Applied Quantitative Logistics

Well-Solved Problems Part I: Network Flows

Simulated Annealing (SA)

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January 31, 2025

Properties of Easy Problems

For the moment we will be very imprecise and say that an algorithm on a graph $G = (V, E)$ with n nodes and m edges is **efficient** if, in the **worst** case, the algorithm requires $O(m^p)$ elementary calculations (such as additions, divisions, comparison, etc) for some integer p, where we assume that $m \geq n$.

Properties of Easy Problems

Consider the following combinatorial optimization problem

$$\text{COP} \qquad \min\{cx : x \in X \subseteq R^n\}$$

Definition: The Separation Problem

The Separation Problem associated with COP is the problem: Given a point $x^* \in R^n$, is $x^* \in conv(X)$? If not find an inequality $\pi x \leq \pi_0$ satisfied by all points in X, but violated by the point x^*.

Properties of Easy Problems

When examining a problem to see if it has an efficient algorithm, we will see that the following four properties often go together:

- **Efficient Optimization Property**: for a given class of problems P, there exists an efficient (polynomial) algorithm.
- **Strong Dual Property**: For the given problem class, there exists a strong dual problem $(D)\ max = \{w(u) : u \in U\}$ allowing us to obtain optimality conditions that can be quickly verified: $x^* \in X$ is optimal in P if and only if there exists $u^* \in U$ with $cx^* = w(u^*)$.
- **Efficient Separation Property**: there exists an efficient algorithm for the separation problem associated with the problem class.
- **Explicit Convex Hull Property**: A compact description of the convex hull $conv(X)$ is known, which in principle allows us to replace every instance by the linear program $\min\{cx : x \in conv(X)\}$.

Properties of Easy Problems

When considering integer programming:

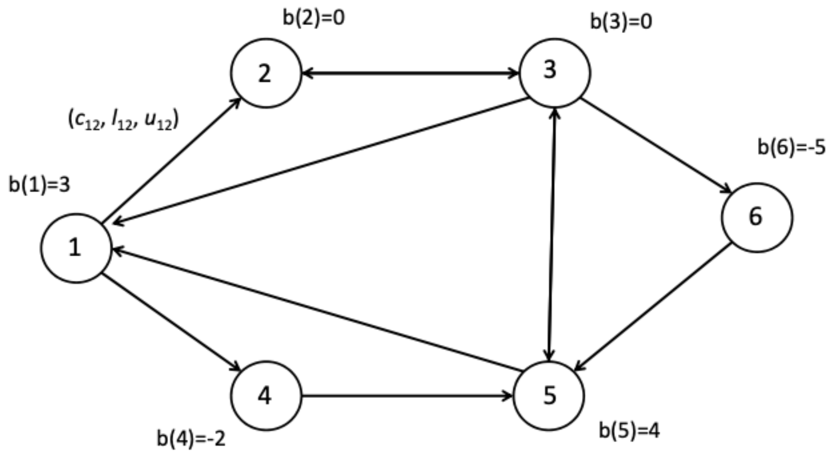
$$\text{(IP)} \qquad min\{cx: Ax \leq b, x \in Z_+^n\},$$

With integral data (A, b), it is important to know when the linear programming relaxation

$$\text{(LP)} \qquad min\{cx: Ax \leq b, x \in R_+^n\},$$

Will have an optimal solution that is integral.

Minimum Cost Flow Problem



Input:

- $G = (N, A)$: directed graph
- c_{ij}: cost per unit flow on $(i\ j) \in A$
- h_{ij} : maximum flow on $(i\ j) \in A$
- b_i: supply / demand of $i \in N$

The minimum Cost Flow Problem:

- If bi > 0, node i is a supply node, bi < 0, node i is a demand node, and bi = 0, node i is a transshipment node
- Objective: minimize the total flow cost for sending the products from the suppliers to the customers through the network
- Capacity constraints on the arcs must me respected

Minimum Cost Flow Problem

Input:

- x_{ij}: amount of flow in arc $(i\ j) \in A$

The minimum cost flow problem can be formulated as:

$$\text{minimize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{j\in V^+(i)} x_{ij} - \sum_{j\in V^-(i)} x_{ji} = b_i \qquad i \in N \qquad (1)$$

$$0 \le x_{ij} \le h_{ij} \qquad (i,j) \in A \qquad (2)$$

- Constraints (1) are known as flow conservation constraints
- A feasible solution will exist if $\sum_{i\in N} b_i = 0$

Minimum Cost Flow Problem

Proposition:

The constraint matrix A arising in a minimum cost network flow problem is totally unimodular.

Corollary:

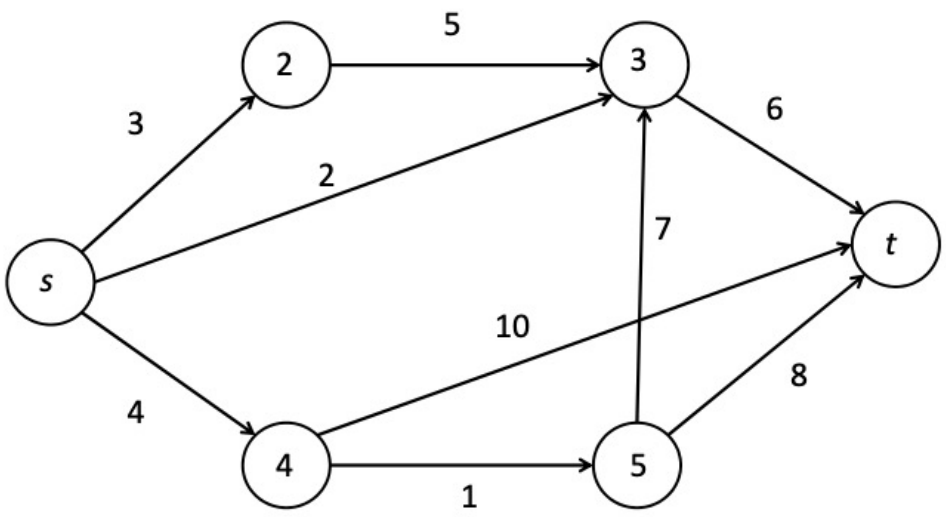
In a minimum cost network flow problem, if the demands bi and the capacities hij are integral,

- Each extreme point is integral
- The constraints describe the convex hull of integral feasible flows

Theorem: (Pioncare, 1900)

Let A be a ± 1 valued matrix, where each column has at most one $+1$ and at most -1. Then A is totally unimodular.

Shortest Path Problem



Input:

- $G = (N, A)$: directed graph
- $c_{ij} \geq 0$: length of arc $(i\ j) \in A$
- s: source node
- t: sink node

The Shortest Path Problem:

- Objective: Find the shortest path between node s and node t

Shortest Path Problem

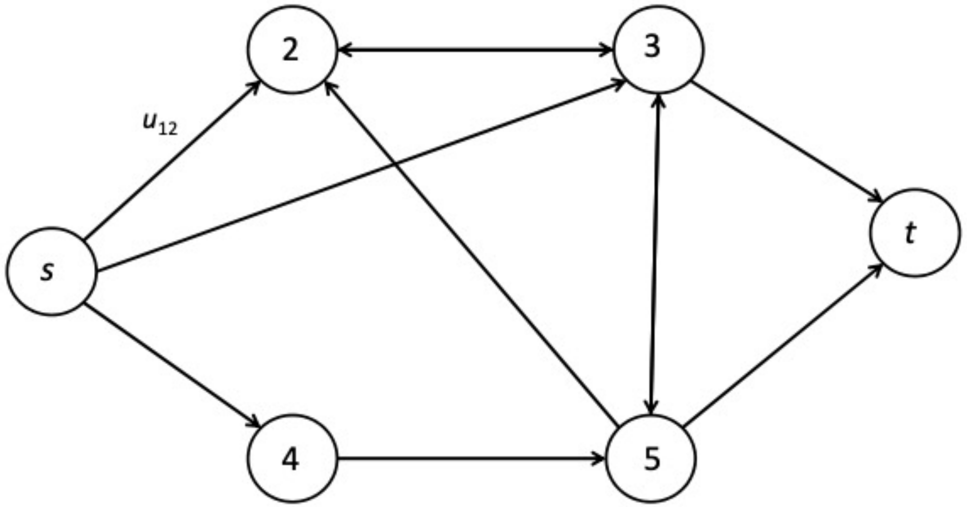
Path variables

$$x_{ij} = \begin{cases} 1 & if\ arc\ (i,j) \in A\ is\ in\ the\ shortest\ s-t\ path; \\ 0 & Otherwise \end{cases}$$

The shortest path problem can be formulated as:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j\in V^+(i)} x_{ij} - \sum_{j\in V^-(i)} x_{ji} = 1 \qquad i = s \\
& \sum_{j\in V^+(i)} x_{ij} - \sum_{j\in V^-(i)} x_{ji} = 0 \qquad i \in N \setminus \{s,t\} \\
& \sum_{j\in V^+(i)} x_{ij} - \sum_{j\in V^-(i)} x_{ji} = -1 \qquad i = t \\
& x_{ij} \in \{0,1\} \qquad (i,j) \in A
\end{aligned}$$

The Maximum Flow Problem



Input:

- $G = (N, A)$: directed graph
- h_{ij}: maximum flow on $(i\ j) \in A$
- s: source node
- t: sink node

The maximum $s - t$ Flow Problem:

- Objective: Find the maximum flow between node s and node t that satisfies the arc capacities

The Maximum Flow Problem

Flow variables

- x_{ij} = amount of flow in arc $(i, j) \in A$
- v = flow sent from source node s to sink node t

The maximum $s - t$ flow problem can be formulated as:

$$
\begin{array}{lll}
\text{maximize} & v & \\
\text{subject to} & \displaystyle\sum_{j \in V^+(i)} x_{ij} - \sum_{j \in V^-(i)} x_{ji} = v & i = s \\
& \displaystyle\sum_{j \in V^+(i)} x_{ij} - \sum_{j \in V^-(i)} x_{ji} = 0 & i \in N \setminus \{s, t\} \\
& \displaystyle\sum_{j \in V^+(i)} x_{ij} - \sum_{j \in V^-(i)} x_{ji} = -v & i = t \\
& 0 \leq x_{ij} \leq h_{ij} \qquad (i, j) \in A &
\end{array}
$$

The Maximum Flow Problem

Adding a backward arc from t to s, the maximum $s-t$ flow problem can be formulated as:

$$\begin{array}{lll} \text{maximize} & x_{ts} & \\ \text{subject to} & \displaystyle\sum_{j\in V^+(i)} x_{ij} - \sum_{j\in V^-(i)} x_{ji} = 0 & i \in N \\ & 0 \leq x_{ij} \leq h_{ij} & (i,j) \in A \end{array}$$

The dual is:

$$\begin{array}{lll} \text{minimze} & \displaystyle\sum_{(i,j)\in A} h_{ij} w_{ij} & \\ \text{subject to} & u_i - u_j + w_{ij} \geq 0 & (i,j) \in A \\ & u_t - u_s \geq 1 & \end{array}$$

Simulated Annealing

Refer to the notes

Thank you!

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