# Lab 5: Eigenvalue and Eigenvector Solutions

**Goals:**

1. Introduce generalized linear differential equation nomenclature
2. Relate Eigenvalue/Eigenvector structure to system properties such as stability and equilibrium convergence time.
3. Introduce how to form and simulate an Eigenvalue/Eigenvector solution

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## <u>Prelab:</u> Generalized System Nomenclature

Let's take our system from the prelab of Lab 4, as shown below:

$$\begin{aligned} x' &= -2x - y - 11 \\ y' &= -5x - 4y - 35 \end{aligned}, \quad \begin{aligned} x(0) &= 0 \\ y(0) &= 0 \end{aligned}$$

which we also note is a ***linear system***. We desire to find a way to optimize how we represent such systems of second order and higher. For linear systems specifically, we showed that we are always able to write them in the form of a ***matrix equation***. Specifically, if we try to write the above in the form of

$$XY' = [A]XY + [B]u$$

we have:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -11 \\ -35 \end{bmatrix} [1], \quad \begin{aligned} x(0) &= 0 \\ y(0) &= 0 \end{aligned}$$

The matrix **A** is known as the ***system matrix***. As we shall see, the ***eigenvalues*** and ***eigenvectors*** obtained from **A** directly relate to the Eigenvalue/Eigenvector solution of the system. The matrix **B** interprets how the input vector **u** is integrated into the dynamics of the system. The input for the above case has **u = 1** for **t ≥ 0**. This input is known as a ***step input*** and the corresponding solution is known as the ***step response*** of the system. Table 1 shows a few possible combinations of system inputs that we will be investigating over the remainder of the term.

**Table 1. System response types for different input types.**

| Name | Description | Input Representation |
|---|---|---|
| Natural Response | Response due to initial conditions of system | $u = 0 \quad \forall\, t \in \Re$ |
| Step Response | Response due to a step input | $u = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ |
| Impulse Response | Response due to an impulse | $u = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$ |

## <u>Prelab:</u> Eigenvalue/Eigenvector Structure and System Responses

From our work in lecture, we know that the eigenvalues of the **A** matrix can be directly related back to the system dynamics that are observed. To find the eigenvalues and eigenvectors of a matrix, we use the MATLAB function ***eig()***. Two usage examples are shown below.

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**<u>Ex 1A:</u>**

```
>> A = [13 5; 2 4]

A =

    13     5
     2     4

>> eig(A)

ans =

    14
     3
```

---

In Example 1A, the function ***eig()*** by itself returns the eigenvalues of the matrix. We can also use ***eig()*** to return the eigenvalues a set of eigenvectors as shown in the following.

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**<u>Ex 1B:</u>**

```
>> A = [13 5; 2 4];

>> [EigVects EigVals]=eig(A)

EigVects =

    0.9806   -0.4472
    0.1961    0.8944


EigVals =

    14     0
     0     3
```

---

In Example 1B, there are two outputs of the eigenvalue function. The first output (named here as ***EigVects*** returns a set of valid eigenvectors for the system as a set of column vectors. Remember, eigenvectors are not unique because any scalar multiple of an eigenvector is still an eigenvector. The eigenvectors produced by ***eig()*** are unique because they each have ***unit magnitude.*** The second output (named here as ***EigVals***) returns the system eigenvalues in a diagonal matrix. Notice that the first column in ***EigVects*** is the eigenvector that corresponds to the eigenvalue in the first diagonal position in ***EigVals***.

For more information on how ***eig()*** operates, reference the MATLAB help files and MATLAB product help.

Recall that for a linear system, the general form of the ***natural response*** of a system is given by:

$$XY = \sum_{i=1}^{n} c_i(\mathbf{v_i} e^{\lambda_i t})$$

where $\lambda_i$ and $\mathbf{v_i}$ correspond to an eigenvalue/eigenvector pair and the constants, $\mathbf{c_i}$, are numerical constants that are defined so that the initial conditions of the system are satisfied. For a ***second order system***, this simplifies to:

$$\boxed{XY = c_1(\mathbf{v_1} e^{\lambda_1 t}) + c_2(\mathbf{v_2} e^{\lambda_2 t})}$$

From the above, we see that the eigenvalues of a linear system correspond to the coefficients of the exponential functions in the system solution. With this in mind, we can now characterize the different system responses that may occur. Differential equations can ***converge to***, ***diverge from***, or ***oscillate around*** equilibrium points. The possible eigenvalue combinations that relate to these cases are shown below in Table 2.

**Table 2. Possible equilibrium types for different eigenvalues. For simplicity sake, eigenvalues are assumed to be distinct and non zero.**

| Eigenvalue Condition | Name of Response | Description of Response | Example |
|---|---|---|---|
| $\lambda_1, \lambda_2$ are Real<br>$\lambda_1, \lambda_2 < 0$ | Stable Node | System response converges to equilibrium | $\lambda_1 = -1$<br>$\lambda_2 = -2$ |
| $\lambda_1, \lambda_2$ are Real<br>$\lambda_1, \lambda_2 > 0$ | Unstable Node | System response diverges from equilibrium | $\lambda_1 = 1$<br>$\lambda_2 = 2$ |
| $\lambda_1, \lambda_2$ are Real<br>$\lambda_1 < 0, \lambda_2 > 0$ | Saddle Point | One component converges to equilibrium while the other components diverges from equilibrium. | $\lambda_1 = 1$<br>$\lambda_2 = -2$ |
| $\lambda_1, \lambda_2$ are Imaginary | Marginally Stable Node | Both components oscillate sinusoidally about the equilibrium point. | $\lambda_1 = 2i$<br>$\lambda_2 = -2i$ |
| $\lambda_1, \lambda_2$ are Complex | Spiral Node | The real part of the eigenvalues determines the stability of the node and the imaginary part brings about oscillation. | $\lambda_1 = -1+2i$<br>$\lambda_2 = -1-2i$ |

In addition to Table 2, the ***real parts*** of the ***eigenvalue magnitudes*** dictate ***the speed of the response*** and the ***imaginary parts*** of the ***eigenvalue magnitudes*** dictate ***the frequency of oscillation***.

**Prelab Assignment:**

Given the ***natural response*** of a system defined below as:

$$x' = Ax, \qquad A = \begin{bmatrix} \alpha & 2 \\ -2 & 0 \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

whose equilibrium is centered at the origin.

1. Vary the parameter alpha from -10 to 10 and determine the system eigenvalues. ***Use a for loop and save all of your results in memory***. Use the function ***eig()*** to help you find the eigenvalues for the system.

   *HINT: Save your results in a 2xN matrix, where each eigenvalue pair is stored in the Nth column, corresponding to the Nth value of alpha.*

2. Create a ***2 x 1 subplot*** to graph the eigenvalues you generated in (1). The first plot should be the real part of the eigenvalues vs. alpha and the second plot should be the imaginary part of the eigenvalues vs. alpha. If done correctly, you should see similar results to Figure 1.

   Make sure that you add a legend and label all axis for both plots. ***You do not have to add textboxes as shown.***

   *HINT: See the functions* ***real()*** *and* ***imag().***

3. Notice that the equilibrium point type changes 4 times between the given values of alpha. These regions are labeled on the graph in Figure 1. ***Pick one value of alpha from each region and simulate the phase portrait using PPLANE.*** You should have four graphs that correspond to the four regions shown in Figure 1.

   ***Each graph should have a single solution plotted that corresponds to the initial condition given above. DO NOT choose alpha = 0, as this will produce an oscillatory/semistable response.***

   Print out your PPLANE solutions and attach them as an addendum to your prelab report.

If done correctly, your eigenvalue vs. alpha plots should resemble those on the next page. You should have PPLANE plots that give corresponding systems responses as given in Table 2, however they will be specific to the values of alpha that you chose.

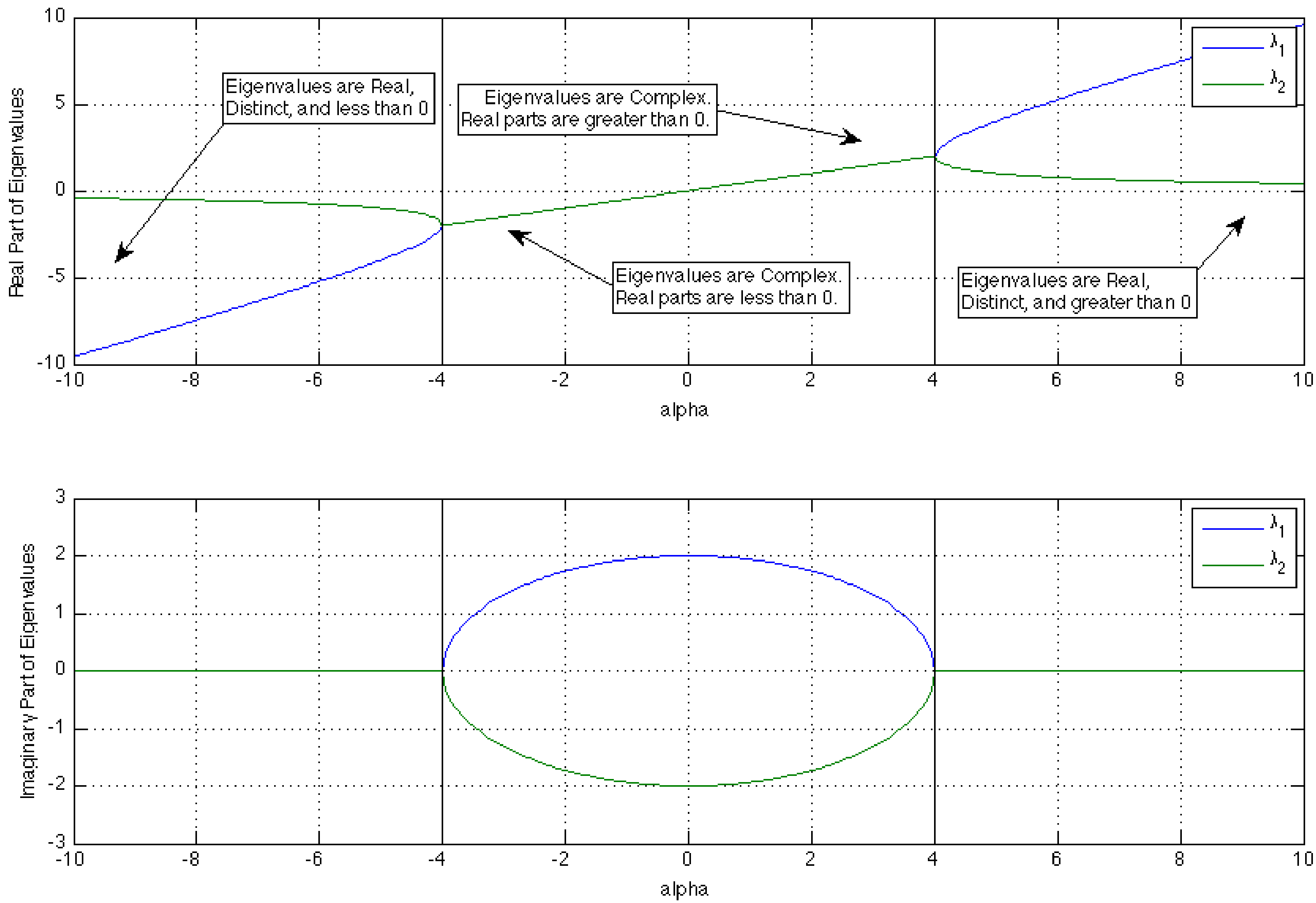


**Figure 1. Correct Results for Lab 5 Prelab. Figures resized and annotated for clarity.**

**<u>In Class Exercise:</u> Generating an Eigenvalue/Eigenvector Solution**

Recall that for a second order system, the analytical solution for the natural response is given by the following equation:

$$\boxed{XY = c_1(\mathbf{v_1}e^{\lambda_1 t}) + c_2(\mathbf{v_2}e^{\lambda_2 t})}$$

In the above, $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are ***vectors*** (2x1), whereas everything else is a scalar. Using the ***eig()*** function on some system matrix, it is possible for us to easily obtain all parameters of this equation except for the constants $\boldsymbol{c_2}$ and $\boldsymbol{c_2}$. However, these constants can be found easily through a matrix equation.

For any initial condition at time $\boldsymbol{t = 0}$, we have the following:

$$\mathbf{IC} = c_1(\mathbf{v_1}e^{\lambda_1(0)}) + c_2(\mathbf{v_2}e^{\lambda_2(0)}) = c_1\mathbf{v_1} + c_2\mathbf{v_2} = [\mathbf{v_1} \quad | \quad \mathbf{v_2}]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where ***IC*** is a column vector (2x1) of the initial conditions. In the last step, we have taken advantage of the fact that $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are vectors, so we have converted the system from a vector equation to a matrix equation. From here, we can just solve for the constants as follows:

$$\boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{v_1} \quad | \quad \mathbf{v_2}]^{-1}\mathbf{IC}}$$

The in class assignment this week will involve simulating a second order system two ways: via ODE45 and through the analytical solution. You should be able to do the following:

1. Formulate the differential equation and solve it via ODE45 as done in ***Lab 4***

2. Implement the analytical solution as described above. Specifically, this involved the following steps:
    a. Find the eigenvalues and eigenvectors of the system matrix through the function ***eig()***.
    b. Use the boxed equation above to find the constants of integration based on the intial conditions.
    c. Use the values from (2a) and (2b) to implement the equations of line via the techniques shown in ***Lab 1***.

ENGR 232
Dynamic Engineering Systems

Lab 5 – Eigenvalue and Eigenvector Solutions
**Verification Sheet**

Student name_______________________________________________Section_____________

1. **Prelab Assignment –** Eigenvalue/Eigenvector Structure (6 points total)
**STAPLE YOUR PRELAB TO THIS VERIFICATION SHEET WHEN SUBMITTING!**

Successful Generation of Eigenvalues (2 pts) ________

Correctly Labeled and Annotated plots (1 pts) ________

Quality of Published Report (1 pts) ________

PPLANE Generated Solutions Addendum (2 pts) ________

Verified ______________________________Date/Time _______________

2. **In Class Assignment –** Eigenvalue/Eigenvector Solutions (4 points total)

In Class Assignment (3 pts)

Prelab Question (1 pts) ________

Verified ______________________________Date/Time _______________

***TOTAL POINTS*** ___________ / 10 pts