, $y(0.5)=2$ - SQRT (+1 , -1)	(-1 1) y(.5) - SQRT OUTSIDE
Roots (t-1)(t-2) 1, 2 $(t^2-3t+2)y'+y=0$, $y(0)=1$ $\dot{y}+\frac{1}{t^2-3t+2}y=0$	1 2 y(0) (-∞, 1)
$(t^2+1)y'+(t^2-1)y=0$, $y(0)=0$ $\dot{y}+\frac{t^2-1}{t^2+1}y=0$	(-∞ ∞) y(0)
$y'+\sec^2(t)y-y=0$, $y(0)=1$	$(-\frac{\pi}{2}$ $\frac{\pi}{2})$ y(0)
$y'+\frac{y}{t^2}=0$, $y(1)=1$	0 y(1) (0 , ∞)

$y^2+y(\sec^2-1)=0$

$\sec^2-1=\tan^2$

$y'+y\tan^2(t)=0$

Scoring

½ correct if range is valid but does not include IC

2. **Eigenvalues and Eigenvectors:**

a. Find the Eigenvalues for the following system ***by hand*** (10 points).

$$\frac{dx}{dt} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$$

$|\lambda I - A| = 0$

CHARACTERISTIC EQ

$$\begin{vmatrix} \lambda-3 & 2 \\ -4 & \lambda+1 \end{vmatrix} = \lambda^2 + \lambda - 3\lambda - 3 + 8 = \underline{\lambda^2 - 2\lambda + 5} = 0$$

$$\lambda_i = \frac{-(-2) \pm \sqrt{4-20}}{2} = 1 \pm i2$$

SCORING
CE CORRECT 4
ROOTS 4

8 POINTS

$\lambda_1 =$ $1+i2$ $\lambda_2 =$ $1-i2$

-1 IF NO EXPLANATION

2 POINTS

STABLE? EXPLAIN.

UNSTABLE
REAL PARTS OF λ_1 λ_2
ARE NOT NEGATIVE

b. Verify which of the following is ***NOT*** a suitable eigenvector given the system defined below. ***Circle and justify your answer*** (10 points).

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x, \quad \lambda = \{3, -1\}$$

i. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ii. $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ iii. $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}}_{\times 3}$ $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\times -1}$ $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}}$

YES YES NO

(3) POINTS (3) POINTS

NOT MULTIPLES

$Ax \neq \lambda x$

$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ IS NOT MULTIPLE OF $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

MUST HAVE ✓ (4 POINTS)

3. **Equivalent First Order System:** Convert the following 2^{nd} order differential equation into an equivalent system of first order differential equations. Write your final answer in matrix form. Also show the initial conditions converted (10 points).

$$4y'' + 16y' + 20y = 20 + 4\tan(t), \quad y(0) = 1, \quad y'(0) = 2$$

$$\ddot{y} + 4\dot{y} + 5y = 4 + \tan(t)$$

$\boxed{\text{Let} \quad x_1 = y \quad x_2 = \dot{y}}$ $\Rightarrow$

$$\dot{x}_1 = \dot{y} = x_2$$
$$\dot{x}_2 = \ddot{y} = 4 + \tan(t) - 4\dot{y} - 5y$$
$$\dot{x}_2 = 4 + \tan(t) - 4x_2 - 5x_1$$

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

SCORING
1 POINTS $x_1 = y$ $x_2 = \dot{y}$
4 POINTS CORRECT MATRIX
5 POINTS CORRECT IC'S

5 POINTS
MATRIX FORM:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 + \tan(t) \end{bmatrix}$$

5 POINTS
INITIAL CONDITION:

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. **Autonomous Systems Sketching:** Sketch dy(t)/dt versus y(t) and use this plot to estimate the sketch of y(t) vs. t near the equilibrium points. For y(t) vs. t, sketch three solutions given the initial conditions ***y(0) = {10, 0, -5}***. ***Make sure you clearly label the equilibrium points and initial conditions on each plot*** (10 points).

$$\frac{dy(t)}{dt} = (y-7)^2(y+2)$$

Roots $7, 7, -2$

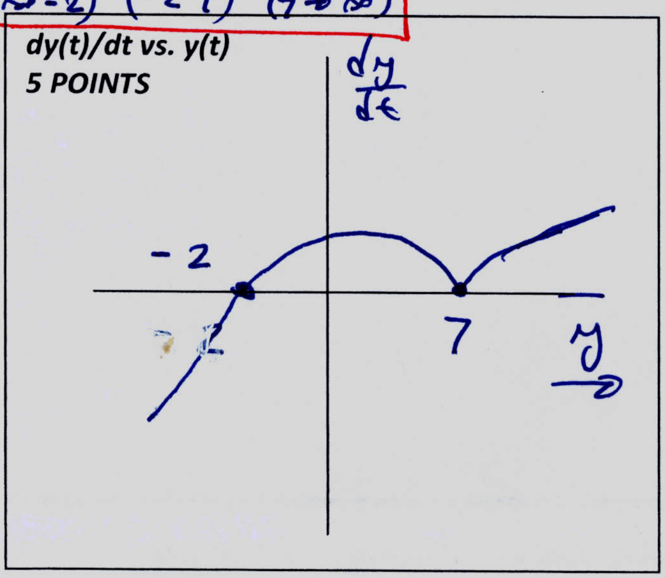
@ 0 $\frac{dy}{dt} = (49)(2) = 98 > 0$

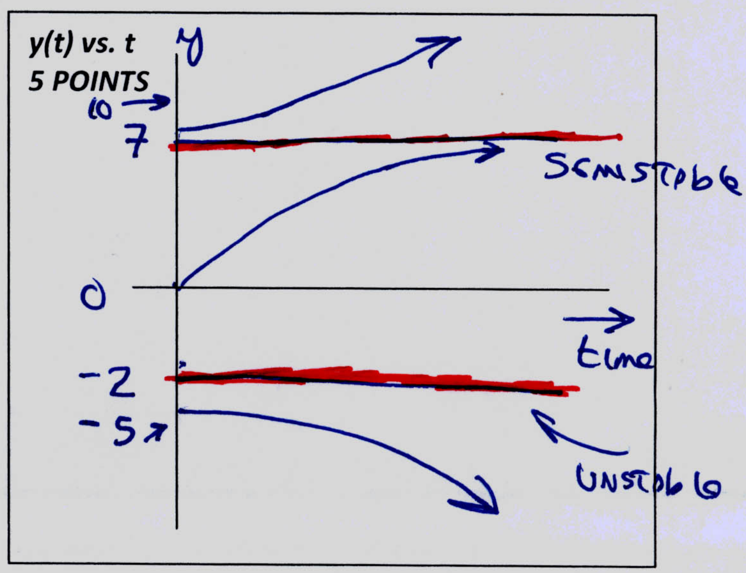
@ 10 $\frac{dy}{dt} = (3)^2(12) = 108 > 0$

@ -7 $\frac{dy}{dt} = (-14)^2(-5) < 0$

Scoring Regions
$[2 \quad 2 \quad 1]$
$(-\infty\ -2)$ $(-2\ 7)$ $(7 \to \infty)$

Scoring
5 points
1 point each line





5. **System Modeling:** Consider two interconnected tanks. Tank 1 initially contains 60 gal of water with a salt concentration of 0.15 oz/gal. Tank 2 initially contains 100 gal of water with a salt concentration of 0.035 oz/gal. The two tanks are configured in the following manner:

   a. There is 3 oz/gal of salt flowing into tank 1 at a rate of 3 gal/min.

   b. There is 4 oz/gal of salt flowing into tank 2 at a rate of 1 gal/min.

   c. Approximately 4 gal/min flows out of tank 1. Only 2 gal/min flows into tank 2, where the remaining fluid exits the system at a rate of 2 gal/min.

   d. Approximately 3 gal/min flows out of tank 2. Only 1 gal/min flows into tank 1, where the remaining fluid exits the system at a rate of 2 gal/min.

Notice that the volume of liquid in each tank in the system remains constant. Write down the differential equations and initial conditions of the system that models the ***amount of salt*** in tank 1 and tank 2 (i.e. $Q_1(t)$ and $Q_2(t)$). ***DO NOT SOLVE THE SYSTEM!*** (20 points)

$$\frac{dQ}{dt} = \text{RATE IN} - \text{RATE OUT}$$

$$\dot{Q}_1 = \overbrace{\frac{3\,oz}{gal} \times \frac{3\,gal}{min} + \frac{Q_2}{100} \times \frac{1\,gal}{min}}^{IN} - \overbrace{\frac{Q_1}{60}\,\frac{4\,gal}{min}}^{OUT}$$

$$\dot{Q}_2 = \overbrace{\frac{4\,oz}{gal} \times \frac{1\,gal}{min} + \frac{Q_1}{60}\,\frac{2\,gal}{min}}^{IN} - \overbrace{\frac{Q_2}{100}\,\frac{3\,gal}{min}}^{OUT}$$

SCORING CORRECT TERMS

	RATE	EQ	1
TERMS	1	3	3
CORRECT	1	3	3

IC's $60\,gal \times .15\,\frac{oz\ salt}{gal}$

$100\,gal \times .035$

DIFFERENTIAL EQUATIONS, 15 POINTS

$\dot{Q}_1(t) = 9 + \frac{Q_2}{100} - \frac{Q_1}{15}$

$\dot{Q}_2(t) = 4 + \frac{Q_1}{30} - \frac{3Q_2}{100}$

INITIAL CONDITIONS, 5 POINTS

$Q_1(0) = 9$

$Q_2(0) = 3.5$

Q1 CORRECT 1
Q2
Q1 Q2 VALUES 2, 2

6. **Eigenvalue/Eigenvector Solutions:** Given the system and its corresponding eigenvalues/eigenvectors, form the eigenvalue/eigenvector solution. Use the initial conditions to solve for the constants C_1 and C_2 using any linear algebra technique of your choosing. You may use a calculator to help you solve this problem (20 points).

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} x(t), \quad \lambda = \{-1, -5\}, \quad v = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \}, \quad x(0) = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$$

$$x^V = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

@ $t=0$

$$\begin{bmatrix} 4 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{\begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}}{-4} \begin{bmatrix} 4 \\ 16 \end{bmatrix} = \frac{\begin{bmatrix} -36 \\ 20 \end{bmatrix}}{-4} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

Scoring correct terms
form 4
terms 1 each

10 POINTS
GENERAL SOLUTION, IN TERMS OF C_1 AND C_2

$x(t)$ = $c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

6 points
Scoring solution C_1 C_2
correct values
2 points each

10 POINTS
VALUES OF C_1 AND C_2

C_1 = 9 **C_2 =** -5