ENGR 232 Dynamic Engineering Systems

Lecture 8

Second Order Differential Equations

The next generation!

Announcements

Final on Thursday

Review: Solve Homogeneous 2^{nd} Order Equation

Equation and initial value:

$$y'' + y' = 0, \quad y(0) = 2, \quad y'(0) = 1$$

Auxiliary equation and roots:

$r^2 + r = 0$ $\quad$ $r(r+1)$

$r_1 = \underline{\ 0\ }$ $\quad$ $r_2 = \underline{\ -1\ }$

General solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 + c_2 e^{-t}$$

Apply initial conditions

$$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 2$$

$$y'(0) = r_1 c_1 e^0 + r_2 c_2 e^0 = -c_2 = 1$$

$c_1 = 3$

$c_2 = -1$

$y(t) = 3 - e^{-t}$

Today's Lecture

- Section 8(4).3 Auxiliary Equations with Complex Roots
- ~~Section 8(4).4 Nonhomogeneous Equations~~

Example

- Differential Equation With Complex Roots

$$y'' + 6y' + 25y = 0,\ y(0) = 2,\ y'(0) = 2$$

- Auxiliary Equation

$r^2 + 6r + 25 = 0$ $\quad D = b^2 - 4ac = 36 - 100 = -64 < 0$

- $r_1 = \underline{-3 + i4}$ $\quad r_2 = \underline{-3 - i4}$
- General solution

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = \qquad y' = r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t} =$$

- Apply initial values

$c_1 = 1 - i$

$c_2 = 1 + i$

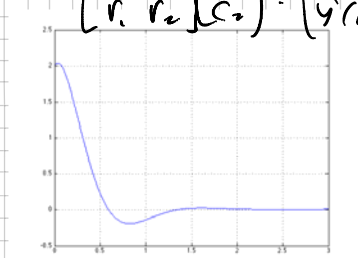
$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = r_1 c_1 + r_2 c_2 = (-3 + i4)c_1 + (-3 - i4)c_2 = 2$$

Go To MATLAB

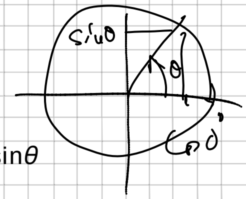
```
%% Second order differential equation example.
clear, clc, close all

% The roots
r1=-3+4i; r2 = -3-4i;
% Matrix for matching initial conditions
M = [1 1; r1 r2]
% The initial conditions
IC = [2; 2]
% c1 and c2, the coefficients
c1c2 = inv(M)*IC
c1 = c1c2(1); c2 = c1c2(2);

%% Plot the solution
t = 0:0.01:3;
y = c1*exp(r1*t) + c2*exp(r2*t);
plot(t, y)
grid on
```

$$\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$



What's Going On?

$e^{x+y} = e^x e^y$



- $r = \alpha + \beta i \quad e^{rt} = ???$

$$e^{rt} = e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t}$$

- What is $e^{i\beta t}$?
  Euler's (Oiler's) formula $e^{i\theta} = \cos\theta + i\sin\theta$

- The whole formula

$$e^{rt} = e^{(\alpha+i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}\underbrace{\left(\cos\beta t + i\sin\beta t\right)}_{e^{i\beta t}}$$

The Solution!

$$y'' + 6y' + 25y = 0,\ y(0) = 2,\ \ y'(0) = 2$$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},\ \boxed{c_1 = 1 - i,}\ r_1 = -3 + 4i,\ \boxed{c_2 = 1 + i,}\ r_2 = -3 - 4i$$

The first term, $c_1\exp(r_1 t)$

$$c_1 e^{r_1 t} = (1-i)e^{(-3+4i)t} = (1-i)e^{-3t}(\cos 4t + i\sin 4t)$$

Multiplication of complex numbers:
$(a + ib)(c + id) = ac - bd + i(ad + bc)$

$$c_1 e^{r_1 t} = e^{-3t}\left[\cos 4t + \sin 4t + i\left(\sin 4t - \cos 4t\right)\right]$$

In a similar way,

$$c_2 e^{r_2 t} = e^{-3t}\left[\cos 4t + \sin 4t - i\left(\sin 4t - \cos 4t\right)\right]$$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = 2e^{-3t}(\cos 4t + \sin 4t)$$

$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $\qquad y(t) = e^{\alpha t}(d_1 \cos\beta t + d_2 \sin\beta t)$

The Complex Case

$$ay'' + by' + cy = 0,\ y(0) = Y_0,\ \ y'(0) = Y_1$$

$$a \neq 0;\ a,\ b,\ c,\ Y_0,\ Y_1 \text{ real};\ D = b^2 - 4ac < 0$$

Solution method:

(1) Find r_1, r_2, roots of auxiliary equation

(2) Set up initial condition equation

$$c_1 + c_2 = Y_0$$
$$r_1 c_1 + r_2 c_2 = Y_1$$

(3) Solve for c_1 and c_2.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Same as for real roots.

Old Fashioned Way

$$ay'' + by' + cy = 0,\ y(0) = Y_0,\ \ y'(0) = Y_1$$

$$a \neq 0;\ a,\ b,\ c,\ Y_0,\ Y_1 \text{ real};\ D = b^2 - 4ac < 0$$

(1) Find r_1, r_2, roots of auxiliary equation

(2) Choose $r_1 = \alpha + i\beta$,

(3) Solution has the form

$$y(t) = e^{\alpha t}\left[d_1 \cos \beta t + d_2 \sin \beta t\right]$$

(4) Initial conditions

$$y(0):\quad d_1 = Y_0$$
$$y'(0):\ \alpha d_1 + \beta d_2 = Y_1$$

8(4) Section 3 Example 3

- Spring-mass-friction system, $my'' + by' + ky = 0$
- m = 36, b = 12, k = 37, $y(0)$ = 0.7, $y'(0)$ = 0.1

(1) Auxiliary equation $36r^2 + 12r + 37 = 0$

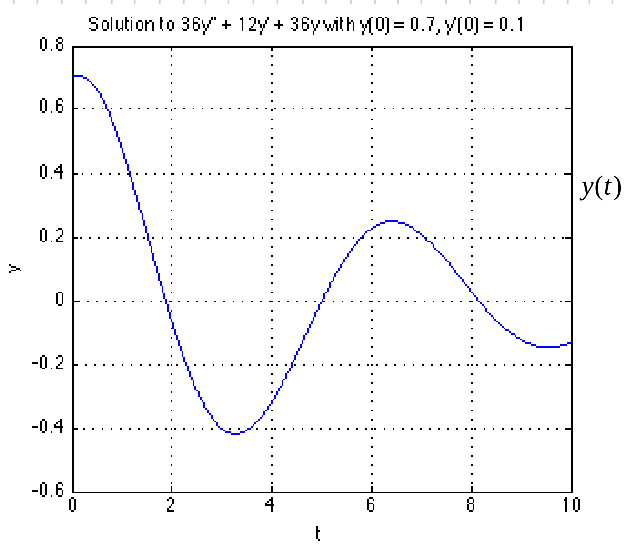
(2) Roots $r = -\frac{1}{6} \pm i$ α $\beta = 1$

(3) General solution: $y(t) = e^{-t/6}(d_1 \cos t + d_2 \sin t)$

(4) Initial value equation: $y(0): \quad d_1 = 0.7$

$y'(0): -\frac{1}{6}d_1 + d_2 = 0.1$

(5) Solution: $y(t) = e^{-t/6}(0.7\cos t + 0.2167\sin t)$



$y(t) = e^{-t/6}(0.7\cos t + 0.2167\sin t)$

Overview of 2nd Order Linear Homogeneous equation

real

complex

Assume a = 1, $y'' + by + c = 0,\ D = b^2 - 4c$

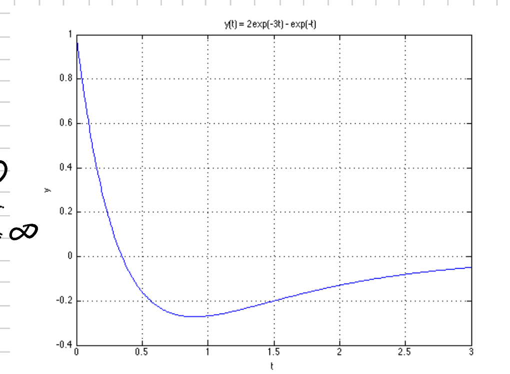
D > 0, real roots

2 negative roots

$y(t) = 2e^{-3t} - e^{-t}$

as $t \to \infty$ $y \to 0$

as $t \to -\infty$ $y \to +\infty$



Overview of 2nd Order Linear Homogeneous equation

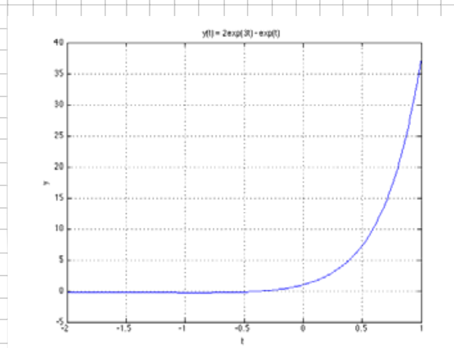
Assume a = 1, $y'' + by + c = 0,\ D = b^2 - 4c$

D > 0, real roots

2 positive roots

$y(t) = 2e^{3t} - e^{t}$

$t \to \infty$ $y \to \infty$



Overview of 2nd Order Linear Homogeneous equation

Assume a = 1, $y'' + by + c = 0,\ D = b^2 - 4c$

D > 0, real roots

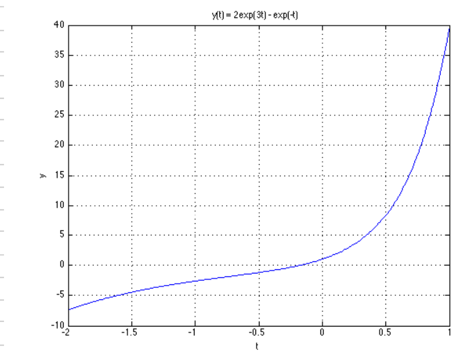
Negative and positive roots

$+ \quad -$

$y(t) = 2e^{3t} - e^{-t}$

$t \to \infty \quad y(t) \to \infty$

$t \to -\infty \quad |y(t)| \to \infty$

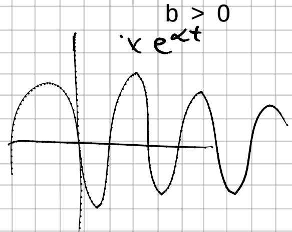


Overview of 2nd Order Linear Homogeneous equation

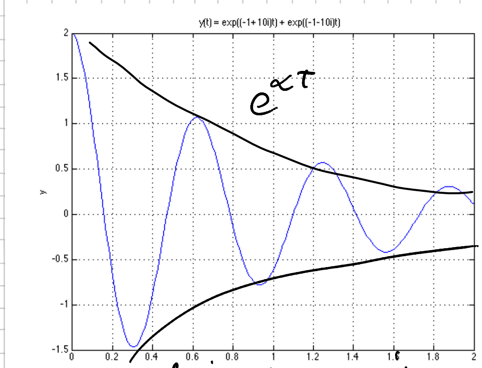
Assume a = 1, $y'' + by + c = 0,\ D = b^2 - 4c$ $\quad \alpha = -\frac{b}{2} < 0$

D < 0, complex roots

b > 0



decaying exponential



string on guitar

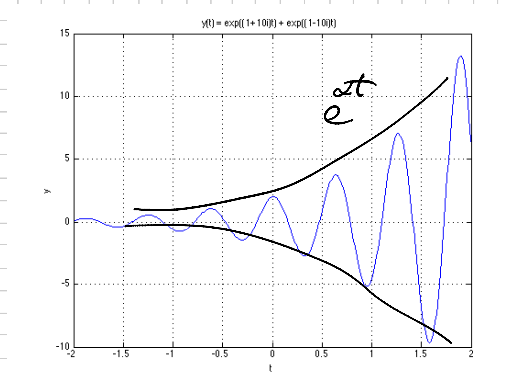
Overview of 2nd Order Linear Homogeneous equation

Assume a = 1, $y'' + by + c = 0,\ D = b^2 - 4c$

D < 0, complex roots
b < 0
$\alpha > 0$

Growing exponent.

Hurricane.



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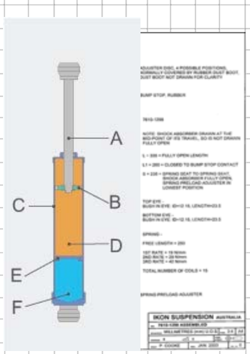
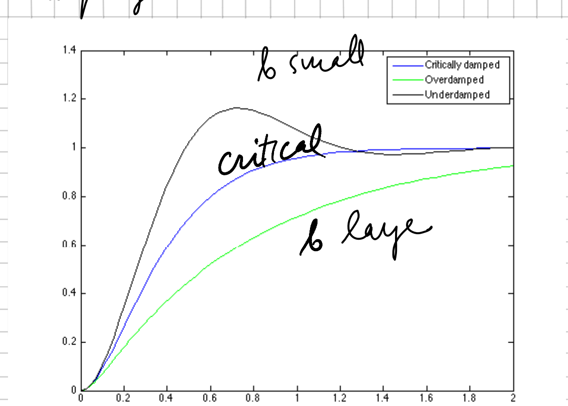
Variable Damping

Assume a = 1, $y'' + by + 25 = 0,\ D = b^2 - 4c$

↑damping

critical $b^2 = 4ac$





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Review

2nd order systems.

1) Auxiliary equ. Roots.

2) General solution, Match initial conditions.

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Complex roots

$$y = e^{\alpha t}(d_1 \cos\beta t + d_2 \sin\beta t)$$