ENGR 232 Dynamic Engineering Systems

Lecture 7

2nd Order Linear Differential Equations

Announcements

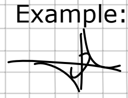
- **Homework 4 on Tuesday**
- Lab Exam this week in recitation
- Midterm will be returned during recitation

Review: Existence of Solutions of Frist Order Differential Equations

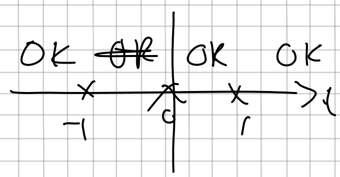
$$\frac{dy}{dt}+P(t)y=Q(t)$$

A solution exists and is unique for $y(t_0) = y_0$ if both $P(t)$ and $Q(t)$ are continuous at t_0. The existence or uniqueness does not depend on the value of $y_{0.}$

Example:

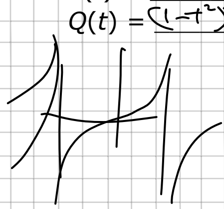


$$y'+\frac{1}{t}y=\frac{1}{1-t^2}$$



$P(t)$ = t^{-1}, not continuous at t = 0

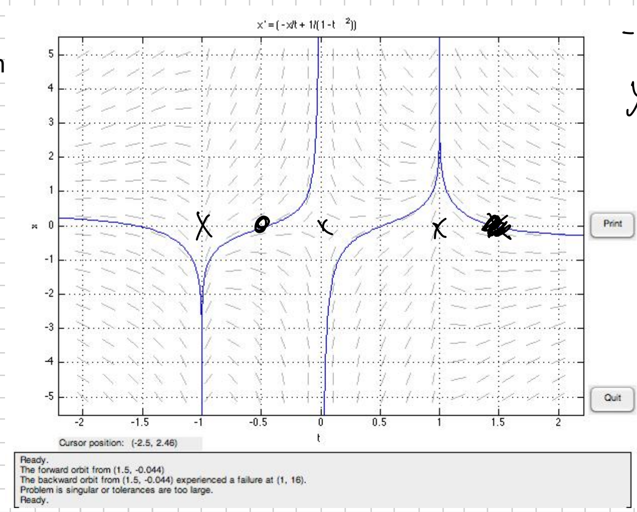
$Q(t)$ = $(1-t^2)^{-1}$, not continuous at t = ± 1



$$y(t)=-\frac{1}{2t}\ln\left|t^2-1\right|+\frac{C}{t}$$

$$y'+\frac{1}{t}y=\frac{1}{1-t^2} \qquad y(t)=-\frac{1}{2t}\ln\left|t^2-1\right|+\frac{C}{t}$$

- m



$y(-0.5) = 0$

$-1 < t < 0$

$y(1.5) = 0$

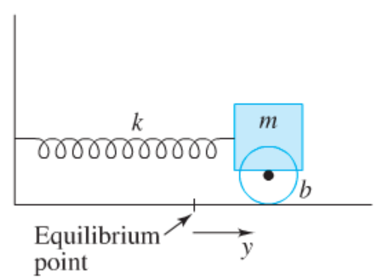
$1 < t < \infty$

Plan For Next Three Weeks

- Linear Second Order Differential Equations and Their Solutions
  - Chapter 8(4) Linear Differential Equations
  - Laplace Transform (notes)
  - Multivariate Systems

Introduction to the Mass-Spring Oscillator 8(4).1

- The mass is attached to a spring and is free to move horizontally along one dimension.
- y is the displacement from equilibrium. Force exerted by the spring is proportional to y (Hooke's law).



$$F_{\text{spring}} = -ky$$

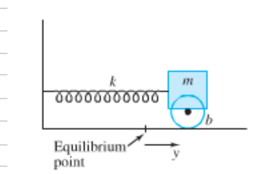
- We assume that the friction force is proportional to the velocity.

$$F_{\text{friction}} = -bv, \quad v = \frac{dy}{dt}$$

Mass-Spring System (continued)

$$F_{\text{spring}} = -ky \qquad F_{\text{friction}} = -bv,\ v = \frac{dy}{dt}$$

v, y



- Newton's third law of motion

$$m\frac{dv}{dt} = F_{\text{spring}} + F_{\text{friction}} = -ky - bv$$

- Write with *t* as independent variable and *y* as dependent (eliminate *v*).

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} \qquad m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

- Special case, *b* = 0

$$my'' = -ky,\ y'' + \omega^2 y = 0,\ \ \omega = \sqrt{k/m}$$

Lossless Spring Mass System

$$my'' = -ky,\ y'' + \omega^2 y = 0,\ \ \omega = \sqrt{k/m}$$

- Solution by the ~~guess and try~~ undetermined parameter method.

$$y(t) = \cos at,\quad y' = -a\sin at,\quad y'' = -a^2\cos at$$

$$a^2 = \omega^2$$

$$y'' + \omega^2 y = -a^2\cos at + \omega^2\cos at = (-a^2 + \omega^2)\cos at, \ = 0$$

- The formula we assumed for the solution is valid when

$$a = \omega = \sqrt{\frac{k}{m}}$$

Another Case

$$y'' + (b/m)y' + (k/m)y = 0$$

- m = 1, k = 25, b = 6.

$y'' + 6y' + 25y = 0$ $\qquad y(t) = e^{-3t}\cos 4t$

Does this $y(t)$ satisfy the differential equation?

$$y'(t) = \underline{-3}\, e^{-3t}\cos 4t + \underline{-4}\, e^{-3t}\sin 4t$$

$$y''(t) = \underline{9}\, e^{-3t}\cos 4t + \underline{12}\, e^{-3t}\sin 4t + \underline{12}\, e^{-3t}\sin 4t + \underline{-16}\, e^{-3t}\cos 4t$$

$9-16$

$$= \underline{-7}\, e^{-3t}\cos 4t + \underline{24}\, e^{-3t}\sin 4t$$

Substitute into the equation,

$$y'' + 6y' + 25y = \underline{(\underbrace{-7}_{y''}\underbrace{-18}_{y'}\underbrace{+25}_{y})}\, e^{-3t}\cos 4t + \underline{24-24}\, e^{-3t}\sin 4t = 0$$

Third Case

$$y'' + (b/m)y' + (k/m)y = 0 \qquad y'' + 10y + 25y = 0$$

- m = 1, k = 25, b = 10. $\qquad y(t) = e^{-5t}$

Does this $y(t)$ satisfy the differential equation?

$$y'(t) = -5e^{-5t}$$

$$y''(t) = 25e^{-5t}$$

Substitute into the equation,

$$y'' + 10y' + 25y = 25e^{-5t} - 50e^{-5t} + 25e^{-5t} = 0$$

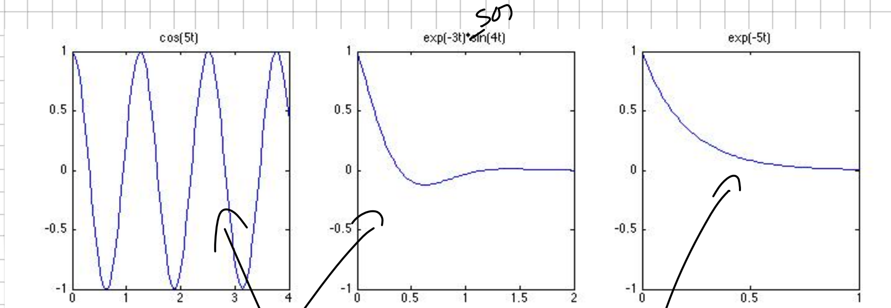
$ay'' + by' + cy = 0 \quad a \neq 0$

Summary: Thee Cases

$y'' + 25y = 0$ | $y'' + 6y' + 25y = 0$ | $y'' + 10y' + 25y = 0$

$y(t) = \cos 5t$ | $y(t) = e^{-3t}\cos 4t$ | $y(t) = e^{-5t}$

son



must 2nd order.

1st order

Homogeneous Linear Differential Equations of 2nd Order With Constant Coefficients 8(4).2

Standard equation: $ay'' + by' + cy = f(t),\ a \neq 0$ forcing function

Homogeneous equation: $ay'' + by' + cy = 0$

This equation always has a solution of the type $y(t) = e^{rt}$. $\to 0$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$ $\quad ar^2 e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0$

Auxiliary equation: $ar^2 + br + c = 0$

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Discriminant: $D = b^2 - 4ac$

If $D > 0$, 2 real roots | If $D < 0$, 2 complex roots | If $D = 0$ repeated root.

8(4).2 Example 1

ODE:

$$y'' + 5y' - 6y = 0$$

Auxiliary equation:

$r^2 + 5r - 6 = 0 \quad (r+6)(r-1) = 0$

General solution:

$-6 \times y(t) = c_1 e^{-6t} + c_2 e^{t}$ linear combinato of e^{-6t}, e^{t}

Check: $5 \times y'(t) = -6c_1 e^{-6t} + c_2 e^{t}$

$1 \times y'' = 36 c_1 e^{-6t} + c_2 e^{t}$

$c_1 e^{-6t}(36 - 30 - 6) + c_2 e^{t}(1 + 5 - 6) = 0$

8(4).2 Example 2

$t=0$

ODE IVP:

$$y'' + 2y' - y = 0, \; y(0) = 0, \; y'(0) = -1$$

Auxiliary equation:

$r^2 + 2r - 1 = 0 \quad r = \frac{-4 \pm \sqrt{4+4}}{2} = -2 \pm \sqrt{2}$

General solution:

$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{(-2+\sqrt{2})t} + c_2 e^{(-2-\sqrt{2})t}$

Apply initial conditions:

$y(0) = c_1 + c_2 = 0 \quad c_2 = -c_1 \quad y'(t) = (-2+\sqrt{2})c_1 e^{(-2+\sqrt{2})t} + c_2(-2-\sqrt{2}) c_2 e^{(\;)t}$

$y'(0) = (-2+\sqrt{2})c_1 + c_2(-2-\sqrt{2})c_2 = -1$

$(\quad)c_1 - c_1(\quad) = 2\sqrt{2}c_1 = -1 \quad c_1 = \frac{-1}{2\sqrt{2}}$

The Existence Theorem

The Question: When does a homogenous linear second-order equation have a unique solution?

Theorem 1. The following differential equation in $y(t)$ and the following initial conditions

$$ay'' + by' + cy = 0, \;\; a \neq 0, \;\; y(t_0) = Y_0, \;\; y'(t_0) = Y_1$$

there is a unique solution defined for $-\infty < t < \infty$.

The Way to the Solution

$$ay'' + by' + cy = 0, \;\; a \neq 0, \;\; y(t_0) = Y_0, \;\; y'(t_0) = Y_1$$

If we find two function $y_1(t)$ and $y_2(t)$ that satisfy the equation, and if furthermore the ~~function~~ neither function is identically 0 and is not a multiple of the other function then we can satisfy the above equation with a function

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t)}$$

and appropriate values of c_1 and c_2.

Is this true? Two questions:

1. Does $y(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfy the equations?

2. Can we satisfy the initial condition?

Answer Questions 1.

$$ay''+by'+c=0,\ \ a\neq 0,\ \ y(t_0)=Y_0,\ \ y'(t_0)=Y_1$$

$c\times$

Does $y(t)=c_1y_1(t)+c_2y_2(t)$ satisfy the equations?

Know (hypothesis) y_1, y_2 satisfy eqn

$b\times\ \ y'(t)=c_1y_1'(t)+c_2y_2'(t)$

$a\times\ \ y''(t)=c_1y_1''(t)+c_2y_2''(t)$

$$=0$$

$$c_1(ay_1''\cancel{+by_1'}+cy_1)$$

$$+\ c_2(ay_2''\cancel{+by_2'}+cy_2)=0$$

Answer Questions 2.

$$ay''+by'+cy=0,\ \ a\neq 0,\ \ y(t_0)=Y_0,\ \ y'(t_0)=Y_1$$

Can $y(t)=c_1y_1(t)+c_2y_2(t)$ satisfy the initial conditions?

Write the equations:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

This linear system of equations for c_1 and c_2 has a unique solution if and only if the two columns are not linearly dependent. For a 2x2 system, the columns are linearly dependent if and only if column is proportional to the other. If

$$\begin{bmatrix} y_1(t_0) \\ y_1'(t_0) \end{bmatrix}=k\begin{bmatrix} y_2(t_0) \\ y_2'(t_0) \end{bmatrix}$$

that is the case, by uniqueness $y_1(t) = ky_2(t)$, which violates our assumption about y_1 not being proportional to y_2.

Two Independent Solutions

$$ay'' + by' + cy = 0,\ \ a \neq 0,\ \ y_1(t),\ y_2(t)$$

$y = e^{rt}$ solution, $ar^2 + br + c = 0$ auxiliary equation.

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Discriminant: $D = b^2 - 4ac$

If D > 0, 2 distinct real roots, if D < 0, two distinct complex roots, if D = 0 two identical real roots.

$D > 0$ r_1, r_2 $y_1 = e^{r_1 t}$ $y_2 = e^{r_2 t}$

$D = 0$ $r_1 = r_2$ $y_1 = e^{r_1 t}$ $y_2 = t e^{r_1 t}$ does this satisfy the DE?

$D < 0$ On Wednesday

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Repeated Root

$$y'' + 4y' + 4y = 0,\ \ y(0) = 1,\ \ y'(0) = 3$$

- Auxiliary equation

$b^2 - 4ac = 16 - 4 \cdot 4 = 0$

$r^2 + 4r + 4 = (r+2)^2 = 0$

- The two homogeneous solutions

$y_1 = e^{-2t}$ $y_2 = t e^{-2t}$

- Does $y(t) = te^{-2t}$ satisfy the equation?

$y_2' = e^{-2t} + t(-2)e^{-2t} = e^{-2t} - 2t e^{-2t}$

$y_2'' = -2e^{-2t} - 2e^{-2t} + 4te^{-2t} = -4e^{-2t} + 4t e^{-2t}$

$y'' + 4y' + 4y = e^{-2t}(\underbrace{-4}_{y''} + \underbrace{4}_{4y'}) + te^{-2t}(\underbrace{4 - 8 + 4}_{y'' + 4y' + 4y})$

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Satisfy the Initial Values

$$y'' + 4y' + 4y = 0, \;\; y(0) = 1, \;\; y'(0) = 3$$
$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$
$$y'(t) = -2c_1 e^{-2t} + c_2(-2t+1)e^{-2t}$$

Initial value equations:

$$y(0) = c_1 \qquad = 1$$
$$y'(0) = -2c_1 + c_2 = 3$$

Solution:

$$c_2 = 3 + 2c_1 = 5$$
$$y(t) = e^{-2t} + 5te^{-2t}$$

Summary

$$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0$$
$$y(t) = e^{rt}$$
$$a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$$

auxiliary equat'n

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t}$$