Name (please print): Solution February 21, 2008

ENGR 232: Dynamic Engineering Systems I: Exam 2

Section No.:

Instructions:

1. Please write your name and section number in the space provided.
2. This exam is closed book. Calculators may be used. Students are allowed to use one page of HAND-WRITTEN notes.
3. Points will be deducted if the work is unclear and/or the answers are not justified.

Solve the following differential equations

1. (5 points) Given the linear differential equation $y' + (\frac{1}{x} + 1)y = x^{-1}e^{-x}\sin 2x$,
   (a) (2 points) Find the integrating factor, $\mu(x)$
   (b) (3 points) Solve the equation for $y(x)$

Comparing the given eq'n with the standard form

$$y' + p(x)y = g(x), \quad p(x) = \frac{1}{x} + 1$$

$$g(x) = x^{-1}e^{-x}\sin 2x$$

$$\text{I.F.} = \mu(x) = e^{\int p(x)dx}$$

$$= e^{\int(\frac{1}{x}+1)dx} = e^{\ln|x| + x} = \boxed{x \cdot e^{x}}$$

$$(\because e^{\ln|x|+x} = e^{\ln|x|} \cdot e^{x} = xe^{x})$$

Multiplying $\mu(x)$,

$$\Rightarrow (xe^{x})y' + (e^{x} + xe^{x})y = \sin 2x$$

The solution is

$$\Rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x) + c\right]$$

$$= \frac{1}{xe^{x}}\left[\int \sin 2x + c\right] = \frac{1}{xe^{x}}\left[-\frac{\cos 2x}{2} + c\right]$$

$$\boxed{y(x) = \frac{-\cos 2x}{2xe^{x}} + \frac{c}{xe^{x}}}$$

Version (B)

$$\boxed{y(x) = -\frac{\cos 3x}{3xe^{x}} + \frac{c}{xe^{x}}}$$

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2. (3 points) Given the autonomous differential equation

$$\frac{dy}{dx} = y^2 - 15y + 50$$

   (a) (2 points) Find the equilibrium solutions $Q_1(x)$ and $Q_2(x)$.

   (b) (1 point) Characterize the equilibrium solutions as stable or unstable. Explain your answers (use sketches if necessary)

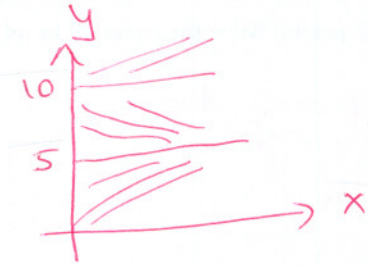
(a) $\frac{dy}{dx} = f(y) = y^2 - 15y + 50$

$\Rightarrow f(y) = 0 \Rightarrow (y-5)(y-10) = 0$

$\Rightarrow y = 5, 10 \Rightarrow Q_1(x) = 5$

$\Rightarrow Q_2(x) = 10$

(b)



$\Rightarrow Q_1(x) = 5 \Rightarrow$ asymptotically stable

$\Rightarrow Q_2(x) = 10 \Rightarrow$ unstable

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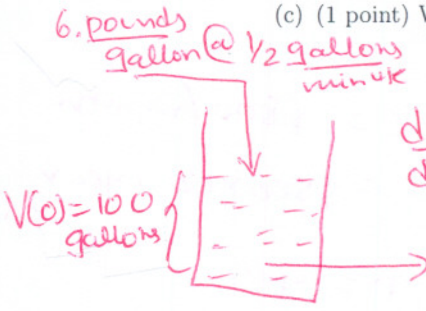
3. (6 points) A 200 gallon tank is filled with 100 gallons of pure water. A mixture containing salt is pumped into the tank at a rate of 1/2 gallons per minute. The concentration of the mixture flowing into the tank is 6 pounds per gallon. The well mixed solution is pumped out at 1/2 gallon per minute.

   (a) (2 points) Write a differential equation model for the amount of salt, S(t), in the tank at any time.

   (b) (3 points) Solve the model for S(t).

   (c) (1 point) What is the concentration of salt in the tank after an hour ? /half an hour ?)



$$\frac{dS}{dt} = \text{In-flow amount} - \text{Outflow amount}$$

$$\Rightarrow \frac{dS}{dt} = 6 \times \frac{1}{2} - \left(\frac{S}{V(t)}\right) \times \frac{1}{2}$$

Volume

$V(t) = V(0) + (\text{in-rate} - \text{outrate})t$

$= 100 + \cancel{(\tfrac{1}{2} - \tfrac{1}{2})t}^{\,0}$

$= 100$

$$= \frac{dS}{dt} = 3 - \frac{S}{100} \cdot \frac{1}{2} = 3 - \frac{S}{200}$$

$$\Rightarrow \boxed{\frac{dS}{dt} + \frac{S}{200} = 3} \rightarrow \boxed{\text{Part (a)}}$$

$p(t) = \frac{1}{200}$

$g(t) = 3$

$\mu(t) = e^{\int \frac{dt}{200}}$

$= e^{t/200}$

$$\Rightarrow S(t) = \frac{1}{\mu(t)}\left[\int g(t)\mu(t) + C\right]$$

$S(0) = 0$ ($\because$ pure water)

$$= \frac{1}{e^{t/200}}\left[\int 3 \cdot e^{t/200} + C\right]$$

$$= \frac{1}{e^{t/200}}\left[3 \cdot \frac{e^{t/200}}{1/200} + C\right]$$

$$\boxed{S(t) = 600 + Ce^{-t/200}}$$

@ $t = 0 \Rightarrow 0 = 600 + C \Rightarrow C = -600$

$$\Rightarrow \boxed{S(t) = 600 - 600e^{-t/200}} \quad \boxed{\text{Part (b)}}$$

Part(c) $S(30) = 83.57$ pounds $\Rightarrow$ Concentration $= S(30)/\text{Volume} = \frac{83.57}{100} = 0.83$

$S(60) = 155.5$ pounds $\Rightarrow$ $\Rightarrow S(60)/\text{Volume} = \frac{155.5}{100} = 1.55$

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4. (6 points) A home buyer can afford to spend no more than \$800 per month on mortgage payments. Suppose that the interest rate is 9% (per year) and that the term of the mortgage is 20 years.

   (a) (2 points) Write a differential equation model for the amount of money, S(t), owed at any time.

   (b) (3 points) Solve the model for S(t).

   (c) (1 point) Determine the maximum amount that this buyer can afford to borrow.

(a) $\frac{dS}{dt} = rS - K$

$= 0.09S - 9600$

$K = \$800/\text{month}$
$= \$800 \times 12 /\text{year}$

$r = 9\%$
$= 0.09$

$T = 20$ years.

$S(20) = 0$

$S(0) = ??$

(b) $S(t) = S(0)e^{rt} - \frac{K}{r}(e^{rt} - 1)$

For $t = T = 20$

$\Rightarrow S(20) = S(0)e^{(0.09)(20)} - \frac{9600}{0.09}(e^{(0.09)(20)} - 1)$

$\Rightarrow 0 = S(0)e^{1.8} - 106666.67(e^{1.8} - 1)$

$\Rightarrow S(0) = \frac{(106666.67)(5.049)}{6.049}$

$\boxed{S(0) = \$89,032.89}$